Bradshaw - Dokchitser - 041309

Dokchitser's Algorithm

consider a motivic L-function

Assume it satisified $L(s) = \epsilon L^*(w - s)$ where

$$L^*(s) = \left(\frac{\sqrt{N}}{\pi^{d/2}}\right)^s \prod_{i=1}^{d} \Gamma\left(\frac{s+\lambda_i}{2}\right) L(s)$$

where

$$L(s) = \sum \frac{a_n}{n^s}$$

converges for $\mathrm{Re}(s) >> 0$.

Assume $L*$ is meromorphic, and has finitely many poles.

EXAMPLES

	w	d	λ_i	N	ϵ	poles
$\zeta(s)$	1	1	0	1	1	1
\zeta_K(s)	1	$[K:\mathbf{Q}]$	0,..,0, 1, ..., 1 (r_1 0's and $2r_2$ 1's)	$\lvert D_F\rvert$	1	1
L(E,s)	2	2	$0, 1$	N	\pm 1	

Why? Can use to compute root number, conductor, etc.

Discussion about poles: The algorithm takes as input the finitely many poles.

Mellin Transforms

Given $f(t): R^+ \to R$, the mellin transform of $f(t)$ is

$$g(s) = \int_0^\infty f(t) t^s \frac{dt}{t}.$$

The inverse Mellin transform is

$$f(t) = \int_{c-i\infty}^{c+i\infty} g(s) t^{-s} ds$$

for any c with $\mathrm{Re}(c) >$ poles of$g$.

Let $\gamma(s) = \prod_{j=1}^{d} \Gamma\left(\frac{s+\lambda_j}{2}\right)$.

Define $\phi(t)$ to be the inverse Mellin transform of $\gamma(s)$, which is

$$\gamma(s) = \int_0^\infty \phi(t) t^s \frac{dt}{t}$$

Let

$$\Theta(t) = \sum_{n=1}^{\infty} a_n \phi(\frac{nt}{A})$$

where $A = \pi^{d/2}$.

We have

$$\int_0^\infty \theta(t) t^s \frac{dt}{t} = \int_0^\infty \sum_{n=1}^\infty a_n \phi(\frac{nt}{A}) t^s \frac{dt}{t} = \sum_{n=1}^\infty a_n \int_0^\infty \phi(t) (\frac{At}{n})^s \frac{dt}{t} = \sum_{n=1}^\infty A^s \frac{a_n}{n^s} \int_0^\infty \phi(t) t^s \frac{dt}{t} = L^*(s).$$

We have

$$\theta(\frac{1}{t}) = \int_{c-i\infty}^{c+i\infty} L^*(s) t^s ds = t^{-w} \int_{c-i\infty}^{c+i\infty} \epsilon L^*(w-s) t^{s-w} ds = t^w \epsilon \int_{w-c+i\infty}^{w-c-i\infty} L^*(s) t^{-s} ds = t^w \epsilon \theta(t) \; - \sum_j r_j t^{P_j}$$

where we sum over the poles p_j with residues r_j.

Now we compute $L^*(s)$ in terms of θ.

$$L^*(s) = \int_0^\infty \theta(t) t^s \frac{dt}{t} = \left(\int_0^1 + \int_1^\infty \right) \theta(t) t^s \frac{dt}{t} = \int_1^\infty \theta(t) t^s \frac{dt}{t} + \int_1^\infty \theta(\frac{1}{t}) t^{-s} \frac{dt}{t} = \int_1^\infty \theta(t) t^s \frac{dt}{t} + \int_1^\infty t^w \epsilon \theta(t) - \epsilon \sum r_j t^{P_j} t^{-s} \frac{dt}{t} \; = \sum_1^\infty \theta(t) t^s \frac{dt}{t}.$$

Let

$$G_s(t) = t^{-s} \int_t^\infty \phi(x) x^s \frac{dx}{x}$$

Then with some more similar manipulation we get

$$\int_1^\infty \theta(t) t^s \frac{dt}{t} = \cdots = \sum_{n=1}^\infty A^s a_n G_s(n/A).$$

So computing L boils down to computing these G-functions.

Computing $G_s(t)$:

Note that $G_s(-)$ is a function of s and the λ_j.

Meier G-function: $\phi(t) = 2G_{0,d}^{d,0}(t^2;;\frac{\lambda_j}{2})$.

Using $s\Gamma(s) = \Gamma(s+1)$ we get horrendous recurrance involving 5-level nested sums, etc. Can do explicitly.

For small t, we have $G_s(t) = \frac{\gamma(s)}{t^s} - F_s(t)$, where $F_s(t) \in \mathbf{C}[\log(t)][[t]]$.

Problem with this approach: $F_s(t)$ behaves like the Taylor series of e^{-t}. Horrible cancellation. Have to do things to very high precision. Bad.

```
show(exp(-x).taylor(x,0,15))
```

$$1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{5040} + \frac{x^8}{40320} - \frac{x^9}{362880} + \frac{x^{10}}{3628800} - \frac{x^{11}}{39916800} + \frac{x^{12}}{479001600} - \frac{x^{13}}{6227020800} + \frac{x^{14}}{87178291200} - \frac{x^{15}}{1307674368000}$$

Better idea to compute $G_s(t)$?

$$G_s(t^{d/2}) \sim \frac{(2\pi)^{(d-1)/2}}{\sqrt{d}} e^{-dt} t^{k-1} \sum_{n=1}^{\infty} \mu_n(s) t^{-n}$$

and

$k = (1 - d + \sum \lambda_j)/2$.

Example. For an elliptic curve E, we have $\lambda_1 = 0, \lambda_2 = 1$, and $\mu_n(s) = \prod_{a=1}^{n} \frac{(s-a)}{2}$ ... so in fact the above formula for G_s is a divergent series!!!

The above is an *asymptotic expansion*, so instead of comparing as $n \to \infty$, compare as $x \to \infty$ for fixed n:

$$F(x) \sim \sum_{n=1}^{\infty} a_n x^{-n}$$

if and only if for each n,

$$\lim_{x \to \infty} \left(F(x) - \sum_{n=1}^{N} a_n x^{-n} \right) x^N = 0$$

Given $\sum M_N x^n$ as a formal power series, consider the continued fraction $b_0 + \frac{x^{k_0}}{b_1 + \frac{x^{k_1}}{b_2 + \frac{x^{k_2}}{\cdots}}}$.

Let $c_n(x)$ be the truncated continued fraction. Then $c_n(x)$ "converges" "quickly" "to $\Psi_s(t)$".