LATTICES IN VECTOR SPACES OVER $\mathbb{R}$, $\mathbb{C}$, AND $\mathbb{H}$

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1. Introduction

Sphere packings in n-dimensional Euclidean space are configurations of congruent non-overlapping spheres. Sphere packings in which the sphere centers form a lattice are referred to as lattice packings. The best or densest packings are characterized by having a packing density, the proportion of space occupied by the spheres, maximal among all sphere packings in the same dimension. The problem of finding and proving optimality of any lattice packing remains open for dimensions larger than eight except for dimension 24 [CS].

For the dimensions in which lattice packings have been found and proven optimally dense, many of the lattices of sphere centers have an additional algebraic structure. This structure allows them to be viewed as Eisenstein lattices in $\mathbb{C}^n$ or Hurwitz lattices in $\mathbb{H}^n$, $\mathbb{C}$ and $\mathbb{H}$ denoting the complex numbers and the Hamiltonian quaternions respectively. Such lattices include the Leech lattice Λ_{24}, Barnes-Wall lattice Λ_{16}, Coxeter-Todd lattice Λ_{12}, and the root lattices D_4, E_8. Given the algebraic structure exhibited by these optimal low-dimensional lattice packings, an interesting sphere packing problem variant is to determine the densest Eisenstein and Hurwitz lattice packings in $\mathbb{C}^n$ and $\mathbb{H}^n$ respectively. However, to address this new problem we need to define what it means to be a such a one of these two types of lattices. Ideally the definition of lattices in these spaces would also generalize to include the lattices in $\mathbb{R}^n$ over $\mathbb{Z}$, referred to in this paper as $\mathbb{Z}$-lattices.

This paper is divided into two parts. The first half of this paper only pertains to $\mathbb{Z}$-lattices in $\mathbb{R}^n$. Here some basic definitions and propositions are presented for such lattices. In the second half of this paper, a general construction of lattices in $\mathbb{R}^n$, $\mathbb{C}^n$, and $\mathbb{H}^n$ is presented. This construction follows one outlined in Jacques Martinet's book, *Perfect Lattices in Euclidean Spaces*, and will be used to explicitly define Eisenstein and Hurwitz lattices.

2. $\mathbb{Z}$-Lattices in Euclidean spaces

Throughout this section E will always denote a n-dimensional real vector space endowed with a Euclidean structure. Recall that a Euclidean structure on E is just a positive definite bilinear form $(x, y) \to \langle x, y\rangle$, commonly referred to as a scalar product.

Definition 2.1. A *$\mathbb{Z}$-lattice* in E is a free sub-$\mathbb{Z}$-module with basis $\{b_1, ..., b_n\}$ which generates E as a vector space over $\mathbb{R}$. A $\mathbb{Z}$-lattice in any subspace of E is called a *relative $\mathbb{Z}$-lattice.* If Λ is a $\mathbb{Z}$-lattice with the property that $\langle x, y\rangle \in \mathbb{Z}$ for every pair of lattice vectors x and y, then Λ is called an *integral $\mathbb{Z}$-lattice.*

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Due to a result commonly referred to as the Jacobi-Bravais theorem, relative $\mathbb{Z}$-lattices can be characterized completely as discrete subgroups of E with rank at most n, E endowed with its natural Euclidean topology [Ma]. This characterization is useful in lattice sphere packing problems because in order for a lattice to even define a sphere packing with non-zero density, the lattice points must be discrete.

2.1. **Lattice Norms and Determinants.** There is a natural norm defined on E via its Euclidean structure. For each vector in v in E, its norm is defined to be the non-negative real number, $\mathrm{N}(v) = \langle v, v\rangle$.

Definition 2.2. Let Λ be a $\mathbb{Z}$-lattice in E. A *minimal vector* in Λ is a non-zero vector contained in Λ of minimal norm. The *norm of* Λ , denoted by $\mathrm{N}(\Lambda)$, is the quantity,

$$\mathrm{N}(\Lambda) = \min_{v\in\Lambda\setminus\{0\}}\mathrm{N}(v).$$

The norm of any lattice is the length of its minimal vectors. This is a well defined non-zero quantity because the points of a lattice are discrete in the topology defined by the vector norm N defined on E. In context of the sphere packing problem, the densest sphere packing determined by a $\mathbb{Z}$-lattice Λ is the packing of congruent spheres with radius $\frac{1}{2}\mathrm{N}(\Lambda)$ whose centers are the lattice points of Λ. This particular sphere packing called the *sphere packing determined by* Λ and its packing density is called the *density of* Λ. The density of a lattice can be computed in terms of its lattice norm and the volume of a fundamental parallelotope, the latter quantity being defined below.

Definition 2.3. Let Λ be a $\mathbb{Z}$-lattice with basis $B = \{b_1, ..., b_n\}$. The *fundamental parallelotope of* Λ *with respect to* B is the set,

$$P = \{\sum_i \alpha_i b_i : 0 \le \alpha_i < 1\}.$$

One interesting property of P in the definition above is that the Euclidean space E can be tiled with infinitely many copies of P. More explicitly,

$$E = \coprod\nolimits_{x\in\Lambda}\{x + p : p \in P\}.$$

Observe that this tiling of E is dependent on the basis B for Λ. However, the volume of each fundamental region is not [Eb]. This leads to a nice method for computing the packing density of Λ, denoted by $\rho(\Lambda)$, with only its lattice norm, the volume of P, and the volume V_n of the n-dimensional unit sphere [CS]. The packing density of Λ is the quantity,

$$\rho(\Lambda) = \frac{\mathrm{N}(\Lambda)^n V_n}{\mathrm{Volume}(P)}.$$

Let Λ be the $\mathbb{Z}$-lattice in definition 2.3. The volume of P used in the formula for $\rho(\Lambda)$ can be easily computed by taking the determinant of a matrix in $\mathrm{GL}_n(\mathbb{R})$. To compute this value, first fix an orthonormal basis for E and construct a matrix $M \in \mathrm{GL}_n(\mathbb{R})$ whose i^{th} row is the vector b_i represented in this basis. This matrix M is called the *generating matrix for* Λ. Observe that the construction of M is dependent on the basis B for Λ and the choice of orthonormal basis for E. Thus M may not be unique as *the generating matrix for* Λ. However, M is unique up to conjugation by a matrix in $GL_n(\mathbb{Z})$ as a generating matrix for Λ [Ma]. Hence the the any volume fundamental parallelotope P is unique and is this is the absolute value of the determinant of any generating matrix M. To avoid absolute values (and for simplifying proofs of propositions), the determinant of the gram matrix,

$$(\langle b_i, b_j\rangle)_{1\leq i,j\leq n} = MM^t,$$

is usually computed instead to calculate the squared volume of P. This quantity is called the *determinant of* Λ and will be denoted by $\det(\Lambda)$.

2.2. **Duality.** For the $\mathbb{Z}$-lattice Λ in E, using Euclidean structure of E one can define a new lattice Λ^* by setting,

$$\Lambda^* = \{x \in \mathbb{R}^n : \langle x, \Lambda\rangle \subseteq \mathbb{Z}\}.$$

The lattice Λ^* is called the *dual lattice of* Λ. This lattice may also be constructed by finding a $\mathbb{Z}$-basis B for Λ and then computing its dual as a basis for E. The dual basis B^* for E is then a $\mathbb{Z}$-basis for Λ^*. Observe that since $(B^*)^* = B$, the lattice $(\Lambda^*)^*$ is just the original lattice Λ. The following proposition is provides a useful relation between the determinants of lattice and its dual.

Proposition 2.4. *Let Λ be a $\mathbb{Z}$-lattice in n-dimensional Euclidean space E. The real numbers* $\det\Lambda$ *and* $\det\Lambda^*$ *satisfy the equation,*

$$\det(\Lambda)\det(\Lambda^*) = 1.$$

Proof. Let M_1 be a generating matrix for Λ (with respect to any fixed orthonormal basis for E). Let M_2 be the matrix whose rows are the dual basis vectors corresponding to each of the row vectors in M_1. The matrix M_2 is a generating matrix for the dual lattice Λ^*. Moreover, the inner product of a row vector vector b_i of M_1 with a row vector ${b_j}^*$ in M_2 is equal to $\delta_{i,j}$. Thus the two generating matrices satisfy the equation $M_1{M_2}^t = I_n$. Now since $\det(\Lambda) = \det(M_1{M_1}^t)$ and $\det(\Lambda^*) = \det(M_2{M_2}^t)$, using the properties of the determinant function,

$$\begin{aligned}
\det(\Lambda)\det(\Lambda^*) &= \det(M_1{M_1}^t)\det(M_2{M_2}^t)\\
&= \det(M_1)\det({M_1}^t)\det(M_2)\det({M_2}^t)\\
&= \det(M_1{M_2}^t)^2\\
&= 1
\end{aligned}$$

□

2.3. **Sub-Lattices.** Continue to let Λ denote a $\mathbb{Z}$-lattice in E. Observe that any sub-$\mathbb{Z}$-module $\Lambda' \subseteq \Lambda$ of free rank r satisfies the definition of a relative lattice in E. Moreover, the relative lattice Λ' is a lattice in the r-dimensional Euclidean space generated by its lattice vectors and is called a *sub-lattice* of Λ. Sub-lattices are a useful tool when working with larger lattices.

For the following series of propositions, let Λ be a $\mathbb{Z}$-lattice in the the n-dimensional Euclidean space E and let F be any r-dimensional subspace of E. It is always the case that the set of vectors in $\Lambda \cap F$ form a relative lattice in both E and F. (This is because Λ is a discrete sub-$\mathbb{Z}$-module in E and so Λ' is a discrete sub-$\mathbb{Z}$-module in F). However, $\Lambda \cap F$ is not always a (full) lattice in F. The following is an attempt to give some criteria for when $\Lambda \cap F$ is a lattice in F. The statements of the propositions can be found in [Ma]

Lemma 2.5. *Let $\{b_1, ..., b_{r'}\}$ be a basis for $\Lambda \cap F$ as a $\mathbb{Z}$-lattice. This basis can which can be extended to a basis $\{b_1, ..., b_{r'}, b_{r'+1}, ..., b_n\}$ for Λ.*

Proof. Let Λ' denote the relative $\mathbb{Z}$-lattice $\Lambda \cap F$ in F. Consider the quotient module Λ/Λ' which is a finitely generated torsion free $\mathbb{Z}$-module and hence a free $\mathbb{Z}$-module of finite rank $n - r'$. Any basis for Λ' can be extended to a basis for Λ by adjoining the vectors obtained by lifting in Λ a basis for Λ/Λ'. □

Let $F^{\perp}$ denote the subspace perpendicular to F with respect to the Euclidean structure endowed on E. The projection of Λ onto this subspace will be denoted by $\pi_{F^{\perp}(\Lambda)}$. This set $\pi_{F^{\perp}(\Lambda)}$will not always be a discrete subset of $F^{\perp}$ and hence in not necessarily a lattice in F^{perp}.

Proposition 2.6. *The relative lattice $\Lambda \cap F$ is a lattice in F if and only if $\pi_{F^{\perp}}(\Lambda)$ is a lattice in $F^{\perp}$*

Proof. Let r' denote the rank of the kernel of the projection $\pi_{F^{\perp}}(\Lambda)$ as a free $\mathbb{Z}$-module. In particular r' is the rank of the relative lattice $\Lambda \cap F$ in E with $r' \leq r$. The projection $\pi_{F^{\perp}}(\Lambda)$ is discrete if and only if $r' \leq n - r$ which happens if and only if $r' = r$. In particular, $\Lambda \cap F$ is a free sub-$\mathbb{Z}$-module in F of rank r if and only if $\pi_{F^{\perp}}(\Lambda)$ has rank $n - r$ as a free sub $\mathbb{Z}$-module in $F^{\perp}$ and so the proposition now follows. □

Proposition 2.7. *$\Lambda \cap F$ is an $\mathbb{Z}$-lattice in F if and only if $\Lambda^* \cap F^{\perp}$ is a $\mathbb{Z}$-lattice in $F^{\perp}$*

Proof. First suppose that $\Lambda \cap F$ is an $\mathbb{Z}$-lattice in F. Let $B^* = \{b_1, ..., b_n\}$ be a basis for Λ such that $\{b_1, ..., b_r\}$ is a basis for $\Lambda \cap F$. The $n - r$ vectors $\{b^*_{r+1}, ..., b^*_n\}$ of the dual basis B^* belong to the relative $\mathbb{Z}$-lattice $\Lambda^* \cap F^{\perp}$ and since $F^{\perp}$ is a $(n - r)$-dimensional real vector space, $\Lambda^* \cap F^{\perp}$ is a lattice in $F^{\perp}$.

Now using the fact that $(\Lambda^*)^* = \Lambda$ and interchanging the roles of Λ and Λ^* in the above paragraph the converse now follows. □

3. Lattices in Vector Spaces over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$

Throughout this section K will always denote a skew field of either real, complex, or quaternionic numbers, and E will denote a n-dimensional vector space over K. All of the three choices for K are finite-dimensional $\mathbb{R}$-algebras on which we can define an anti-involution map, $x \mapsto \overline{x}$. If $K = \mathbb{R}$ this map will simply be the identity on K. If $K = \mathbb{C}$ for each $x = a + bi$ in K, define $\overline{x} = a - bi$. Finally, if $K = \mathbb{H}$ for each $x = a + bi + cj + dk$ in K, define $\overline{x} = a - bi - cj - dk$. Using this anti-involution on K a Hermitian structure can be endowed on E as a vector space over K. This is done by fixing any basis for E and naturally extending the the multiplication map $h(x, y) = x\overline{y}$ defined on $\mathbb{H}$.

Recall that the $\mathbb{Z}$-lattices in the previous section were defined with respect to a Euclidean structure on the ambient space. The same will be done for E in this section. If r is the rank of K as a finite dimensional $\mathbb{R}$-algebra, E has an algebraic structure as a $(2r)$-dimensional real vector space. Let $f : K \to \mathbb{R}$ be a linear functional on K such that the map

$$T_f : K \times K \longrightarrow \mathbb{R}$$

$$(x, y) \longmapsto f(x\bar{y})$$

defines an Euclidean structure on K as a real vector space. Then fix any basis $\{b_1, ...b_n\}$ for E and define a Euclidean structure T by

$$T(\sum_i \alpha_i b_i, \sum_i \beta_i b_i) = \sum_i T_f(\alpha_i, \beta_i)$$

Note that the trace form on E as a vector space over $\mathbb{R}$ may be used to define T.

Next, let $\mathcal{O}$ be an order in K which is invariant under the anti-involution, i.e $\overline{\mathcal{O}} = \mathcal{O}$. An order in K is a subring which is also is a free $\mathbb{Z}$-module with $rank_{\mathbb{Z}}\mathcal{O} = rank_{\mathbb{R}}K$. Observe that any order $\mathcal{O}$ in K is a $\mathbb{Z}$-lattice with respect to the Euclidean structure T endowed on K as an (rn)-dimensional real vector space.

Now we are finally ready to give a formal definition of a lattice in a vector space over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

Definition 3.1. Let K be either $\mathbb{R}$, $\mathbb{C}$, or the skew field $\mathbb{H}$ and let $\mathcal{O}$ be an order in K. Let E be a n-dimensional vector space over K with a Euclidean structure T constructed as above. A *$\mathcal{O}$-lattice* is a free sub-$\mathcal{O}$-module of E with free basis $\{b_1, b_2, ..., b_n\}$ which generates E as a vector space over K. If Λ is a *$\mathcal{O}$-lattice* in any subspace of E, Λ is called a *relative $\mathcal{O}$-lattice* in E.

The definition given above for $\mathcal{O}$-lattices is a natural generalization of the definition previously given for $\mathbb{Z}$-lattices in any n-dimensional Euclidean space. Definition 2.1 is recovered by setting $K = \mathbb{R}$ and $\mathcal{O} = \mathbb{Z}$. Also, since any order $\mathcal{O}$ is a $\mathbb{Z}$-module, any $\mathcal{O}$-lattice in E has the structure as a $\mathbb{Z}$-lattice. So as in the previous section a $\mathcal{O}$-lattices, we can define the to use the invariants $\mathrm{N}(\Lambda), \det(\Lambda), and \rho(\Lambda)$ for Λ to just be those previously defined.

Two interesting types of $\mathcal{O}$-lattices in $\mathbb{C}^n$ and $\mathbb{H}^n$ which contain optimally dense lattice packings are the Eisenstein lattices and the Hurwitz lattices. Each of these two types of lattices are described in more detail below.

3.1. **Eisenstein Lattices.** In $\mathbb{C}$, the *Eisenstein integers* are the elements in the subring,

$$\mathcal{E} = \{a + (\frac{1 - i\sqrt{3}}{2})b : a, b \in \mathbb{Z}\}.$$

This subring is a maximal order in $\mathbb{C}$ (with respect to inclusion). If E is a n-dimensional complex vector space, an *Eisenstein lattice* is a $\mathcal{E}$-lattice in E. Examples of Eisenstein lattices include the hexagonal lattice, the 4-dimensional root lattice D_4, the 12-dimensional Coxeter-Todd lattice, the 16-dimensional Barnes-Wall lattice, and the 24-dimensional Leech lattice. Each of these examples is proven or conjectured to achieve optimal density in the lattice sphere packing problem [CS].

3.2. **Hurwitz Lattices.** In the skew field $\mathbb{H}$, the Hurwitz integers are the elements in the subring,

$$\mathcal{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in \mathbb{Z} + \frac{1}{2}\}$$

The subring $\mathcal{H}$ is a maximal order in $\mathbb{H}$. If E is a now a n-dimensional quaternionic vector space, a *Hurwitz lattice* in E is a $\mathcal{H}$-lattice. Examples of Hurwitz lattices include the 4-dimensional root lattice D_4, the 8-dimensional root lattice E_8, the 16-dimensional Barnes-Wall lattice, and the 24-dimensional Leech lattice. As stated above, each of these examples is proven or conjectured to achieve optimal density in the lattice sphere packing problem [CS].

3.3. **Duality.** Let Λ be a $\mathcal{O}$-lattice for some order $\mathcal{O}$ in K. As a $\mathbb{Z}$-lattice, Λ possess a dual lattice $\Lambda^* = \{x \in E : \langle x, \Lambda\rangle \subseteq \mathbb{Z}\}$. However the construction of the dual lattice Λ^* does not use any of the additional algebraic structure that Λ has as a $\mathcal{O}$-lattice. This leaves the possibility that Λ^* may not even be a $\mathcal{O}$-lattice. Below is a more general defition for dual lattices which does incorporate the $\mathcal{O}$-lattice structure of Λ.

Definition 3.2. Let Λ be the $\mathcal{O}$-lattice in definition 3.1. The *dual $\mathcal{O}$-lattice of* Λ is the $\mathcal{O}$-lattice

$$\Lambda = \{x \in E : T(x, \Lambda) \subseteq \mathcal{O}\}.$$

If Λ is a $\mathbb{Z}$-lattice in a real vector space, definition 2.1 and definition 3.1 for its dual yield the same lattice Λ^*. Also, the construction using dual bases to define a dual $\mathbb{Z}$-lattices in the previous section applies to this general setting. However in the quaternionic case, one has to take care to consistently use left (or right) bases and their corresponding left (or right) dual bases.

3.4. **sub-lattices.** Sub-lattices of $\mathcal{O}$-lattices are defined the same way as sub-lattices of $\mathbb{Z}$-lattices were in the previous section. The only modification needed is to replace $\mathbb{Z}$ with $\mathcal{O}$. With this change the lemma and two propositions for sub-lattices of $\mathbb{Z}$-lattices still hold for sub-lattices of $\mathcal{O}$-lattices.

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