Zeta Functions of Number Fields and the Class Number Formula

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I. Introduction

The primary goal of this paper is to introduce the reader to the concept of the zeta function of a number field and describe some applications of these functions in number theory. One of the most well known applications is the analytic class number formula which expresses a relationship between several invariants of a number field and the residue of the pole of the zeta function of the field. In particular, we will outline a proof of the class number formula and look at specific applications of the formula quadratic extensions of the rationals.

II. Dirichlet Series and the Zeta Function of a Number Field

This section assumes some knowledge of complex variables. Much of what is discussed here is found detailed in [Gam01].
Zeta functions of number fields occur as a special case in a more general class of functions known as Dirichlet series. Consequently, we will begin by introducing these series and describing some of their basic properties before discussing zeta functions.

Definition: A **Dirichlet series** is a series of the form $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ where $\{a_n\}$ is a sequence of complex numbers, and s is a complex variable.

As is often the case when working with series, it is useful to find sets on which the series is known to converge. With this in mind, we will prove the following:

Theorem: If a Dirichlet series converges absolutely for some $s_0 = \sigma_0 + i\tau_0$ then the series converges absolutely and uniformly for all complex numbers $s = \sigma + i\tau$ such that $\sigma \geq \sigma_0$.

Proof: Suppose we have a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ that converges absolutely for $s = s_0$. To see that the series must also converge absolutely when $\sigma \geq \sigma_0$, we observe that the modulus of the nth term is given by $|a_n n^{-s}| = |a_n| n^{-\sigma}$. Thus the modulus is independent of τ. If $\sigma \geq \sigma_0$ then $|a_n n^{-s}| = |a_n| n^{-\sigma} \leq |a_n| n^{-\sigma_0} = |a_n n^{-s_0}|$. It follows then that the series converges absolutely when $\sigma \geq \sigma_0$. The series converges uniformly in the same region by the Weierstrass M-test: $\sum |a_n| n^{-\sigma_0}$ converges by assumption and as noted, $|a_n| n^{-\sigma} \leq |a_n| n^{-\sigma_0}$ so that $\sum_{n=1}^{\infty} a_n n^{-s}$ thus the M-test yields the desired result.

Now that we have a method for finding a half-plane in which a Dirichlet series converges, we want to define a notion of the largest region in which the series converges.

Definition: The **abscissa of absolute convergence** of a Dirichlet series, denoted by σ_a, is defined to be the infimum of the set of all σ such that $\sum |a_n| n^{-\sigma}$ converges.

There are several results concerning convergence of Dirichlet series, and we will simply state one of them.
Theorem: σ_a is unique and the series converges absolutely when $\sigma > \sigma_0$ and unformily on any compact subset of this region. Furthermore, $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ defines an analytic function for $\sigma > \sigma_a$.

Example: The well known Riemann zeta function given by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is a Dirichlet series with $a_n = 1$ for all n and $\sigma_a = 1$.
Before defining the zeta function of a number field, we want to make note of one other application of Dirichlet series to number theory.
Example: The Prime Number Theorem: The function $\pi(x)$ defined to be the number of primes less than or equal to x behaves asymptotically like $x/\log(x)$ as $x \to \infty$. In order to prove the theorem, one may consider the Dirichlet series $\sum_p \log p / p^s$ where the sum is taken over all primes p. We will not prove the statement here; however, the reader may refer to [Gam01] for detailed discussion and proof.

Definition: The **zeta function** of a number field K is defined to be

$$\zeta_K(s) = \sum_{n=1}^{\infty} a_K(n) n^{-s}$$

Where $a_K(n)$ denotes the number of ideals of K with norm n.
The reason we have selected this definition is that we can clearly see here that $\zeta_K(s)$ is a Dirichlet series; however, we will immediately make the following observation:
Observation: $\zeta_K(s) = \sum_{\mathrm{I}} N\mathrm{I}^{-s}$ where the sum is over all integral ideals of K and $N\mathrm{I}$ denotes the norm of the ideal I. Furthermore, using Euler products, we may also write $\zeta_K(s) = \prod_{\mathrm{P}} \frac{1}{1 - \frac{1}{N\mathrm{P}^s}}$ where the product is taken over all prime ideals.

Example: As a somewhat trivial example, we note that $\zeta_Q(s) = \sum_{n=1}^{\infty} n^{-s}$, i.e. the zeta function of the rationals is the Riemann zeta function.

III. The Class Number Formula

Before stating the class number formula, we need the following definitions:
For a number field K, let r_1 denote the number of real embeddings of K, and similarly, let r_2 denote the number of pairs of complex conjugate embeddings. As usual, h_K represents the class number of K and D_K the discriminant of the field. We will define ω_K to be the number of roots of unity in K. Finally, we make one last definition:
Definition: First consider the matrix whose ijth entry is $N_i \log|u_i^j|$ where $\{u_i\}$ is a set of generators for the group of units modulo the roots of unity and u_i^j is the image of u_i under an embedding of K into the complex numbers. (Thus the matrix in question is $r_1 + r_2 - 1$ by $r_1 + r_2$ dimensional where we only consider one of each of the pairs of complex conjugate embeddings). Then the **regulator** of the number field K, denoted here by R, is the determinant of the submatrix obtained by deleting any single column of the matrix.
It can be shown that the value of the regulator is independent of which column is selected.
Theorem: (The Class Number Formula): For a number field K, $\zeta_K(s)$ converges for all complex numbers s such that σ>1. Furthermore, at s=1 the function has a simple pole with residue given by

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K R}{\omega_K \sqrt{|D_K|}}.$$

Outline of Proof: The outline given here is based on the detailed proof given by [Siv05] and that found in [FT93].
The proof relies on the following lemmas:

Lemma 1: Suppose that C is a cone in $\mathbb{R}^n$ and F is a function from C to $\mathbb{R}^+$ such that $F(ax) = a^n F(x)$ when $x \in C$ and $a \in \mathbb{R}^+$ and the set $S = \{x \in C \mid F(x) \le 1\}$ is bounded with positive volume V. If $\mathbb{L}$ is a lattice in $\mathbb{R}^n$ with volume L then the following series converges for $\sigma = \mathrm{Re}(s) > 1$:

$$\zeta_{F,\mathbb{L}}(s) = \sum_{x \in C \cap \mathbb{L}} F(x)^{-s}.$$

Furthermore, $\lim_{s\to 1}(s-1)\zeta_{F,\mathbb{L}}(s) = \frac{V}{L}$.
Thus the defined function has residue equal to the volume of S divided by the volume of the lattice at the point s=1.

To state the next lemma, we need to make a choice of a specific cone C. For points $x = (x_1, \ldots x_{r_1}, \ldots, x_{r_1+r_2})$ where the first r_1 coordinates are real and the remaining coordinates are complex, we may define a function $l(x)$ that maps x to a linear combination of the

images of the u_i (as in the definition of the regulator) under the embeddings of K (see [FT93] and [Siv05]).
Then we want to define our cone to be the set of x with nonzero norm such that the coefficients of the linear combination $l(x)$ are nonnegative and strictly less than one and such that x_1 satisfies $0 \le \arg(x_1) < \frac{2\pi}{\omega_K}$.
With C defined in this way, we state our second lemma:
Lemma 2: If A_α is defined to be the set of all associates of α (i.e. elements that differ from α by a unit), then exactly one element of A_α has image in C under than map l.

So far, we may make the following steps toward the result in the class number formula:
The recall that we may write the zeta function of a number field as $\zeta_K(s) = \sum_{\mathrm{I}} N\mathrm{I}^{-s}$.
Then equivalently, we may break this into a double sum where the outer sum is taken over ideal classes in the class group C_K and the inner sum is taken over all ideals in the given class. i.e. $\zeta_K(s) = \sum_{\mathrm{R}\in C_K} \sum_{\mathrm{I}\in} N\mathrm{I}^{-s}$. It is clear that these two representations are equal.
Now if we select J such that JI is a principal ideal for all I in the class R (i.e. choose J to be in R^{-1}) then we may write the inner sum as:

$$\sum_{\mathrm{I}\in\mathrm{R}} N\mathrm{I}^{-s} = N(J)^s \sum_{\alpha\in\mathrm{B}} |N(\alpha)|^{-s}$$

Where B is the set of $x = (x_1, \ldots x_{r_1}, \ldots, x_{r_1+r_2})$ (as above) with x=sig(b) for some b
The idea here is to use the second lemma to write this sum as

$$N(J)^s \sum_{\alpha\in C\cap\mathrm{L}} N(\alpha)^{-s}$$

Where the lattice L is taken to be the set of all $x = (x_1, \ldots x_{r_1}, \ldots, x_{r_1+r_2})$ such that x=sig(b) for some b in the ideal J.
The proof then proceeds by showing that L, the volume of the lattice, is given by $L = N(J)\sqrt{|D_K|}$, and furthermore, $V = \frac{2^{r_1+r_2}\pi^{r_2}R}{\omega_K}$. Using these two volumes and applying the first lemma to the summation above yields the class number formula.
As noted in [FT93], we can also state the class number formula in terms of another property of number fields. We have the following theorem:
Theorem: $\lim_{x\to\infty} \frac{M_K}{x} = \frac{2^{r_1+r_2}\pi^{r_2}h_K R}{\omega_K}$ where M_K denotes the number of ideals in the ring of integers of K with absolute norm less than x.

As an application of the class number formula, we consider specifically the case of quadratic extensions. We have the following theorem from[AW04]:

Theorem: Let $K = \mathbb{Q}(\sqrt{m})$ where m is a squarefree integer. If p is a rational prime and $(\frac{m}{p})$ denotes the Legendre symbol of m mod p then:

(a) If p>2 and $(\frac{m}{p}) = 1$ then (p) splits as a product of two primes.

(b) If p>2 and p|m then (p) is the square of some prime ideal.

(c) If p>2 and $(\frac{m}{p}) = -1$ then (p) is inert (prime in the ring of integers).

Using this result together with the theory of L-series, we find that we can write the zeta function of a quadratic number field as the product of the Riemann zeta function with the L-series of the field. Since the residue of the pole of the Riemann zeta function is known to be one, the problem reduces to finding the value of the L-series at one.

One of the problems frequently encountered in applying the class number formula is that the regulator of a number field can be quite difficult to calculate (even in some quadratic extensions); however, the formula provides a very elegant way to connect several of the most important quantities associated to a number field.

IV. References

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