Bernoulli numbers and the unity of mathematics

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March 14, 2008

(Very rough notes for the Bartlett Lecture)

Contents

1 Daniel Bartlett; personal reminiscences; general introduction; Bernoulli numbers as "fundamental numbers"

to be written

Bernoulli numbers are the signs of a very strong bond between these pillars of mathematics:

1. *elementary number theory*: congruences;
2. *complex analytic number theory* : values of zeta-functions;
3. *homotopy theory* : the J-homomorphism, and stable homotopy groups of spheres;
4. *differential topology*: differential structures on spheres;
5. *the theory of modular forms* : Eisenstein series;
6. *p-adic analytic number theory* : the p-adic L-function.

2 Bernoulli Numbers ... in Elementary Number Theory

Here is a picture of their founding mathematician:

PICTURE OF BERNOULLI HERE

and here are the first few Bernoulli numbers referred to in the title, dripping down the left hand side of the page.

$$
\begin{aligned}
B_0 &= +1\\
B_1 &= -1/2\\
B_2 &= +1/6\\
B_4 &= -1/30\\
B_6 &= +1/42\\
B_8 &= -1/30\\
B_{10} &= +5/66\\
B_{12} &= -691/2730\\
B_{14} &= +7/6\\
B_{16} &= -3617/510\\
B_{18} &= +43867/798\\
B_{20} &= -174611/330\\
B_{22} &= +854513/138\\
B_{24} &= -23634091/2730\\
B_{26} &= +8553103/6\\
B_{28} &= -23749461029/870\\
B_{30} &= +8615841276005/14322\\
B_{32} &= -7709321041217/510\\
&\dots
\end{aligned}
$$

These Bernoulli numbers are rational numbers. You'll notice that except for B_1 the odd number indices are missing as entries of the above table. This is because $B_k = 0$ for $k > 1$ an odd number. Also the even-indexed Bernoulli numbers alternate in sign.

People who work with these numbers sometimes make personal attachments to them; for example, my favorites in this table are B_{12} and B_{32} (in that order). We'll see why, below.

You might wonder how a mere sequence of rational numbers can possibly be a "unifying force" in mathematics as the title of my lecture is meant to suggest. *Theories*, of course, can unify: *category theory*, for example, or *set theory*; physicists have their quest for a "unified theory of everything." But how can a bunch of numbers have the effect of unifying otherwise seemingly disparate branches of our subject?

As we'll see, for starters, Bernoulli numbers sit in the center of this block diagram of mathematical fields, and whenever, for a given index k the Bernoulli number B_k exhibits some particular behavior, all six of these mathematical fields seem to feel the consequences, each in their own way.

BERNOULLI NUMBERS IN THE CENTER WITH SIX FIELDS AROUND

The "Bernoulli Number" Website http://www.mscs.dal.ca/ dilcher/bernoulli.html offers a bibliography of a few thousand articles giving us a sense *that* these numbers pervade mathematics, but to get a more vivid sense of *how* they do so, we will survey the pertinence of Bernoulli numbers in just a few subjects, those listed in the *Table of Contents* above.

There may have been early appearances of the sequence of numbers referred to as Bernoulli numbers, but it is traditional to think of them as originating in Jacob Bernoulli's posthumous manuscript *Ars Conjectandi* (published 1713).

PICTURE OF FRONTISPIECE OF ARS CONJECTANDI

The text *Ars Conjectandi* itself might stand for the unity inherent in mathematics. It ostensibly focusses on *combinatorics* which, as Bernoulli says, corrects our most frequent error (counting things incorrectly) and is an art "most useful, because it remedies this defect of our minds and teaches how to enumerate all possible ways in which several things can be combined, transposed, or joined with another."[1]I am thankful to Edith Sylla for providing me with a manuscript of her new English translation (in progress) of Bernoulli's treatise; all quotations from that treatise, given below, are from her translation. Bernoulli continues by claiming that this art is so important that

"neither the wisdom of the philosopher nor the exactitude of the historian, nor the dexterity of the

[1] *

physician, nor the prudence of the statesman can stand without it."

He goes on to say that the work of these people depend upon "*conjecturing* and every conjecture involves weighing complexions or combinations of causes." For Bernoulli, *conjecturing* means quantitatively assessing the likelihood of an outcome, given one's current partial knowledge; in other words, "figuring the odds." Indeed *Ars Conjectandi* is viewed as one of the founding texts in probability, but it roams wide. For example, Bernoulli's notion of probability, including the famous law of large numbers whose origin is in this treatise, is not entirely without theological overtones. Bernoulli suggests by some of his terminology that, in his view, the law exhibits an overarching sense of *pre-destination*, for events are constrained to occur in specific ironclad frequencies, even though, from our finite viewpoint, it might appear as if things were random. Here is how *Ars Conjectandi* ends:

"Whence at last this remarkable result is seen to follow, that if the observations of all events were continued for the whole of eternity (with the probability finally transformed into perfect certainty) then everything in the world would be observed to happen in fixed ratios and with a constant law of alternation. Thus in even the most accidental and fortuitous we would be bound to acknowledge a certain quasi necessity and, so to speak, fatality. I do not know whether or not Plato already wished to assert this result in his dogma of the universal return of things to their former positions [apocatastasis], in which he predicted that after the unrolling of innumerable centuries everything would return to its original state."

Bernoulli initiates his discussion, though, by concentrating on the combinatorics of what we call *binomial coefficients*–i.e., "Pascal's triangle,"–and what he calls his table of "figurate numbers."[2] The terminology *figurate numbers* takes off from the fact that the numbers $\frac{n\cdot(n-1)}{2}$ are *triangular numbers*; i.e., they count the number of dots in an orderly array forming a right-angle triangle. Similarly the higher binomial coefficients fill out elementary polytopes in higher dimensions. He writes:

"This Table has clearly admirable and extraordinary properties, for beyond what I have already shown of the mystery of combinations hiding within it, it is known to those skilled in the more hidden parts of geometry that the most important secrets of all the rest of mathematics lie concealed within it."

This, of course, is a serious claim.

The numbers that will eventually be attached to his name enter Bernoulli's treatise only briefly, and in the discussion of closed forms for the sums of k-th powers of consecutive integers.

[2] ∗∗

PICTURE OF BERNOULLI'S TABLE (p.115) HERE

The Bernoulli numbers in question are the coefficients of the linear terms of these polynomial expressions. His predecessors had already made some computations of the polynomials. In particular, Johann Faulhaber (1580-1635) of Ulm computed the formulas up to $k = 17$ in his *Mysterium Arithmeticum* published in 1615. But Bernoulli chides them (Wallis included) for first laboriously working out closed expressions for the sums of consecutive k-th powers and then trying to understand "figurate numbers" in terms of these formulas, rather than what Bernoulli himself does which is to reverse the procedure; namely, he bases his analysis on the formula

$$\sum_{k=1}^{n-1} \frac{k \cdot (k-1) \cdot \ldots (k-c+1)}{1 \cdot 2 \cdot \ldots (c-1)} \;=\; \frac{(n \cdot (n-1) \cdot \ldots (n-c)}{1 \cdot 2 \cdot \ldots (c)}$$

and he derives the formulas for power sums from this, and then goes on to explain why this is philosophically, as well as practically, the better method.

He proclaims that one can continue his table without, as he puts it, "digressions," by deriving the formula that he writes[3]Bernoulli sometimes uses $\int$ for $\sum$, when he is summing over consecutive integers. He tends to be somewhat terse in his notation for summation, and rarely gives explicit "limits of summation." In the formula to be quoted above, we are summing from 1 to n. as

$$(*) \qquad \sum n^c \;=\; \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c(c-1)(c-2)}{4!}Bn^{c-6} +$$

$$+\frac{c(c-1)(c-2)(c-3)(c-4)}{6!}Cn^{c-5} + \frac{c(c-1)(c-2)(c-3)(c-4)(c-5)(c-6)}{8!}Dn^{c-7} + \ldots$$

[3]*

where $A = \frac{1}{6}, B = -\frac{1}{30}, C = \frac{1}{42}, D = -\frac{1}{30}, \ldots$ These, of course, are the numbers $B_2, B_4, B_6, B_8, \ldots$ that will bear his name. Bernoulli explains how to rapidly compute them (specifically, by induction: for example if you know A, B, C you can get D by setting $n = 1$ and $c = 8$ in the above formula, etc.) and is not above taunting his predecessors:

> I have found in less than a quarter of an hour that the tenth powers (or the *quadrate-sursolids*) of the first thousand numbers beginning from 1 added together equal
>
> $$91,409,924,241,424,243,424,241,924,242,500,$$
>
> from which it is apparent how useless should be judged the works of Ismael Bullialdus, recorded in the thick volume of his *Arithmeticae Infinitorum*, where all he accomplishes is to show that with immense labor he can sum the first six powers–part of what we have done in a single page.

With that salvo, Bernoulli makes no further mention, in his treatise, of the numbers we will be concentrating on, and turns his attention to other things.

Bernoulli doesn't indicate specifically the limits of summation in his formula **(*)** but let us officially define

$$S_k(n) := 1^k + 2^k + 3^k + \ldots + (n-1)^k,$$

noting that $S_k(n)$ is a polynomial in n of degree $k+1$ with no constant term, and leading term is simply

$$\frac{1}{k+1} n^{k+1} \;=\; \int_{x=0}^{x=n} x^k dx.$$

So we might write:

$$\begin{aligned}
S_1(n) &= 1 + 2 + 3 + \ldots + (n-1) = \frac{n(n-1)}{2} = \int_{x=0}^{x=n} x dx - \frac{1}{2} \cdot n, \\
S_2(n) &= 1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 = \int_{x=0}^{x=n} x^2 dx + \ldots - \frac{1}{6} \cdot n, \\
S_3(n) &= 1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 = \int_{x=0}^{x=n} x^3 dx + \ldots - 0 \cdot n, \\
S_4(n) &= 1^4 + 2^4 + 3^4 + \ldots + (n-1)^4 = \int_{x=0}^{x=n} x^4 dx + \ldots - \frac{1}{30} \cdot n, \\
&\ldots
\end{aligned}$$

$$S_k(n) \;=\; \sum_{x=0}^{n-1} x^k \;=\; \int_{x=0}^{x=n} x^k dx + \ldots + B_k \cdot n,$$

the Bernoulli numbers occurring successively as coefficients in each of the further error terms in formula **(*)**. Indeed, we might think of this formula as a way of estimating the errors in passing from integrals to discrete sums (of powers); or, inverting the procedure, we can as well go the other way: from discrete Riemann sums to integrals. Bernoulli numbers, then, are the coefficients that mediate between the discrete and the continuous. What we can do for powers alone, we can do for appropriately convergent *power series* and Euler and Maclaurin did exactly that. Nowadays, the Euler-Maclaurin formula that connects sums over discrete variables to integrals over domains (via Bernoulli numbers) is still the focus of interesting activity (cf. *The Euler-Maclaurin formula for simple integral polytopes*, Y. Karshon, S. Sternberg, J. Weitsman, PNAS **100** no. 2 (2003) 426-433).

Bernoulli suggested, as we discussed above, a recurrent procedure for calculating the $B_k's$ but there is no difficulty producing some straight "explicit formulas" such as:

$$B_k \;=\; \frac{(-1)^k k}{2^k-1}\sum_{i=1}^{k} 2^{-i}\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}(j+1)^{k-1}.$$

This formula was published some 170 years after *Ars Conjectandi* by J. Worpitsky (for the history and the derivation of this and other explicit formulas, see articles by H.W. Gould, *Explicit formulas for Bernoulli numbers* American Mathematical Monthly, **79** (1972) 44-51, and G. Rządkowsi, *A short proof of the Explicit Formula for Bernoulli numbers*, American Mathematical Monthly, **111** (2004) 432-434).

More telling for our story is the standard definition given nowadays. Namely, the Bernoulli number B_k is the coefficient of $\frac{x^k}{k!}$ in the power series expansion

$$\frac{x}{e^x-1} \;=\; 1-\frac{x}{2}+\sum_{k=2}^{\infty} B_k\frac{x^k}{k!}.$$

That these numbers form the coefficients of the Taylor expansions of the trigonometric function $\frac{x}{e^x-1}$ is a hint that Bernoulli numbers play a somewhat basic role in the arithmetic study of the algebraic group $\mathbf{C}^*$, the group of nonzero complex numbers under multiplication. Therefore it should not come as too much of a surprise if these numbers show up ubiquitously in the Taylor expansions of trigonometric functions. For example,

$$tan(x) \;=\; \sum_{k=1}^{\infty}(-4)^k(1-4^k)B_{2k}x^{2k-1}/(2k)!,$$

and if that is not enough, you can work out the Taylor expansion at the origin of any of these trigonometric functions

$$coth(x),\ cosh(x),\ tanh(x),\ x/sin(x),\ x/sinh(x),\ldots$$

to find Bernoulli numbers, combined with more elementary factorials, as coefficients.

3 ... in Complex Analytic Number Theory

That Bernoulli numbers are firmly embedded in analytic number theory is guaranteed by their relationship to *reciprocal* power sums, otherwise known as values of the Riemann Zeta function. *Half* of this relationship was already known to Euler. Namely, Euler proved formulas for the summation of the reciprocals of the squares, fourth powers, sixth powers, etc., of all positive integers,

$$\begin{aligned} 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots &= \frac{1}{6}\pi^2, \\ 1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\dots &= \frac{1}{90}\pi^4, \\ 1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\dots &= \frac{1}{945}\pi^6, \\ &\cdots \\ 1+\frac{1}{2^{2k}}+\frac{1}{3^{2k}}+\frac{1}{4^{2k}}+\dots &= \frac{2^{2k-1}}{(2k)!}|B_{2k}|\pi^{2k}. \end{aligned}$$

The first of these formulas has its own history and life within the theory of Fourier series. But all of these formulas are extraordinary, and for quite a few reasons, not the least of which is that they have astounded generations of students of elementary number theory by providing a curious *infinite sequence of proofs of the infinitude of prime numbers* if you believe that π is transcendental. Briefly, here is how. Define the Riemann zeta-function,

$$\zeta(s) := \sum_{m=0}^{\infty} m^{-s},$$

this sum being uniformly convergent in compact sets in the complex half-plane $Re(s) > 1$ (and therefore the infinite sum defines a complex analytic function of the variable s ranging through that half-plane). Euler already explicitly considered this function for real values of s that are > 1. Our formulas displayed above can be written as

$$\zeta(2k) = \frac{2^{2k-1}}{(2k)!}|B_{2k}|\pi^{2k}$$

for $k = 1, 2, 3, \dots$.

Thanks to the unique factorization theorem, $\zeta(s)$ can also be given as the infinite product

$$\zeta(s) = \prod_p (1-\frac{1}{p^s})^{-1}$$

where this product is taken over all prime numbers p, the right-hand side being convergent and equal to the left-hand side, for $Re(s) > 1$. Given the unique factorization theorem, this is an exercise in elementary manipulation of infinite series, once you expand $(1 - \frac{1}{p^s})^{-1}$ as a geometric series.

This too, was known to Euler, at least for real values of s. In particular, using the first of the sequence of summation formulas displayed above, we have

$$\zeta(2) \;=\; \prod_p (1 - \frac{1}{p^2})^{-1} \;= (1 - \frac{1}{2^2})^{-1}(1 - \frac{1}{3^2})^{-1}(1 - \frac{1}{5^2})^{-1} \cdots (1 - \frac{1}{p^2})^{-1} \cdots \;=\; \frac{1}{6}\pi^2.$$

From this we see that if there were only finitely many primes, the left-hand side of our formula, which would be a *finite* product of factors, each a rational number, would itself be a rational number. But the right-hand side is equal to a rational number times the square of π. Since π is transcendental, we have a contradiction; ergo there are infinitely many prime numbers. Of course, we get a similar proof with any of the infinitely many formulas on the list, granted the further fact that our Bernoulli numbers are rational–which they are. One is left to puzzle over the queries: are these proofs giving us *basically the same information*? If so, why? If not, what is the combined information that, taken all together, they are providing us?

In any event, the fact that

$$(**) \qquad \zeta(2k) \;=\; \frac{2^{2k-1}}{(2k)!}|B_{2k}|\pi^{2k}$$

for $k = 1, 2, 3, \dots$ speaks volumes. Indeed, as I will be hinting at towards the end of my lecture, it may bespeak volumes in the less figurative sense that, after all, $\pi^2, \pi^4, \dots$ are perhaps most immediately encountered as volumes of various spaces. That Bernoulli numbers are somehow implicated in volumes of ... what? ... may indeed be the key to their unifying role. But I am getting ahead of myself.

I suggested above that the formulas given by **(**)** are only *half* of the relationship that Bernoulli numbers enjoy with this aspect of analytic number theory. The other half comes along with the functional equation that the Riemann zeta function satisfies. Namely, the complex analytic function $\zeta(s)$ has a meromorphic extension to the entire complex plane, with a single (simple) pole at $s = 1$ and $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is invariant under the transformation $s \mapsto 1 - s$. Here, $\Gamma(z)$ is the classical "Gamma-function." This functional equation relates the values $\zeta(s)$ at positive even integers to its values at negative odd integers. One easily does the arithmetic to produce, from **(**)** and this functional equation, the formulas

$$(**) \qquad \zeta(1 - 2k) \;=\; (-1)^k B_{2k}/2k$$

for $k = 1, 2, 3, \dots$ which is, perhaps, an even cleaner manifestation of Bernoulli numbers within the framework of analytic number theory. We will be taking up this connection in some depth later on in this lecture. For now let us only make a quick acknowledgment of the fact that we have not accounted for all *values of the Riemann zeta-function at integers*; namely, there are its values at s ranging through odd integers > 1, these being quite mysterious numbers, about which we have the beginnings of a beautiful story (specifically, the value $\zeta(3)$; cf. [**]); finally, there are the ζ-values at the negative even integers, where the zeta-function vanishes, the residues at these integers being linked, via the functional equation, to the "quite mysterious numbers" just referred to.

It is time to get to the centripetal and centrifugal forces in the block diagram

BERNOULLI NUMBERS IN THE CENTER WITH SIX FIELDS AROUND

where, as we shall be discussing, the Bernoulli numbers are being tugged along the three axes.

Let us begin with the topological axis (differential topology/ homotopy theory):

Diagram:

$$B_{2k}/2k = \text{numerator}/\text{denominator}$$

$$\begin{aligned}\text{numerator} &\longrightarrow \text{important for differential topology}\\ \text{denominator} &\longrightarrow \text{important for homotopy theory.}\end{aligned}$$

The denominator of $B_{2k}/2k$ is the less mysterious part, and there are a number of fairly elementary ways of computing it. For example the *odd* part of that denominator is given by the formula

$$\text{odd part of denominator}(\mathrm{B_{2k}/2k}) \;=\; \prod_{p>2 \quad (p-1)\ \mid\ 2k} p^{1+v_p(2k)},$$

where $v_p(N)$ denotes the largest exponent e such that $p^e \mid N$.

4 ... in Stable Homotopy Theory

Stable homotopy theory studies the properties of continuous mappings of spaces ("spaces" for us may be taken to mean topological spaces realizable as finite simplicial complexes)

$$X \longrightarrow Y$$

where these continuous mappings are taken "up to" the natural equivalence relationship generated by homotopy after iterated suspensions. The **suspension** SX of a space X may be visualized this way: think of X as sort of an "equator" (e.g., as a topological subspace in a Euclidean space $\mathbf{R}^{N-1} \subset \mathbf{R}^N$); now put a point ("North pole") above X in $\mathbf{R}^N$, a point ("South pole ") below it, and draw all straight line-segments in $\mathbf{R}^N$ between North and South pole to the points of X. The locus of all these line segments in $\mathbf{R}^N$ is the topological space SX. Any continuous mapping $f : X \to Y$ generates its **suspension**, which is a map from SX to SY that brings equators to equators (via f), North pole to North pole, South pole to South pole, and is linear on the drawn line-segments. Two mappings are **homotopic** if one can be continuously deformed to the other. The continuous mappings $X \to Y$ taken "up to" the natural equivalence relationship generated by homotopy after iterated suspensions, are called the **stable homotopy classes** of mappings from X to Y and the collection of these stable homotopy classes, from X to Y, defined so topologically, has a natural abelian group structure, which is only the beginning hint that there is profound algebra here, much of which is still hidden from us.

The core collection of spaces for which the computation of these abelian groups of stable homotopy classes is particularly important are the spheres S^m for $m = 0, 1, 2, 3, \ldots$. Since there are no nontrivial homotopy classes of mappings $S^n \to S^m$ for $n < m$ and since the suspension operation induces an isomorphism between the group of stable homotopy classes of mappings $S^n \to S^m$ and $S^{n+1} \to S^{m+1}$, it makes sense to organize the stable homotopy of spheres as follows. For each $j \geq 0$ put

$$\Pi_j := \text{ the group of stable homotopy classes of continuous maps from } S^{m+j} \text{ to } S^m.$$

Equivalently, we could have defined Π_j to be the group of homotopy classes of maps $S^{m+j} \to S^m$ for $m >> 0$. The *degree* gives an isomorphism $\Pi_0 \to \mathbf{Z}$, and the groups Π_j for $j > 0$ are finite abelian groups. If we put them all together, we get a natural graded ring (multiplication being given by composition of maps)

$$\Pi_* := \bigoplus_{j \geq 0} \Pi_j,$$

and in an evident sense the entire *stable homotopy theory* is a module over this graded ring. Perhaps the single most surprising thing about this ring is how complicated it is.

The graded ring Π_* is nontrivial in degree 1. More specifically, Π_1 is a cyclic group of order two, the generator being the stable homotopy class of the *Hopf map* $h : S^3 \to S^2$, namely the mapping that can be realized as mapping $S^3 =$ the group of quaternions of norm 1 to its the quotient space modulo a circle subgroup. The fibers of h are all circles– in fact, the mapping exhibits S^3 as a circle bundle over S^2– and any two fibers are *linked* circles in S^3 with linking number equal to 1.

The stable homotopy class of the Hopf mapping $S^{m+1} \to S^m$ for $m \geq 2$, has the property that it can be represented by a C^∞ mapping from $f : S^{m+1} \to S^m$ such that for one point (in fact, for most points) $s \in S^m$ the fiber $f^{-1}(s)$ is a smooth circle, and such that the jacobian matrix of the restriction of f to that circle has maximal rank. This condition, generalized to Π_j for $j \geq 1$, cuts out a very important subgroup of Π_j studied in depth by Frank Adams and others, a subgroup that can be analyzed with real precision as we shall shortly see. Namely, let $C_j \subset \Pi_j$ denote the stable homotopy classes of mappings $S^{m+1} \to S^m$ for $m \geq 2$, has the property that it can be represented by a C^∞ mapping from $f : S^{m+j} \to S^m$ (for $m >> 0$) such that for one point $s \in S^m$ the fiber $f^{-1}(s)$ is diffeomorphic to a smooth (standard) j-dimensional sphere, S^j, and such that the jacobian matrix of the restriction of f to any point of that S^j has maximal rank. The subgroup C_j is called in the literature *the image of the J-homomorphism* for indeed, it is the image of a natural mapping

$$J : \pi_j(SO(m)) \to \Pi_j$$

(where $SO(m)$ is the special orthogonal group in $GL(m)$, π_j is the standard j-dimensional homotopy group, and $m >> 0$.)

The homotopy groups $\pi_j(SO(m))$ are periodic with period eight

$$\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}/2\mathbf{Z}, 0, \mathbf{Z}, 0, 0, 0, \mathbf{Z},$$

the first $\mathbf{Z}/2\mathbf{Z}$ in this series occurring for $j \equiv 0$ modulo 8. Therefore the subgroups $C_j \subset \Pi_j$ are also all cyclic (and even have a *preferred generator*) and the particularly interesting C_j's occur for $j = 4k-1$, $k = 1, 2, 3, \ldots$. To illustrate how the denominator of the Bernoulli numbers enters into the study of Π_j, consider this table:

INSERT TABLE OF STABLE HOMOTOPY GROUPS

Generally, we have that

$$2 \cdot \text{denominator}(\frac{B_{2k}}{2k}) \;=\; |C_{4k-1}|,$$

and that C_{4k-1} splits off as a direct summand in Π_{4k-1}. A nontrivial complement of C_{4k-1} in Π_{4k-1} finally occurs (as $\mathbf{Z}/2\mathbf{Z}$) in the last entry of the above table, but from this modest beginning, the complement to the image of the J-homomorphism will exhibit extraordinary complexity as k gets large.

5 . . . in Differential Topology

Sandpaper the sharp edges and corners of a cube to make a smooth surface, and that surface will be differentiably equivalent to the standard two-dimensional sphere. It is hard to imagine getting different differential structures by such a beveling procedure. Well, one of the great mathematical surprises of the early 60's (at least for some of us) was John Milnor's discovery that there was more than one smooth structure compatible with the standard combinatorial structure on the seven-dimensional sphere. In fact, Milnor proved that there were exactly 28 different such differential structures.

Milnor, along with Michel Kervaire, considered a group Θ_j in their classic article *Groups of Homotopy Spheres I* which has a number of equivalent definitions, the simplest being that it is the group of diffeomeorphism classes of differentiable j-dimensional manifolds whose underlying combinatorial manifold is isomorphic to the standard combinatorial j-dimensional sphere. The group structure is given by connected sum of differentiable manifolds. The group Θ_j is abelian, and as Kervaire-Milnor showed, it is finite.

I offer \$100 to anyone who can come up with a single formula that involves more disparate branches of mathematics than the formula that Milnor and Kervaire proved for the order of the group Θ_{4k-1}. Here is the **Kervaire-Milnor Formula** each term of which connects with a different field in mathematics:

$$\text{card } (\Theta_{4k-1}) = R(k) \cdot \text{card } (\Pi_{4k-1}) \cdot B_{2k}/2k$$

where

$$R(k) := 2^{2k-3}(2^{2k-1} - 1)$$

if k is even, and

$$R(k) := 2^{2k-2}(2^{2k-1} - 1)$$

if k is odd.

Reading it from left to right, Θ_{4k-1} is in *differential topology*, Π_{4k-1} is in *stable homotopy theory* and $B_{2k}/2k$, well, is something of a universalist–after all, that's the point of this lecture– but let's tag it as in *number theory* for the purposes of this inventory. I skipped the elementary-seeming factor $R(k)$ in this tally but shouldn't have done so: $R(k)$ is an indication of a number of things, one of them being the role that the formula will play in yet another field of mathematics, *the theory of modular forms* where Θ_{4k-1} connects to modular forms of level 2.

Note that the appearance of Π_{4k-1} as a factor in this formula conveniently annihilates the denominator of $B_{2k}/2k$: the product

$$\text{card } (\Pi_{4k-1}) \cdot B_{2k}/2k \quad = \quad \text{card } (\Pi_{4k-1}/C_{4k-1}) \cdot \text{numerator } (B_{2k}/2k)$$

is an integer, and in fact, combines the more mysterious part of Π_{4k-1} with the more mysterious part (i.e., the numerator) of $B_{2k}/2k$.

The famous result of Milnor that there are exactly 28 different differential structures on the seven-sphere computes out, from this formula, as follows.

$$\Theta_7 \;=\; 2 \cdot (2^3 - 1) \cdot 240 \cdot \frac{1}{30}/4 \;=\; 28.$$

6 ... in the Theory of Modular Forms

It is impressive how much of the theory we are about to discuss rests on *Fermat's Little Theorem* and Euler's extension of it:

(**Euler's Theorem.**) Let p be a prime number, $n > 0$, and d an integer not divisible by p. Then

$$d^{\phi(p^n)} = d^{(p-1)p^{n-1}} \equiv 1 \text{ modulo } p^n.$$

Here, for $m > 0$, $\phi(m)$ is *Euler's ϕ-function* evaluated at m, i.e., the number of positive integers $\leq m$ that are relatively prime to m.

Let us return to the *power sums* that Bernoulli worked with,

$$S_k(n) := \sum_{m=1}^{n-1} m^k,$$

and note that if we modify its definition ever so slightly, we get an expression very well suited to Euler's theorem[4]This type of elementary modification (which in the theory of modular forms corresponds to guaranteeing that the level of the modular from you are working with is divisible by p, and if it isn't, performing an appropriate 'level-raising" operation to replace that modular form that has p dividing its level) is the key to achieving a plentiful supply of *congruences modulo high powers of* p (otherwise known as *p-adic interpolation*).. Namely, define:

$$S_k^{(p)}(n) := \sum_{m<n;\ m\not\equiv 0 \bmod p} m^k,$$

so that

$$S_k^{(p)}(n) = S_k(n) - p^k S_k([n/p]).$$

Fro Euler's Theorem, we have the intriguing congruences,

$$S_k^{(p)}(n) \equiv S_{k+\phi(p^r)}^{(p)}(n) \text{ modulo } p^r,$$

which tells us that the value of $S_k^{(p)}(n)$ mod p^r depends only on the index k modulo $\phi(p^r)$. This allows us to pass to limits in the p-adic numbers, and to define, for $\kappa \in \mathbf{Z}_p$ any p-adic integer,

$$S_\kappa^{(p)}(n) := \lim_{r=1,2,3,\dots} S_{\kappa_r}^{(p)}(n) \bmod \phi(p^r)$$

[4]*

and offers us a computation, which, if developed, eventually allows us (or more precisely, Claussen-Von Staudt and Kummer, respectively) to establish the famous classical theorems:

Claussen-Von Staudt: If $2k \not\equiv 0 \bmod p-1$ then $B_{2k}/2k$ is a p-integer. If $2k \equiv 0 \bmod p-1$ then $B_{2k}/2k + \frac{1}{p}$ is a p-integer.

Kummer: If $n \not\equiv 0 \bmod p-1$ and n, n' are positive integers such that $n' \equiv n$ modulo $(p-1)$ then

$$B_{n'}/n' \;\equiv\; B_n/n \quad \bmod p.$$

Let p be an odd prime number and let us begin to examine the "easier" coefficients first; namely, the functions,

$$\sigma_r(n) = \sum_{d|n} d^r.$$

consider the classical facts about the p-adic nature of these rational numbers.

1. The p-adic interpolation of Bernoulli Numbers and of Eisenstein series; first formulation.

To any p-ordinary eigenform (for the Hecke operators T_ℓ, for all prime numbers ℓ) f of level 1 we may associate a unique p-ordinary eigenform (for the Hecke operators T_ℓ, for all prime numbers $\ell \neq p$ and for the Atkin-Lehner operator U_p)of level p f' in the standard way. Starting with the Eisenstein series $f = E_{2k}$ which is an eigenform of level 1 one gets the eigenform of level p,

$$f' = E'_{2k} = \quad (-1)^k(1-p^{k-1})\frac{B_{2k}}{2k} \;+\; \sum_{n=1}^{\infty} \sigma^{(p)}_{2k-1}(n)q^n,$$

where $\sigma_r(n)^{(p)} := \sum_{d|n} d^r$, the summation being over all positive divisors of n which are prime to p.

2. The J-homomorphism.

Let $r >> 0$. Recall that for m a positive integer, we have a canonical identification $\mathbf{Z} = \pi_{4m-1}(SO(r))$. The J-homomorphism

$$J : \mathbf{Z} = \pi_{4m-1}(SO(r)) \to \pi_{4m-1+r}(S^r)$$

is the mapping defined by the following recipe. Embed S^{4m-1} in S^{4m-1+r} in the standard manner. We obtain a mapping ν from S^{4m-1+r} onto the Thom space of the normal bundle of $S^{4m-1} \subset S^{4m-1+r}$ by pulling everything outside a tubular neighborhood of S^{4m-1} to the "infinite" point of the Thom space. Viewing an element $\gamma \in \pi_{4m-1}(SO(r))$ as giving us a framing (up to homotopy) of the stable r-dimensional normal bundle of S^{4m-1} we obtain a projection p_γ of that Thom space onto S^r defined in the evident way via this framing. The element $J(\gamma) \in \Pi_{4m-1} := \pi_{4m-1+r}(S^r)$ is the homotopy class of the composition

$$p_\gamma \cdot \nu : S^{4m-1+r} \to S^r.$$

The image of J is then a cyclic subgroup of Π_{4m-1}, is a direct summand of that group, and is (up to a factor of 2) of order $2d_{2m}$:= the denominator of the Bernoulli number B_{2m}. In particular,

- for $m = 1$ we have $2d_2 = 24$, and Π_3 is cyclic of order 24,

- for $m = 2$ we have $2d_4 = 240$, and Π_7 is cyclic of order 240,

- for $m = 3$ we have $2d_6 = 504$, and Π_{11} is cyclic of order 504,

- for $m = 4$ we have $2d_8 = 480$, and Π_{15} is isomorphic to the direct sum of a a cyclic group of order 480 with a group of order 2.

Definition/Theorem. For m a positive integer, let $d(m)$:= the largest non-negative odd integer d such that, equivalently:

(i) $\phi(d) \mid m$,

(ii) the natural map $\iota_m : (\mathbf{Z}/d\mathbf{Z})(m) \to (\mathbf{Z}/d\mathbf{Z})(0)$ which sends the generator $1 \in (\mathbf{Z}/d\mathbf{Z})(m)$ to $1 \in (\mathbf{Z}/d\mathbf{Z})(0)$ is an isomorphism of $\mathbf{\Lambda}$-modules,

(iii) $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ acts trivially on $\mu_d^{\otimes m}$.

Also:

(iv)

$$d(m) = \prod_{p>2,\ p-1 \ \mid \ m} p^{1+v_p(m)}.$$

(v) $d(m)$ = the odd part of the denominator of B_{2m}.

(vi) $d(m)$ = the odd part of order of the image of the J-homomorphism ...

7 ... in p-adic Analytic Number Theory

The Riemann zeta-function $\zeta(s)$ has Bernoulli numbers as values at odd negative integers. Explicitly:

$$\zeta(1-2k) \quad = \quad (-1)^k B_{2k}/2k,$$

for $k \geq 1$, and the values of Dirichlet L-functions $L(\chi, s)$ at odd negative integers s lie in $\mathbf{Q}(\chi)$ and enjoy similar formulas:

$$L(1-2k, \chi) = (-1)^k B_{2k,\chi}/2k$$

where $B_{2k,\chi}$ is the generalized Bernoulli number attached to the character χ. The classical result of Leopold-Kubota is that, thanks to the "Kummer congruences," slightly modified versions of these values, *interpolate p-adically* to produce p-adic meromorphic functions of a variable s often denoted $L_p(\omega^i, s)$ (where i is an even integer modulo $p-1$). These functions are defined in the *extended* p-adic disc $\{|(1+p)^s - 1|_p < 1\}$. Explicitly, the *Leopold-Kubota p-adic L-functions* are uniquely determined by the following formulas, for integer $k \geq 1$,

$$L_p(\omega^{2k}, 1-2k) = (-1)^k (1-p^{2k-1}) \frac{B_{2k,\omega^{2k}}}{2k}.$$

If $i \not\equiv 0 \bmod p-1$ then $L_p(\omega^i, s)$ is holomorphic in the extended disc, but if $i \equiv 0 \bmod p-1$ then $L_p(\omega^i, s) = L_p(\omega^0, s)$ has a simple pole at $s = 1$ and is holomorphic away from $s = 1$. The appearance of this pole is connected to the fact (*Claussen-von Staudt*) that the denominators of Bernoulli numbers B_{2k} are divisible by p^n when $2k \equiv 0 \bmod (p-1)p^{n-1}$.

If you think through the standard construction of these p-adic L-functions (e.g., via distributions formed from Bernoulli polynomials) you discover that what is actually constructed is a canonical element (let us call it the **canonical Leopold-Kubota element**)

$$L \in I^{-1} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1).$$

The element L is characterized by

$$L \quad \longmapsto \quad (-1)^k (1-p^{2k-1}) \frac{B_{2k,\omega^{2k}}}{2k}$$

under the ring-homomorphism

$$\chi_{2k} : \mathbf{\Lambda} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \to \mathbf{Q}_p$$

which sends $[\gamma] \in \mathbf{Z}_p^* \subset \mathbf{\Lambda}$ to $\gamma^{2k} \in \mathbf{Z}_p^* \subset \mathbf{Z}_p \subset \mathbf{Q}_p$ for integers k.

The "I^{-1}" accomodates the pole of the L-function at $(i, s) = (0, 1)$.

For any prime number $\ell \neq p$ define $L^{(\ell)} \in I^{-1} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1)$, the **Kubota-Leopold element of level** ℓ, by the formula

$$L^{(\ell)} := \quad (1 - [\ell]/\ell) \cdot L \; \in \; I^{-1} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1).$$