Exercises for Part 2, Section 1.4: The Real Period

Math 582e, Winter 2009, University of Washington

Due Wednesday March 4, 2009

Let a and b be two positive real numbers. The AGM of a and b is the common limit of the two sequences a_n and b_n defined by $a_0 = a$, $b_0 = b$, and

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad \text{and} \qquad b_{n+1} = \sqrt{a_n b_n}.$$

1. (Exercise 10 of Chapter 7 of Cohen GTM 138)
   (a) Prove that the AGM of two positive real numbers exists, i.e., that the two sequences a_n and b_n given above both converge and to the same limit.
   (b) Show also that the convergence is quadratic.
2. (Exercise 11 of Chapter 7 of Cohen GTM 138) The goal of this exercise is to relate $\mathrm{AGM}(a, b)$ to an elliptic integral.
   (a) Set
   $$I(a,b) = \int_0^{\pi/2} \frac{dt}{\sqrt{a^2\cos^2(t) + b^2\sin^2(t)}}.$$
   By making the change of variables
   $$\sin(t) = \frac{2a\sin(u)}{(a+b) + (a-b)\sin^2(u)}$$
   show that $I(a,b) = I((a+b)/2, \sqrt{ab})$.
   (b) Deduce from this the formula
   $$I(a,b) = \frac{\pi}{2 \cdot \mathrm{AGM}(a,b)}.$$