Exercises for Part 2, Section 1.3: Minimal Model

Math 582e, Winter 2009, University of Washington

Due Wednesday March 4, 2009

1. Let E be an elliptic curve over $\mathbb{Q}$.

   (a) Prove that E has a reduced minimal model, i.e., one that has minimal $|\Delta|$ such that $a_1, a_3 \in \{0, 1\}$ and $a_2 \in \{-1, 0, 1\}$.

   (b) If one does not make the reduced condition on a_1, a_2, a_3, how many global minimal models does E have?

   (c) Prove that the reduced minimal model is unique.

2. Let $c_4, c_6 \in \mathbb{Z}$ with $0 \neq \Delta = (c_4^3 - c_6^2)/1728 \in \mathbb{Z}$. Prove that there exists an elliptic curve E (defined over $\mathbb{Z}$, i.e., all $a_i \in \mathbb{Z}$) with invariants c_4, c_6 if and only if

   (a) $c_6 \not\equiv \pm 9 \pmod{27}$, and

   (b) either $c_6 \not\equiv -1 \pmod{4}$ or $c_4 \equiv 0 \pmod{16}$ and $c_6 \equiv 0, 8 \pmod{32}$.

   [See Prop 3.1.1 of Cremona's book for hints.]

3. Given any $c_4, c_6 \in \mathbb{Z}$ with $c_4^3 \neq c_6^2$, does there exist an elliptic curve over $\mathbb{Q}$ with those c_4 and c_6 as c-invariants?