Exercises for Section 4: Computing the Class Group

Math 582e, Winter 2009, University of Washington

Due Wednesday February 4, 2009

1. Given α that is a root of some irreducible $f \in \mathbb{Q}[x]$ give an algorithm to find a $\beta \in \mathbb{Q}(\alpha)$ such that β is a root of an irreducible monic $g(x) \in \mathbb{Z}[x]$ and $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$.

2. Compute the Minkowski and Bach bounds for the following fields.
   - $K = \mathbb{Q}(\sqrt{-389})$.
   - $K = \mathbb{Q}(\alpha)$, where $\frac{\alpha^{37}}{2} + \frac{\alpha}{3} = -1$.
   - $K = \mathbb{Q}(\zeta_{389})$, where ζ_{389} is a 389th root of unity. You may assume that $\mathcal{O}_K = \mathbb{Z}[\zeta_{389}]$.)

3. What is the rank of the group $\mathcal{O}_{K,T}^*$ of T-units of $\mathbb{Q}(\sqrt[3]{2})$ for $T = \{(7), (-(\sqrt[3]{2})^2 - 1)\}$?

4. (a) What is the rank of the group of T-units of $\mathbb{Q}(\sqrt{-7})$ where $T = \{(13)\}$?

   (b) Give explicit generators for $\mathcal{O}_{K,T}^*$. You may assume that $\mathrm{Cl}(K) = 1$.

5. Let $K = \mathbb{Q}(\alpha)$ with $\alpha = \sqrt[4]{5}$.

   (a) The following units generate a subgroup U' of finite index in $\mathcal{O}_K^*$:

   $$u_1 = \alpha^2 - 2, \qquad u_2 = -48a^3 + 72a^2 - 108a + 161.$$

   Compute the regulator of U', i.e., the absolute value of the determinant of the log embedding of U'.

   (b) It is a fact that $\mathrm{Reg}_K = 1.55616\ldots$. Use this to compute $[\mathcal{O}_K^* : U']$. (Hint: Don't forget $-1 \in \mathcal{O}_K^*$.)