Exercises for Section 3: The Maximal Order

Math 582e, Winter 2009, University of Washington

Due Wednesday January 28, 2009 (note: extra week)

1. Prove that if $\mathcal{O}$ is an order in $\mathcal{O}_K$, then $[\mathcal{O}_K : \mathcal{O}]^2 = \mathrm{disc}(\mathcal{O})/\,\mathrm{disc}(\mathcal{O}_K)$.
2. Prove that every order $\mathcal{O} \subset \mathbb{Z}[i] = \mathbb{Z}[\sqrt{-1}]$ is of the form $\mathbb{Z} + fi\mathbb{Z}$ for some $f \in \mathbb{Z}_{\geq 1}$. Prove that if $\mathbb{Z} + fi\mathbb{Z} = \mathbb{Z} + gi\mathbb{Z}$ for $f, g \in \mathbb{Z}_{\geq 1}$, then $f = g$.
3. Use the *naive algorithm* to compute the maximal order of $K = \mathbb{Q}(\sqrt{5})$.
4. Prove that the radical of an ideal I in a commutative ring R is an ideal, where

$$\mathrm{rad}(I) = \{x \in R : x^j \in I \text{ for some } j > 0\}.$$

5. Compute a 3-maximal order in $\mathbb{Q}(\sqrt[3]{2})$ using the "round 2" algorithm from class (but without the Dedekind termination condition). Explicitly compute each matrix that comes up in the algorithm, etc., just like in class. You can use a computer to do the actual linear algebra.
6. Compute $\mathcal{O}_K$ for $K = \mathbb{Q}(\alpha)$. where $\alpha^3 + \alpha^2 - 2\alpha + 8 = 0$ using the "round 2" algorithm from class (but without the Dedekind termination condition). Explicitly compute each matrix that comes up in the algorithm, etc., just like in class. You can use a computer to do the actual linear algebra.
7. Create a general implementation of round 2 (without early termination) to compute a p-maximal order in any number field. Use your program to check your answers for the previous two problems.