Exercise for Section 2: The Birch and Swinnerton-Dyer Conjecture

Math 582e, Winter 2009, University of Washington

Due Wednesday January 14, 2009

1. Let E be an elliptic curve over $\mathbb{Q}$. For $P \in E(\mathbb{Q})$, prove that if $P \in E(\mathbb{Q})_{\mathrm{tor}}$, then $\hat{h}(P) = 0$, where $\hat{h}$ is the Néron-Tate canonical height. * (Note: The converse is also true, but more difficult to prove.)

2. Using a computer, compute every quantity in the BSD formula, except $\#Ш(E/\mathbb{Q})$ for each of the following elliptic curves:

   (a) $y^2 + xy + y = x^3 + 4x - 6$

   (b) $y^2 + y = x^3 + x^2$

   (c) $y^2 + xy = x^3 + 1$

   (d) $y^2 + xy = x^3 + x^2 - 1154x - 15345$

3. The congruent number problem has been called the oldest specific open problem in mathematics (over 1000 years old).

   Congruent Number Problem: *Give an algorithm to decide whether or not a given integer n is the area of a right triangle with rational side lengths.*

   (a) Explain why if the rank part of the Birch and Swinnerton-Dyer conjecture were known (i.e., that $\mathrm{rank}(E(\mathbb{Q})) = \mathrm{ord}_{s=1} L(E, s)$), then we would also have a solution to the congruent number problem. [Hint: I want a few paragraphs summary as an answer. You can find everything you need to easily answer this question by looking, e.g., in Koblitz's book *Introduction to Elliptic Curves and Modular Forms*. Alternatively, see the end of section 1 of Andrew Wiles' article on the Clay Math Website about the Birch and Swinnerton-Dyer Conjecture.]

   (b) Is 2009 a congruent number?