Exercises for Part 2, Section 1.7: The L-series

Math 582e, Winter 2009, University of Washington

Due Wednesday March 11, 2009

1. Let E be an elliptic curve over $\mathbb{Q}$. Prove that $\mathrm{ord}_{s=1} L(E,s)$ is even if and only if the sign ε_E in the functional equation is $+1$.

2. Suppose E is an elliptic curve such that $\varepsilon_E = +1$.

   (a) Is $L^{(n)}(E,1) = 0$ for all even integers n?

   (b) Is $\Lambda^{(n)}(E,1) = 0$ for all even integers n?

3. Verify that if $f(z) = \sum a_n e^{2\pi i n z}$ then (ignoring all questions of convergence), we have

$$(2\pi)^s \Gamma(s)^{-1} \int_0^{i\infty} (-iz)^s f(z) \frac{dz}{z} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

   Here $f(z) = \sum a_n e^{2\pi i n z}$ is a cuspidal eigenform of weight 2.

4. Prove (ignoring all questions of convergence) that if $z_0 = x_0 + i y_0$, then

$$\int_{i\infty}^{z_0} 2\pi i f(z) dz = \sum_{n=1}^{\infty} \frac{a_n}{n} e^{2\pi i n x_0} e^{-2\pi n y_0}.$$

   Here $f(z) = \sum a_n e^{2\pi i n z}$ is a cuspidal eigenform of weight 2.