Exercise for Section 1:
The Analytic Class Number Formula

Math 582e, Winter 2009, University of Washington

Due Wednesday January 14, 2009

1. A field K is called *totally real* if $r_2 = 0$. Prove that if K is totally real then $\#(U_K)_{\rm tor} = 2$. Is the converse true?

2. Prove that complex $s \gg 0$,
$$\prod_{\text{primes } \mathfrak{p} \text{ in } \mathcal{O}_K} \frac{1}{1 - N(\mathfrak{p})^{-s}} = \sum_{\text{ideals } I \neq 0 \text{ of } \mathcal{O}_K} \frac{1}{N(I)^s}.$$

3. Use a computer to compute every quantity in the class number formula for each of the following fields: $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\zeta_5)$, $\mathbb{Q}(\sqrt[5]{11})$.

4. List all prime ideals of residue characteristic ≤ 7 in the fields $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\zeta_5)$.

5. Prove that Dirichlet's Tnit Theorem about the structure of U_K^* implies that for any squarefree $d \geq 2$, *Pell's equation*
$$x^2 - dy^2 = 1$$
has infinitely many integer solutions (x, y).

6. Use the functional equation for $\zeta_K(s)$ to prove that the two formulations of the analytic class number formula in terms of $\zeta_K^*(1)$ and $\zeta_K^*(0)$ are equivalent. (Hint: See Theorem 4.9.12 of Cohen's GTM 138).