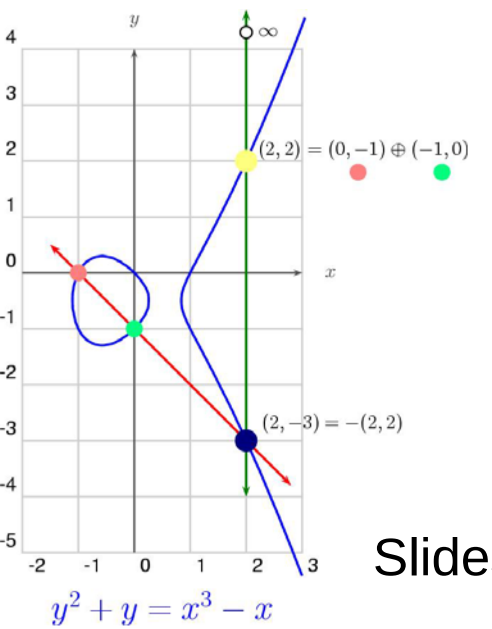
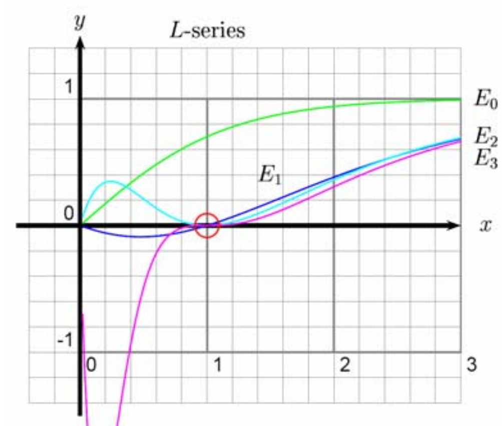
An Introduction to the Birch and Swinnerton-Dyer Conjecture

April 1, 2004



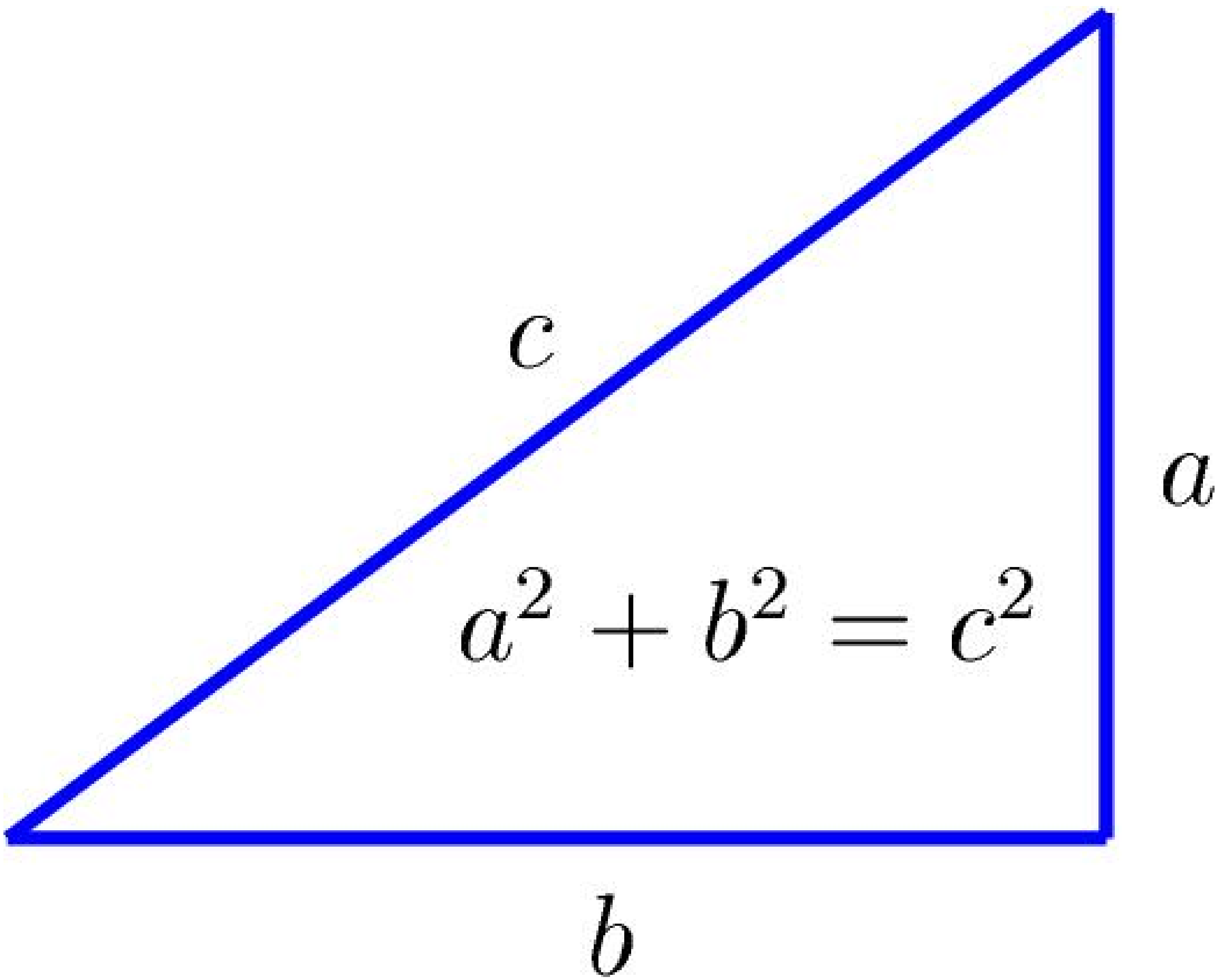
William Stein

http://modular.fas.harvard.edu



Slides: **http://modular.fas.harvard.edu/talks/uconn**

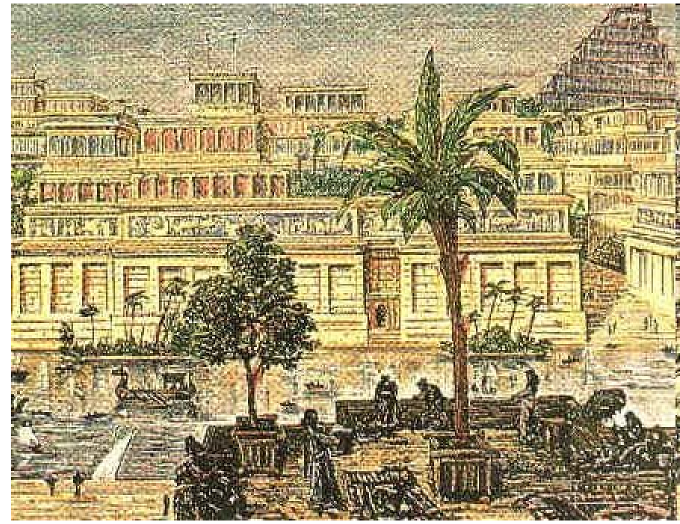
Pythagorean Theorem

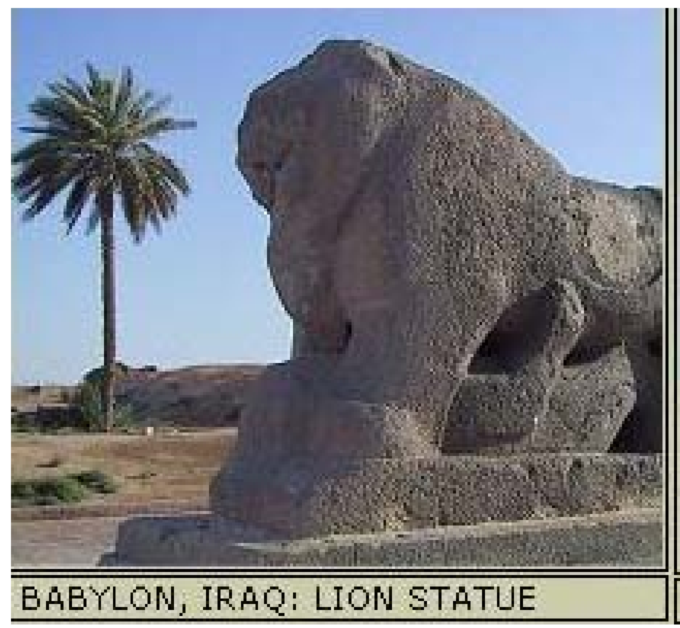


Pythagoras
approx 569-475 B.C.

Babylonians

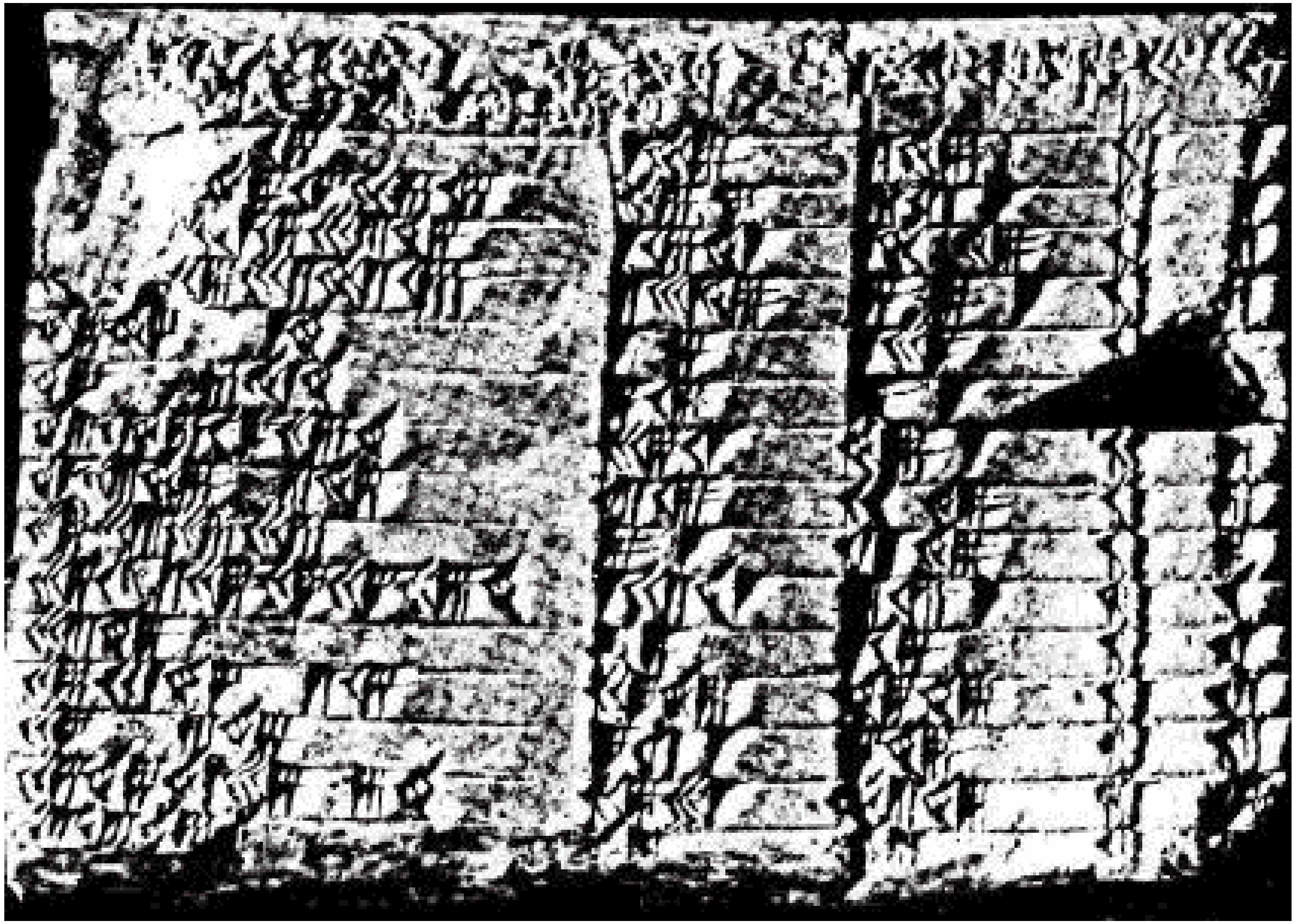
1800-1600 B.C.





$(3,4,5)$
$(5,12,13)$
$(7,24,25)$
$(9,40,41)$
$(11,60,61)$
$(13,84,85)$
$(15,8,17)$
$(21,20,29)$
$(33,56,65)$
$(35,12,37)$
$(39,80,89)$
$(45,28,53)$
$(55,48,73)$
$(63,16,65)$
$(65,72,97)$
$(77,36,85)$
$\vdots$

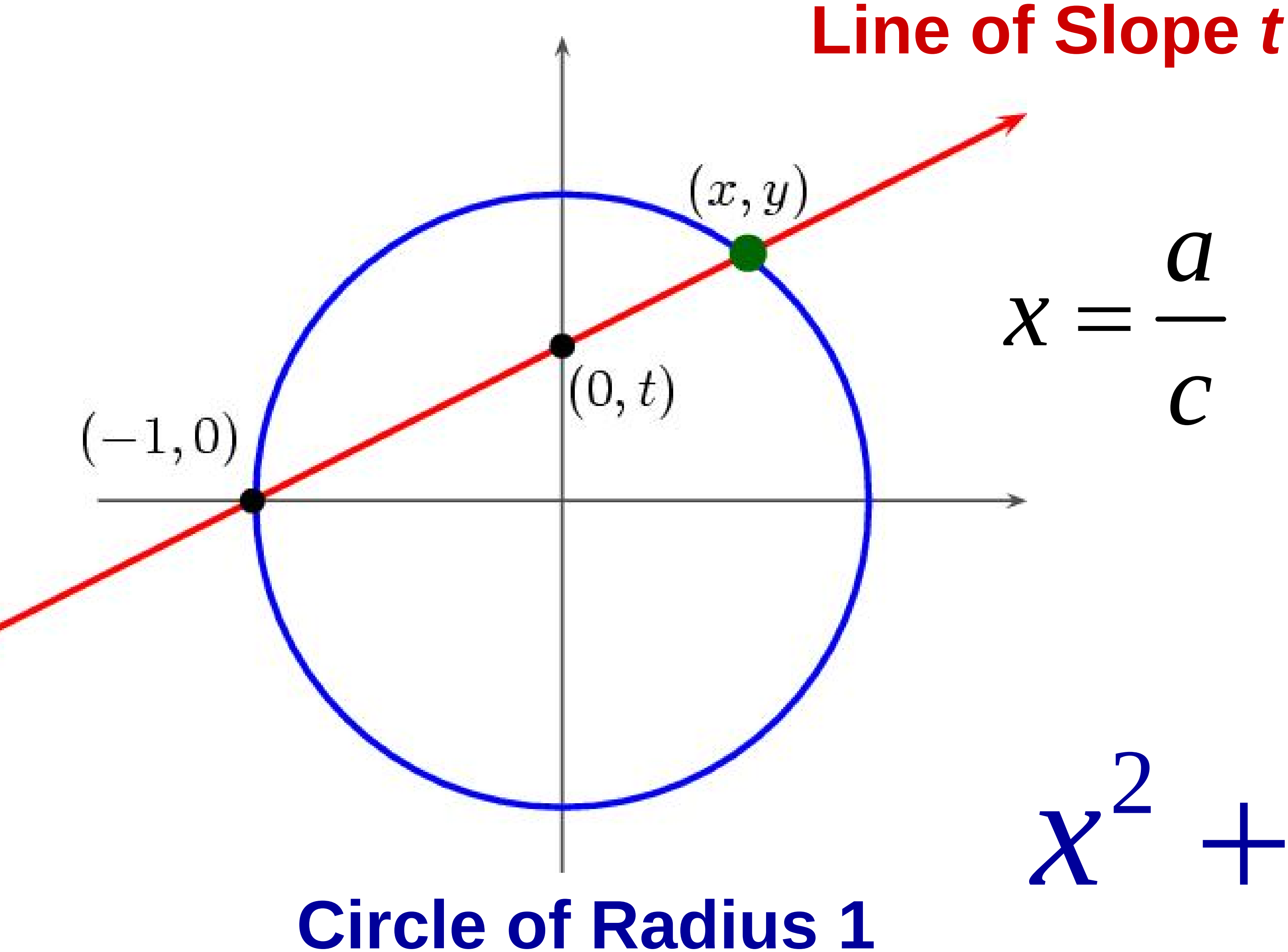
Pythagorean Triples



Triples of whole numbers *a*, *b, c* such that

$$a^2 + b^2 = c^2$$

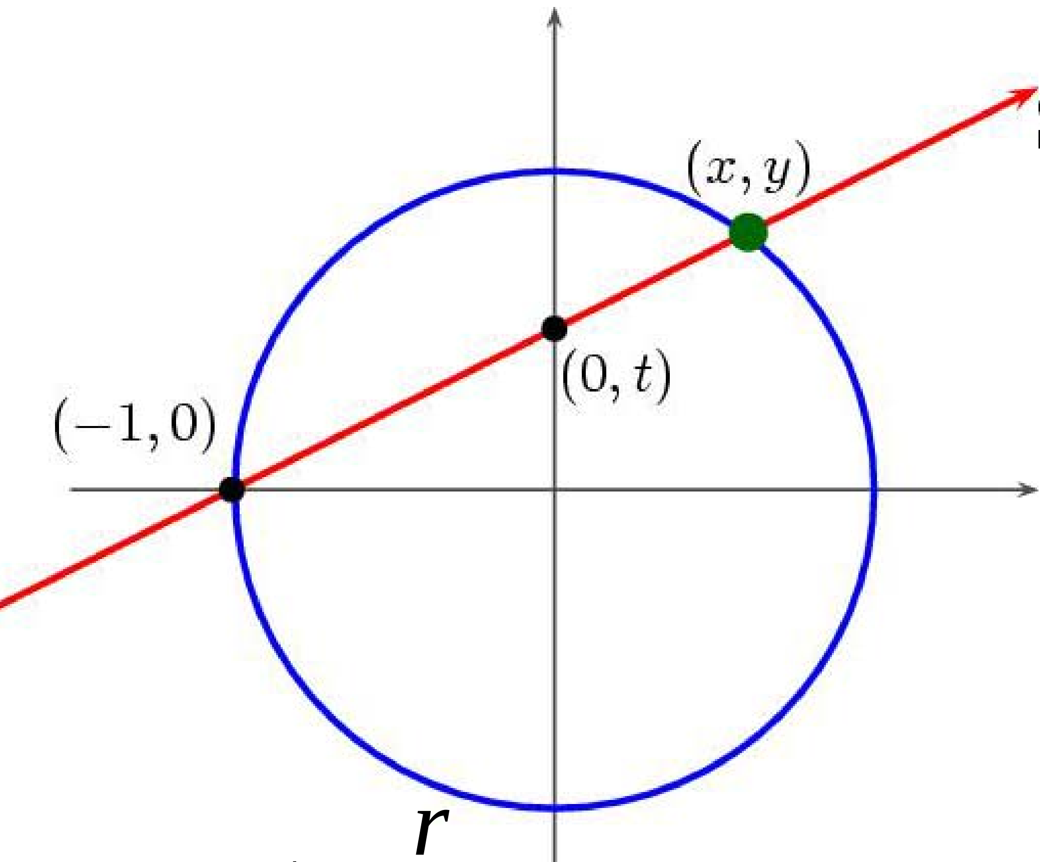
Enumerating Pythagorean Triples



$$x = \frac{a}{c} \qquad y = \frac{b}{c}$$

$$x^2 + y^2 = 1$$

Enumerating Pythagorean Triples



$$\text{Slope} = t = \frac{y}{x+1}$$

$$x = \frac{1-t^2}{1+t^2}$$

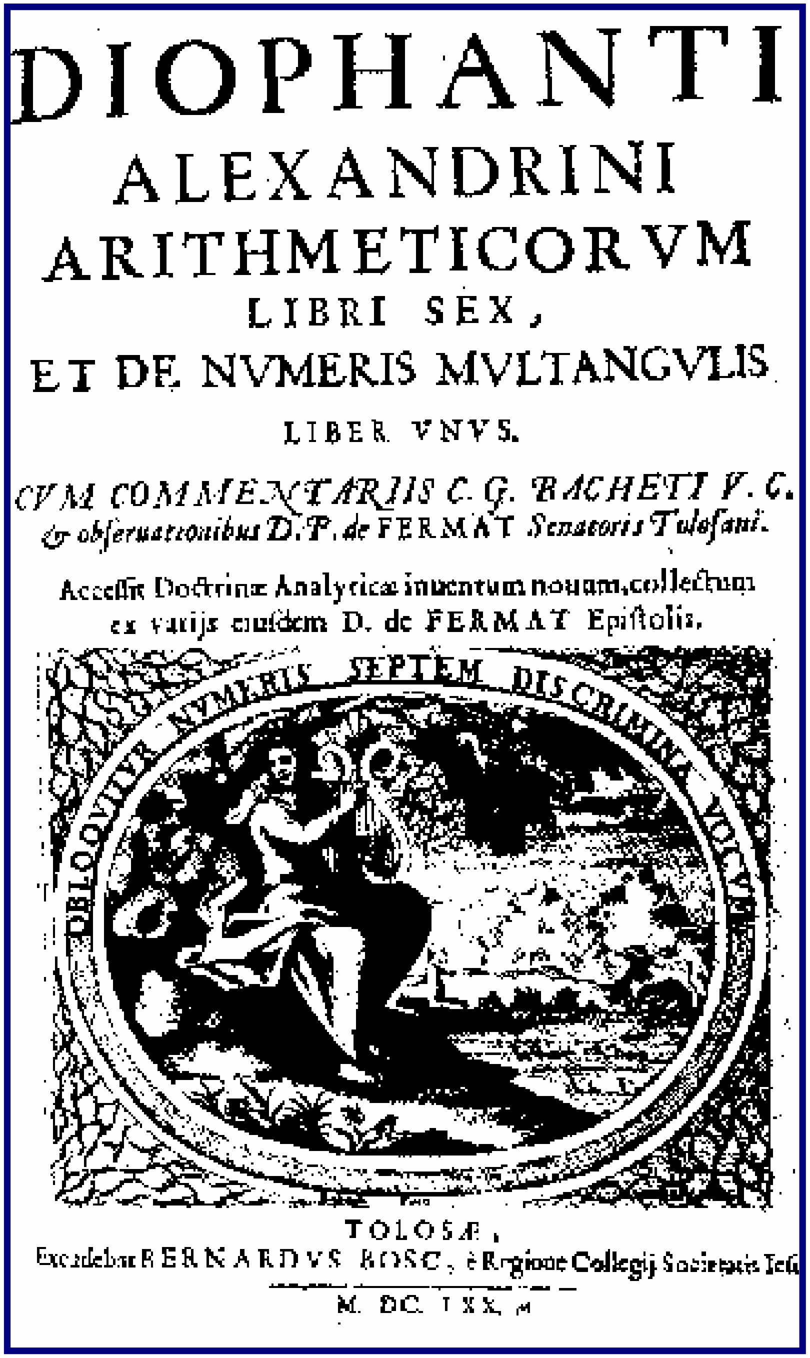
$$y = \frac{2t}{1+t^2}$$

If $t = \dfrac{r}{s}$ then

$$a = s^2 - r^2 \qquad b = 2rs \qquad c = s^2 + r^2$$

is a Pythagorean triple.

Integer and Rational Solutions

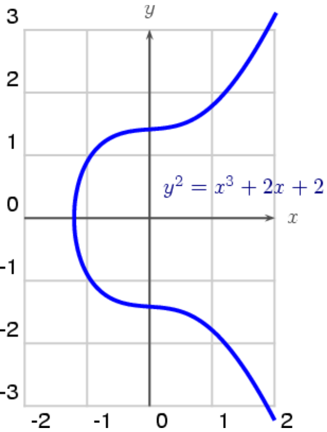


DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.

CVM COMMENTARIIS C. G. BACHETI V. C.
& obseruationibus D. P. de FERMAT Senatoris Tolosani.

Accessit Doctrinæ Analyticæ inuentum nouum, collectum
ex varijs eiusdem D. de FERMAT Epistolis.

TOLOSÆ,
Excudebat BERNARDVS BOSC, è Regione Collegij Societatis Iesu.
M. DC. LXX.

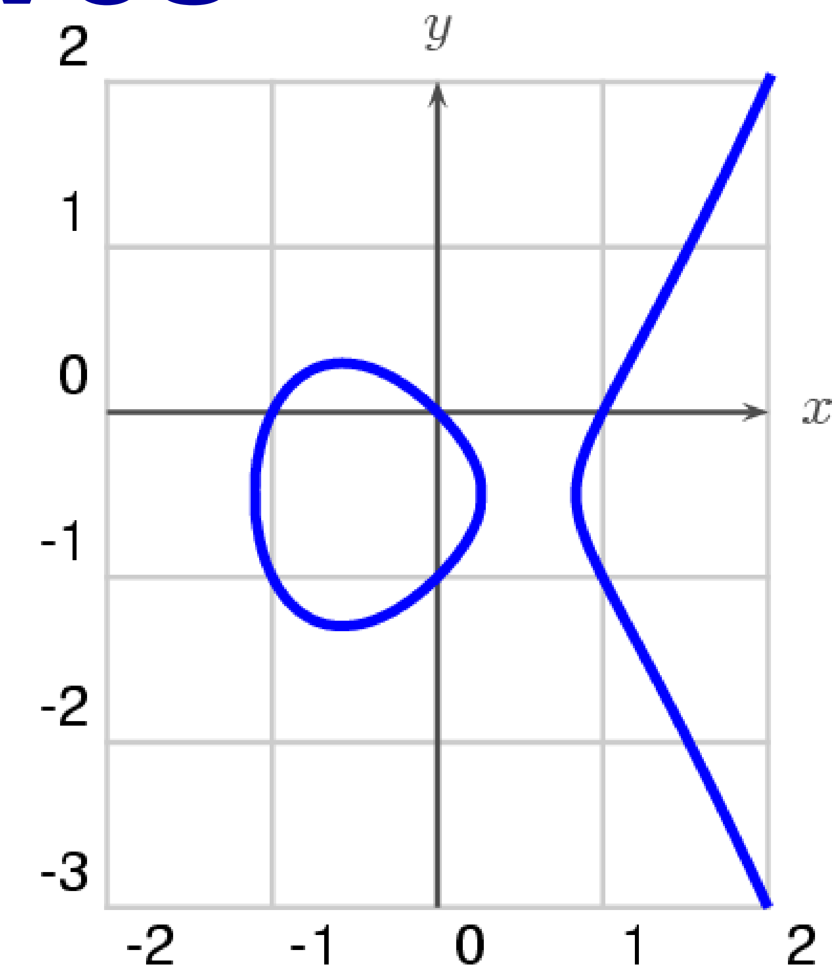


Elliptic Curves

$$x^3 + y^3 = 1$$

$$y^2 = x^3 + ax + b$$

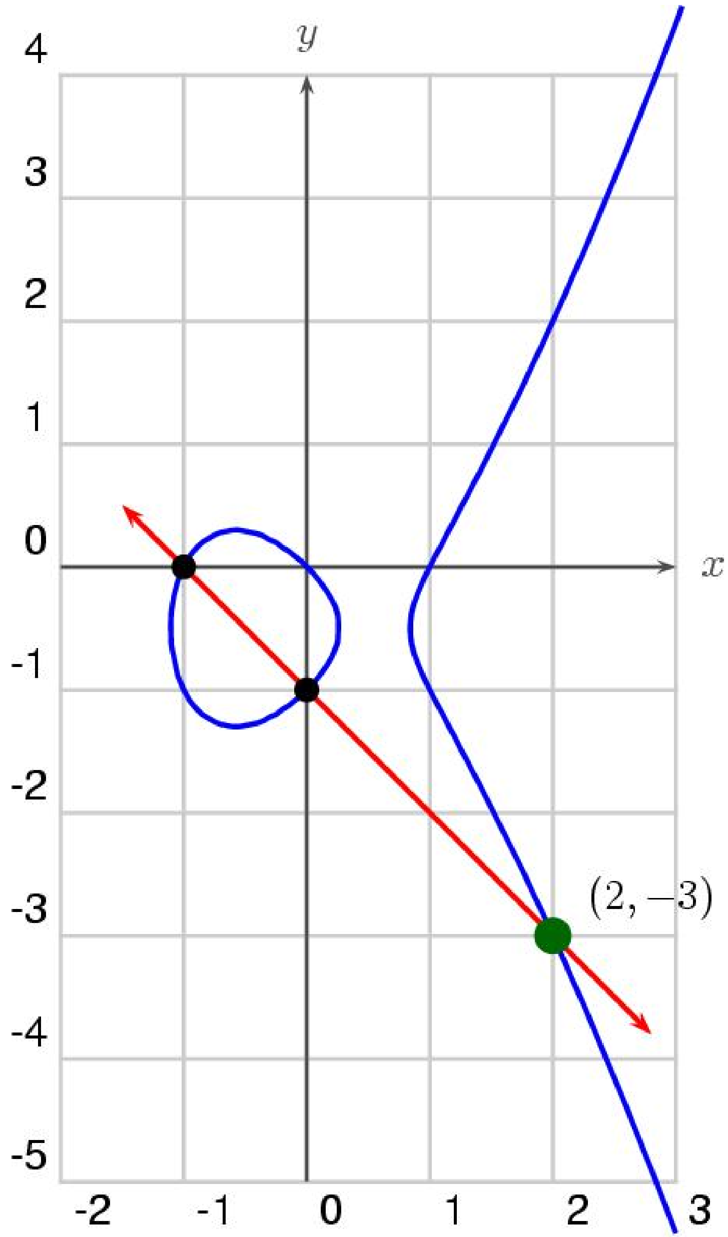
$$\cancel{3x^3 + 4y^3 + 5 = 0}$$



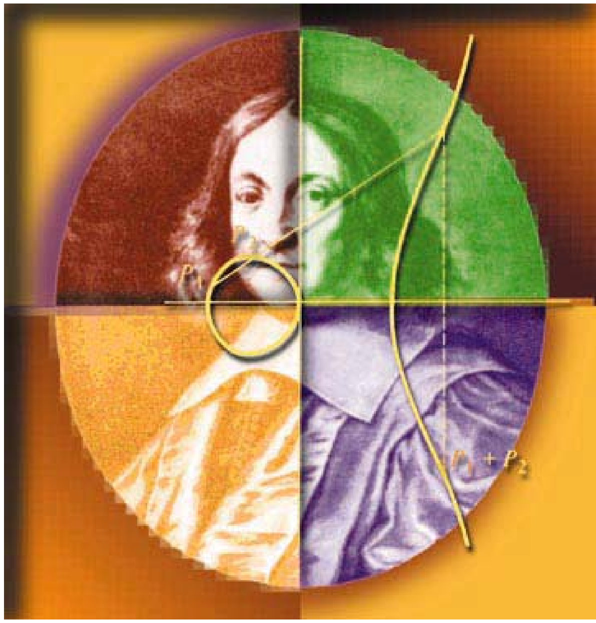
$$y^2 + y = x^3 - x$$

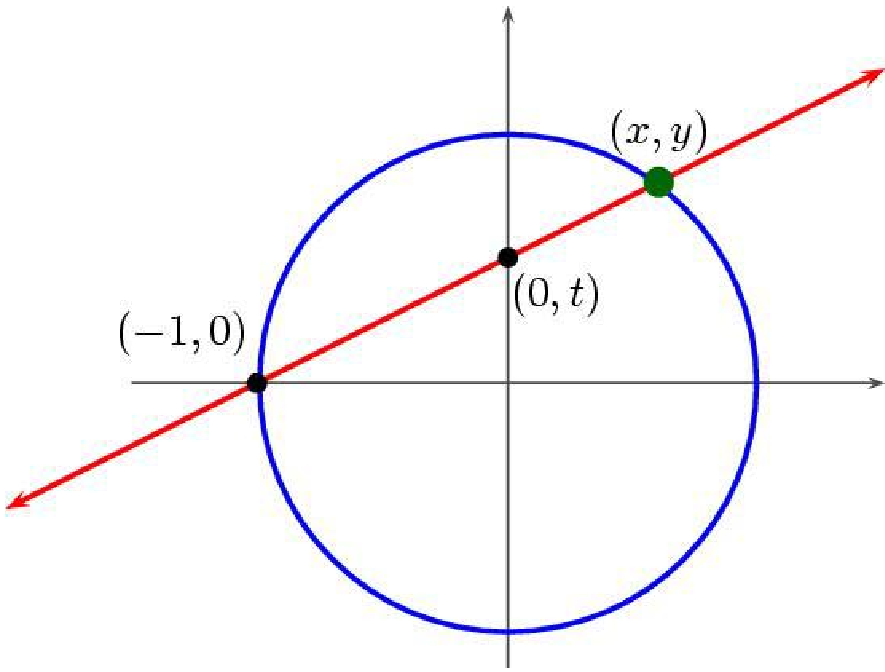
Cubic algebraic equations in two unknowns *x* and *y*.
Exactly the 1-dimensional compact algebraic groups.

The Secant Process

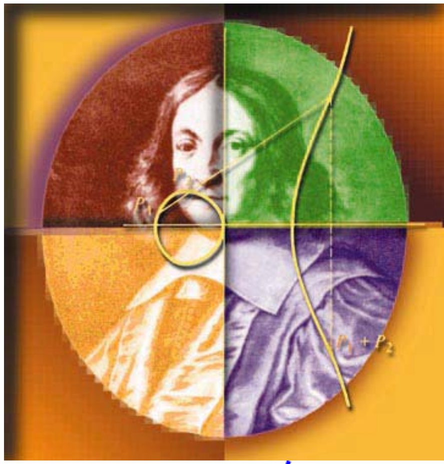


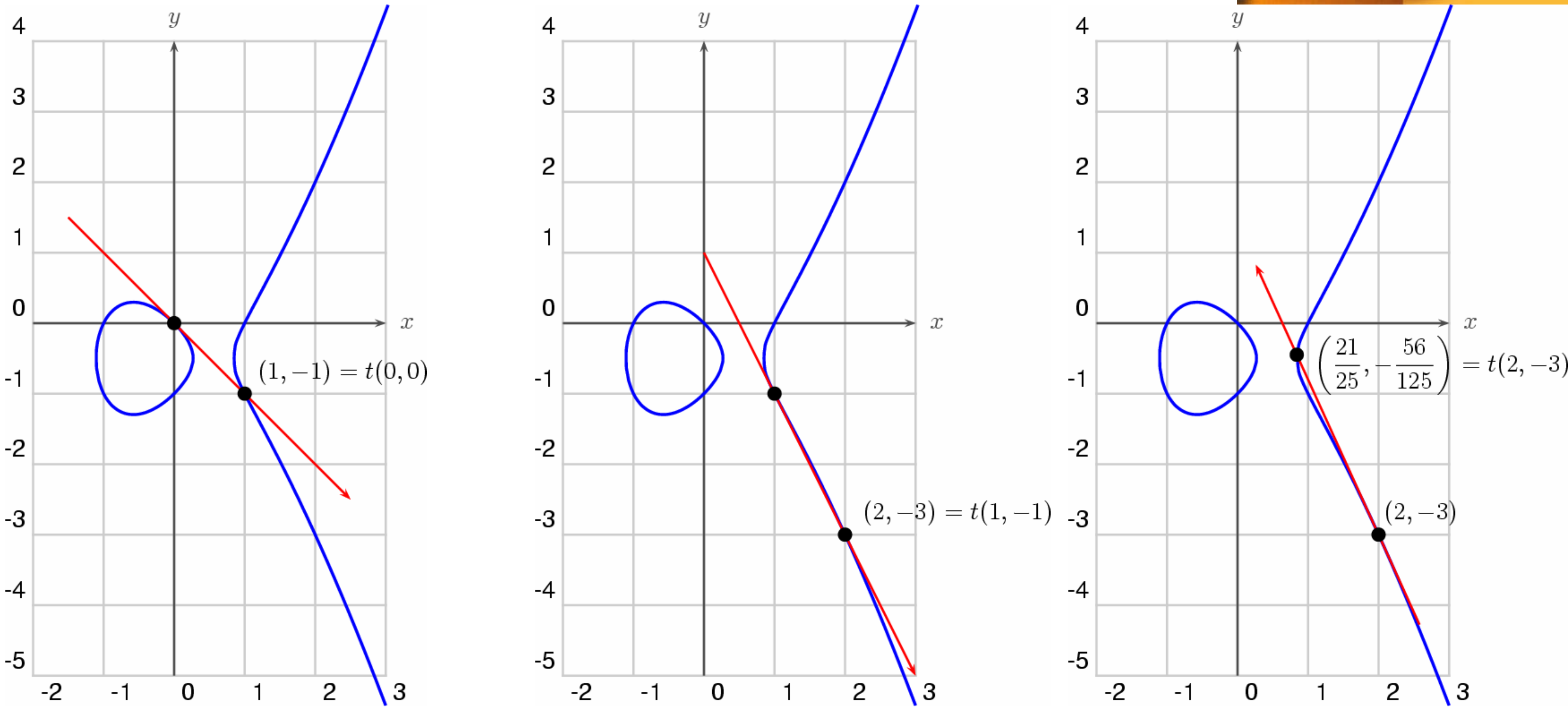
$$y^2 + y = x^3 - x$$





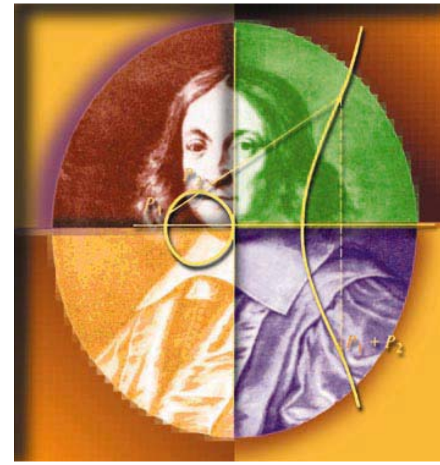
The Tangent Process

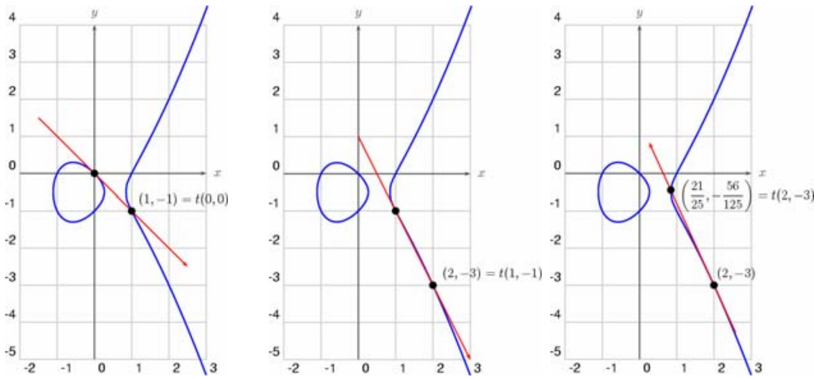




$$y^2 + y = x^3 - x$$

Big Points From Tangents





$$(0,0)$$

$$(1,-1)$$

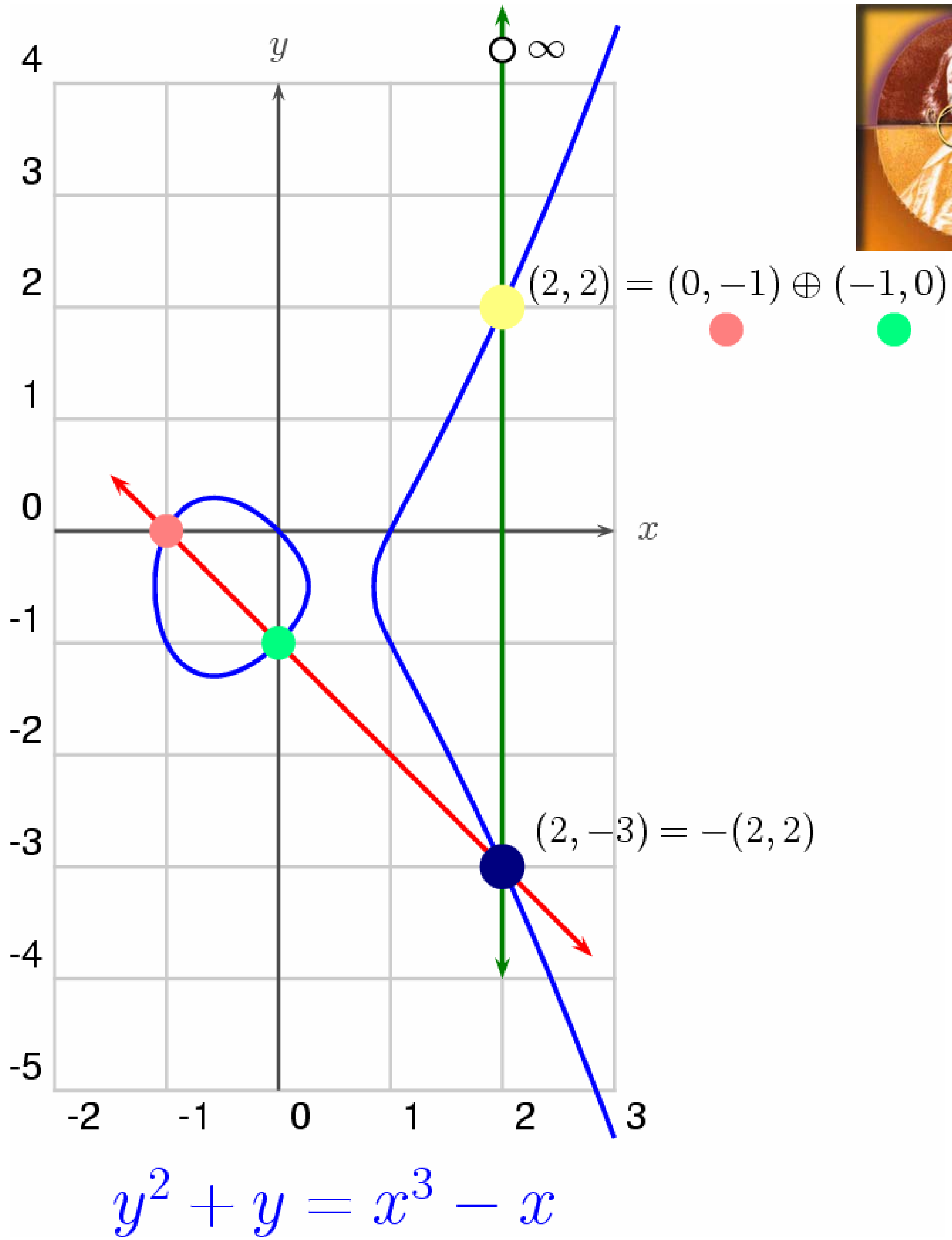
$$(2,-3)$$

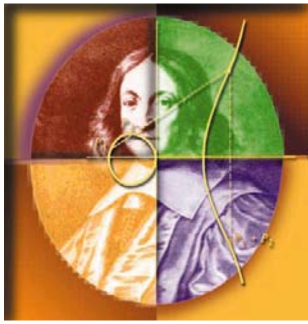
$$\left(\frac{21}{25}, -\frac{56}{125}\right)$$

$$\left(\frac{480106}{4225}, \frac{332513754}{274625}\right)$$

$$\left(\frac{53139223644814624290821}{1870098771536627436025}, -\frac{12282540069555885821741113162699381}{80871745605559864852893980186125}\right)$$

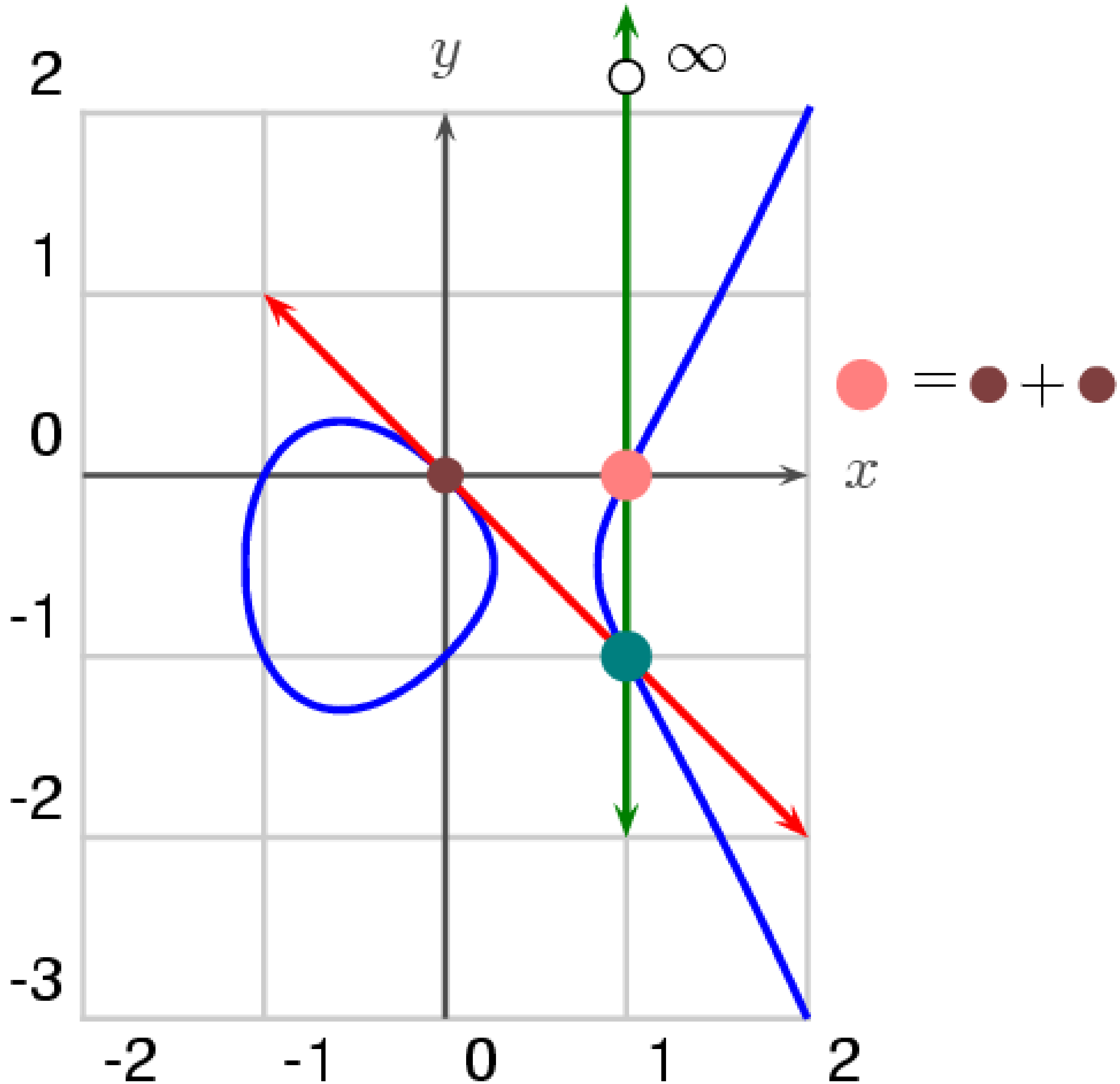
The Group Operation





$$\boxed{E(\mathbf{Q}) \cong \mathbf{Z}}$$

Group Law When P=Q



Mordell's Theorem

The group $E(\mathbf{Q})$ of rational points on an elliptic curve is finitely generated. Thus every ***rational*** point can be obtained from a ***finite*** number of solutions, using some combination of the secant and tangent processes.

1888-1972

The Simplest Solution Can Be Huge

Stolls

Simplest solution to $y^2 = x^3 + 7823$:

$$x = \frac{2263582143321421502100209233517777}{143560497706190989485475151904721}$$

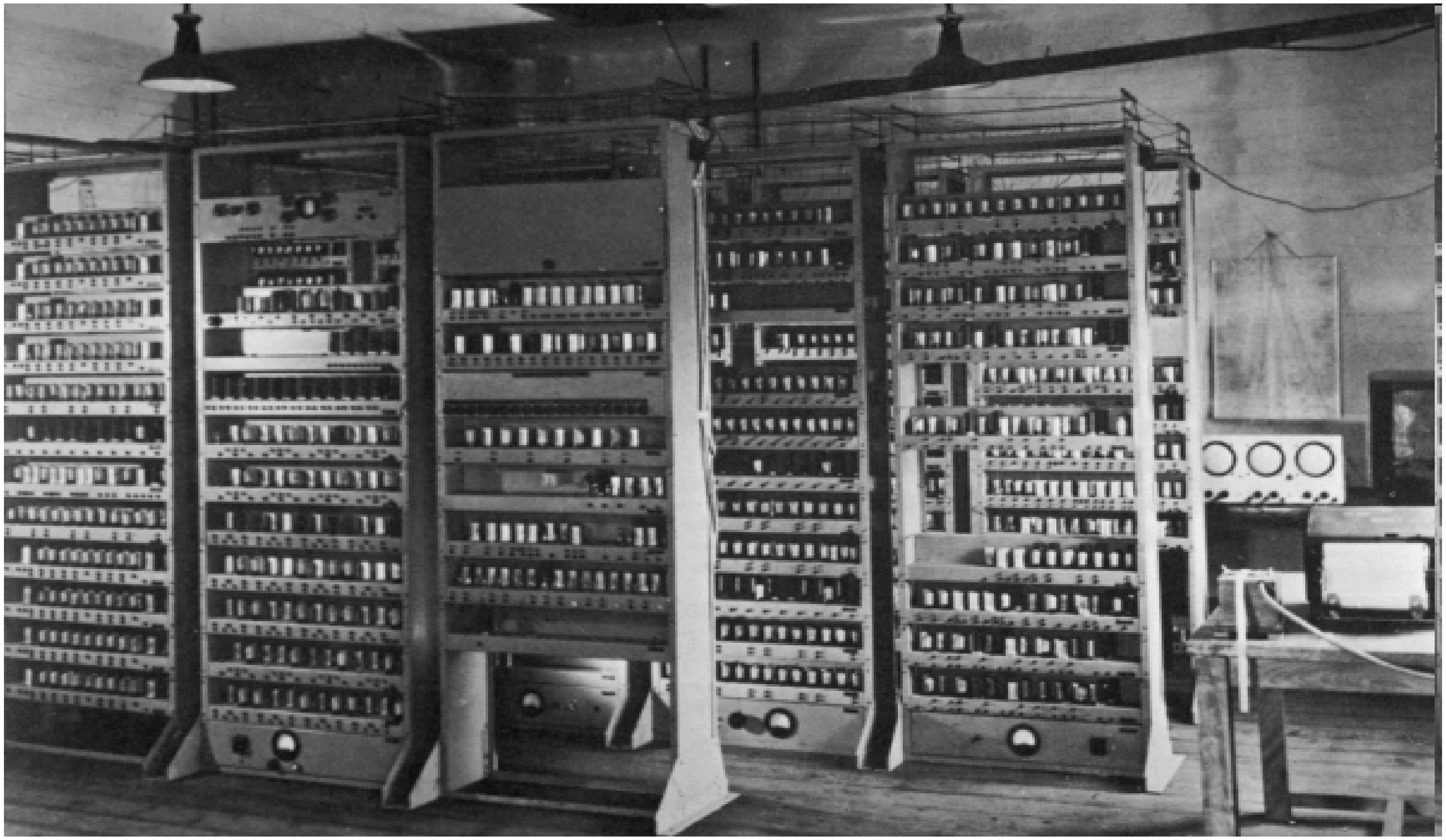
$$y = \frac{186398152584623305624837551485596770028144776655756}{1720094998106353355821008525938727950159777043481}$$

(Found by Michael Stoll in 2002)

Central Question

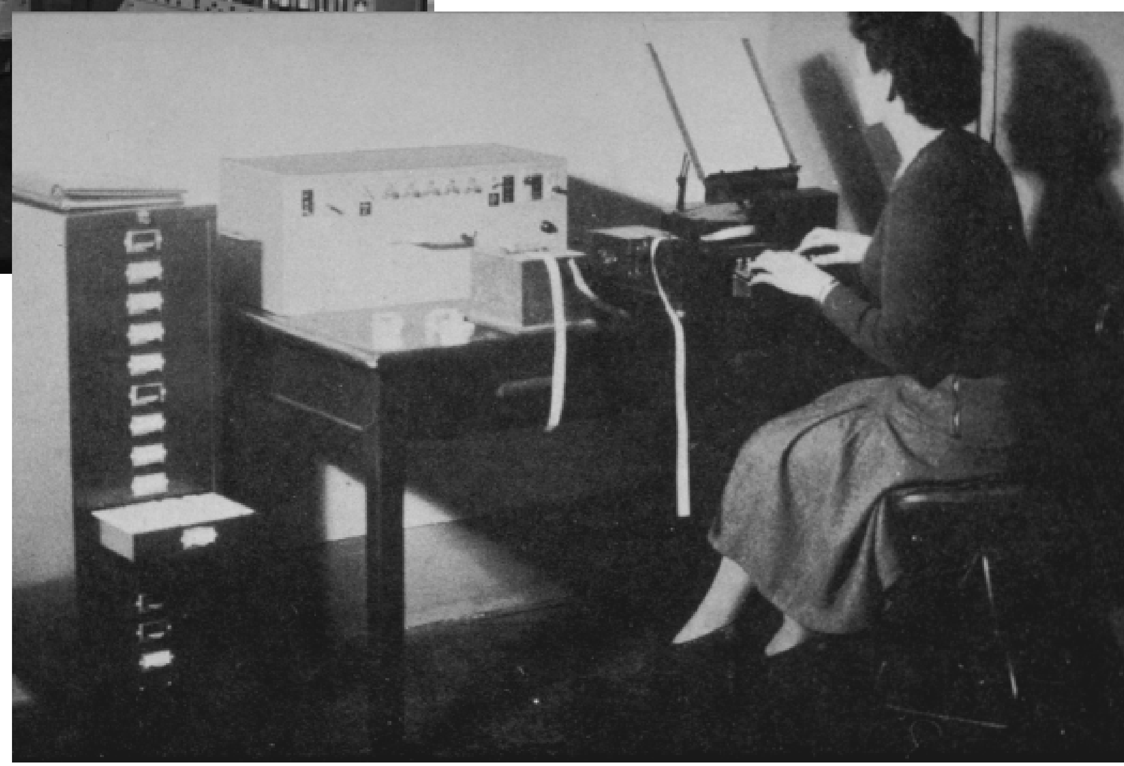
How many solutions are needed to generate all solutions to a cubic equation?

Birch and Swinnerton-Dyer

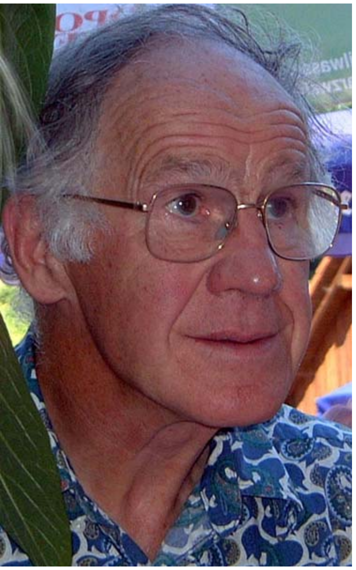


EDSAC in Cambridge, England

More EDSAC Photos



Electronic **D**elay
Storage **A**utomatic
Computer



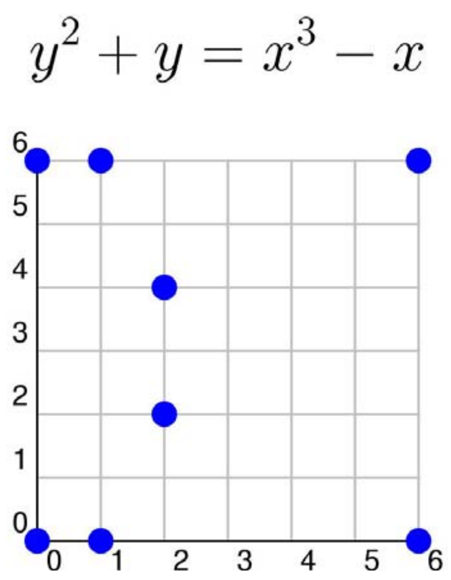
Conjectures Proliferated

Conjectures Concerning Elliptic Curves

By B.J. Birch, pub. 1965

“The subject of this lecture is rather a special one. I want to describe some computations undertaken by myself and Swinnerton-Dyer on EDSAC, by which we have calculated the zeta-functions of certain elliptic curves. As a result of these computations we have found an analogue for an elliptic curve of the Tamagawa number of an algebraic group; and conjectures (due to ourselves, due to Tate, and due to others) have proliferated. […] though the associated theory is both abstract and technically complicated, the objects about which I intend to talk are usually simply defined and often machine computable; experimentally we have detected certain relations between different invariants, but we have been unable to approach proofs of these relations, which must lie very deep.”

Solutions Modulo p



Consider solutions modulo a prime number:

$$p = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \ldots$$

We say that (a,b), with a,b integers, is a **solution modulo p** to

$$y^2 + y = x^3 - x$$

if

$$b^2 + b \equiv a^3 - a \pmod{p}.$$

For example,

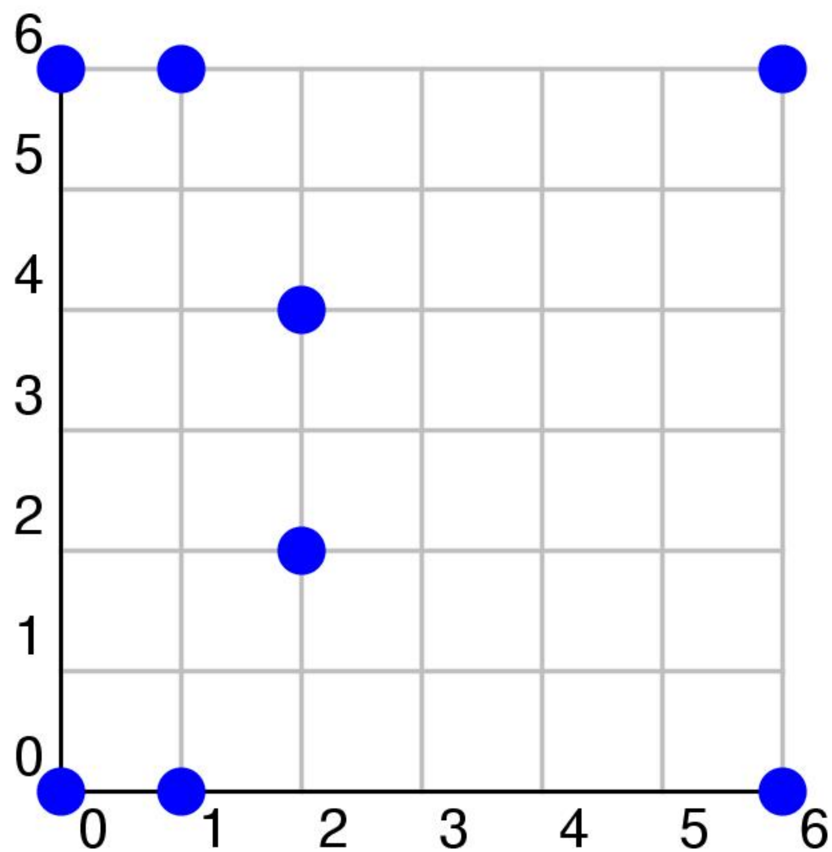
$$4^2 + 4 \equiv 2^3 - 2 \pmod{7}.$$

This idea generalizes to any cubic equation.

Counting Solutions

$$N(p) = \# \text{ of solutions } \pmod p \leq p^2$$

$$y^2 + y = x^3 - x$$



$$N(7) = 8$$

The Error Term (Hasse's Bound)

1898-1979

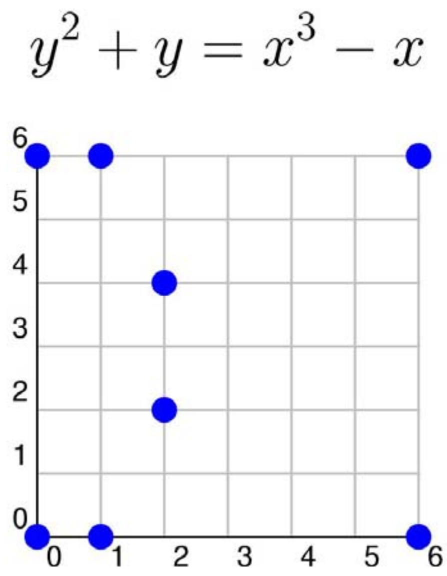
Write $N(p) = p + A(p)$ with error term

$$\boxed{|A(p)| \le 2\sqrt{p}}$$

For example, $N(7) = 8$ so $A(7) = 1.$

More Primes

$$y^2 + y = x^3 - x$$



$$A(2) = 2$$
$$A(3) = 3$$
$$A(5) = 2$$
$$A(7) = 1$$
$$A(11) = 5$$
$$\vdots$$

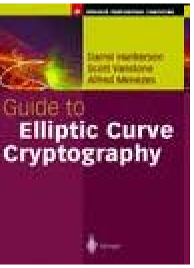
$N(p) =$ number of soln's

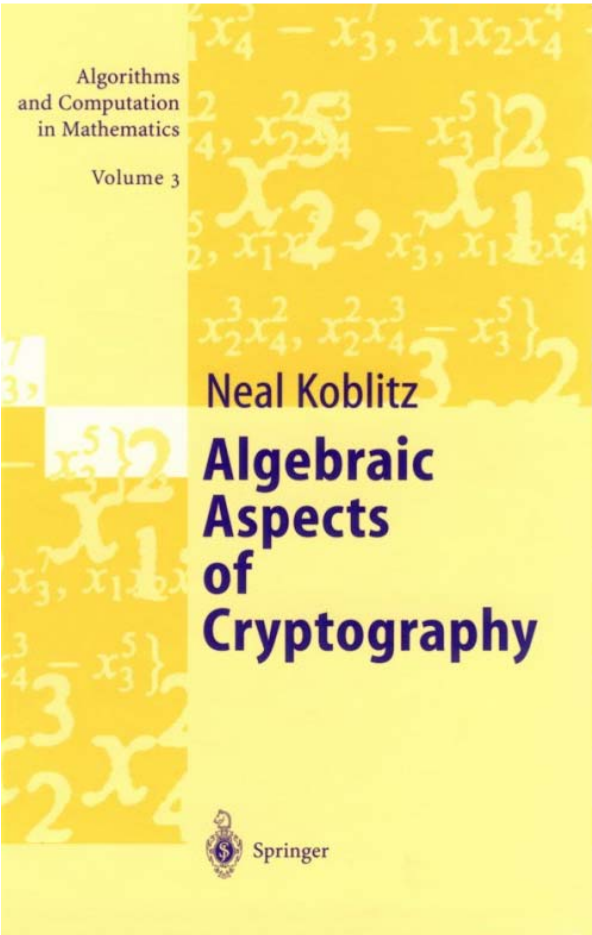
$N(p) = p + A(p)$

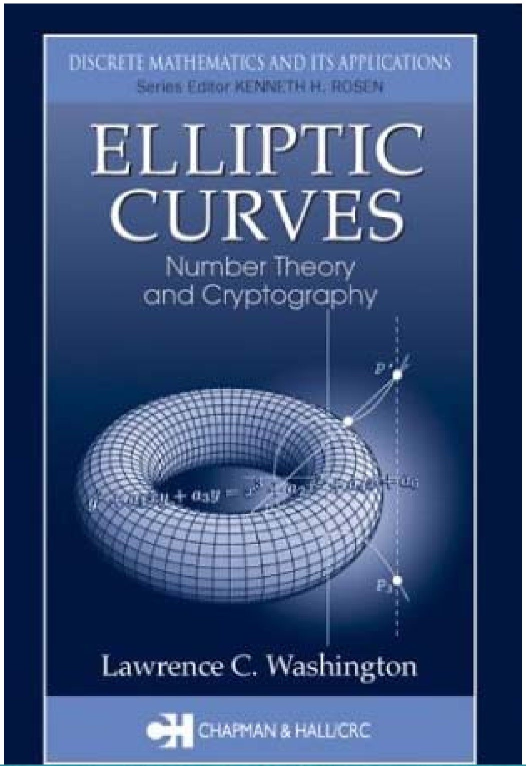
Thus $N(p) > p$ for these primes p.

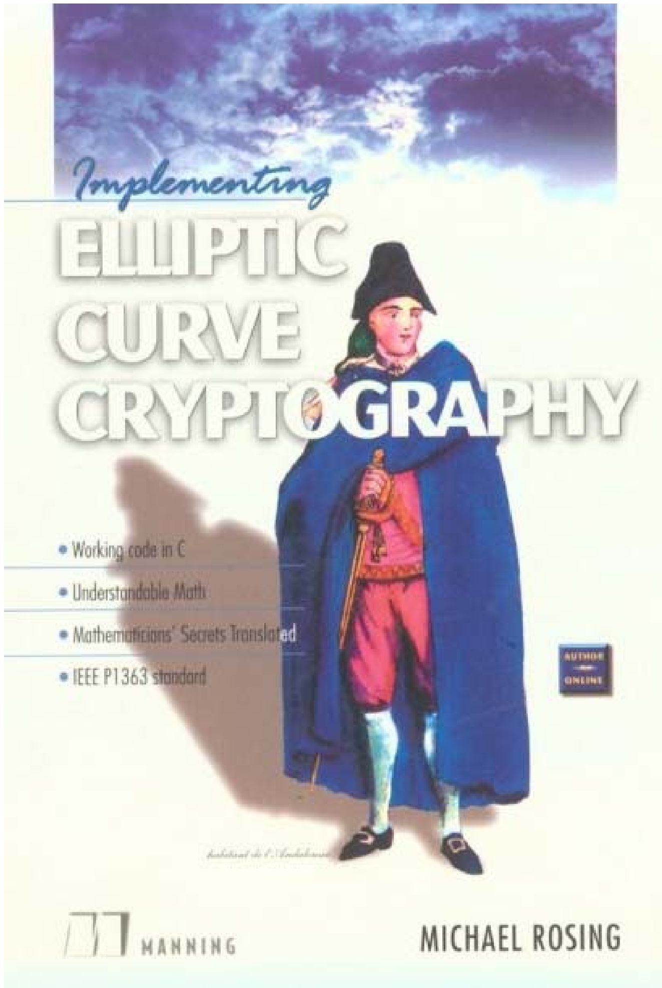
Continuing: $A(13) = 2,\ A(17) = 0,\ A(19) = 0,\ A(23) = \text{-}2,\ A(29) = \text{-}6,\ A(31) = 4,\ \ldots.$

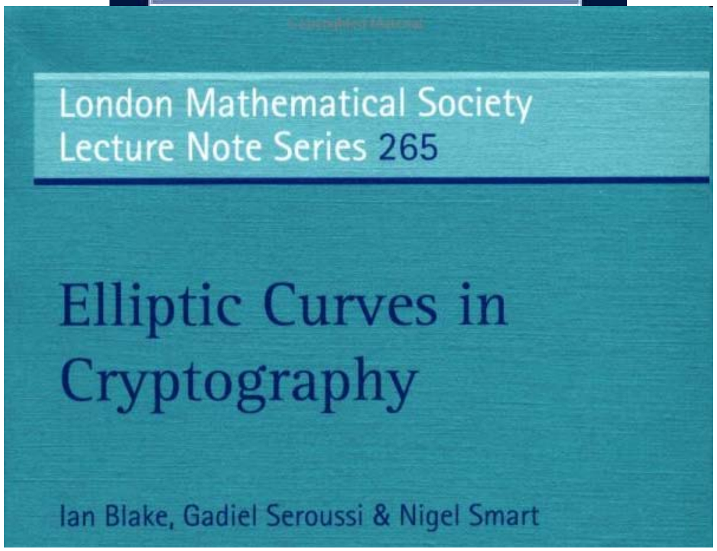
Commercial Break: Cryptographic Application











Guess

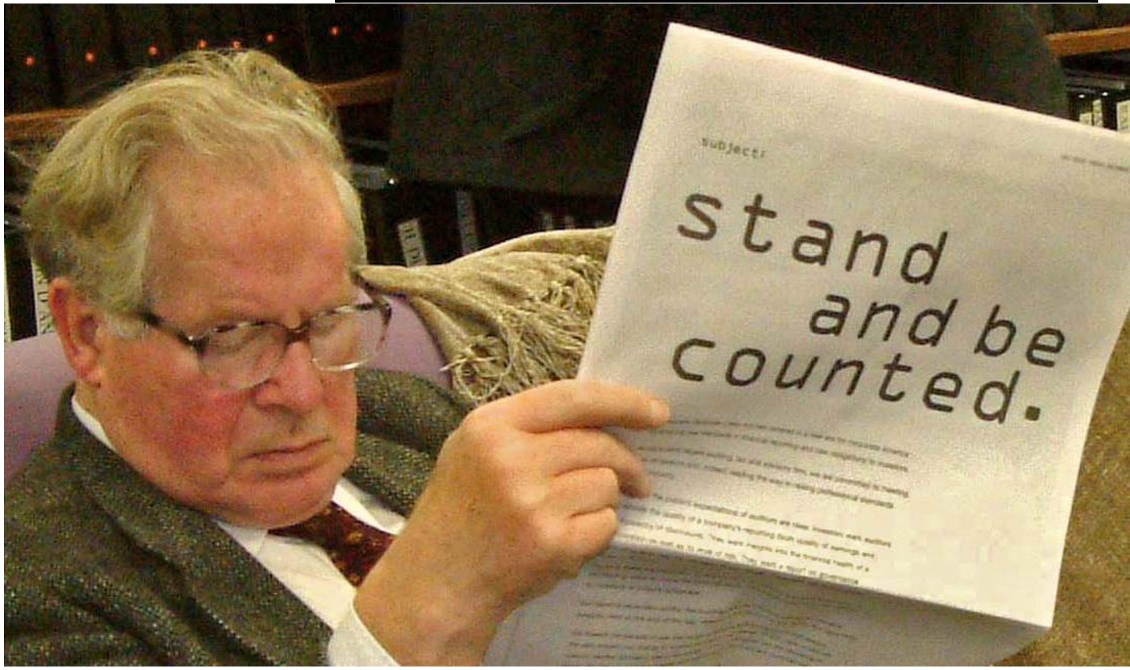
M	$\prod_{p \le M} \frac{p}{N(p)}$
10	0.083…
100	0.032…
1000	0.021…
10000	0.013…
100000	0.010…

If a cubic curve has infinitely many solutions, then probably $N(p)$ is **larger** than p, for many primes p.

Thus maybe the product of terms

$$\prod_{p \le M} \frac{p}{N(p)}$$

will tend to 0 as *M* gets larger.



Swinnerton-Dyer at AIM

The *L*-function

$$L(E,s)=\prod\frac{1}{1-A(p)\cdot p^{-s}+p\cdot p^{-2s}}$$

The product is over all primes *p*. (At a finite number of primes the factor must be slightly adjusted.)

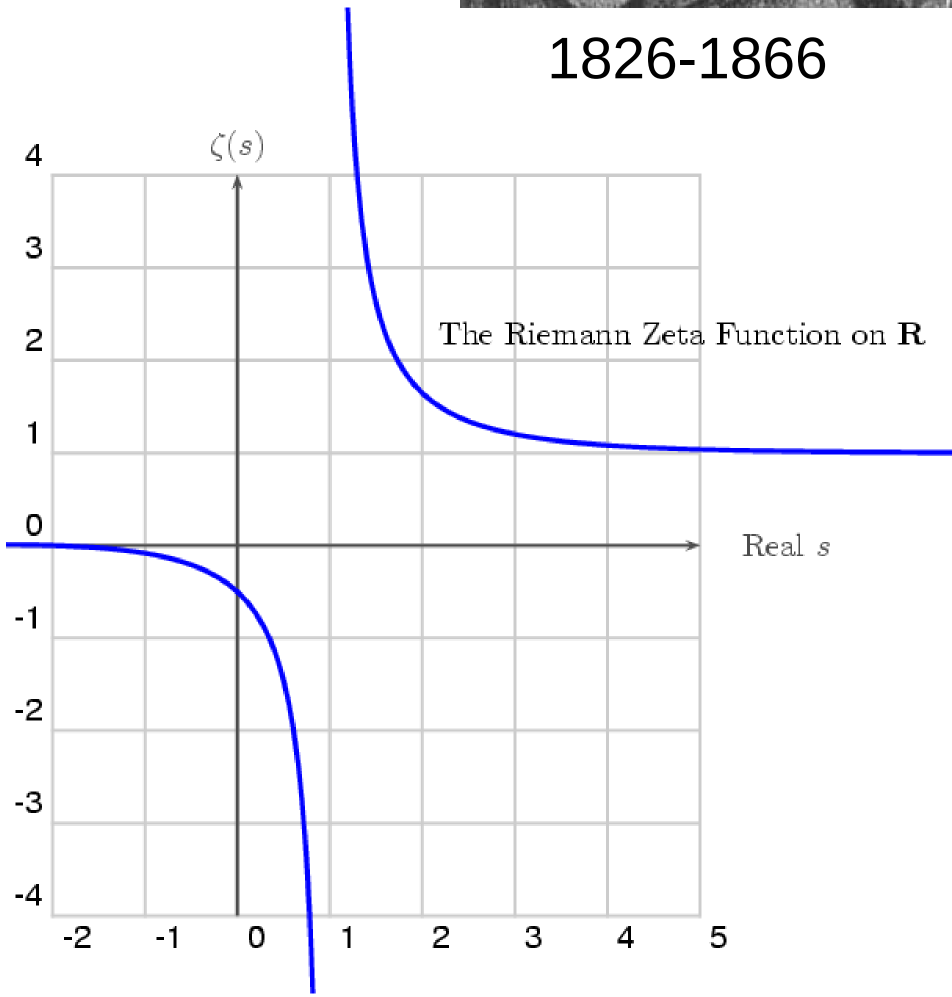
Product converges for $\mathrm{Re}(s)>\frac{3}{2}$

The Riemann Zeta Function

$$\zeta(s) = \prod_{\text{all primes } p} \frac{1}{1-p^{-s}}$$

1826-1866

Zeta extends to an analytic function everywhere but at 1.



An Analytic Function

Swinnerton-Dyer

Thus Bryan Birch and Sir Peter Swinnerton-Dyer defined an analytic function $L(E,s)$ such that formally:

$$L(E,1) = "\prod \frac{p}{N(p)}"$$

The Birch and Swinnerton-Dyer Conjecture

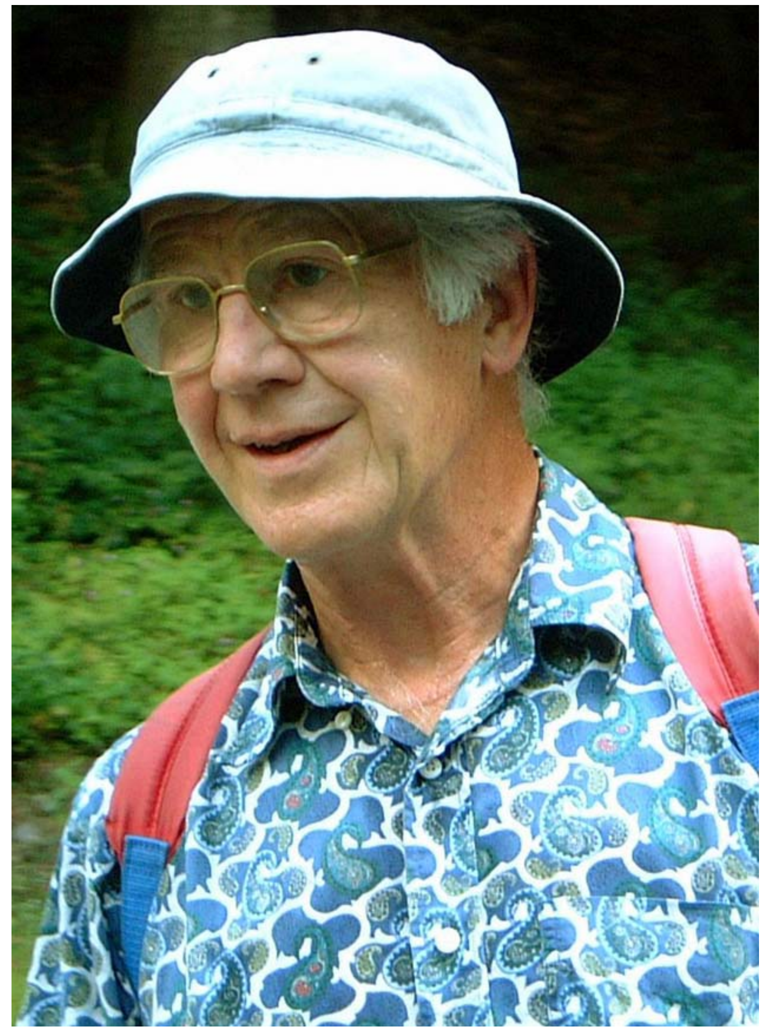
The order of vanishing of

$$L(E,s)$$

at 1 equals the rank of the group $E(\mathbf{Q})$ of all rational solutions to E:

$$\mathrm{ord}_{s=1} L(E,s) = \mathrm{rank}\, E(\mathbf{Q})$$

CMI: $1000000 reward for a proof.



Bryan Birch

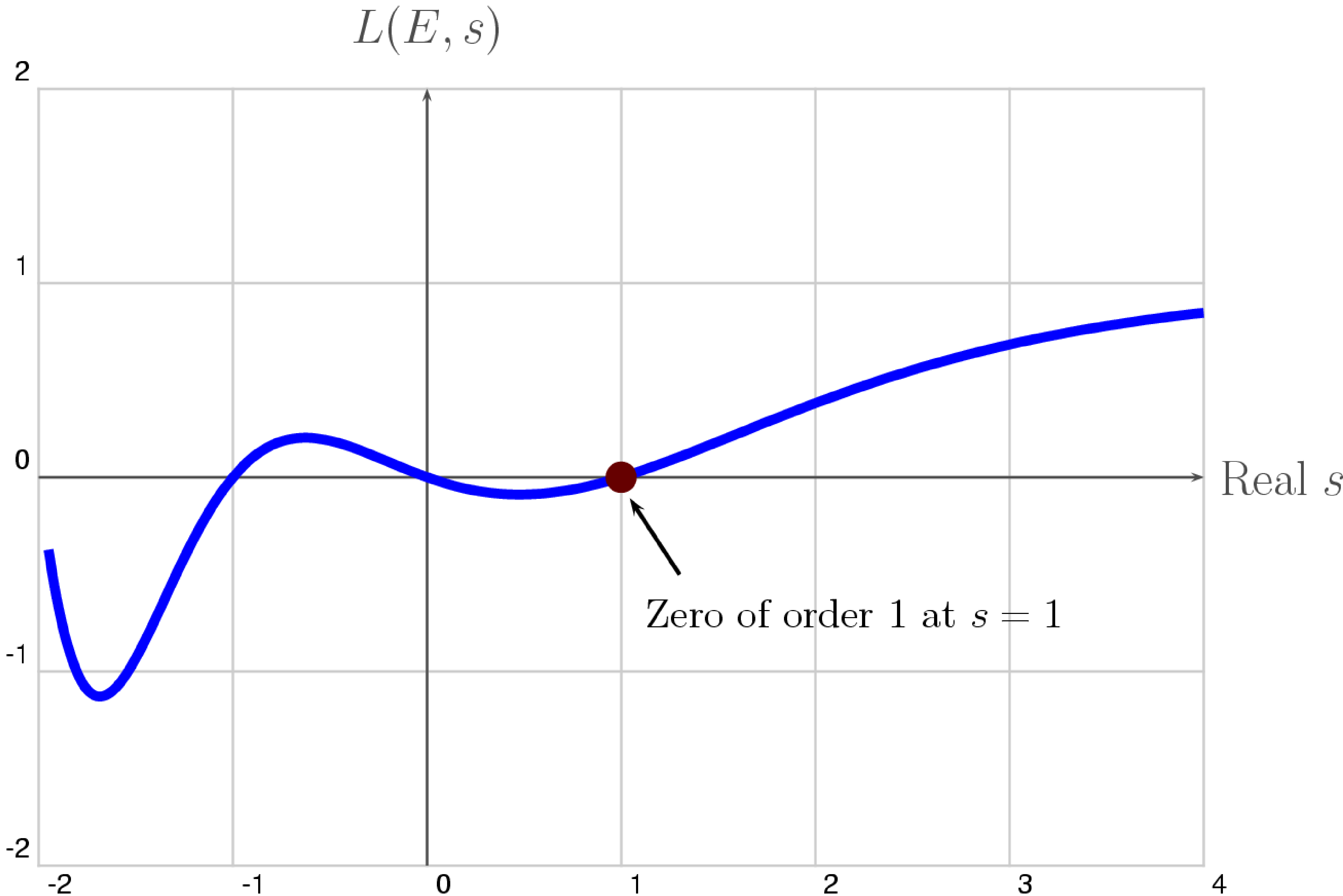
The **Modularity Theorem**

A. Wiles

Theorem (2000, Wiles, Taylor, and Breuil, Conrad, Diamond) *The curve E arises from a "modular form", so* $L(E,s)$ *extends to an analytic function on the whole complex plane.*

(This modularity is the key input to Wiles's proof of Fermat's Last Theorem.)

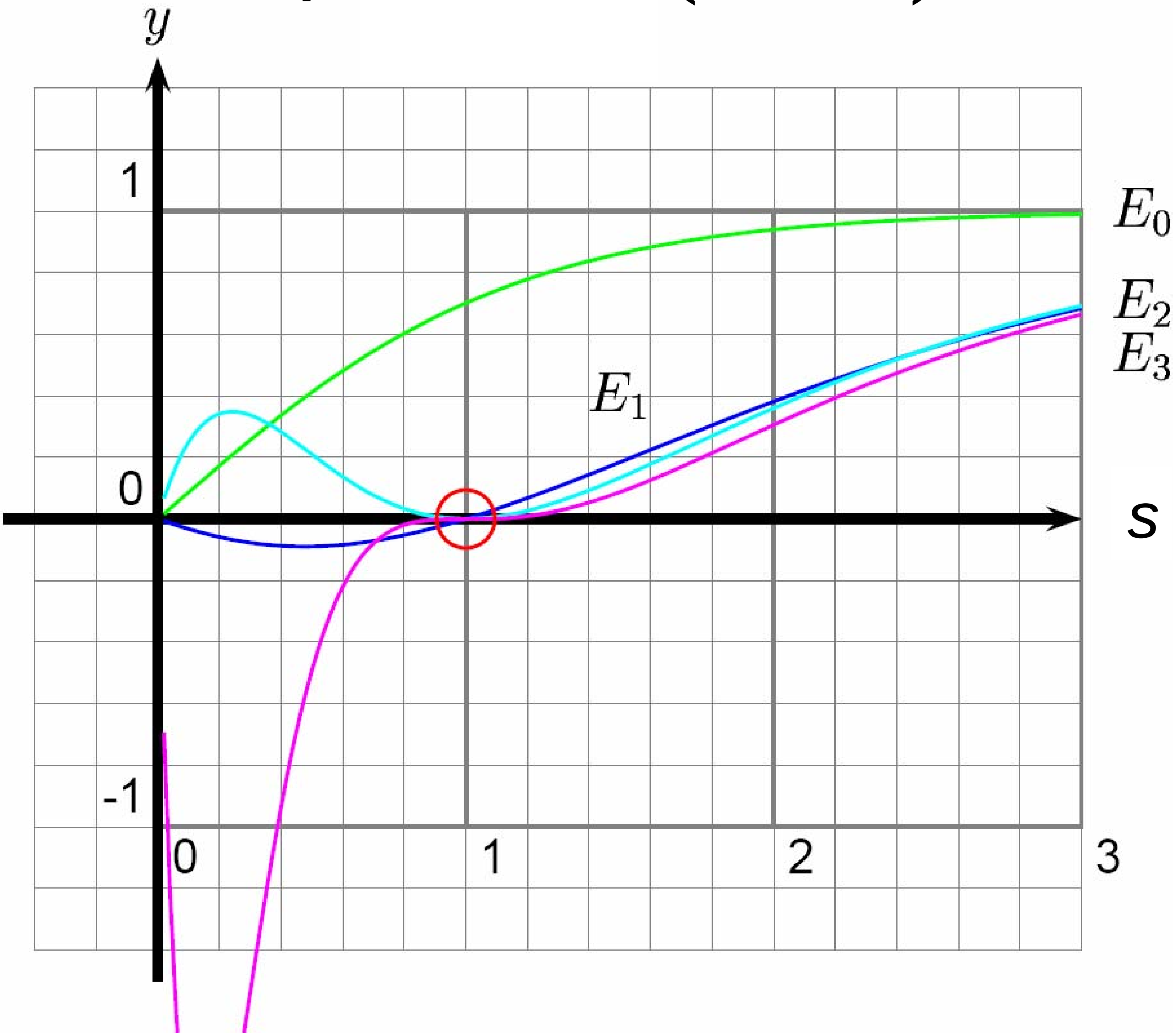
R. Taylor



L-series for $y^2 + y = x^3 - x$

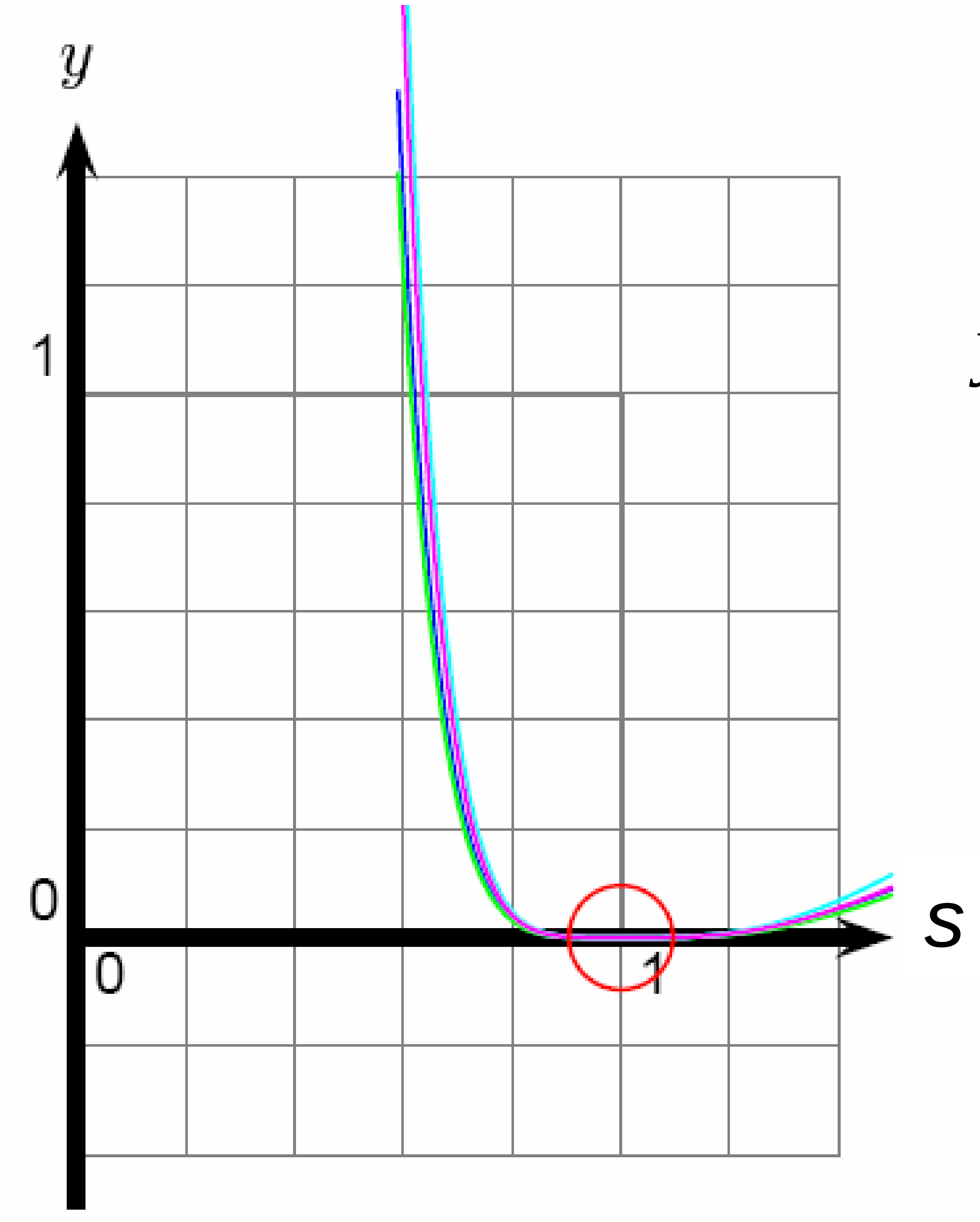
Birch and Swinnerton-Dyer

Some Graphs of $L(E, s)$ for s real



The graph of $L(E_r, s)$ vanishes to order *r*.

Examples of $L(E,s)$ that ***appear to*** vanish to order 4



$$y^2 + xy = x^3 - x^2 - 79x + 289$$

Congruent Number Problem

Open Problem: Decide whether an integer *n* is the area of a right triangle with rational side lengths.

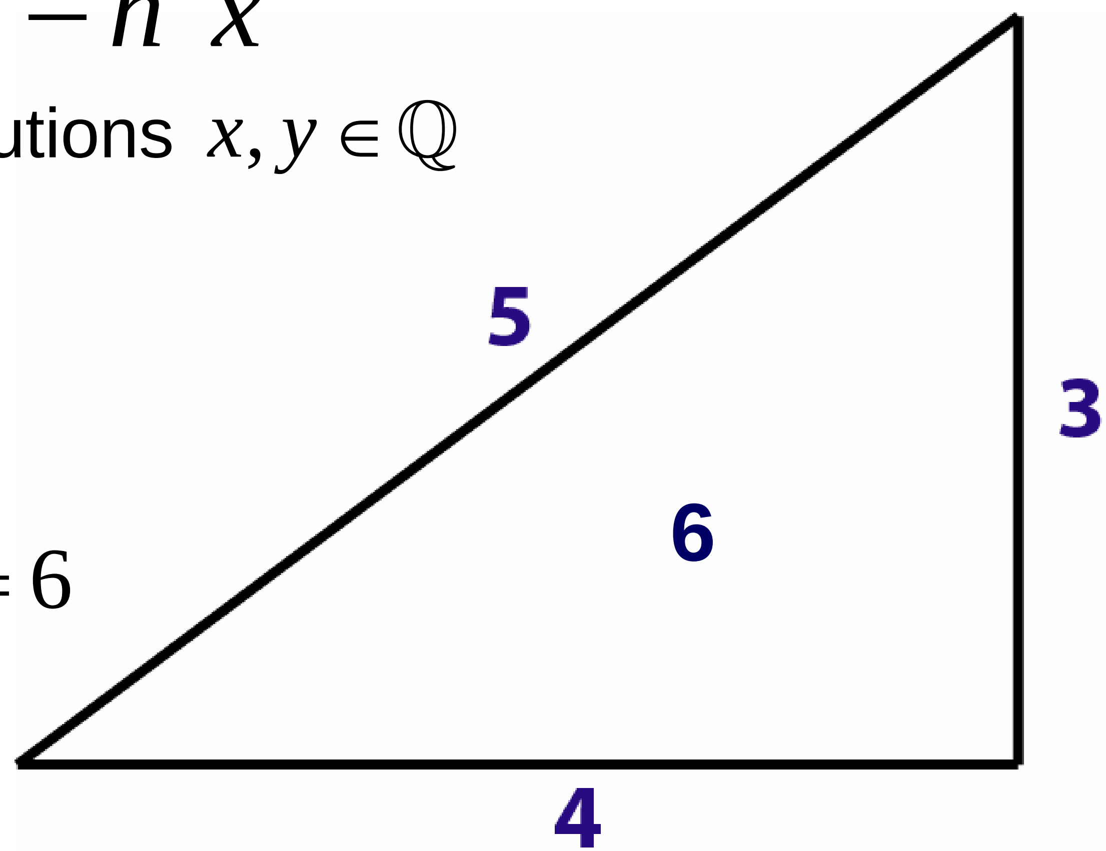
Fact: Yes, precisely when the cubic equation

$$y^2 = x^3 - n^2 x$$

has infinitely many solutions $x, y \in \mathbb{Q}$

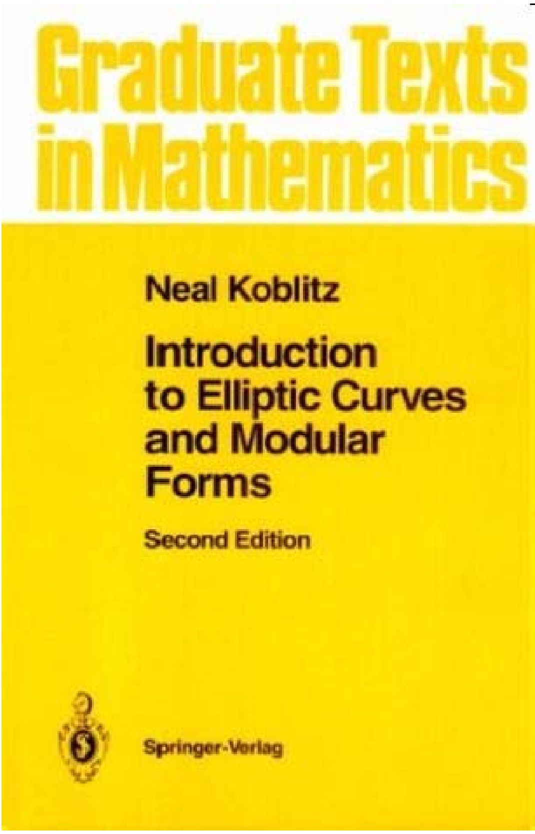
$$n = 6$$

$$A = \frac{1}{2} b \times h = \frac{1}{2} 3 \times 4 = 6$$



Connection with BSD Conjecture

Theorem (Tunnell): The Birch and Swinnerton-Dyer conjecture implies that there is a simple algorithm that decides whether or not a given integer n is a congruent number.



See [Koblitz] for more details

Benedict Gross

Gross-Zagier Theorem

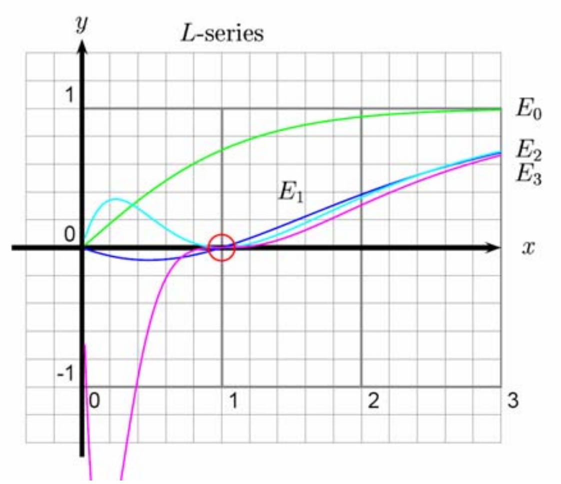
Don Zagier

When the order of vanishing of $L(E,s)$ at 1 is exactly 1, then E has rank at least 1.

Subsequent work showed that if the order of vanishing is exactly 1, then the rank equals 1, so the Birch and Swinnerton-Dyer conjecture is true in this case.

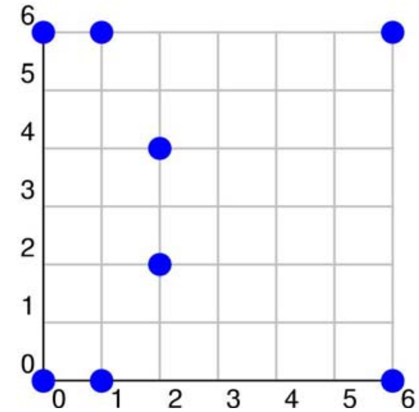
Kolyvagin's Theorem

Theorem. If $L(E,1)$ is nonzero then the rank is zero, i.e., $E(\mathbf{Q})$ is finite.

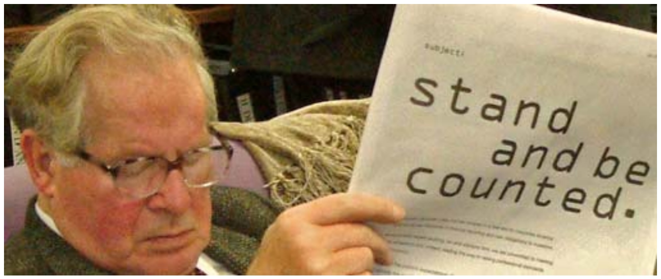


$$\operatorname{ord}_{s=1} L(E,s) = \operatorname{rank} E(\mathbf{Q})$$

$$y^2 + y = x^3 - x$$



Thank You



Acknowledgments

- Benedict Gross
- Keith Conrad
- Ariel Shwayder (graphs of $L(E,s)$)

Mazur's Theorem

For any two rational a, b, there are at most 15 rational solutions (x,y) to

$$y^2 = x^3 + ax + b$$

with finite order.

Theorem **(8)**. — *Let* Φ *be the torsion subgroup of the Mordell-Weil group of an elliptic curve defined over* $\mathbf{Q}$. *Then* Φ *is isomorphic to one of the following* 15 *groups*:

$$\mathbf{Z}/m.\mathbf{Z} \quad \textit{for} \quad m \leqslant 10 \quad \textit{or} \quad m = 12$$

or:

$$(\mathbf{Z}/2.\mathbf{Z}) \times (\mathbf{Z}/2\nu.\mathbf{Z}) \quad \textit{for} \quad \nu \leqslant 4.$$