

Computing Modular Abelian Varieties

Computing Modular Abelian Varieties over \mathbb{Q}

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1. The Challenge -- Extend Cremona

Challenge: Create a table of data like [Cremona's book](#), but for all modular abelian varieties A_f with $f \in S_2(\Gamma_0(N))$ for $N \leq 1000$.

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TABLE 1: ELLIPTIC CURVES 90C–99A

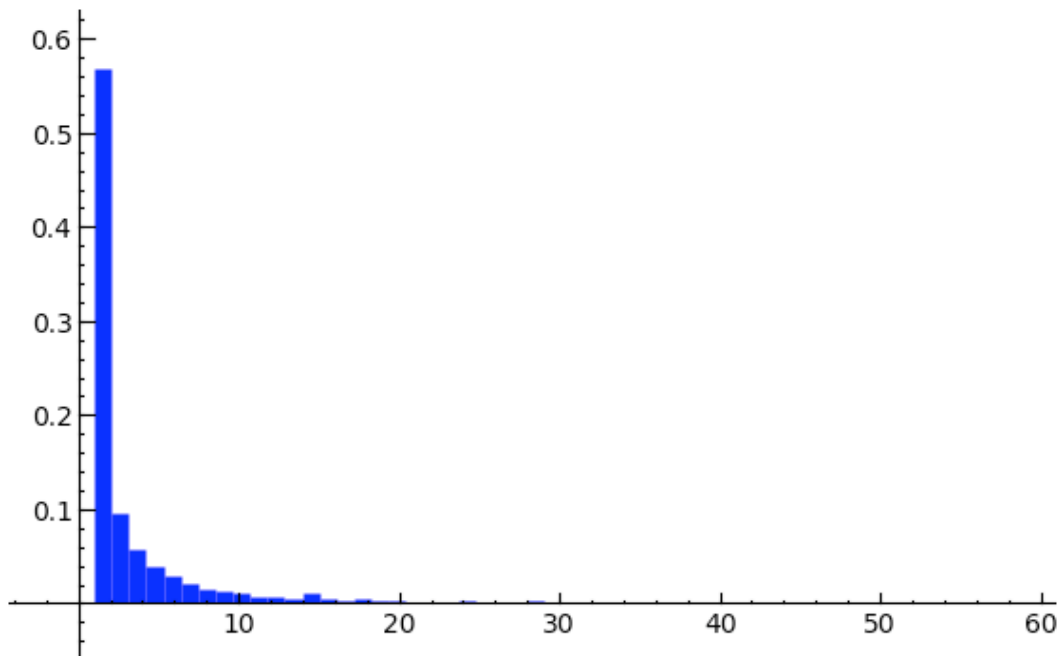
	a_1	a_2	a_3	a_4	a_6	r	$ T $	s	$\text{ord}(\Delta)$	$\text{ord}_-(j)$	c_p	Kodaira	Is
90 $N = 90 = 2 \cdot 3^2 \cdot 5$ (continued)													
C1(E)	1	-1	1	13	-61	0	4	-	4, 9, 1	4, 3, 1	4, 4, 1	I_4, I_3^*, I_1	2 : 2
C2(F)	1	-1	1	-167	-709	0	4	+	2, 12, 2	2, 6, 2	2, 4, 2	I_2, I_6^*, I_2	2 : 1
C3(G)	1	-1	1	-122	1721	0	12	-	12, 7, 3	12, 1, 3	12, 4, 3	I_{12}, I_1^*, I_3	2 : 6
C4(I)	1	-1	1	-2597	-50281	0	2	+	1, 9, 4	1, 3, 4	1, 2, 4	I_1, I_3^*, I_4	2 : 2
C5(H)	1	-1	1	-617	5231	0	2	+	1, 18, 1	1, 12, 1	1, 4, 1	I_1, I_{12}^*, I_1	2 : 2
C6(J)	1	-1	1	-3002	63929	0	12	+	6, 8, 6	6, 2, 6	6, 4, 6	I_6, I_2^*, I_6	2 : 3
C7(L)	1	-1	1	-4082	14681	0	6	+	3, 7, 12	3, 1, 12	3, 2, 12	I_3, I_1^*, I_{12}	2 : 6
C8(K)	1	-1	1	-48002	4059929	0	6	+	3, 10, 3	3, 4, 3	3, 4, 3	I_3, I_4^*, I_3	2 : 6
91 $N = 91 = 7 \cdot 13$ (2 isogeny classes)													
A1(A)	0	0	1	1	0	1	1	-	1, 1	1, 1	1, 1	I_1, I_1	
B1(B)	0	1	1	-7	5	1	3	-	1, 1	1, 1	1, 1	I_1, I_1	3 : 2
B2(C)	0	1	1	13	42	1	3	-	3, 3	3, 3	3, 3	I_3, I_3	3 : 1
B3(D)	0	1	1	-117	-1245	1	1	-	9, 1	9, 1	9, 1	I_9, I_1	3 : 2

...

There are 5951 new Simple Modular Abelian Varieties Factors of $J_0(N)$ up to Level 1000. The largest dimension is 55.

Dim	Count	Dim	Count	Dim	Count	Dim	Count
1	2463	15	35	29	6	43	2
2	1195	16	33	30	4	44	1
3	616	17	25	31	5	45	3
4	373	18	32	32	5	46	2

5	255
6	182
7	132
8	95
9	85
10	72
11	49
12	50
13	32
14	39
19	15
20	19
21	11
22	8
23	10
24	17
25	7
26	10
27	7
28	9
33	5
34	3
35	4
36	4
37	4
38	6
39	2
40	2
41	5
42	2
47	3
49	1
50	1
51	2
53	1
54	1
55	1
≥ 56	0



Dimensions of simple factors

Level N	Dimensions
11	[1]
14	[1]
15	[1]

17	[1]
19	[1]
20	[1]
21	[1]
23	[2]
...	...
388	[3, 5]
389	[1, 2, 3, 6, 20]
390	[1, 1, 1, 1, 1, 1, 1, 2]
...	...
990	[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2]
991	[33, 49]
992	[2, 2, 3, 3, 4, 4, 6, 6]
993	[2, 3, 10, 10, 15, 15]
994	[1, 1, 1, 1, 1, 1, 1, 2, 3, 3, 3, 3, 3, 4, 7]
995	[1, 1, 3, 9, 13, 19, 21]
996	[1, 1, 1, 2, 4, 5]
997	[1, 1, 1, 4, 5, 5, 23, 42]
998	[2, 2, 2, 3, 3, 4, 4, 9, 13]
999	[1, 1, 2, 2, 6, 6, 6, 6, 8, 10]
1000	[2, 2, 2, 2, 4, 4, 4, 4]

2. Defining Data for a Modular Abelian Variety

Many attempts to generalize Cremona's tables have got hung up on the a -invariants (the Weierstrass equation), or special cases, and don't get very far toward our stated goal. They always restrict to low dimension. That said, these attempts are all fascinating contributions:

- X. Wang, 1995, 2-dimensional simple factors
- Hasegawa, 1995, quotients of modular curves
- Galbraith, 1996, thesis on models for modular curves and their quotients
- Students of Frey
- Quer, Lario, Gonzalez, et al.: the \mathbf{Q} -curves group in Barcelona

We will thus **replace defining equations by something else**. For any positive integer N , the group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \text{ and } N \mid c \right\}$$

acts via linear fractional transformations on the extended upper half plane $\mathbf{h}^* = \mathbf{h} \cup \mathbf{P}^1(\mathbf{Q})$. Let

$$X_0(N) = \Gamma_0(N) \backslash \mathbf{h}^*$$

and

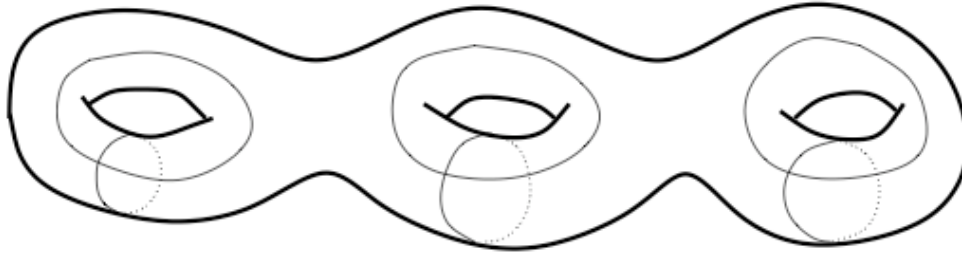
$$J = J_0(N) = \text{Jac}(X_0(N)),$$

which are both defined over \mathbf{Q} .

Definition (Modular Abelian Variety): An abelian variety A for which there is a finite degree map $A \rightarrow J_0(N)$ for some N is called Γ_0 -modular.

Notes:

1. In this talk to fix ideas we only consider $\Gamma_0(N)$ and $J_0(N)$, but most things we say make sense for $\Gamma_1(N)$ and $J_1(N)$.
2. Wiles, Taylor, et al. proved that every elliptic curve over \mathbf{Q} is modular.
3. If A is simple and $\text{End}(A/\mathbf{Q}) \otimes \mathbf{Q}$ is a totally real number field of degree $\dim(A)$, then Serre's conjecture and a theorem of Ribet imply that A is $\Gamma_0(N)$ -modular. In fact, Serre's conjecture was recently proved by Khare and Wintenberger.



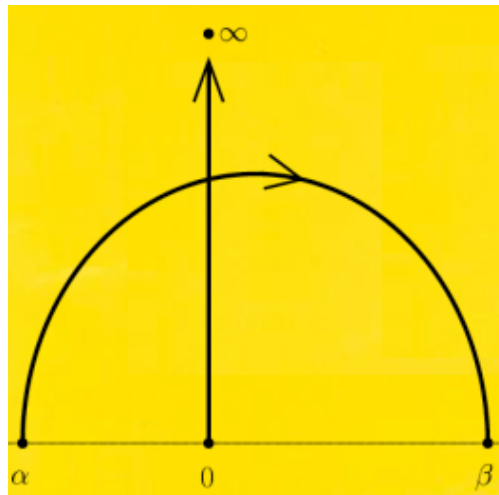
$$H_1(X_0(39), \mathbf{Z}) \cong \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$$

Figure 1: The homology of $X_0(39)$.

Consider the integral and rational homology of the modular curve $X_0(N)$:

$$L_J = H_1(X_0(N), \mathbf{Z}) \subset V_J = H_1(X_0(N), \mathbf{Q}) \cong \mathbf{Q}^{2d},$$

where $d = \dim(J)$. We compute L_J explicitly using **modular symbols**:



Modular symbols give an explicit presentation for L_J as sums of symbols $\{\alpha, \beta\}$ with $\alpha, \beta \in \mathbf{P}^1(\mathbf{Q})$ modulo

$$\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0.$$

This presentation also gives an explicit description of the action of the Hecke algebra

$$\mathbf{T} = \mathbf{Z}[T_1, T_2, \dots, T_n, \dots].$$

Abel-Jacobi Theorem: We have

$$J(\mathbf{C}) = (V_J \otimes \mathbf{R})/L_J.$$

Thus we can understand the group $J(\mathbf{C})$ by understanding L_J with its extra structure!

Here is how we **explicitly represent** absolutely any modular abelian variety A over \mathbf{Q} . There is a map $\varphi : A \rightarrow J$ with finite kernel. The image $B = \varphi(A)$ is an abelian subvariety of J . It induces a homomorphism $V_B \subset V_J$ on rational homology. The vector space V_B determines B . Using the dual isogeny we find an isogeny $B \rightarrow A$ with some kernel G . Then $A = B/G$, and we give G by giving a lattice $L_A \subset V_B$ that contains L_B .

Defining Data:

- We specify a modular abelian variety A by giving a lattice $L_A \subset V_J$.
- We specify a morphism $A \rightarrow C$ of modular abelian varieties by giving the induced homomorphism $L_A \rightarrow L_C$ on lattices.

Thus instead of giving a Weierstrass equation, we give modular abelian varieties by giving lattices in V_J . Every single modular abelian variety can be specified this way, up to isomorphism. Also, in general we replace J by any finite product of Jacobians $J_0(N)$, $J_1(N)$, and $J_H(N)$.

In terms of this defining data L_A for A , I have algorithms to:

1. Explicitly write L_A up to isogeny as a product of simples (corresponding to newforms).
2. Compute intersections and sums of A 's in a common ambient Jacobian J .
3. Compute complements (poincare reducibility); uses the intersection pairing in general.
4. Quotients by subvarieties and finite subgroups.
5. Compute the n -torsion subgroup, the cuspidal subgroup, the rational torsion subgroup (in some cases), etc.
6. Compute kernels and images of morphisms.
7. Compute $\text{Hom}(A, C)$ for any A, C . (Also, can do this over any number field.)

Demo

```
D = J0(39).decomposition(); D
```

```
[
  Simple abelian subvariety 39a(1,39) of dimension 1 of J0(39),
  Simple abelian subvariety 39b(1,39) of dimension 2 of J0(39)
]
```

```
D[1].lattice()
```

```
Free module of degree 6 and rank 4 over Integer Ring
Echelon basis matrix:
[ 1  0  0  1 -1  0]
[ 0  1  1  0 -1  0]
[ 0  0  2  0 -1  0]
[ 0  0  0  0  0  1]
```

```
G = D[1].rational_torsion_subgroup(); G
```

```
Torsion subgroup of Simple abelian subvariety 39b(1,39) of dimension
2 of J0(39)
```

```
G.order()
```

```
28
```

```
G.gens()
```

```
[[ (1/14, 2/7, 0, 1/14, -3/14, 1/7) ], [ (0, 1, 0, 0, -1/2, 0) ], [ (0,
0, 1, 0, -1/2, 0) ]]
```

```
B, phi = D[1]/G
```

```
B
```

```
Abelian variety factor of dimension 2 of J0(39)
```

```
phi
```

```
Abelian variety morphism:
```

```
From: Simple abelian subvariety 39b(1,39) of dimension 2 of J0(39)
To:   Abelian variety factor of dimension 2 of J0(39)
```

```
phi.kernel()
```

```
(Finite subgroup with invariants [2, 14] over QQ of Simple abelian
subvariety 39b(1,39) of dimension 2 of J0(39), Abelian subvariety of
```

```
dimension 0 of J0(39))
```

```
J = J0(91); J
```

```
Abelian variety J0(91) of dimension 7
```

```
D = J.decomposition(); D
```

```
[
Simple abelian subvariety 91a(1,91) of dimension 1 of J0(91),
Simple abelian subvariety 91b(1,91) of dimension 1 of J0(91),
Simple abelian subvariety 91c(1,91) of dimension 2 of J0(91),
Simple abelian subvariety 91d(1,91) of dimension 3 of J0(91)
]
```

```
print D[0], '\n'
D[0].lattice()
```

```
Simple abelian subvariety 91a(1,91) of dimension 1 of J0(91)
```

```
Free module of degree 14 and rank 2 over Integer Ring
```

```
Echelon basis matrix:
```

```
[ 1 0 0 -2 -2 0 2 0 1 -1 0 2 0 0]
[ 0 1 0 -1 -1 0 1 -1 1 0 -1 1 0 1]
```

```
print D[1], '\n'
D[1].lattice()
```

```
Simple abelian subvariety 91b(1,91) of dimension 1 of J0(91)
```

```
Free module of degree 14 and rank 2 over Integer Ring
```

```
Echelon basis matrix:
```

```
[ 1 0 0 0 0 0 0 0 0 -1 -1 0 0 0 0]
[ 0 1 -2 -1 -1 -2 1 1 1 2 -1 1 0 -1]
```

```
print D[2], '\n'
D[2].lattice()
```

```
Simple abelian subvariety 91c(1,91) of dimension 2 of J0(91)
```

```
Free module of degree 14 and rank 4 over Integer Ring
```

```
Echelon basis matrix:
```

```
[ 1 0 0 0 -2 2 0 -2 1 -1 2 0 -2 2]
[ 0 1 0 0 -1 1 0 -2 1 0 1 1 -2 1]
[ 0 0 1 0 0 1 -1 -2 1 0 1 0 -1 1]
[ 0 0 0 1 0 1 -1 -1 0 0 0 0 0 0]
```

```
print D[3], '\n'
D[3].lattice()
```

```
Simple abelian subvariety 91d(1,91) of dimension 3 of J0(91)
```

```
Free module of degree 14 and rank 6 over Integer Ring
```

```
Echelon basis matrix:
```

```
[ 1 0 0 0 0 0 0 0 0 0 0 0 0 0]
[ 0 1 0 0 -1 -1 0 0 0 0 0 0 1 0]
[ 0 0 1 0 0 1 -1 0 0 0 0 -1 0 0]
```

```
[ 0 0 0 1 0 -1 1 -1 0 0 0 0 0 0]
[ 0 0 0 0 0 0 0 0 1 0 -1 1 -1 1]
[ 0 0 0 0 0 0 0 0 0 1 -1 1 -1 1]
```

```
# Example with lattice not just a vector space intersect $L_J$.
```

```
A = D[2].dual()[0]
```

```
L = A.lattice(); L
```

```
Free module of degree 14 and rank 4 over Integer Ring
```

```
Echelon basis matrix:
```

```
[ 1/2  0  0  0  -1  1  0  -1  1/2 -1/2  1  0  -1
1]
```

```
[  0  1/2  0  1/2 -1/2  1 -1/2 -3/2  1/2  0  1/2  1/2  -1
1/2]
```

```
[  0  0  1  0  0  1  -1  -2  1  0  1  0  -1
1]
```

```
[  0  0  0  1  0  1  -1  -1  0  0  0  0  0
0]
```

```
D[2].lattice().index_in(L)
```

```
4
```

3. Enumeration of all Modular Abelian Varieties of Given Level

We have

$$J_0(N) \sim J_0(N)_{\text{new}} \times J_0(N)_{\text{old}}$$

where $J_0(N)_{\text{old}}$ is the sum of the images of $J_0(M)$'s under all natural "degeneracy" maps for all proper divisors of N .

Theorem (Atkin-Lehner-Shimura): $J_0(N)_{\text{new}}$ is isogenous to a product of distinct simple abelian varieties that are in bijection with the $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ -orbits of newforms

$$f = \sum a_n q^n \in S_2(\Gamma_0(N)).$$

Thus the isogeny classes of simple abvars of level N are in bijection with the conjugacy classes of newforms of level N . The conjugacy classes can be enumerated using linear algebra (various interesting tricks to make this fast...).

Demo

```
for f in ModularForms(91).newforms('alpha'):
    view(f)
```

$$q - 2q^2 + 2q^4 - 3q^5 + O(q^6)$$

$$q - 2q^3 - 2q^4 - 3q^5 + O(q^6)$$

$$q + a_2q^2 - a_2q^3 + (a_2 + 3)q^5 + O(q^6)$$

$$q + a_3q^2 + (-a_3^2 + a_3 + 2)q^3 + (a_3^2 - 2)q^4 + (-a_3 + 1)q^5 + O(q^6)$$

```
for A in J0(91).new_subvariety().decomposition():
    print A
```

```
Simple abelian subvariety 91a(1,91) of dimension 1 of J0(91)
Simple abelian subvariety 91b(1,91) of dimension 1 of J0(91)
Simple abelian subvariety 91c(1,91) of dimension 2 of J0(91)
Simple abelian subvariety 91d(1,91) of dimension 3 of J0(91)
```

```
A,B,C,D = J0(91)
```

```
A.intersection(B)
```

```
(Finite subgroup with invariants [2, 2] over QQ of Simple abelian
subvariety 91a(1,91) of dimension 1 of J0(91), Simple abelian
subvariety of dimension 0 of J0(91))
```

```
B.intersection(D)
```

```
(Finite subgroup with invariants [2, 2] over QQ of Simple abelian
subvariety 91b(1,91) of dimension 1 of J0(91), Simple abelian
subvariety of dimension 0 of J0(91))
```

```
C.intersection(D)
```

```
(Finite subgroup with invariants [2, 2, 2, 2] over QQ of Simple
abelian subvariety 91c(1,91) of dimension 2 of J0(91), Simple
abelian subvariety of dimension 0 of J0(91))
```

```
t2 = J0(91).hecke_operator(2); t2
```

```
Hecke operator T_2 on Abelian variety J0(91) of dimension 7
```

```
show(t2.matrix())
```

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & -2 & 1 & -1 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & -1 & 2 & -2 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 2 & 1 & 0 & -1 & 0 & -2 & 1 & 0 & -1 & 1 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

```
show(t2.fcp())
```

$$x \cdot (x + 2) \cdot (x^2 - 2) \cdot (x^3 - x^2 - 4x + 2)$$

```
t2.kernel()
```

(Finite subgroup with invariants [2, 2] over $\overline{\mathbb{Q}\mathbb{Q}}$ of Abelian variety $J_0(91)$ of dimension 7, Abelian subvariety of dimension 1 of $J_0(91)$)

```
(t2+2).cokernel()
```

(Abelian subvariety of dimension 1 of $J_0(91)$, Abelian variety morphism:

From: Abelian variety $J_0(91)$ of dimension 7

To: Abelian subvariety of dimension 1 of $J_0(91)$)

```
(t2+2).image()
```

Abelian subvariety of dimension 6 of $J_0(91)$

Challenge: Given a practical algorithm that...

INPUT: A positive integer N

OUTPUT: All simple modular abelian varieties of level N divided up into isogeny classes with each isomorphism class listed exactly once.

By practical I mean that it will run for every $N < 1000$ in a reasonable amount of time. I'm OK with assuming GRH for algebraic number theory computations, e.g., class groups.

As suggested above we can already construct one representative of each isogeny class, following Shimura. Given a newform f , let

$$I_f = \text{Ann}_{\mathbf{T}}(f)$$

and let

$$A_f = J_0(N)[I_f]^0$$

be the connected component of the part of $J_0(N)$ annihilated by all

elements of I_f . Then A_f is a new simple modular abelian variety with defining data the lattice $L = L_J[I_f]$. There are various other ways to construct abelian varieties B in the isogeny class of A_f (e.g., make B the dual of A_f). But can one tell if B is abstractly isomorphic to A ? YES!

Endomorphisms

Let A be a simple new factor of $J_0(N)$.

Proposition (Shimura): $\text{End}(A/\mathbf{Q}) \otimes \mathbf{Q}$ equals the image of $\mathbf{T}_{\mathbf{Q}}$ in $\text{End}(A) \otimes \mathbf{Q}$.

We view $\text{End}(A/\mathbf{Q}) \otimes \mathbf{Q}$ as a subring of $\text{Mat}_{2d \times 2d}(\mathbf{Q})$.

Proposition (--): We have

$$\text{End}(A/\mathbf{Q}) = (\text{End}(A/\mathbf{Q}) \otimes \mathbf{Q}) \cap \text{Mat}_{2d \times 2d}(\mathbf{Z}).$$

Remark: The above proposition generalizes to $\text{Hom}(A, B)$.

Thus we can explicitly compute $\text{End}(A)$ on the nose (not just up to finite index).

```
A = J0(91)[2]; A
```

```
Simple abelian subvariety 91c(1,91) of dimension 2 of J0(91)
```

```
R = End(A); R
```

```
Endomorphism ring of Simple abelian subvariety 91c(1,91) of dimension 2 of J0(91)
```

```
for x in R.gens(): print x.matrix(),'\n'
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

```
[ 0  4 -2  0]
[-1  5 -2  1]
[-1  2  0  2]
[-1  1  0  3]
```


Given simple modular abelian varieties A and B we can decide if A is isomorphic to B as follows:

1. Check if A is isogenous to B by finding an explicit isogeny from each to an A_f . If not isogenous, return false.
2. Let $\varphi : B \rightarrow A$ be an isogeny. We want to know if φ is "really an endomorphism".
3. Compute $d = \deg(\varphi)$. If d is not a square, return false, since degrees of endomorphisms are squares of norms.
4. Compute the number field $K = \text{End}(A) \otimes \mathbf{Q}$ explicitly.
5. Compute $\text{Hom}(A, B) \subset \text{Mat}_{2d \times 2d}(\mathbf{Z})$ explicitly.
6. Compute the image H_φ of $\text{Hom}(A, B)$ in K got by composing with φ .
7. Compute the order \mathcal{O} in K that is the image of $\text{End}(A)$.
8. Find representative solutions (up to units of \mathcal{O}) of the norm equation $\text{Norm}(x) = \pm\sqrt{d}$ in \mathcal{O} . If no solutions, return false.
9. For each solution (up to units), check if it lies in H_φ .
10. If a solution x lies in H_f return true and $x \circ \varphi^{-1}$
11. If no solution lies in H_f return false.

Open Problem: *I do not know how to determine if two arbitrary non-simple modular abelian varieties are isomorphic.*

If $A = A_f$ is a newform abelian subvariety of $J_0(N)$ and \mathcal{O}_A is the order $\text{End}(A)$, then we have a well-defined map

$$\text{Cl}(\mathcal{O}_A) \rightarrow \{ \text{isogeny class of } A \}$$

that sends a maximal ideal \mathfrak{m} that represents an ideal class to the abelian variety $B = A/A[\mathfrak{m}]$. This map is well defined.

Proposition: If $A \rightarrow B$ is an isogeny with irreducible kernel G whose support is a multiplicity one irreducible maximal ideal \mathfrak{m} , then B is in the image of the above map.

Also, one can consider quotients of A by simple submodules of A intersected with the cuspidal and Shimura subgroups of $J_0(N)$. For irreducible \mathfrak{m} of multiplicity greater than 1, one can probably take all simple submodules of $A[\mathfrak{m}]$.

Open Question: Taking all the constructions above together, does one fill out the isogeny class of A ?

4. Invariants of Modular Abelian Varieties

Here is a list of invariants we *would like to compute* for each A . We can compute some in all cases, some in some cases, something about some in other cases, and sometimes nothing:

- Leading coefficient $L^*(A, 1)$
- Real volume Ω_A
- Tamagawa numbers $c_{A,p}$
- Real component group $c_\infty = \#(A(\mathbf{R})/A(\mathbf{R})^0)$
- The regulator $\text{Reg}(A)$.
- The order $\#\text{Sha}(A)$ of the Shafarevich-Tate group.
- The order $\#A(\mathbf{Q})_{\text{tor}}$ of the torsion subgroup.
- The period lattice $\Lambda \subset \mathbf{C}^d$.
- For each good p , the p -adic L -series $\mathcal{L}_p(A, T) \in \overline{\mathbf{Q}}_p[[T]]$.
- For each good p , the p -adic Regulator $\text{Reg}_p(A)$.
- The modular degree m_A of A .
- The minimal degree of an isogeny $A \rightarrow A^\vee$, and whether or not A is principally polarized.

5. Addendum

In fact compared to what I have in mind, *very very little* for modular abelian varieties is implemented in Sage compared to what is needed for our project, and I have to greatly optimize the current implementation. Almost nothing interesting is implemented yet and a massive amount of work remains in order to turn the theory of this talk into a practical implementation available in Sage. (My Magma implementation has more, but is still fairly limited too, and is built on a shaky foundation.)

Fixing this is right now a very high priority for me. I do not recommend anybody but me seriously use this particular code right now, unless they are seriously interested in helping finish it. I created Sage for this project.

Things to do include:

- **Pizer's algorithm** for computing Brandt matrices (Hecke operators), which is needed for my algorithm for computing **Tamagawa numbers**
- Isomorphism testing for simple modular abelian varieties (embed endomorphism ring in number field, solve norm equations, pull back solutions).
- Period lattice
- Complex and p -adic L -functions
- Heegner points of ring class fields (Numerical)
- Tons of optimization, bug fixing, deal with things working only for $\Gamma_0(N)$ when they should work in general.