An Introduction to the Modular Forms Database Project:

My Dream Computation (not a toy problem!)

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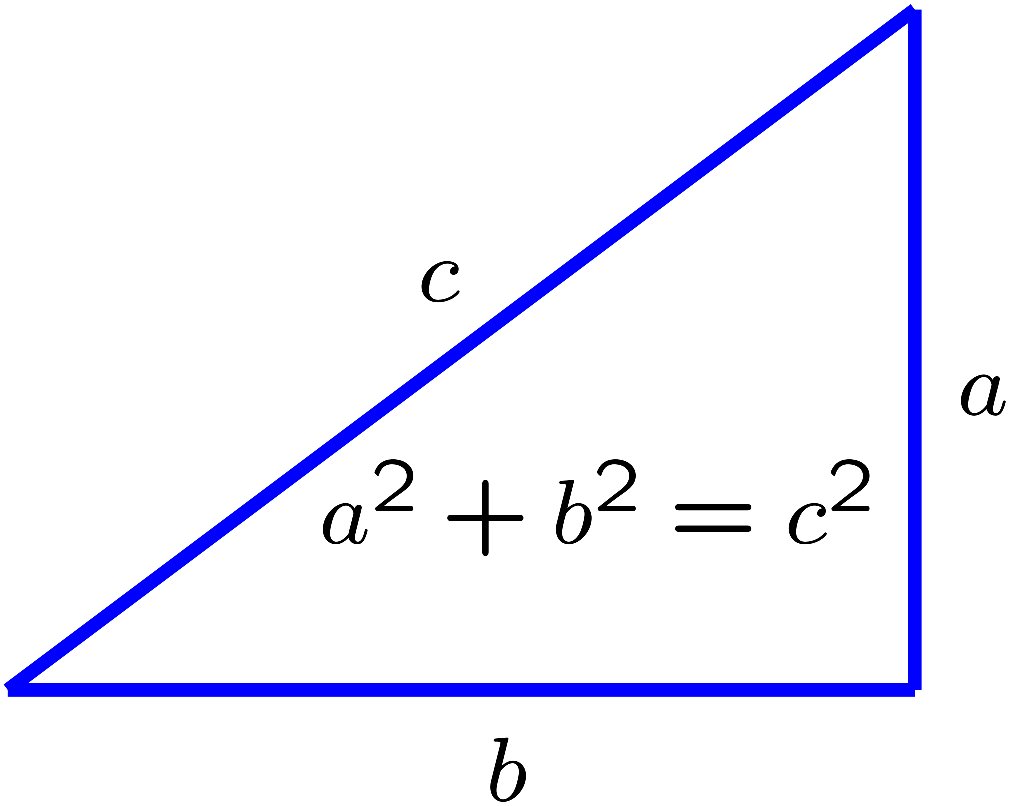
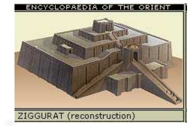
The Pythagorean Theorem

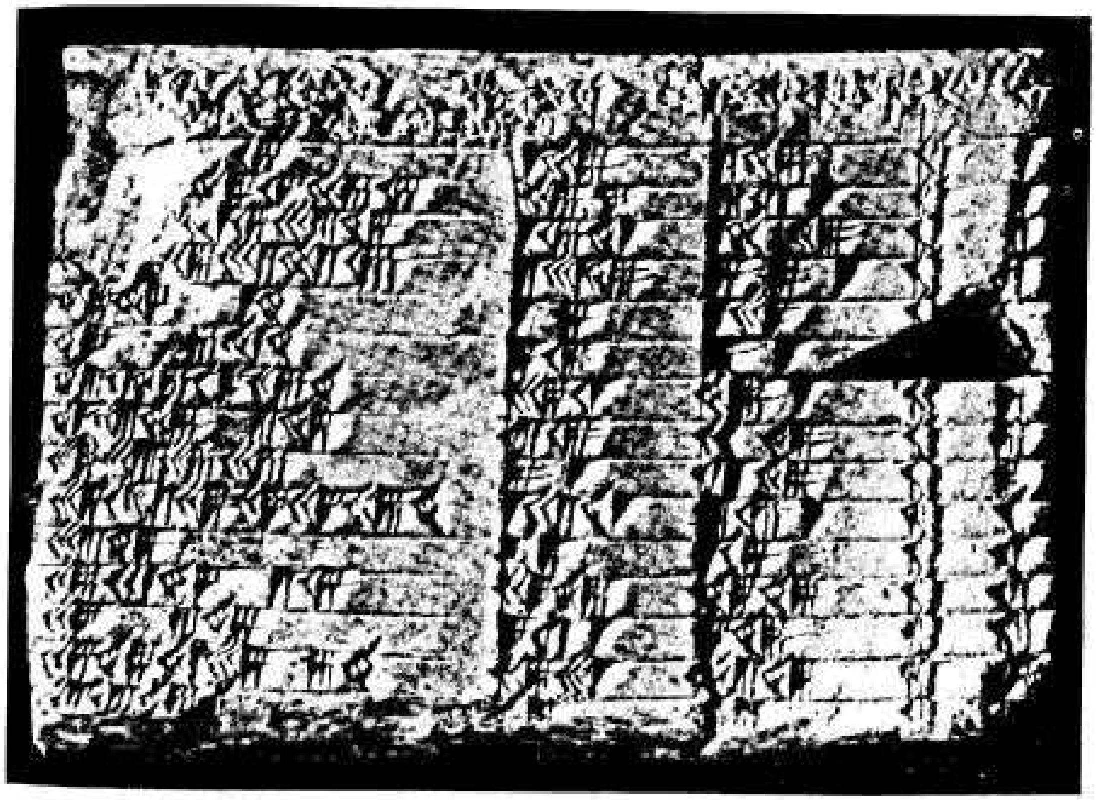


Pythagoras
Approx 569–475BC

Pythagorean Triples



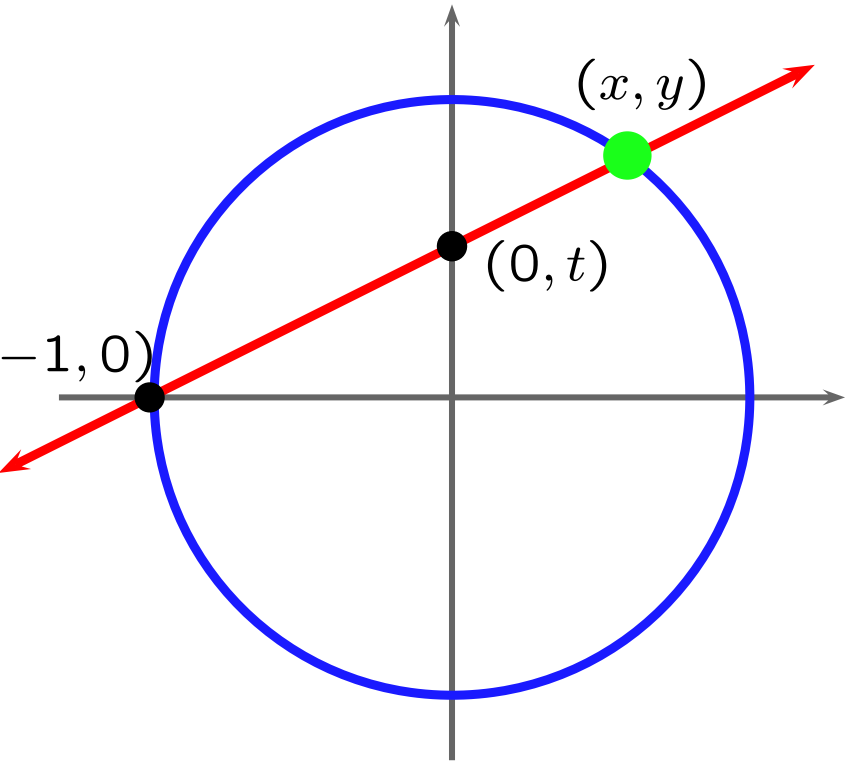
(3, 4, 5)
(5, 12, 13)
(7, 24, 25)
(9, 40, 41)
(11, 60, 61)
(13, 84, 85)
(15, 8, 17)
(21, 20, 29)
(33, 56, 65)
(35, 12, 37)
(39, 80, 89)
(45, 28, 53)
(55, 48, 73)
(63, 16, 65)
(65, 72, 97)
(77, 36, 85)
⋮



Triples of integers a, b, c such that

$$a^2 + b^2 = c^2$$

Enumerating Pythagorean Triples



$$\text{Slope} = t \quad = \quad \frac{y}{x+1}$$

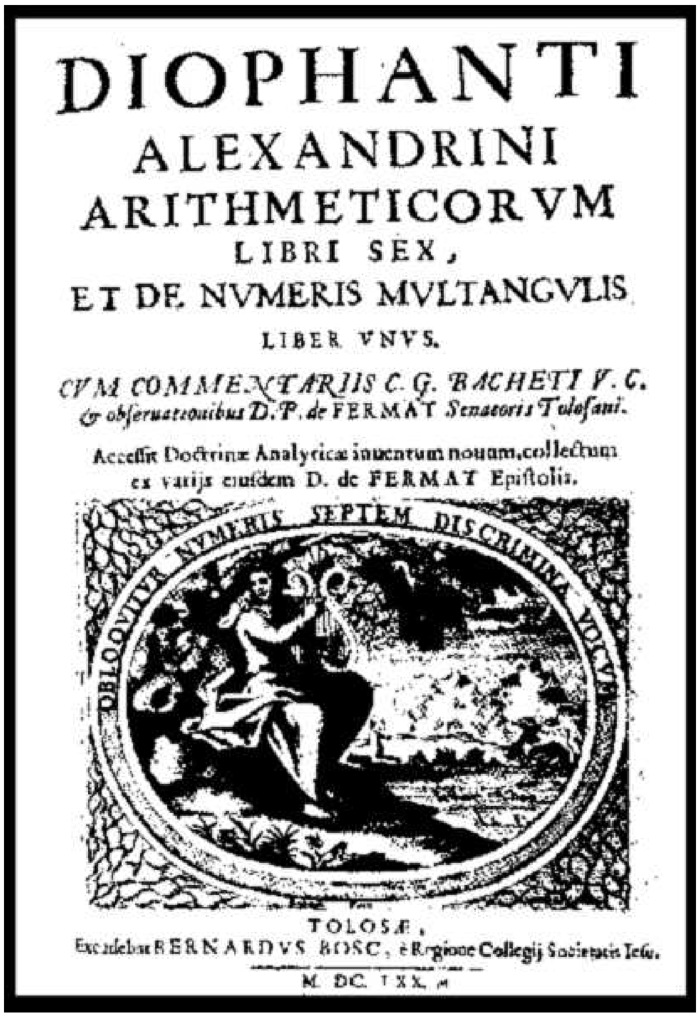
$$x \quad = \quad \frac{1-t^2}{1+t^2}$$

$$y \quad = \quad \frac{2t}{1+t^2}$$

If $t = \frac{r}{s}$, then $\quad a = s^2 - r^2, \quad b = 2rs, \quad c = s^2 + r^2$
is a Pythagorean triple, and all primitive unordered triples arise in this way.

Fermat's "Last Theorem"

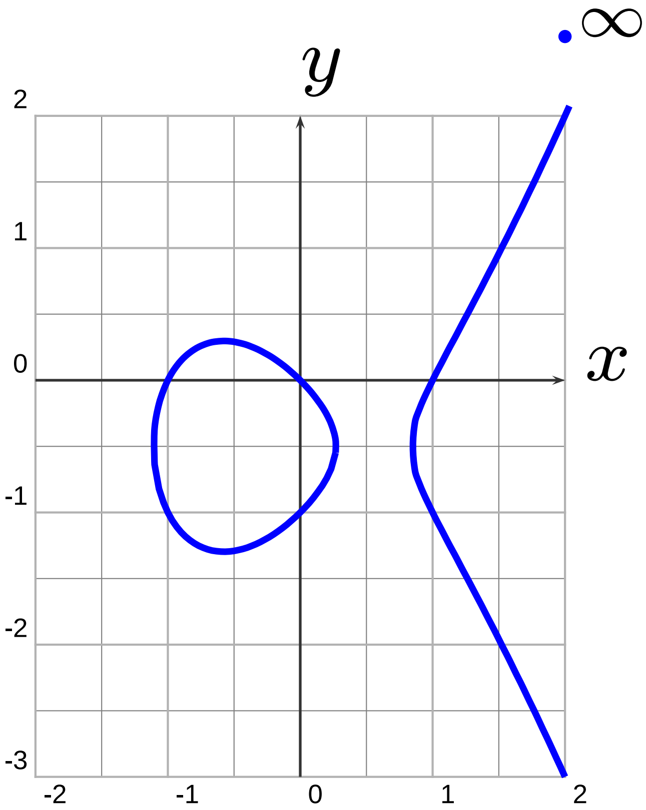
No "Pythagorean triples" with exponent 3 or higher.



DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
CVM COMMENTARIIS C. G. BACHETI V. C.
& obseruationibus D. P. de FERMAT Senatoris Tolosani.
Accessit Doctrinæ Analyticæ inuentum nouum, collectum
ex varijs eiusdem D. de FERMAT Epistolis.
OBLOQVITVR NVMERIS SEPTEM DISCRIMINA VOCVM
TOLOSÆ,
Excudebat BERNARDVS BOSC, è Regione Collegij Societatis Iesu.
M. DC. LXX.

Wiles's Proof of FLT Uses Elliptic Curves

An **elliptic curve** is a nonsingular plane cubic curve with a rational point (possibly "at infinity").



$$y^2 + y = x^3 - x$$

EXAMPLES

$$y^2 + y = x^3 - x$$

$$x^3 + y^3 = 1 \text{ (Fermat cubic)}$$

$$y^2 = x^3 + ax + b$$

~~$$3x^3 + 4y^3 + 5 = 0$$~~

The Frey Elliptic Curve

Suppose Fermat's conjecture is **FALSE**. Then there is a prime $\ell \geq 5$ and coprime positive integers a, b, c with $a^\ell + b^\ell = c^\ell$.

Consider the corresponding Frey elliptic curve:

$$y^2 = x(x - a^\ell)(x + b^\ell).$$

Ribet's Theorem: This elliptic curve is not *modular*.

Wiles's Theorem: This elliptic curve is *modular*.

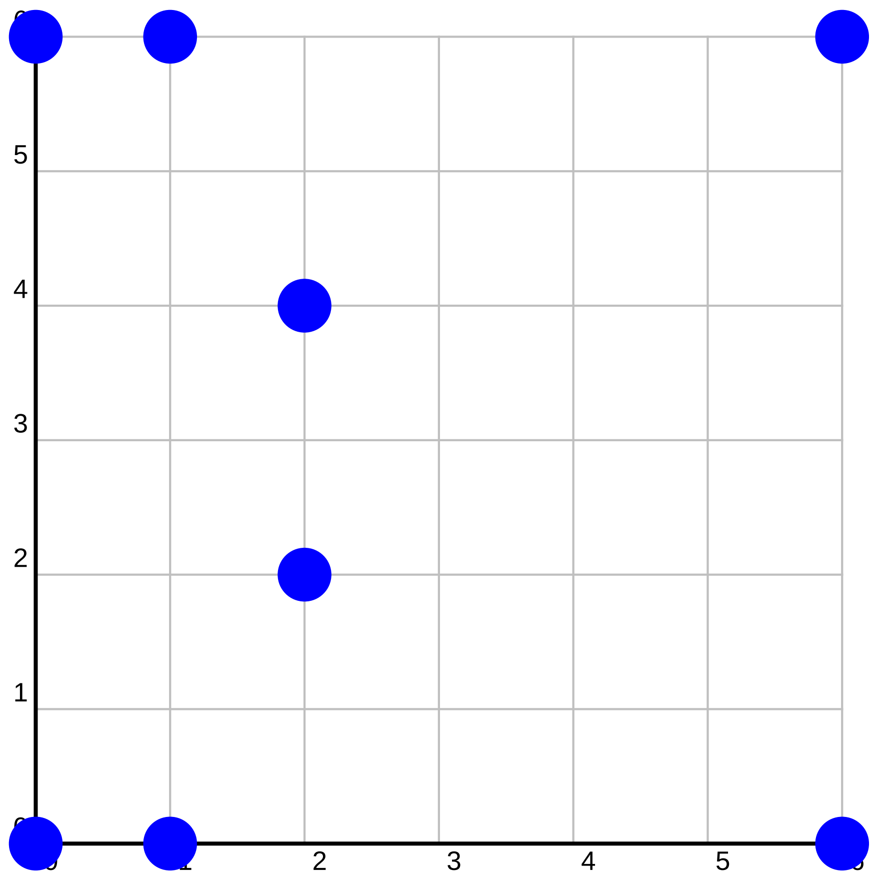
Conclusion: Fermat's conjecture is true.

Counting Solutions Modulo p

$$N(p) = \#\ \text{of solutions} \pmod{p}$$

∞

$$y^2 + y = x^3 - x \pmod{7}$$

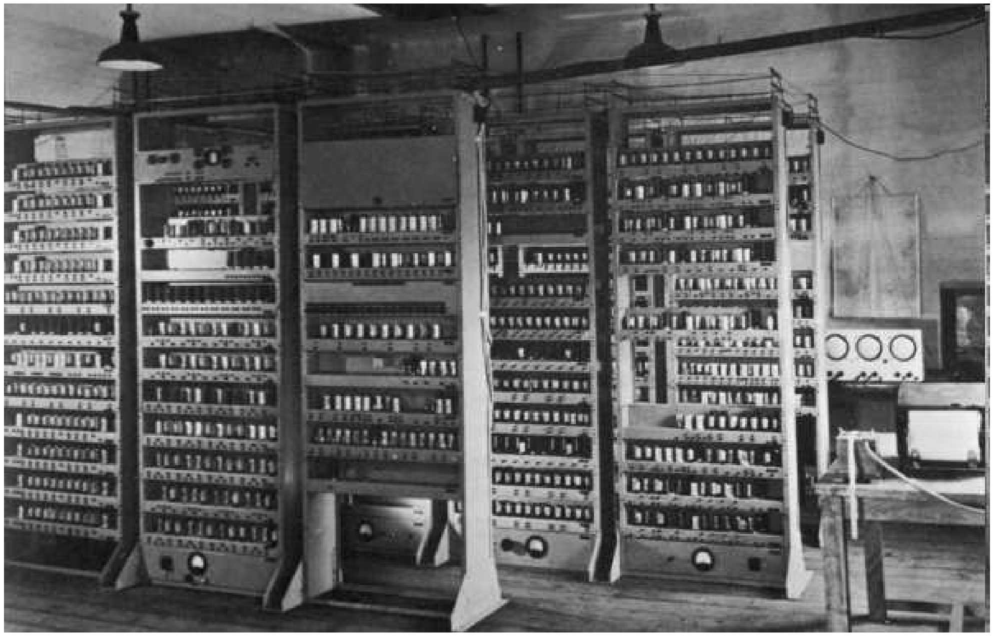


$$N(7) = 9$$

Counting Points

Cambridge **EDSAC**: The first point counting supercomputer...

Birch and Swinnerton-Dyer



The Hecke Eigenvalue

Hasse

Let

$$a_p = p + 1 - N(p).$$

Hasse proved that

$$|a_p| \le 2\sqrt{p}.$$

For $y^2 + y = x^3 - x$:

$$a_2 = -2, \quad a_3 = -3, \quad a_5 = -2, \quad a_7 = -1, \quad a_{11} = -5, \quad a_{13} = -2,$$

$$a_{17} = 0, \quad a_{19} = 0, \quad a_{23} = 2, \quad a_{29} = 6, \quad \ldots$$

Elliptic Curves are "Modular"

An elliptic curve is **modular** if the numbers a_p are coefficients of a "modular form".

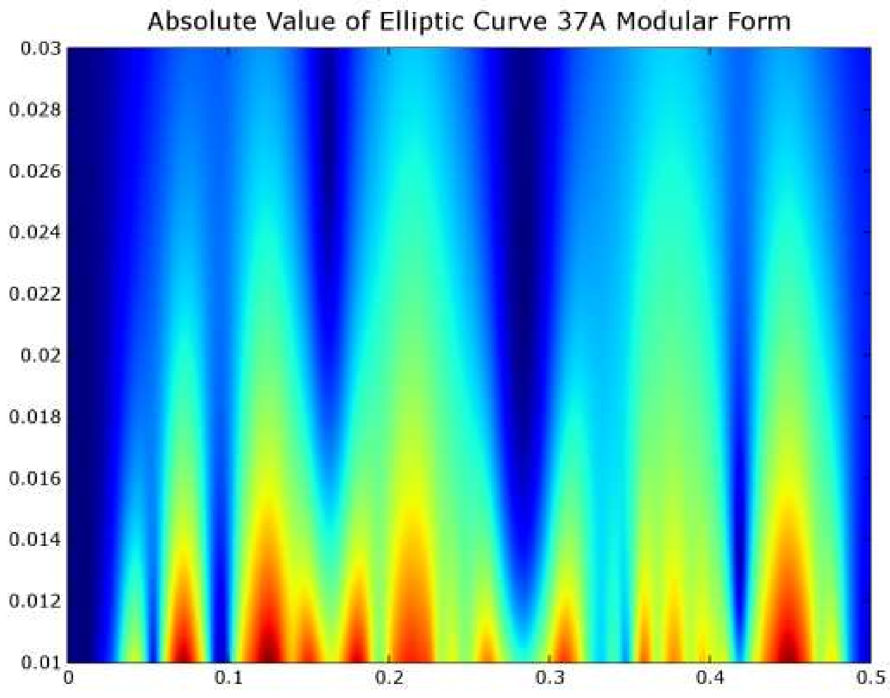
Theorem (Wiles et al.): *Every elliptic curve over the rational numbers is modular.*

Wiles at the Institute for Advanced Study

Modular Forms

The definition of modular forms as holomorphic functions satisfying a certain equation is very abstract.

I will skip the abstract definition, and instead give you an explicit "engineer's recipe" for producing modular forms. In the meantime, here's a picture:



Computing Modular Forms: Motivation

Motivation: Data about modular forms is **extremely** useful to many research mathematicians (e.g., number theorists, cryptographers). This data is like the astronomer's telescope images.

I want to compute modular forms on a **huge** scale using the SDSC resources, and make the resulting database widely available. I have done this on a small scale during the last 5 years — see `http://modular.fas.harvard.edu/Tables/`

What to Compute: Newforms

For each positive integer N there is a finite list of **newforms** of level N. E.g., for $N = 37$ the newforms are

$$f_1 = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + \cdots$$
$$f_2 = q + q^3 - 2q^4 - q^7 + \cdots,$$

where $q = e^{2\pi i z}$.

The newforms of level N determine all the modular forms of level N (like a basis in linear algebra). The coefficients are algebraic integers. *Goal: compute these newforms.*

Bad idea – write down many elliptic curves and compute the numbers a_p by counting points over finite fields. No good – this misses most of the interesting newforms, and gets newforms of all kinds of random levels, but you don't know if you get everything of a given level.

An Engineer's Recipe for Newforms

Fix our positive integer N. For simplicity assume that N is prime.

1. Form the $N+1$ dimensional $\mathbf{Q}$-vector space V with basis the symbols $[0],\ldots,[N-1],[\infty]$.

2. Let R be the suspace of V spanned by the following vectors, for $x = 0,\ldots,N-1,\infty$:

$$[x] - [N-x]$$
$$[x] + [x.S]$$
$$[x] + [x.T] + [x.T^2]$$

   $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$, and $x.\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (ax+c)/(bx+d)$.

3. Compute the quotient vector space $M = V/R$. This involves "intelligent" **sparse Gauss elimination** on a matrix with $N+1$ columns.

4. Compute the matrix T_2 on M given by

$$[x] \mapsto [x.\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)] + [x.\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)] + [x.\left(\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right)] + [x.\left(\begin{smallmatrix} 1 & 0 \\ 1 & 2 \end{smallmatrix}\right)].$$

This matrix is unfortunately not sparse. Similar recipe for matrices T_n for any n.

5. Compute the **characteristic polynomial** f of T_2.

6. **Factor** $f = \prod g_i^{e_i}$. Assume all $e_i = 1$ (if not, use a random linear combination of the T_n.)

7. Compute the **kernels** $K_i = \ker(g_i(T_2))$. The **eigenvalues** of T_3, T_5, etc., acting on an **eigenvector** in K_i give the coefficients a_p of the newforms of level N.

Implementation

- I implemented code for computing modular forms that's included with **MAGMA**:
  `http://magma.maths.usyd.edu.au/magma/`.

- Unfortunately, MAGMA is expensive and closed source, so I'm reimplementing everything as part of **SAGE**:
  `http://modular.fas.harvard.edu/sage/`.

- I'm teaching a **course** on this topic at UCSD this Fall.

- I'm finishing a **book** on these algorithms that will be published by the American Mathematical Society.

The Modular Forms Database Project

- Create a database of all newforms of level N for each $N < 100000$. This will require many gigabytes to store. (50GB?)

- So far this has only been done for $N < 7000$ (and is incomplete), so 100000 is a **major challenge**.

- Involves sparse linear algebra over $\mathbf{Q}$ on spaces of dimension up to 200000 and dense linear algebra on spaces of dimension up to 25000.

- Easy to parallelize – run one process for each N.

- Will be very useful to number theorists and cryptographers.

- John Cremona has done something similar but only for the newforms corresponding to elliptic curves (he's at around 84000 right now).