Visibility of Shafarevich-Tate Groups at Higher Level

William Stein
http://modular.fas.harvard.edu

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Abstract

I will begin by introducing the Birch and Swinnerton-Dyer conjecture in the context of abelian varieties attached to modular forms, and discuss some of the main results about it. I will then introduce Mazur's notion of visibility of Shafarevich-Tate groups and explain some of the basic facts and theorems. Cremona, Mazur, Agashe, and myself carried out large computations about visibility for modular abelian varieties of level N in $J_0(N)$. These computations addressed the following question: If A is a modular abelian variety of level N, how much of the Shafarevich-Tate group $\text{Ш}(A)$ is modular of level N, i.e., visible in $J_0(N)$. The results of these computations suggest that often much of the Shafarevich-Tate group is not modular of level N. It is then natural to ask if every element of $\text{Ш}(A)$ is modular of level M, for some multiple $M = NR$, and if so, what can one say about the set of such M? I will finish the talk with some new data and a conjecture about this last question, which is still very much open.

1 Modular Abelian Varieties

Let N be a positive integer and consider the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \text{ such that } N \mid c \right\}.$$

(Almost everything in this talk also makes sense with $\Gamma_0(N)$ replaced by $\Gamma_1(N)$.) The *modular curve*

$$X_0(N) = \Gamma_0(N) \setminus (\{z \in \mathbf{C} : \mathrm{Im}(z) > 0\} \cup \mathbf{Q} \cup \{\infty\})$$

is a Riemann surface that is the set of complex points of an algebraic curve over $\mathbf{Q}$. We will not use that

$$X_0(N)(\mathbf{C}) = \{ \text{ isomorphism classes of } (E, C) \} \cup \{ \text{ cusps } \}.$$

Our primary interest is the Jacobian

$$J_0(N) = \mathrm{Jac}(X_0(N))$$

which is an abelian variety over $\mathbf{Q}$ of dimension equal to the genus of $X_0(N)$. The points on the Jacobian parametrize, in a natural way, the divisor classes of degree 0 on $X_0(N)$.

Let $S_2(\Gamma_0(N))$ be the cusp forms of weight 2 for $\Gamma_0(N)$. This is the finite-dimensional complex vector space of holomorphic functions on the upper half plane such that

$$f(z)dz = f(\gamma(z))d(\gamma(z))$$

for all $\gamma \in \Gamma_0(N)$, and which "vanish at the cusps". The map $f(z) \mapsto f(z)dz$ induces

$$S_2(\Gamma_0(N)) \cong \mathrm{H}^0(X_0(N)_{\mathbf{C}}, \Omega^1)$$

so $S_2(\Gamma_0(N))$ has dimension the genus of $X_0(N)$.

The *Hecke algebra* is a commutative ring

$$\mathbf{T} = \mathbf{Z}[T_1, T_2, T_3, \ldots]$$

which acts on $S_2(\Gamma_0(N))$ and $J_0(N)$. A *newform*

$$f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N))$$

is an eigenvector for every element of $\mathbf{T}$ normalized so $a_1 = 1$, which does not "come from" any lower level. Attached to f there is an ideal

$$I_f = \mathrm{Ann}_{\mathbf{T}}(f) = \mathrm{Ker}(\mathbf{T} \to \mathbf{Z}[a_1, a_2, \ldots]),$$

and (following Shimura) to this ideal we attach an abelian variety A_f and an L-function $L(A_f, s)$.

Let

$$A_f = J_0(N)[I_f]^0 = \left(\bigcap_{\varphi \in I_f} \mathrm{Ker}(\varphi) \right)^0$$

be the connected component of the intersections of the kernels of elements of I_f. Then A_f has dimension $[K_f : \mathbf{Q}] = [\mathbf{Q}(a_1, a_2, \ldots) : \mathbf{Q})]$, and is define over $\mathbf{Q}$.

Let

$$L(A_f, s) = \prod_{i=1}^{d} L(f_i, s)$$

where $d = [K_f : \mathbf{Q}]$ and the f_i are the Galois conjugates of f. Also,

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Hecke proved that $L(f, s)$ is entire and satisfies a functional equation.

The abelian varieties A_f are a rich class of abelian varieties. The elliptic curves over $\mathbf{Q}$ are all isogenous to some A_f (the Wiles-Breuil-Conrad-Diamond-Taylor modularity theorem).

2 The Birch and Swinnerton-Dyer Conjecture

2.1 Conjecture

Conjecture 2.1 (Birch and Swinnerton-Dyer).

1. $\mathrm{rank}\, A_f(\mathbf{Q}) = \mathrm{ord}_{s=1} L(A_f, s)$

2. $$\frac{L^{(r)}(A_f, 1)}{r!} = \frac{\prod c_p \cdot \Omega_{A_f} \cdot \mathrm{Reg}_{A_f} \cdot \#\text{Ш}(A_f)}{\#A_f(\mathbf{Q})_{\mathrm{tor}} \cdot \#A_f^{\vee}(\mathbf{Q})_{\mathrm{tor}}}.$$

Remarks: Part of the conjecture is that $Ш(A_f)$ is finite. There is also a conjecture for arbitrary abelian varieties over global fields. Clay Math Problem: $1000000 prize for proof of (1) in case $\dim(A_f) = 1$

Here:

- c_p is the *Tamagawa number* at the prime p, and the product is over the prime divisors of N.
- Ω_{A_f} is the canonical Néron measure of $A_f(\mathbf{R})$.
- Reg_{A_f} is the regulator (absolute value of Néron-Tate canonical height pairing matrix).
- $A_f(\mathbf{Q})_{\mathrm{tor}}$ is the torsion subgroup of $A_f(\mathbf{Q})$.
- $Ш(A_f)$ is the Shafarevich-Tate group.

2.2 Evidence

- Rubin: results in CM Case
- Kolyvagin, Logachev, Gross-Zagier, et al.: If $\mathrm{ord}_{s=1} L(f, s) = 0$ or 1, then (1) true and $Ш(A_f)$ finite.
- Cremona: Compute $Ш(A_f)_?$ (=conjectural order) for tens of thousands of A_f of dimension 1 and get approximate square order. (Theorem of Cassels: if E an elliptic curve and $Ш(E)$ finite then order a perfect square. Note that the analogue for abelian varieties is false; for exampe, I've constructed examples for each odd prime $p < 25000$ of abelian varieties A of dimension $p - 1$ such that $Ш(A) = p \cdot n^2$.)

In this talk I will focus on A_f of possibly large dimension with $L(A_f, 1) \neq 0$, since computation of Reg_{A_f} is difficult (impossible?) when one can't even reasonably hope to write down A_f explicitly with equations.

3 Visibility of Shafarevich-Tate Groups

3.1 Definitions

It is easy to write down a point on an elliptic curve E. You simply write down a pair of rational numbers, which are a solution to a Weierstrass equation. In contrast, imagine describing explicitly an element of $Ш(E)$ of order 2003. The most direct way would be to give a genus one curve (with principal homogeneous space structure), embedded in $\mathbf{P}^3$ of degree at least 2003 (!), hence very complicated.

The idea of visibility of Shafarevich-Tate groups was introduced by Barry Mazur around 1998 to unify various constructions of elements of Shafarevich-Tate groups.

Definition 3.1 (Shafarevich-Tate Group).

$$Ш(A) = \mathrm{Ker}\left(\mathrm{H}^1(K, A) \to \bigoplus_v \mathrm{H}^1(K_v, A)\right).$$

Here $\mathrm{H}^1(K, A)$ is the first Galois cohomology, which can be interpreted geometrically as the Weil-Chatalet group

$$\mathrm{WC}(A/K) = \{ \text{ principal homogenous spaces } X \text{ for } A \}/\sim .$$

Then $Ш(A)$ is the subgroup of locally trivial classes of homogenous spaces. For example

$$3x^3 + 4y^3 + 5z^3 = 0 \in Ш(x^3 + y^3 + 60z^3 = 0)[3].$$

Fix an inclusion $i : A \hookrightarrow B$ of abelian varieties and let $\pi : B \to C$ be the quotient of B by the image of A, so we have an exact sequence

$$0 \to A \to B \to C \to 0$$

of abelian varieties.

Definition 3.2 (Visible Subgroup).

$$\begin{aligned} \mathrm{Vis}_i(\mathrm{H}^1(K, A)) &= \mathrm{Ker}\left(\mathrm{H}^1(K, A) \to \mathrm{H}^1(K, B)\right) \\ &= \mathrm{Coker}(B(K) \to C(K)) \end{aligned}$$

and

$$\mathrm{Vis}_i(Ш(A)) = \mathrm{Ker}(Ш(A) \to Ш(B)).$$

1. The visible subgroup is finite because $B(K)$ is finitely generated and $\mathrm{Vis}_i(\mathrm{H}^1(K, A))$ is torsion.
2. If $c \in \mathrm{Vis}_i(\mathrm{H}^1(K, A))$, then c is also "visible" in the sense that if c is the image of a point $x \in C(K)$, and if $X = \pi^{-1}(x) \subset B$, then $[X] \in \mathrm{WC}(A)$ corresponds to c.
3. The visibile subgroups depends on the choice of embedding $i : A \hookrightarrow B$. I've also considered defining $\mathrm{Vis}_B(\mathrm{H}^1(K, A))$ to be the subgroup generated by all visible subgroups with respect to all embeddings $A \to B$, but I'm not sure what properties this definition has.

3.2 Theorems

"Everything is visible somewhere."

Theorem 3.3 (Stein). *If $c \in \mathrm{H}^1(K, A)$ then there exists $B = \mathrm{Res}_{L/K}(A_L)$ such that $i : A \hookrightarrow B$ and $c \in \mathrm{Vis}_i(\mathrm{H}^1(K, A))$. (Here L is such that $\mathrm{res}_{L/K}(c) = 0$.)*

"Visibility construction."

Theorem 3.4 (Agashe-Stein). *Suppose $A, B \subset C$ over $\mathbf{Q}$, that $A + B = C$, that $A \cap B$ is finite. Suppose N is divisible by all bad primes for C, and p is a prime such that*

- $B[p] \subset A$
- $p \nmid 2 \cdot N \cdot \#B(\mathbf{Q})_{\mathrm{tor}} \cdot \#(C/B)(\mathbf{Q})_{\mathrm{tor}} \cdot \prod_{p \mid N} c_{A,p} \cdot c_{B,p}.$

If A has rank 0, then there is a natural inclusion

$$B(\mathbf{Q})/pB(\mathbf{Q}) \hookrightarrow \mathrm{Vis}_C(Ш(A)).$$

(And certain generalizations...)

3.3 Example

Example 3.5. For $N = 389$, take B the (first ever) rank 2 elliptic curve, and A the 20-dimensional rank 0 factor.

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ A & \longrightarrow & J_0(389) \end{array}$$

Gives

$$(\mathbf{Z}/5\mathbf{Z})^2 \cong B(\mathbf{Q})/5B(\mathbf{Q}) \hookrightarrow Ш(A).$$

Part 2 of the Birch and Swinnerton-Dyer conjecture predicts that

$$Ш(A) = 5^2 \cdot 2^?,$$

so this gives evidence.

4 Visibility in Modular Jacobians

Suppose now $A = A_f \subset J_0(N)$ is attached to a newform.

Definition 4.1 (Modular of level M). An element $c \in Ш(A)[p]$ is *modular of level M* if $c \in \mathrm{Vis}^p_M(Ш(A))$, where $\mathrm{Vis}^p_M(Ш(A))$ is the subgroup generated by all kernels of maps $Ш(A)[p^\infty] \to Ш(J_0(M))[p^\infty]$ induced by homomorphisms $A \to J_0(M)$ of degree coprime to p.

Note that M must be a multiple of N.

Question 4.2 (Mazur). Suppose $E \subset J_0(N)$ is an elliptic curve of conductor N. How much of $Ш(E)$ is modular of level N?

Answer: In examples, surprisingly much. Expect not all visible, since

$$\mathrm{Vis}_N(Ш(E)) \subset Ш(E)[\text{modular degree}],$$

and modular degree annihilates symmetric square Selmer group (work of Flach).

4.1 Data and Experiments

- **Cremona-Mazur:** There are 52 elliptic curves $E \subset J_0(N)$ with $N < 5500$ such that $p \mid \#Ш(E)_?$. Cremona-Mazur show that for 43 of these that $Ш(E)$ "probably" is modular of level N, and for 3 that it is definitely not: $N = 2849, 4343, 5389$. ("Probably" was made "provably" in many cases in subsequent work.)
- **Agashe-Stein:** Same question as Cremona-Mazur for $A_f \subset J_0(N)$ of any dimension. Using results of my Ph.D. thesis, MAGMA packages, etc. I computed a divisor and multiple of $\#Ш(A_f)_?$ for the following:
  - 10360 abelian varieties $A_f \subset J_0(N)$ with $L(A_f, 1) \neq 0$.
  - Found 168 with $\#Ш(A_f)_?$ definitely divisible by an odd prime.

- For 39 of these, prove that all $\#\text{Ш}(A_f)^{\text{odd}}_?$ elements are modular of level N, and 106 probably are. This gives strong evidence for the BSD conjecture, and a sense that maybe something further is going on.
- Of these 168, at least 62 have odd conjectural Ш that is definitely *not* modular of level N. Big mystery? Where is this Ш modular? Is it modular at all? Is it even there?? (Perhaps a good place to look for counterexample to BSD.)

5 Visibility at Higher Level

Definition 5.1. Let $c \in \text{Ш}(A_f)$. The *modularity levels* of c are the set of integers

$$\mathcal{N}(c) = \{M : c \in \mathrm{Vis}_M(\text{Ш}(A_f))\}.$$

Conjecture 5.2 (Stein). *For any $c \in \text{Ш}(A_f)$ we have*

$$\mathcal{N}(c) \neq \emptyset,$$

i.e., every element of $\text{Ш}(A_f)$ is modular.

Motivation: This is a working hypothesis that makes *computing* with modular abelian varieties easier. Also, if there were a common level at which all of $\text{Ш}(A_f)$ were modular, then $\text{Ш}(A_f)$ would be finite, and conversely (assuming the conjecture).

5.1 Ribet Level Raising

Suppose that $f = \sum a_n q^n \in S_2(\Gamma_0(N))$ is a newform and $\mathfrak{p}$ is a nonzero prime ideal of $\mathbf{Z}[a_1, a_2, \ldots]$ such that $A_f[\mathfrak{p}]$ is irreducible. If

$$a_\ell + \ell + 1 \equiv 0 \pmod{\mathfrak{p}}$$

then there exists an ℓ-newform $g \in S_2(\Gamma_0(N\ell))$ such that $i(A_f[\mathfrak{p}]) = A_g[\mathfrak{p}]$ for an appropriate $i : J_0(N) \to J_0(N\ell)$ of degree coprime to $\mathrm{char}(\mathfrak{p})$ and the sign of the functional equations for $L(f,s)$ and $L(g,s)$ are the same.

If we instead require that $a_\ell - (\ell+1) \equiv 0 \pmod{\mathfrak{p}}$ then there is such a g, but the sign of the functional equation changes, and the new Tamagawa numbers of A_g at ℓ will (or tends to be?) divisible by $\mathfrak{p}$.

5.2 Evidence for Conjecture

I defined a precise notion of "probably modular" motivated by Theorem 3.4 and what I can compute. In many cases I could do extra work and actually prove modularity; however, at this stage it is more interesting to gather data to see what is going on, in order to have a sense for what to conjecture.

Mazur proved that everything in $\text{Ш}(E)[3]$, for E an elliptic curve, is visible in an abelian surface, which, together with the modularity theorem, *might* imply modularity of $\text{Ш}(E)[3]$ at higher level. Same for 2, proved by me and by a different method by Thomas Klenke.

6 Some Tables

The first two pages of tables below give some of the data that I computed about visibility of Shafarevich-Tate groups at level N. The third table gives the new data about visibility at higher level.

Nontrivial Odd Parts of Shafarevich-Tate Groups

A	dim	S_l	S_u	$\mathrm{moddeg}(A)^{\mathrm{odd}}$	B	dim	$A^\vee \cap \tilde{B}^\vee$	Vis
389E$*$	20	5^2	$=$	5	**389A**	1	$[20^2]$	5^2
433D$*$	16	7^2	$=$	$7\cdot 111$	**433A**	1	$[14^2]$	7^2
446F$*$	8	11^2	$=$	$11\cdot 359353$	**446B**	1	$[11^2]$	11^2
551H	18	3^2	$=$	169	NONE			
563E$*$	31	13^2	$=$	13	**563A**	1	$[26^2]$	13^2
571D$*$	2	3^2	$=$	$3^2\cdot 127$	**571B**	1	$[3^2]$	3^2
655D$*$	13	3^4	$=$	$3^2\cdot 9799079$	**655A**	1	$[36^2]$	3^4
681B	1	3^2	$=$	$3\cdot 125$	**681C**	1	$[3^2]$	–
707G$*$	15	13^2	$=$	$13\cdot 800077$	**707A**	1	$[13^2]$	13^2
709C$*$	30	11^2	$=$	11	**709A**	1	$[22^2]$	11^2
718F$*$	7	7^2	$=$	$7\cdot 5371523$	**718B**	1	$[7^2]$	7^2
767F	23	3^2	$=$	1	NONE			
794G$*$	12	11^2	$=$	$11\cdot 34986189$	**794A**	1	$[11^2]$	–
817E$*$	15	7^2	$=$	$7\cdot 79$	**817A**	1	$[7^2]$	–
959D	24	3^2	$=$	583673	NONE			
997H$*$	42	3^4	$=$	3^2	**997B**	1	$[12^2]$	3^2
					997C	1	$[24^2]$	3^2
1001F	3	3^2	$=$	$3^2\cdot 1269$	**1001C**	1	$[3^2]$	–
					91A	1	$[3^2]$	–
1001L	7	7^2	$=$	$7\cdot 2029789$	**1001C**	1	$[7^2]$	–
1041E	4	5^2	$=$	$5^2\cdot 13589$	**1041B**	2	$[5^2]$	–
1041J	13	5^4	$=$	$5^3\cdot 21120929983$	**1041B**	2	$[5^4]$	–
1058D	1	5^2	$=$	$5\cdot 483$	**1058C**	1	$[5^2]$	–
1061D	46	151^2	$=$	$151\cdot 10919$	**1061B**	2	$[2^2 302^2]$	–
1070M	7	$3\cdot 5^2$	$3^2\cdot 5^2$	$3\cdot 5\cdot 1720261$	**1070A**	1	$[15^2]$	–
1077J	15	3^4	$=$	$3^2\cdot 1227767047943$	**1077A**	1	$[9^2]$	–
1091C	62	7^2	$=$	1	NONE			
1094F$*$	13	11^2	$=$	$11^2\cdot 172446773$	**1094A**	1	$[11^2]$	11^2
1102K	4	3^2	$=$	$3^2\cdot 31009$	**1102A**	1	$[3^2]$	–
1126F$*$	11	11^2	$=$	$11\cdot 13990352759$	**1126A**	1	$[11^2]$	11^2
1137C	14	3^4	$=$	$3^2\cdot 64082807$	**1137A**	1	$[9^2]$	–
1141I	22	7^2	$=$	$7\cdot 528921$	**1141A**	1	$[14^2]$	–
1147H	23	5^2	$=$	$5\cdot 729$	**1147A**	1	$[10^2]$	–
1171D$*$	53	11^2	$=$	$11\cdot 81$	**1171A**	1	$[44^2]$	11^2
1246B	1	5^2	$=$	$5\cdot 81$	**1246C**	1	$[5^2]$	–
1247D	32	3^2	$=$	$3^2\cdot 2399$	**43A**	1	$[36^2]$	–
1283C	62	5^2	$=$	$5\cdot 2419$	NONE			
1337E	33	3^2	$=$	71	NONE			
1339G	30	3^2	$=$	5776049	NONE			
1355E	28	3	3^2	$3^2\cdot 2224523985405$	NONE			
1363F	25	31^2	$=$	$31\cdot 34889$	**1363B**	2	$[2^2 62^2]$	–
1429B	64	5^2	$=$	1	NONE			
1443G	5	7^2	$=$	$7^2\cdot 18525$	**1443C**	1	$[7^1 14^1]$	–
1446N	7	3^2	$=$	$3\cdot 17459029$	**1446A**	1	$[12^2]$	–

Nontrivial Odd Parts of Shafarevich-Tate Groups

A	dim	S_l	S_u	$\mathrm{moddeg}(A)^{\mathrm{odd}}$	B	dim	$A^\vee \cap \tilde{B}^\vee$	Vis
1466H$*$	23	13^2	$=$	$13\cdot 25631993723$	**1466B**	1	$[26^2]$	13^2
1477C$*$	24	13^2	$=$	$13\cdot 57037637$	**1477A**	1	$[13^2]$	13^2
1481C	71	13^2	$=$	70825	NONE			
1483D$*$	67	$3^2\cdot 5^2$	$=$	$3\cdot 5$	**1483A**	1	$[60^2]$	$3^2\cdot 5^2$
1513F	31	3	3^4	$3\cdot 759709$	NONE			
1529D	36	5^2	$=$	535641763	NONE			
1531D	73	3	3^2	3	**1531A**	1	$[48^2]$	–
1534J	6	3	3^2	$3^2\cdot 635931$	**1534B**	1	$[6^2]$	–
1551G	13	3^2	$=$	$3\cdot 110659885$	**141A**	1	$[15^2]$	–
1559B	90	11^2	$=$	1	NONE			
1567D	69	$7^2\cdot 41^2$	$=$	$7\cdot 41$	**1567B**	3	$[4^4 1148^2]$	–
1570J$*$	6	11^2	$=$	$11\cdot 228651397$	**1570B**	1	$[11^2]$	11^2
1577E	36	3	3^2	$3^2\cdot 15$	**83A**	1	$[6^2]$	–
1589D	35	3^2	$=$	6005292627343	NONE			
1591F$*$	35	31^2	$=$	$31\cdot 2401$	**1591A**	1	$[31^2]$	31^2
1594J	17	3^2	$=$	$3\cdot 259338050025131$	**1594A**	1	$[12^2]$	–
1613D$*$	75	5^2	$=$	$5\cdot 19$	**1613A**	1	$[20^2]$	5^2
1615J	13	3^4	$=$	$3^2\cdot 13317421$	**1615A**	1	$[9^1 18^1]$	–
1621C$*$	70	17^2	$=$	17	**1621A**	1	$[34^2]$	17^2
1627C$*$	73	3^4	$=$	3^2	**1627A**	1	$[36^2]$	3^4
1631C	37	5^2	$=$	6354841131	NONE			
1633D	27	$3^6\cdot 7^2$	$=$	$3^5\cdot 7\cdot 31375$	**1633A**	3	$[6^4 42^2]$	–
1634K	12	3^2	$=$	$3\cdot 3311565989$	**817A**	1	$[3^2]$	–
1639G$*$	34	17^2	$=$	$17\cdot 82355$	**1639B**	1	$[34^2]$	17^2
1641J$*$	24	23^2	$=$	$23\cdot 1491344147471$	**1641B**	1	$[23^2]$	23^2
1642D$*$	14	7^2	$=$	$7\cdot 123398360851$	**1642A**	1	$[7^2]$	7^2
1662K	7	11^2	$=$	$11\cdot 16610917393$	**1662A**	1	$[11^2]$	–
1664K	1	5^2	$=$	$5\cdot 7$	**1664N**	1	$[5^2]$	–
1679C	45	11^2	$=$	6489	NONE			
1689E	28	3^2	$=$	$3\cdot 172707180029157365$	**563A**	1	$[3^2]$	–
1693C	72	1301^2	$=$	1301	**1693A**	3	$[2^4 2602^2]$	–
1717H$*$	34	13^2	$=$	$13\cdot 345$	**1717B**	1	$[26^2]$	13^2
1727E	39	3^2	$=$	118242943	NONE			
1739F	43	659^2	$=$	$659\cdot 151291281$	**1739C**	2	$[2^2 1318^2]$	–
1745K	33	5^2	$=$	$5\cdot 1971380677489$	**1745D**	1	$[20^2]$	–
1751C	45	5^2	$=$	$5\cdot 707$	**103A**	2	$[505^2]$	–
1781D	44	3^2	$=$	61541	NONE			
1793G$*$	36	23^2	$=$	$23\cdot 8846589$	**1793B**	1	$[23^2]$	23^2
1799D	44	5^2	$=$	201449	NONE			
1811D	98	31^2	$=$	1	NONE			
1829E	44	13^2	$=$	3595	NONE			
1843F	40	3^2	$=$	8389	NONE			
1847B	98	3^6	$=$	1	NONE			
1871C	98	19^2	$=$	14699	NONE			

Visibility at Higher Level

A_f with odd invisible $Ш_{\mathrm{an}}[\ell]$	All ℓ-congruent $A_g \subset J_0(Np)_{\mathrm{new}}$ with $Np \leq 5000$ and $\mathrm{ord}_{s=1} L(g,s) \geq 0$ (and higher Np if known)
551, dim 18, $\ell = 3$	$\mathbf{p=2}$: dim 1, rank 2 $\mathbf{p=3}$: dim 1, rank 2 $\mathbf{p=5}$: dim 25, rank 0
767, dim 23, $\ell = 3$	$\mathbf{p=2}$: dim 1, rank 2 $\mathbf{p=7}$: dim 1, rank 2 $\mathbf{p=7}$: dim 52, rank 0
959, dim 24, $\ell = 3$	$\mathbf{p=2}$: dim 1, rank 2
1091, dim 62, $\ell = 7$	$\mathbf{p=7}$: dim 2, rank 2
1283, dim 62, $\ell = 5$	$\mathbf{p=3}$: dim 2, rank 2
1337, dim 33, $\ell = 3$	$\mathbf{p=2}$: dim 1, rank 2
1339, dim 30, $\ell = 3$	$\mathbf{p=2}$: dim 1, rank 2
1355, dim 28, $\ell = 3$	$\mathbf{p=2}$: dim 1, rank 2
1429, dim 64, $\ell = 5$	$\mathbf{p=2}$: dim 2, rank 2 $\mathbf{p=3}$: dim 66, rank 0
1481, dim 71, $\ell = 13$	Nothing in range
1513, dim 31, $\ell = 3$	$\mathbf{p=2}$: dim 1, rank 2
1529, dim 36, $\ell = 5$	$\mathbf{p=7}$: dim 1, rank 2
1559, dim 90, $\ell = 11$	Nothing in range
1589, dim 35, $\ell = 3$	Nothing in range
1631, dim 37, $\ell = 5$	$\mathbf{p=2}$: dim 1, rank 2
1679, dim 45, $\ell = 11$	$\mathbf{p=2}$: dim 2, rank 2
1727, dim 39, $\ell = 3$	$\mathbf{p=2}$: dim 1, rank 2
2849, dim 1, $\ell = 3$	$\mathbf{p=3}$: dim 1, rank 2
4343, dim 1, $\ell = 3$	Nothing in range
5389, dim 1, $\ell = 3$	$\mathbf{p=7}$: dim 1, rank 2

When the second column contains an A_g of rank 2, then $Ш(A_f)[\ell]$ is "very likely" to be visible of level $M = Np$. This is the case for most examples. The "Nothing in range" note means that the smallest p for which there exists g of even analytic rank congruent to f is beyond the range of my current tables. The examples of level 2849, 4343, and 5389 are the odd and definitely invisible examples in Cremona and Mazur's original paper on visibility.