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Mean values of derivatives of modular L-series

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1. Introduction

Recently, Kolyvagin [4] proved the finiteness of the Tate-Shafarevic group $\text{III}_{E/\mathbf{Q}}$ of certain modular elliptic curves E over $\mathbf{Q}$. More precisely, let $E/\mathbf{Q}$ be a modular elliptic curve with conductor N and $L(s)$ its associated L-series:

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

Set

$$L_D(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}\left(\frac{D}{n}\right)$$

where (D/n) is the Legendre symbol. Suppose that $L(1) \neq 0$ and there exists a $D < 0$, such that

(i) $L_D(s)$ has a simple zero at $s = 1$ and

(ii) all primes which divide the conductor of E split in the imaginary quadratic field $\mathbf{Q}(\sqrt{D})$.

Under these conditions, Kolyvagin [4] showed that both $E(\mathbf{Q})$ and $\text{III}_{E/\mathbf{Q}}$ are finite. More recently, he extended this theorem to show that if $L(s)$ has a simple zero at $s = 1$ and there is a $D < 0$ satisfying (ii) above and $L_D(1) \neq 0$, then rank $E(\mathbf{Q}) = 1$ and $\text{III}_{E/\mathbf{Q}}$ is finite. Previously, Rubin [11] established the finiteness of $\text{III}_{E/\mathbf{Q}}$ for CM elliptic curves for which $L(1) \neq 0$. The work of Rubin and Kolyvagin represents significant steps toward the resolution of the important conjecture that $\text{III}_{E/\mathbf{Q}}$ is finite.

The purpose of this paper is to establish the existence of a $D < 0$ such that $L_D(s)$ has a simple zero at $s = 1$ and all primes dividing the conductor of E split completely in the quadratic field $\mathbf{Q}(\sqrt{D})$. Thus, the result of Kolyvagin can be stated without any hypothesis on quadratic twists of $L(s)$. We prove our theorems by showing that the mean value of $L'_D(1)$ is non-zero. More precisely, we prove the following.

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THEOREM 1. *Suppose that* $L(1) \neq 0$. *Let*

$$C = \frac{1}{2N} \sum_{n_1, n_2} \frac{a(n_1 n_2^2)}{n_1 n_2^2} \frac{\phi(n_2)}{n_2},$$

where n_1 *ranges over positive integers with the property that* $p \mid n_1$ *implies* $p \mid 4N$ *and* $(n_2, 4N) = 1$ *and* ϕ *denotes Euler's function. Then,* $C \neq 0$ *and*

$$\sum_{\substack{0 < -D \leq Y \\ D \equiv 1 (\mathrm{mod}\, 4N)}} L'_D(1) = CY(\log Y) + o(Y \log Y),$$

as $Y \to \infty$.

The theorem is, in reality, a theorem about the mean values of derivatives of L-series attached to modular forms.

To fix ideas, let $F(z)$ be a cusp form of weight 2 on $\Gamma_0(N)$ which is a normalized eigenform for the Hecke operators. Suppose that F is not a modular form on $\Gamma_0(M)$ for any proper divisor M of N, and write

$$F(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

for its Fourier expansion at the cusp $i\infty$. Let

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

be the associated L-series which satisfies a functional equation:

$$A^s \Gamma(s) L(s) = w A^{2-s} \Gamma(2-s) L(2-s),$$

where $w = \pm 1$ and $A = \sqrt{N}/2\pi$. Let $\chi_D(n) = (D/n)$ be a real character $\mathrm{mod}\, D$. Then, we can consider

$$L_D(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \left(\frac{D}{n} \right).$$

It extends to an entire function of s, and if D is a fundamental discriminant prime to N, then $L_D(s)$ satisfies a functional equation:

$$(A|D|)^s \Gamma(s) L_D(s) = w \chi_D(-N) (A|D|)^{2-s} \Gamma(2-s) L_D(2-s).$$

THEOREM 2. *Suppose that* $L(1) \neq 0$. *Let*

$$C = \frac{1}{2N} \sum_{n_1, n_2} \frac{a(n_1 n_2^2)}{n_1 n_2^2} \frac{\phi(n_2)}{n_2},$$

where n_1 *ranges over positive integers with the property that* $p \mid n_1$ *implies*

$p \mid 4N$ and $(n_2, 4N) = 1$ and ϕ denotes Euler's function. Then $C \neq 0$ and

$$\sum_{\substack{0 < -D \leq Y \\ D \equiv 1 (\mathrm{mod}\, 4N)}} L'_D(1) = CY \log Y + o(Y \log Y),$$

as $Y \to \infty$.

COROLLARY. *Suppose that $L(1) \neq 0$. Then, there are infinitely many fundamental discriminants $D < 0$ such that $L_D(s)$ has a simple zero at $s = 1$ and all primes dividing the conductor of E split completely in the imaginary quadratic field $\mathbf{Q}(\sqrt{D})$.*

This corollary was also established by Bump, Friedberg and Hoffstein [1] utilising the automorphic theory of GSp(4).

Remark 1. The proof will show that we need not assume that $L(1) \neq 0$ but only that the root number of $L(s)$ is $+1$.

Remark 2. We know that

$$w = (-1)^{\mathrm{ord}_{s=1} L(E, s)}.$$

Therefore, the assumption that $w = +1$ and the congruence $D \equiv 1 (\mathrm{mod}\, 4N)$ imply that $w\chi_D(-N) = -1$ and so $L_D(E, s)$ has an odd order zero at $s = 1$.

Remark 3. To see that C is non-zero, we obtain that

$$\sum_{n_1, n_2} \frac{a(n_1 n_2^2)}{(n_1 n_2^2)^s} \frac{\phi(n_2)}{n_2}$$

$$= \left(\sum_{\substack{b=1 \\ p \mid b \Rightarrow p \mid 4N}}^{\infty} \frac{a(b)}{b^s} \right) \prod_{p \nmid 4N} \left(1 + \frac{p-1}{p} \left(\frac{a(p^2)}{p^{2s}} + \frac{a(p^4)}{p^{4s}} + \cdots \right) \right).$$

Now consider the Euler product above. For $p \nmid 4N$, let us write $a(p) = \alpha_p + \beta_p$ with $|\alpha_p| = |\beta_p| = p^{1/2}$. If from the factor at p we factor out $\Sigma a(p^{2\alpha})/p^{2\alpha s}$, we find (for even N)

$$\sum \frac{a(n_1 n_2^2)}{(n_1 n_2^2)^s} \frac{\phi(n_2)}{n_2} = \mathscr{P}(s) \left(\sum_{(n, 4N)=1} \frac{a(n^2)}{n^{2s}} \right) \prod_{p \mid 4N} \left(1 - \frac{a(p)}{p^s} \right)^{-1}$$

where

$$\mathscr{P}(s) = \prod_{p \nmid 4N} \left(1 + \frac{1}{p}\left[\left(1 - \frac{1}{p^{4s-2}}\right)^{-1}\left(1 - \frac{\alpha_p^2}{p^{2s}}\right)\right.\right.$$
$$\left.\left.\times\left(1 - \frac{\beta_p^2}{p^{2s}}\right)\left(1 - \frac{1}{p^{2s-1}}\right) - 1\right]\right).$$

We observe that this product converges absolutely for $\mathrm{Re}(s) > 1/2$ and that none of the Euler factors vanishes at $s = 1$. Since $a(p) = 0, \pm 1$, for $p \mid 4N$,

$$\prod_{p\mid 4N}\left(1 - \frac{a(p)}{p}\right)^{-1} \neq 0.$$

If N is odd, a similar non-zero factor is obtained. Moreover, the Euler product of the series

$$\sum_{\substack{n=1 \\ (n,4N)=1}}^{\infty} \frac{a(n^2)}{n^s}$$

differs from that of $L(s, \mathrm{Sym}^2)\zeta(2s - 2)^{-1}$ at only a finite number of primes and at these primes, none of the Euler factors vanishes at $s = 2$. Thus,

$$C \neq 0,$$

since $L(2, \mathrm{Sym}^2) \neq 0$ (see for example [5] or [8]). Therefore, our theorem shows that there exists D such that (i) and (ii) hold.

Remark 4. Our proof produces an error term of $\mathbf{O}(Y(\log Y)^{1-\rho})$ for an explicit value of ρ.

Remark 5. Our method is applicable to holomorphic modular forms f of any weight $k \geq 2$ for $\Gamma_0(N)$.

Proof of the corollary. Suppose there are only a finite number of D_i's, $D_1, \cdots, D_r$, say, satisfying (i) and (ii). Set

$$\frac{1}{L(s)} = \sum_{n=1}^{\infty} \frac{\tilde{\mu}(n)}{n^s}.$$

Notice that $|\tilde{\mu}(d)| \leq \mathbf{d}(d)\sqrt{d}$ and $\tilde{\mu}(d) = 0$ if $p^3 | d$ for any prime p. (We write $\mathbf{d}(n)$ for the number of positive divisors of n.) Then, fixing an i and writing

$D = D_i \delta^2$, we have the relation

$$L_D(s) = L_{D_i}(s) \sum_{d|\delta^2} \frac{\tilde{\mu}(d)}{d^s} \left(\frac{D_i}{d} \right).$$

We deduce that

$$L'_D(1) = L'_{D_i}(1) \sum_{d|\delta^2} \frac{\tilde{\mu}(d)}{d} \left(\frac{D_i}{d} \right).$$

Thus,

$$\sum_{\substack{0 < -D \leq Y \\ D = D_i \delta^2}} L'_D(1) \ll |L'_{D_i}(1)| \sum_{0 \leq \delta \leq \sqrt{Y}} \sum_{d|\delta^2} \frac{|\tilde{\mu}(d)|}{d}$$

$$\ll \sum_{d \leq Y} \frac{|\tilde{\mu}(d)|}{d} \left(\sum_{\substack{\delta \leq \sqrt{Y} \\ \delta^2 \equiv 0 (\operatorname{mod} d)}} 1 \right).$$

We write $d = d_0 d_1^2$ with d_0, d_1 coprime and squarefree. Then, the inner sum is

$$\frac{\sqrt{Y}}{d_0 d_1} + \mathbf{O}(1).$$

Then our sum is

$$\ll \sum_{d_1 \leq Y} \frac{\mathbf{d}(d_1^2)}{d_1} \sum_{d_0 \leq Y/d_1^2} \frac{\mathbf{d}(d_0)}{\sqrt{d_0}} \left(\frac{\sqrt{Y}}{d_0 d_1} + \mathbf{O}(1) \right)$$

and we easily deduce that this is

$$\ll \sqrt{Y} \log Y.$$

Summing over i produces a contradiction.

There are heuristics that suggest, in fact, that there should be a positive proportion of such D's. To approach the problem of getting an estimate for the *number* of such D's, one should modify the kernel in our integrals to make it more sensitive to the counting problem.

We close this section by introducing some further notation. If D_0 is a fundamental discriminant which is coprime to N, then

$$L_{D_0}(1 + s) = w \chi_{D_0}(-N) |D_0|^{-2s} A^{-2s} \frac{\Gamma(1 - s)}{\Gamma(1 + s)} L_{D_0}(1 - s).$$

Any $D \equiv 1 (\operatorname{mod} 4)$ can be written as $D = D_0 r^2$, where D_0 is a fundamental discriminant and

$$L_D(s) = L_{D_0}(s) \sum_{d|r^2} \frac{\tilde{\mu}(d)}{d^s}\left(\frac{D_0}{d}\right).$$

Let us set

$$\tilde{f}_Y(n, s; a) = \sum_{\substack{0<-D\leq Y \\ D\equiv a(\mathrm{mod}\, 4N)}} \left(\frac{D}{n}\right)|D|^s, \qquad D \text{ unrestricted}$$

$$f_Y(n, s; a) = \sum_{\substack{0<-D_0\leq Y \\ D_0\equiv a(\mathrm{mod}\, 4N)}} \left(\frac{D_0}{n}\right)|D_0|^s, \quad D_0 \text{ fundamental}$$

$$\tilde{f}_Y(n; a) = \tilde{f}_Y(n, 0; a), \qquad f_Y(n; a) = f_Y(n, 0; a).$$

We shall write $\mathbf{d}(n)$ for the number of positive divisors of n.

We stress that the naive approach to the proof of the main theorem works after a few technical details are surmounted. The reader interested in ignoring these details and desiring an intuitive description of the proof can find it in [9].

The next two sections establish the requisite lemmas to estimate the sums we will encounter.

2. Lemmas

Throughout this paper, we will adopt the convention that a natural number n is written as n_1n_2, where n_1 has the property that any prime divisor of it is also a divisor of $2N$ and n_2 is coprime to $2N$. On certain occasions, the same convention applies when we write $m = m_1m_2$.

LEMMA 1. *For* $(a, N) = 1$,

$$\sum_{\substack{n\leq X \\ n=n_1n_2}}{}' \left| \sum_{\substack{0<-h\leq Y \\ h\equiv a(\mathrm{mod}\, N)}} \left(\frac{h}{n}\right) \right|^2 \ll (N^2/\phi(N))XY \log^2 X$$

where the implied constant is absolute and the sum over $n \leq X$ *such that* n_2 *is not a perfect square.*

Proof. This lemma is easily derived from the results of Jutila [3] (for the case $N = 1$) and Fainleib and Saparnijazov [2] (for the general case). They prove:

$$(1) \qquad \sum_{\substack{n\leq X \\ (n,2N)=1}}{}' \left| \sum_{\substack{0<-h\leq Y \\ h\equiv a(\mathrm{mod}\, N)}} \left(\frac{h}{n}\right) \right|^2 \ll NXY \log^2 X,$$

where the prime on the first sum means that n is not a square. The sum in the lemma is seen to be

$$\sum_{n_1 \le X} \sum_{n_2 \le X/n_1}' \left| \sum_{\substack{0 < -h \le Y \\ h \equiv a (\mathrm{mod}\, N)}} \left(\frac{h}{n_2} \right) \right|^2 .$$

By (1), the sum in question is

$$\ll N \sum_{n_1 \le X} \frac{X}{n_1} Y \log^2 X \ll (NXY \log^2 X) \prod_{p|2N} \left(1 - \frac{1}{p}\right)^{-1}$$

which is the desired result.

Lemma 2. *Let d and a be fixed integers, with $a \equiv 1 (\mathrm{mod}\, 4)$, $(ad, 4N) = 1$. Then,*

$$\left| \sum_{n \le U,\, n_2 d \ne b^2} \frac{a(n)}{n} f_Y(nd; a) \right| \ll (Ud)^{1/2} Y^{1/2} \log Y \log(Ud).$$

Proof. By partial summation and Lemma 1,

$$\sum_{m \le U}' \frac{1}{\sqrt{m}} \left| \sum_{\substack{h \le Y \\ h \equiv a (\mathrm{mod}\, 4N)}} \left(\frac{h}{m} \right) \right|^2 \ll U^{1/2} Y \log^2 U.$$

Let $\chi_0^{(a)}$ denote the principal character mod a. The sum in question is

$$\sum_{n \le U,\, n_2 d \ne b^2} \frac{a(n)}{n} \sum_{\substack{0 < -D \le Y \\ D \equiv a (\mathrm{mod}\, 4N)}} \left(\frac{D}{nd} \right) \sum_{j^2 | D} \mu(j)$$

$$= \sum_{j^2 \le Y} \mu(j) \sum_{n \le U,\, n_2 d \ne b^2} \frac{a(n)}{n} \chi_0^{(j)}(nd) \sum_{\substack{0 < -h \le Y/j^2 \\ h \equiv a\bar{j}^2 (\mathrm{mod}\, 4N)}} \left(\frac{h}{nd} \right)$$

$$\ll \sum_{j^2 \le Y} \left(\sum_{n \le U} \frac{|a(n)|^2}{n^{3/2}} \right)^{1/2} \left(\sum_{n \le U,\, n_2 d \ne b^2} \frac{1}{\sqrt{n}} \left| \sum_{\substack{0 < -h \le Y/j^2 \\ h \equiv a\bar{j}^2 (\mathrm{mod}\, 4N)}} \left(\frac{h}{nd} \right) \right|^2 \right)^{1/2}$$

$$\ll U^{1/4} d^{1/4} \sum_{j^2 \le Y} \left(\sum_{m \le Ud}' \frac{1}{\sqrt{m}} \left| \sum_{\substack{0 < -h \le Y/j^2 \\ h \equiv a\bar{j}^2 (\mathrm{mod}\, 4N)}} \left(\frac{h}{m} \right) \right|^2 \right)^{1/2}$$

$$\ll U^{1/2} d^{1/2} Y^{1/2} (\log Y)(\log Ud),$$

as desired.

LEMMA 3. *For* $\Re s = -\eta$, $0 < \eta < 1/2$, $a \equiv 1 \pmod 4$, $(ad, 4N) = 1$,

$$\sum_{n \le U, n_2 d \ne b^2} \frac{a(n)\chi_0^{(j)}(nd)}{n^{1-s}} \sum_{\substack{0 < -h \le Y \\ h \equiv a \pmod{4N}}} \left(\frac{h}{nd}\right) \ll U^{1/2-\eta} d^{1/2} Y^{1/2} \log Ud.$$

Proof. We apply Cauchy's inequality:A

$$\left(\sum_{n \le U} \frac{|a(n)|^2}{n^{3/2+\eta}}\right)^{1/2} \left(\sum_{n \le U, n_2 d \ne b^2} \frac{1}{n^{1/2+\eta}} \left| \sum_{\substack{0 < -h \le Y \\ h \equiv a \pmod{4N}}} \left(\frac{h}{nd}\right)\right|^2\right)^{1/2}$$

$$\ll d^{1/2} U^{1/2-\eta} Y^{1/2} \log Ud$$

by partial summation.

LEMMA 4. *Under the same conditions as in Lemma* 3,

$$\sum_{n \le U, n_2 d \ne b^2} \frac{a(n)}{n^{1-s}} \chi_0^{(j)}(nd) \sum_{\substack{0 < -h \le Y \\ h \equiv a \pmod{4N}}} \left(\frac{h}{nd}\right) h^{-2s}$$

$$\ll (|s| + 1) d^{1/2} U^{1/2-\eta} Y^{1/2+2\eta} \log Ud.$$

Proof. Apply partial summation to Lemma 3.

LEMMA 5. *If* $\Re s = -\eta$, $0 < \eta < 1/2$, $a \equiv 1 \pmod 4$, $(ad, 4N) = 1$, *then*

$$\sum_{n \le U, n_2 d \ne b^2} \frac{a(n)}{n^{1-s}} f_Y(nd, -2s; a) \ll (|s| + 1) d^{1/2} U^{1/2-\eta} Y^{1/2+2\eta} \log Y \log Ud.$$

Proof. The sum in question is

$$\sum_{n \le U, n_2 d \ne b^2} \frac{a(n)}{n^{1-s}} \sum_{\substack{0 < -D \le Y \\ D \equiv a \pmod{4N}}} \left(\frac{D}{nd}\right) \sum_{j^2 | D} \mu(j) D^{-2s}$$

$$= \sum_{j^2 \le Y} \mu(j) j^{-4s} \sum_{n \le U, n_2 d \ne b^2} \frac{a(n)\chi_0^{(j)}(nd)}{n^{1-s}} \sum_{\substack{0 < -h \le Y/j^2 \\ h \equiv a\bar{j}^2 \pmod{4N}}} \left(\frac{h}{nd}\right) h^{-2s}$$

$$\ll \sum_{j^2 \le Y} j^{4\eta} (|s| + 1)(d^{1/2} U^{1/2-\eta} \log Ud) \frac{Y^{1/2+2\eta}}{j^{1+4\eta}}$$

by Lemma 4. Summing over j gives the desired result.

Putting the above lemmas together proves:

LEMMA 6. *For* $\Re s = -\eta$, $0 < \eta < 1/2$, $a \equiv 1 (\mathrm{mod}\, 4)$, $(ad, 4N) = 1$,

(i)

$$\int_{(-\eta)} \sum_{n \le U,\, n_2 d \ne b^2} \frac{a(n)}{n^{1-s}} \tilde{f}_Y(nd, -2s; a) \zeta(1+2s) \frac{\Gamma(1-s)}{\Gamma(1+s)} x^s \Gamma(s)\, ds$$

$$\ll x^{-\eta} d^{1/2} U^{1/2-\eta} Y^{1/2+2\eta} \log Ud,$$

and

(ii)

$$\int_{(-\eta)} \sum_{n \le U,\, n_2 d \ne b^2} \frac{a(n)}{n^{1-s}} f_Y(nd, -2s; a) \zeta(1+2s) \frac{\Gamma(1-s)}{\Gamma(1+s)} x^s \Gamma(s)\, ds$$

$$\ll x^{-\eta} d^{1/2} U^{1/2-\eta} Y^{1/2+2\eta} \log Y \log Ud.$$

We now proceed to handle the terms corresponding to $n > U$.

LEMMA 7. *For* $\Re s = -\eta$, $\eta > 1/2$, $(ad, 4N) = 1$,

$$\sum_{n > U,\, n_2 d \ne b^2} \frac{a(n)}{n^{1-s}} \chi_0^{(j)}(nd) \sum_{\substack{0 < -h \le Y \\ h \equiv a (\mathrm{mod}\, 4N)}} \left(\frac{h}{nd}\right) \ll U^{1/2-\eta} d^{1/2} Y^{1/2} \log Ud.$$

Proof. By Cauchy's inequality, the sum is bounded by

$$\left(\sum_{n > U} \frac{|a(n)|^2}{n^{3/2+\eta}}\right)^{1/2} \left(\sum_{n > U,\, n_2 d \ne b^2} \frac{1}{n^{1/2+\eta}} \left|\sum_{\substack{0 < -h \le Y \\ h \equiv a (\mathrm{mod}\, 4N)}} \left(\frac{h}{nd}\right)\right|^2\right)^{1/2}$$

$$\ll U^{1/4-\eta/2} d^{1/4+\eta/2} \left(\sum_{m > Ud}{}' \frac{1}{m^{1/2+\eta}} \left|\sum_{\substack{0 < -h \le Y \\ h \equiv a (\mathrm{mod}\, 4N)}} \left(\frac{h}{m}\right)\right|^2\right)^{1/2}$$

$$\ll U^{1/4-\eta/2} d^{1/4+\eta/2} \left(\int_{Ud}^{\infty} \frac{Y \log^2 t}{t^{1/2+\eta}}\, dt\right)^{1/2}$$

$$\ll U^{1/2-\eta} d^{1/2} Y^{1/2} \log Ud,$$

as desired.

LEMMA 8. *Under the conditions of Lemma* 7,

$$\sum_{n>U,\, n_2 d \neq b^2} \frac{a(n)}{n^{1-s}} \chi_0^{(j)}(nd) \sum_{\substack{0<-h\leq Y \\ h\equiv a (\mathrm{mod}\, 4N)}} \left(\frac{h}{nd}\right) h^{-2s}$$
$$\ll (|s|+1) Y^{1/2+2\eta} d^{1/2} U^{1/2-\eta} \log Ud.$$

Proof. Apply partial summation to Lemma 7.

LEMMA 9. *If* $\Re s = -\eta$, $\eta > 1/2$, $a \equiv 1 (\mathrm{mod}\, 4)$, $(ad, 4N) = 1$, *then*

$$\sum_{n>U,\, n_2 d\neq b^2} \frac{a(n)}{n^{1-s}} f_Y(nd, -2s; a) \ll (|s|+1) d^{1/2} U^{1/2-\eta} Y^{1/2+2\eta} \log Y \log Ud.$$

Proof. The proof is analogous to that of Lemma 5, except we use Lemma 8 instead of Lemma 4.

Therefore, we deduce by putting these lemmas together:

LEMMA 10. *For* $\Re s = -\eta$, $1 > \eta > 1/2$, $a \equiv 1 (\mathrm{mod}\, 4)$, $(ad, 4N) = 1$,

(i)

$$\int_{(-\eta)} \sum_{n>U,\, n_2 d\neq b^2} \frac{a(n)}{n^{1-s}} \tilde{f}_Y(nd, -2s; a) \zeta(1+2s) \frac{\Gamma(1-s)}{\Gamma(1+s)} x^s \Gamma(s)\, ds$$
$$\ll x^{-\eta} d^{1/2} U^{1/2-\eta} Y^{1/2+2\eta} \log Ud,$$

and

(ii)

$$\int_{(-\eta)} \sum_{n>U,\, n_2 d\neq b^2} \frac{a(n)}{n^{1-s}} f_Y(nd, -2s; a) \zeta(1+2s) \frac{\Gamma(1-s)}{\Gamma(1+s)} x^s \Gamma(s)\, ds$$
$$\ll x^{-\eta} d^{1/2} U^{1/2-\eta} Y^{1/2+2\eta} \log Y \log Ud.$$

This is the series of lemmas needed. The next section establishes a lemma which refines the above estimates.

3. Further lemmas

We begin by proving:

LEMMA 11.

$$\sum_e \frac{1}{e} \exp(-ne^2/X) = \begin{cases} \frac{1}{2}\log\frac{X}{n} + \frac{1}{2}\gamma + \mathbf{O}\left((n/X)^{1+\varepsilon}\right) & \textit{if } n < \frac{1}{2}X \\ \mathbf{O}(\exp(-n/X)) & \textit{otherwise}. \end{cases}$$

Proof. This follows easily by partial summation (see, for example, Jutila [3, Lemma 1]).

LEMMA 12. *There is a* $\rho > 0$ *so that if* $X \le Y(\log Y)^{1+\rho}$, $a \equiv 1 (\mathrm{mod}\, 4)$, $(a, 4N) = 1$, *then*,

$$\sum_{\substack{m,e \\ (m,j)=1}} \frac{a(m)}{me} \tilde{f}_Y(m; a) \exp(-me^2/X) \ll Y \log^2 Y,$$

for any $j \ge 1$.

Remark. Note that in this sum we are not restricting to fundamental discriminants.

Proof. We split the sum into two parts corresponding to m_2 a square and m_2 not a square.

For the first part, it is

$$= \sum_e \frac{1}{e} \sum_{(m_1 m_2, j)=1} \frac{a(m_1 m_2^2)}{m_1 m_2^2} \exp(-m_1 m_2^2 e^2/X)$$

$$\times \left\{ \left(\frac{a}{m_1} \right) \frac{Y}{4N} \frac{\phi(m_2)}{m_2} + \mathbf{O}(\mathbf{d}(m_2)) \right\}.$$

Using Lemma 11, we see that the above sum is

$$\ll Y \sum_{m < \frac{1}{2}X} \frac{|a(m_1 m_2^2)|}{m_1 m_2^2} \log \frac{X}{m} + Y \sum_{m > \frac{1}{2}X} \frac{|a(m_1 m_2^2)|}{m_1 m_2^2} \mathbf{d}(m) \exp(-m/X)$$

which is

$$\ll Y (\log X)^2.$$

For the second part, we use the Pólya-Vinogradov inequality to see that it is

$$\ll \sum_m \frac{|a(m)|}{m} m^{1/2} \log m \sum_e \frac{1}{e} \exp(-me^2/X).$$

Using Lemma 11 for the inner sum, we see that it is

$$\ll \sum_{m < \frac{1}{2}X} \frac{|a(m)|}{m} m^{1/2} \log m \left\{ \log \frac{X}{m} \right\} + \sum_{m > \frac{1}{2}X} \frac{|a(m)|}{m} m^{1/2} \log m \exp(-m/X)$$

and by partial summation and Lemma 17, this is

$$\ll X (\log X)^{1-\rho},$$

for some $\rho > 0$. This completes the proof.

LEMMA 13. *Let $a \equiv 1 (\bmod 4)$ with $(a, 4N) = 1$ and a equal to a square* mod $4N$. *Then,*

$$\sum_{\substack{0 < -D \le Y \\ D \equiv a (\bmod 4N)}} L'_D(1) \ll Y \log^2 Y.$$

Proof. Consider

$$\frac{1}{2\pi i} \int_{(\sigma)} L_D(1+s)\zeta(1+2s) X^s \Gamma(s)\, ds, \qquad \sigma > 1/2.$$

This is

$$\sum_{n,m} \frac{a(n)}{n} \left(\frac{D}{n} \right) \frac{1}{m} \exp(-nm^2/X).$$

(Note that this series converges absolutely.) By the Phragmén-Lindelöf theorem [6],

$$|L_D(1+s)| \ll (|t|+2)^{1/2-\sigma} (\log(|t|+2))^2$$

uniformly for $-1/2 - \varepsilon \le \sigma \le 1/2 + \varepsilon$. (The implied constant depends on D.) We can therefore move the line of integration to $\Re s = -\eta \ge -1/2 - \varepsilon$.

Moving the line of integration to $\Re s = -\eta$, we obtain

$$\sum_{n,m} \frac{a(n)}{n} \left(\frac{D}{n} \right) \frac{1}{m} \exp(-nm^2/X)$$
$$= \frac{1}{2} L'_D(1) + \frac{1}{2\pi i} \int_{(-\eta)} L_D(1+s)\zeta(1+2s) X^s \Gamma(s)\, ds.$$

Writing

$$L_D(1+s) = L_{D_0}(1+s) \sum_{d|\delta^2} \frac{\tilde{\mu}(d)}{d^{1+s}} \left(\frac{D_0}{d} \right)$$

where D_0 is a fundamental discriminant, we can rewrite the integral as

$$\frac{1}{2\pi i} \sum_{d|\delta^2} \frac{\tilde{\mu}(d)}{d} \int_{(-\eta)} L_{D_0}(1+s) \left(\frac{D_0}{d} \right) \zeta(1+2s)(X/d)^s \Gamma(s)\, ds.$$

Applying the functional equation to $L_{D_0}(1+s)$, we find that the above is equal to

$$\frac{w\chi_{D_0}(-N)}{2\pi i} \sum_{d|\delta^2} \frac{\tilde{\mu}(d)}{d} \int_{(-\eta)} \sum_{n=1}^{\infty} \frac{a(n)}{n^{1-s}} \left(\frac{D_0}{nd} \right) \zeta(1+2s) |D_0|^{-2s} A^{-2s}$$
$$\times \frac{\Gamma(1-s)}{\Gamma(1+s)} (X/d)^s \Gamma(s)\, ds.$$

We sum this over $\delta^2 \le Y$, $(\delta, 4N) = 1$, $0 < -D_0 \le Y/\delta^2$, $D_0\delta^2 \equiv a(\text{mod}\, 4N)$. Recall that we have set

$$\tilde{f}_Y(n, s; a) = \sum_{\substack{0 < -D \le Y \\ D \equiv a(\text{mod}\, 4N)}} \left(\frac{D}{n}\right)|D|^s, \qquad D \text{ unrestricted},$$

$$f_Y(n, s; a) = \sum_{\substack{0 < -D_0 \le Y \\ D_0 \equiv a(\text{mod}\, 4N)}} \left(\frac{D_0}{n}\right)|D_0|^s, \quad D_0 \text{ fundamental},$$

$$\tilde{f}_Y(n; a) = \tilde{f}_Y(n, 0; a), \qquad f_Y(n; a) = f_Y(n, 0; a).$$

Then,

$$(\star) \quad \sum_{n,m} \frac{a(n)}{n}\tilde{f}_Y(n; a)\frac{1}{m}\exp(-nm^2/X)$$
$$= \frac{1}{2}\sum_{\substack{0 < -D \le Y \\ D \equiv a(\text{mod}\, 4N)}} L'_D(1) - \frac{1}{2\pi i}\sum_{\substack{\delta^2 \le Y \\ (\delta, 4N) = 1}}\sum_{d | \delta^2}\frac{\tilde{\mu}(d)}{d}\int_{(-\eta)}\sum_{n=1}^{\infty}\frac{a(n)}{n^{1-s}}$$
$$\times f_{Y/\delta^2}(nd, -2s; a\bar{\delta}^2)\zeta(1 + 2s)$$
$$\times \frac{\Gamma(1 - s)}{\Gamma(1 + s)}A^{-2s}(X/d)^s\Gamma(s)\, ds.$$

Here $\bar{\delta}\delta \equiv 1(\text{mod}\, 4N)$. The sum on the left side of $(\star)$ is seen to be

$$\ll Y(\log Y)^2$$

by Lemma 12, provided we take $X \le Y(\log Y)^{1+\rho}$. Now we deal with the integral on the right side of $(\star)$. The integral is first broken up according to whether $n_2 d$ is a square or not. If $n_2 d$ is not a square, then splitting the series at $n \le Y$ and moving the integral to $\Re s = -\eta_1$, $0 < \eta_1 < 1/2$, we can utilise Lemma 6(ii) to get an estimate

$$\sum_{\delta^2 \le Y}\sum_{d | \delta^2}\frac{|\tilde{\mu}(d)|}{d}d^{1/2}\frac{Y}{\delta^{1+3\eta_1}}\left(\frac{Y}{X}\right)^{\eta_1}\log^2 Y \ll Y(Y/X)^{\eta_1}\log^2 Y.$$

The series corresponding to $n \ge Y$ is similarly estimated by moving to $\Re s = -\eta_2$, $\eta_2 > 1/2$ and using Lemma 10(ii). The final contribution when $n_2 d$ is not a square is therefore

$$Y\left\{\left(\frac{Y}{X}\right)^{\eta_1} + \left(\frac{Y}{X}\right)^{\eta_2}\right\}\log^2 Y.$$

This term is $\ll Y \log^2 Y$ if $Y \le X$. Now we consider the contribution when $n_2 d$ is a square.

The series in the integral is seen to be

$$\sum_{n_1, n_2} \frac{a(n_1 n_2)}{(n_1 n_2)^{1-s}} \left(\frac{a}{n_1}\right) f_{Y/\delta^2}(n_2 d, -2s, a\bar{\delta}^2).$$

Let

$$H_d(s) = \sum_{\substack{n_1, n_2 \\ n_2 d = b^2}} \frac{a(n_1 n_2)}{(n_1 n_2)^s} \left(\frac{a}{n_1}\right) \prod_{p | 4Nn_2 d} \left(1 + \frac{1}{p}\right)^{-1}.$$

We see that this is equal to

$$\sum_{\substack{n_1, n_2 \\ n_2 d = b^2}} \frac{a(n_1) a(n_2)}{(n_1 n_2)^s} \left(\frac{a}{n_1}\right) \prod_{p | 4Nn_2 d} \left(1 + \frac{1}{p}\right)^{-1}$$

$$= \left(\sum_{p | n \Rightarrow p | 2N} \frac{a(n_1)}{n_1^s} \left(\frac{a}{n_1}\right)\right) \left(\sum_{n_2 d = b^2} \frac{a(n_2)}{n_2^s} \prod_{p | 4Nn_2 d} \left(1 + \frac{1}{p}\right)^{-1}\right).$$

Thus, in order to estimate the integral, we first need an estimate for $H_d(s)$. Since the first sum is uniformly bounded in the region under consideration, we need only consider the second Dirichlet series. If $d = d_0 d_1^2$, d_0 squarefree, then

$$\sum_{n_2 d = b^2} \frac{a(n_2)}{n_2^s} \prod_{p | 4Nn_2 d} \left(1 + \frac{1}{p}\right)^{-1} = \sum_m \frac{a(d_0 m^2)}{d_0^s m^{2s}} \prod_{p | 4Nmd} \left(1 + \frac{1}{p}\right)^{-1}$$

where the sum over m is such that $(m, 2N) = 1$. Let us now estimate the integral. For this purpose, let us define

$$F_d(s) = \sum_{n=1}^{\infty} \frac{a(d_0 n^2)}{(d_0 n^2)^s} \prod_{p | 4Nnd} \left(1 + \frac{1}{p}\right)^{-1}.$$

By factoring $F_d(s)$ as an Euler product and using simple estimates, we find that

$$|F_d(s)| \ll c^{\nu(d)} d_0^{1/2-\sigma} |L(2s, \mathrm{Sym}^2) \zeta(4s-2)^{-1}|,$$

for $\sigma > 3/4$. Here, c is an absolute positive constant (we can take $c = 200$ for example) and $\nu(d)$ is the number of prime factors of d.

We are now ready to estimate the integral when $n_2 d$ is a square. We move this integral to $\eta = 1/\log Y$ and evaluate it using the above estimate for $F_d(s)$.

By an easy variant of Jutila's result [3, Lemma 1], we find the integral is

$$\frac{1}{2\pi i}\int_{(-\eta)} \sum_{\substack{n=1\\ n_2 d = b^2}}^{\infty} \frac{a(n)}{n^{1-s}}\left(\frac{a}{n_1}\right)A^{-2s}\frac{\Gamma(1-s)}{\Gamma(1+s)}\zeta(1+2s)(X/d)^s\Gamma(s)$$

$$\times\left\{\frac{6}{\pi^2}\frac{1}{\phi(4N)}\prod_{p|4Nn_2d}\left(1+\frac{1}{p}\right)^{-1}\frac{(Y/\delta^2)^{1-2s}}{1-2s}\right.$$

$$\left.+\mathbf{O}\left((|s|+1)\mathbf{d}(4Nn_2d)\frac{Y^{1/2+2\eta}}{\delta^{1+4\eta}}\right)\right\}ds.$$

The error term presents no problem. The main term is

$$\ll \frac{Y}{\delta^2}(d/X)^{\eta}c^{\nu(d)}(\log Y)^2.$$

Summing over $d|\delta^2$ and $\delta^2 \le Y$, we obtain a total estimate of

$$\ll Y\sum_{\delta^2\le Y}\frac{1}{\delta^2}\sum_{d|\delta^2}\frac{|\tilde{\mu}(d)|}{d}(d/X)^{\eta}c^{\nu(d)}(\log Y)^2 \ll Y(\log Y)^2.$$

This completes the proof of Lemma 13.

Utilising Lemma 13, we can now derive an alternate estimate for the sum

$$\sum_{m,e}\frac{a(m)}{me}\tilde{f}_Y(m;a)\exp\left(-\frac{me^2}{X}\right).$$

LEMMA 14. *If $Y \le X$, $a \equiv 1 (\mathrm{mod}\ 4)$, $(ad, 4N) = 1$, and any $j \ge 1$,*

(i) $$\sum_{\substack{m,e\\ (m,j)=1}}\frac{a(m)}{me}\tilde{f}_Y(md;a)\exp(-me^2/X) \ll d^{1/2}Y\log^2 Y.$$

Also

(ii)

$$\sum_{\substack{m,e\\ (m,j)=1}}{}' \frac{a(m)}{me}\tilde{f}_Y(md;a)\exp(-me^2/X) \ll d^{1/2}Y(\log^2 Y + \log X),$$

where the prime on the sum indicates that $m_2 d$ is not a square.

Proof. Consider equation ($\star$) in the proof of Lemma 13, which shows that under the hypothesis $Y \le X$, the integral in ($\star$) is $\mathbf{O}(Y\log^2 Y)$. Lemma 13 itself asserts that the same is true of the sum of the $L'_D(1)$. Our assertion follows for

$j = 1$ and $d = 1$. In general, we consider an analogue of Lemma 13 in which we multiply both sides of the original equation by (D/d) and only sum over $D = \delta^2 D_0$ with $j \mid \delta$. The estimates proceed exactly as before and we obtain assertion (i) of the lemma. For (ii), we need only observe that the contribution from those m, e with $m_2 d$ a square is $\ll d^{1/2} Y(\log X)$.

LEMMA 15. *If* $a \equiv 1 (\mathrm{mod}\, 4)$, $(ad, 4N) = 1$, *then*

$$\sum_{\substack{m, e \\ (m,j)=1}}' \frac{a(m)}{me} \tilde{f}_Y(md; a) \exp(-me^2/X) \ll d^{1/2} X^{1/2} Y^{1/2} \log Xd$$

where the prime indicates that the sum ranges over values of m *such that* $m_2 d$ *is not a square.*

Proof. We have to estimate

$$\sum_{\substack{m, e \\ (m,j)=1}}' \frac{a(m)}{me} \exp(-me^2/X) \sum_{\substack{0 < -h \leq Y \\ h \equiv a (\mathrm{mod}\, 4N)}} \left(\frac{h}{md}\right).$$

We bring the summation over e inside and use Lemma 11. If we truncate the sum over m at $\frac{1}{2}X$, then it is

$$\ll \sum_{\substack{m < \frac{1}{2}X \\ m_2 d \neq b^2}} \frac{|a(m)|}{m} \left| \sum_{\substack{0 < -h \leq Y \\ h \equiv a (\mathrm{mod}\, 4N)}} \left(\frac{h}{md}\right) \right| \left(\log \frac{X}{m} + O(1)\right),$$

and by Lemma 1 and the method of Lemma 2, this is

$$\ll (Xd)^{1/4} \left\{ (Xd)^{1/2} Y (\log Xd)^2 \right\}^{1/2}.$$

The sum over $m > \frac{1}{2}X$ is estimated in the same way. It is

$$\ll \sum_{\substack{m > \frac{1}{2}X \\ m_2 d \neq b^2}} \frac{|a(m)|}{m} \exp(-m/X) \left| \sum_{\substack{0 < -h \leq Y \\ h \equiv a (\mathrm{mod}\, 4N)}} \left(\frac{h}{md}\right) \right|$$

which is

$$\ll (Xd)^{1/4} \left\{ (Xd)^{1/2} Y (\log Xd)^2 \right\}^{1/2}.$$

In both cases, the estimate simplifies to

$$\ll (dXY)^{1/2} (\log Xd).$$

This proves the lemma.

We shall use this to prove the following crucial and penultimate lemma.

LEMMA 16. *Take $a \equiv 1 (\mathrm{mod}\, 4)$, $(ad, 4N) = 1$. For all X satisfying $X \geq Y \log^{-B} Y$,*

$$\sum_{m,e}{}' \frac{a(m)}{me} f_Y(md; a) \exp(-me^2/X) \ll_B d^{1/2} X^{1/2} Y^{1/2} (\log X)(\log\log Y)$$

where the prime indicates that the sum ranges over values of m such that $m_2 d$ is not a square.

Proof. We have

$$\sum_{m,e}{}' \frac{a(m)}{me} f_Y(md; a) \exp(-me^2/X)$$
$$= \sum_{\substack{j^2 \leq Y \\ (j,d)=1}} \mu(j) \sum_{\substack{m,e \\ (m,j)=1}}{}' \frac{a(m)}{me} \exp(-me^2/X) \sum_{\substack{0 < -h \leq Y/j^2 \\ h \equiv a\bar{j}^2 (\mathrm{mod}\, 4N)}} \left(\frac{h}{md}\right).$$

The inner sum is what we have denoted by $\tilde{f}_{Y/j^2}(md; a\bar{j}^2)$. The sum of the terms with $j < \log^B Y$ can be estimated by Lemma 15 and it is seen to be

$$\ll \sum_{j < \log^B Y} \frac{(dXY)^{1/2}}{j} \log Xd$$

which is

$$d^{1/2} X^{1/2} Y^{1/2} \log X \log\log Y.$$

For $j > \log^B Y$, we use the second part of Lemma 14 to get

$$\sum_{j > \log^B Y} d^{1/2} \frac{Y}{j^2} (\log^2 Y + \log X) \ll d^{1/2} Y \log^{-B+2} Y.$$

Since $X \geq Y \log^{-B} Y$, this proves the lemma.

The final lemma is an estimate for a weighted average of the $a(n)$ which will be needed in some of the error term estimates.

LEMMA 17. *There is a $\rho > 0$ so that*

$$\sum_{n \leq x} \frac{|a(n)|}{\sqrt{n}} \ll x (\log x)^{-\rho}$$

and

$$\sum_{n \leq x} \frac{|a(n)|}{\sqrt{n}} \mathrm{d}(n) \ll x (\log x)^{1-2\rho}.$$

Proof. The first estimate is due to Rankin [10]. (It should be remarked that Rankin proved the estimate for $N = 1$. However, his proof carries over for the general level in view of Shahidi's result [12].)

For the second estimate, let $b_n = |a(n)|/n^{1/2}$ and consider the Dirichlet series $F(s) = \sum_{n=1}^{\infty} b_n \mathbf{d}(n)/n^s$. Set $Q(s) = \sum_{n=1}^{\infty} \mu^2(n) b_n \mathbf{d}(n)/n^s$. We are interested in bounding the partial sums of $F(s)$. Consider first the partial sums of $Q(s)$, namely,

$$\sum_{n \le x} \mu^2(n) b_n \mathbf{d}(n).$$

The coefficients of $Q(s)$ are dominated by the coefficients of $(\sum_{n=1}^{\infty} b_n/n^s)^2$. By the first part of the lemma,

$$\sum_{n \le x} b_n \ll x/(\log x)^{\rho}.$$

Hence,

$$\sum_{mn \le x} b_m b_n \ll \sum_{n \le \sqrt{x}} b_n \sum_{m \le x/n} b_m \ll \sum_{n \le \sqrt{x}} b_n (x/n)(\log x)^{-\rho} \ll x(\log x)^{1-2\rho}$$

by partial summation.

Now set $F(s) = Q(s)R(s)$. Write $R(s) = \sum c(n) n^{-s}$. We see that as the Euler product of $R(s)$ converges absolutely for $\mathrm{Re}(s) > 1/2$, the Dirichlet series $\sum c(n)/n$ is absolutely convergent. Using these facts, we see that

$$\begin{aligned} \sum_{n \le x} b_n \mathbf{d}(n) &= \sum_{me \le x} c(m) \mu^2(e) b_e \mathbf{d}(e) \\ &\ll \sum_{m \le x} c(m) \frac{2x}{m} \left(\log \frac{2x}{m} \right)^{1-2\rho} \\ &\ll x(\log x)^{1-2\rho}. \end{aligned}$$

This proves the lemma.

4. The main theorems

Now consider

$$\frac{1}{2\pi i} \int_{(\sigma)} L_{D_0}(1+s) \zeta(1+2s) \sum_{d | \delta^2} \frac{\tilde{\mu}(d)}{d^{1-s}} \left(\frac{D_0}{d} \right) X^s \Gamma(s) \delta^{-4s}\, ds.$$

This is

$$\sum_{d|\delta^2} \frac{\tilde{\mu}(d)}{d} \sum_{m,e} \frac{a(m)}{me} \left(\frac{D_0}{md} \right) \exp(-me^2\delta^4/Xd).$$

On the other hand, moving the line of integration to the left and picking up the residue at $s = 0$, we obtain

$$\frac{1}{2} L'_D(1) + \frac{1}{2\pi i} \int_{(-\eta)} L_{D_0}(1+s)\zeta(1+2s) \sum_{d|\delta^2} \frac{\tilde{\mu}(d)}{d^{1-s}} \left(\frac{D_0}{d} \right) X^s \Gamma(s) \delta^{-4s}\, ds.$$

Writing the functional equation for $L_{D_0}(s)$, we see that the above integral is

$$\frac{w\chi_{D_0}(-N)}{2\pi i} \int_{(-\eta)} |D|^{-2s} L_D(1-s)\zeta(1+2s) \frac{\Gamma(1-s)}{\Gamma(1+s)} A^{-2s} X^s \Gamma(s)\, ds.$$

Let Y be such that $Y \log^{-B} Y \le X \le Y(\log Y)^{1+\nu}$ where $0 < \nu < \rho$, with ρ as in Lemma 17. Now we sum the entire expression above over $\delta^2 \le Y$, $0 < -D_0 \le Y/\delta^2$, $D_0\delta^2 \equiv 1 (\mathrm{mod}\, 4N)$ and obtain the expression:

$$\frac{1}{2} \sum_{\substack{0 < -D \le Y \\ D \equiv 1 (\mathrm{mod}\, 4N)}} L'_D(1)$$

$$= \sum_{\substack{\delta^2 \le Y \\ (\delta, 4N) = 1}} \sum_{d|\delta^2} \frac{\tilde{\mu}(d)}{d} \sum_{n=1}^{\infty} \sum_{me^2 = n} \frac{a(m)}{me} f_{Y/\delta^2}(md; \bar{\delta}^2) \exp(-n\delta^4/Xd)$$

$$+ \frac{1}{2\pi i} \int_{(-\eta)} \sum_{\substack{m=1 \\ m = m_1 m_2}}^{\infty} \frac{a(m)}{m^{1-s}} \tilde{f}_Y(m, -2s; 1)\zeta(1+2s)$$

$$\times \frac{\Gamma(1-s)}{\Gamma(1+s)} A^{-2s} X^s \Gamma(s)\, ds,$$

for $\eta > 1/2$.

The integral is split into two parts. In the first, the sum over m_2 is taken only over non-squares and, in the second, it ranges over squares.

Let us consider the first integral. It is easily estimated in the following way. Truncating the sum over m at U, we move the integral involving the part $m \le U$ to $\Re s = -\eta_1$ where $\eta_1 < 1$. The part corresponding to $m > U$ is moved to $\Re s = -\eta_2$ where $\eta_2 > 1$. Choosing $U = Y^2/X$ and using the Pólya-Vinogradov inequality and Lemma 17, we see that the integral is $\ll X^{-1}Y^2(\log Y)^{1-\nu}$.

The contribution from the squares is easily handled. With a slight change in notation, it is

$$\frac{1}{2\pi i}\int_{(-\eta)}\sum_{m=1}^{\infty}\frac{a(m_1m_2^2)}{(m_1m_2^2)^{(1-s)}}\zeta(1+2s)\sum_{D}|D|^{-2s}\frac{\Gamma(1-s)}{\Gamma(1+s)}X^sA^{-2s}\Gamma(s)\,ds$$

where the sum over D is over

$$0 < -D \le Y, \qquad D \equiv 1 (\operatorname{mod} 4N), \qquad (D, m_2) = 1.$$

First, we move the integral to a line $-\eta$, with $0 < \eta < \frac{1}{2}$. By partial summation, it is easily seen that the sum over D is asymptotic to

$$\frac{1}{\phi(4N)}\frac{\phi(4Nm_2)}{4Nm_2}\frac{Y^{1-2s}}{1-2s} + \mathbf{O}(Y^{2\eta}\mathbf{d}(m_2)(|s|+1)),$$

where, as before, $\mathbf{d}(m_2)$ is the number of divisors of m_2. Inserting this into the integral, we find that the main term is

$$\frac{Y}{\phi(4N)}\frac{1}{2\pi i}\int_{(-\eta)}T(1-s)\left(\frac{X}{A^2Y^2}\right)^s\frac{\Gamma(1-s)}{\Gamma(1+s)}\frac{1}{1-2s}\zeta(1+2s)\Gamma(s)\,ds$$

where

$$T(s) = \sum_{n_1, n_2}\frac{\phi(4Nn_2)a(n_1n_2^2)}{4Nn_2(n_1n_2^2)^s}.$$

The error term is easily seen to be $\mathbf{O}(Y^{2\eta}X^{-\eta})$. Now moving this integral to the right of $\Re s = 0$, and using the expansions

$$\frac{\Gamma(1-s)}{\Gamma(1+s)}\frac{1}{1-2s} = 1 + (2\gamma+2)s + (2\gamma^2+4\gamma+4)s^2 + \cdots,$$

$$\left(\frac{X}{A^2Y^2}\right)^s = 1 + \left(\log\frac{X}{A^2Y^2}\right)s + \frac{1}{2}\left(\log^2\frac{X}{A^2Y^2}\right)s^2 + \cdots$$

and

$$\zeta(1+2s)\Gamma(s) = \left\{\frac{1}{2s} + \gamma + \cdots\right\}\left\{\frac{1}{s} - \gamma + \cdots\right\} = \frac{1}{2s^2} + \frac{\gamma}{2s} + \cdots,$$

we find that it is

$$-\frac{Y}{\phi(4N)}\sum_{n_1, n_2}\frac{a(n_1n_2^2)}{n_1n_2^2}\frac{\phi(4Nn_2)}{4Nn_2}\left(\frac{1}{2}\log\frac{X}{Y^2}\right) + \mathbf{O}(Y).$$

We see that this is

$$-\frac{1}{4}CY\log\frac{X}{Y^2} + \mathbf{O}(Y).$$

Summarizing, we have proved that the integral is

$$-\frac{1}{4}CY\log\frac{X}{Y^2}+\mathbf{O}(Y)+\mathbf{O}\left(\frac{1}{X}Y^2(\log Y)^{1-\nu}\right).$$

Next, we consider the sum which we split into the contribution when m_2d is a perfect square and when m_2d is not a perfect square.

In the latter case, we estimate the sum as follows. First consider the contribution of terms in which $\delta > \log^A Y$, for some large $A > 0$. By using Lemma 2 and partial summation, we see that

$$\sum_e \frac{1}{e}\sum_m{}' \frac{a(m)}{m} f_{Y/\delta^2}(md;\bar{\delta}^2)\exp(-me^2\delta^4/Xd)$$

$$\ll \left(\frac{Xd}{\delta^4}d\right)^{1/2}\frac{Y^{1/2}}{\delta}(\log Y)(\log X).$$

Thus the contribution of these terms to the sum is

$$\ll \sum_{\delta>\log^A Y}\sum_{d|\delta^2}\frac{|\tilde{\mu}(d)|}{d}\left(\frac{Xd}{\delta^4}d\right)^{1/2}\frac{Y^{1/2}}{\delta}(\log Y)(\log X)$$

and this is clearly $\mathbf{O}(Y)$. For $\delta < \log^A Y$, we argue as above, except that we use the estimate of Lemma 16 in place of Lemma 2. We find that these terms contribute an amount

$$\ll X^{1/2}Y^{1/2}\log X\log\log X.$$

In the former case, it is

$$\sum_\delta\sum_{d|\delta^2}\frac{\tilde{\mu}(d)}{d}\sum_{n=1}^{\infty}\sum_{\substack{me^2=n\\ m_2d=r^2}}\frac{a(m)}{me}\left(\sum_{D_0}1\right)\exp(-n\delta^4/Xd)$$

where the sum on δ ranges over

$$\delta^2\le Y,\qquad (\delta,4N)=1$$

and the sum on D_0 ranges over

$$0<-D_0\le\frac{Y}{\delta^2},\qquad D_0\equiv\bar{\delta}^2(\operatorname{mod}4N),\qquad (D_0,m_2d)=1,$$

$$D_0\equiv 1(\operatorname{mod}4),\qquad D_0\text{ fundamental}.$$

The sum over D_0 can be expressed as

$$\sum_{0<-D_0\leq Y/\delta^2} \mu^2(-D_0)\chi_0^{(m_2d)}(D_0)\frac{1}{\phi(4N)}\sum_{\psi \bmod 4N} \bar{\psi}(\bar{\delta}^2)\psi(D_0)$$

which is equal to

$$\sum_{\psi \bmod 4N} \bar{\psi}(\bar{\delta}^2)\frac{1}{\phi(4N)}\sum_{0<-D_0\leq Y/\delta^2} \mu^2(-D_0)\chi_0^{(m_2d)}(D_0)\psi(D_0).$$

Now we consider the inner sum,

$$\frac{1}{\phi(4N)}\sum_{0<-D_0\leq Y/\delta^2}\left(\sum_{a^2|D_0}\mu(a)\right)\chi_0^{(m_2d)}(D_0)\psi(D_0).$$

Rearranging, we find it is

$$\frac{1}{\phi(4N)}\sum_{a\leq\sqrt{Y}/\delta}\mu(a)\chi_0^{(m_2d)}(a^2)\psi(a^2)\sum_{0<-h\leq Y/\delta^2a^2}\chi_0^{(m_2d)}(h)\psi(h).$$

If ψ is nontrivial, then $\psi\chi_0^{(m_2d)}$ is a nontrivial character of conductor dividing $4N$. Thus, by Pólya-Vinogradov, the innermost sum is $\mathbf{O}(1)$. The whole sum is then

$$\mathbf{O}\left(\frac{Y^{1/2}}{\delta}\frac{1}{\phi(4N)}\right).$$

On the other hand, if ψ is the trivial character $\bmod 4N$, we get

$$\frac{1}{\phi(4N)}\sum_{a\leq\sqrt{Y}/\delta}\mu(a)\chi_0^{(m_2d)}(a^2)\chi_0^{(4N)}(a^2)\left\{\frac{Y}{\delta^2a^2}\frac{\phi(4Nm_2d)}{4Nm_2d}+\mathbf{O}(\mathbf{d}(4Nm_2d))\right\}.$$

Inserting this information into our big sum, we find that it is

$$\sum_{\substack{\delta^2\leq Y\\(\delta,4N)=1}}\sum_{d|\delta^2}\frac{\tilde{\mu}(d)}{d}\sum_{n=1}^{\infty}\sum_{\substack{me^2=n\\m_2d=r^2}}\frac{a(m)}{me}\exp(-n\delta^4/Xd)\frac{1}{\phi(4N)}$$

$$\times\left\{\frac{Y}{\delta^2}\frac{6}{\pi^2}\prod_{p|4Nm_2d}\left(1-\frac{1}{p^2}\right)^{-1}\frac{\phi(4Nm_2d)}{4Nm_2d}\right.$$

$$\left.+\mathbf{O}\left(\frac{Y^{1/2}}{\delta}\mathbf{d}(4Nm_2d)\right)\right\}.$$

We observe that the contribution from terms with $\delta > X^{1/5}$ is negligible.

Indeed, it is

$$\ll \sum_{\sqrt{Y}>\delta>X^{1/5}} \frac{Y^{1/2}}{\delta} \sum_{d|\delta^2} \frac{|\tilde{\mu}(d)|}{d} \sum_{m=1}^{\infty} \frac{|a(m)|}{m}$$

$$\times \left(\frac{Y^{1/2}}{\delta} + \mathbf{d}(4Nmd) \right) \left(\sum_e \frac{1}{e} \exp(-me^2\delta^4/Xd) \right).$$

We use Lemma 11 to estimate the sum over e.

The terms with $m > Xd/2\delta^4$ contribute an amount $\mathbf{O}(S)$ where

$$S = \sum_{\sqrt{Y}>\delta>X^{1/5}} \frac{Y^{1/2}}{\delta} \sum_{d|\delta^2} \frac{|\tilde{\mu}(d)|}{d} \sum_{m>Xd/2\delta^4} \frac{|a(m)|}{m}$$

$$\times \left(\frac{Y^{1/2}}{\delta} + \mathbf{d}(4Nmd) \right) \exp\left(-\frac{m\delta^4}{Xd} \right).$$

By Lemma 17 and partial summation, we have

$$\sum_{m>Xd/2\delta^4} \frac{|a(m)|}{m} \mathbf{d}(m) \exp\left(-\frac{m\delta^4}{Xd} \right) \ll \sqrt{\frac{Xd}{\delta^4}} \left(\log \frac{Xd}{\delta^4} \right).$$

Thus,

$$S \ll \sum_{\sqrt{Y}>\delta>X^{1/5}} \frac{Y^{1/2}}{\delta} \sum_{d|\delta^2} \left(\frac{Y^{1/2}}{\delta} + \mathbf{d}(4Nd) \right) \frac{|\tilde{\mu}(d)|}{d} \sqrt{\frac{Xd}{\delta^4}} \left(\log \frac{Xd}{\delta^4} \right)$$

and this is

$$\ll (XY)^{1/2} (\log X) \sum_{\delta>X^{1/5}} \frac{1}{\delta^3} \sum_{d|\delta^2} \left(\frac{Y^{1/2}}{\delta} + \mathbf{d}(4Nd) \right) \mathbf{d}(d).$$

Since $\mathbf{d}(n) \ll n^{\varepsilon}$, it follows that

$$S \ll YX^{-1/10+\varepsilon}.$$

Now consider the terms with $m < Xd/2\delta^4$. Again, by Lemma 11, the sum to be estimated is

$$\ll \sum_{\sqrt{Y}>\delta>X^{1/5}} \frac{Y^{1/2}}{\delta} \sum_{d|\delta^2} \frac{|\tilde{\mu}(d)|}{d} \sum_{m<Xd/2\delta^4} \left(\frac{Y^{1/2}}{\delta} + \mathbf{d}(4Nmd) \right) \frac{|a(m)|}{m}$$

$$\times \left(\log \frac{Xd}{m\delta^4} + \mathbf{O}(1) \right).$$

By an argument analogous to that used in the treatment of S we see that the above is also

$$\ll YX^{-1/10+\varepsilon}.$$

Thus, the sum to be considered is

$$\sum_{\substack{\delta \le X^{1/5} \\ (\delta, 4N)=1}} \sum_{d|\delta^2} \frac{\tilde{\mu}(d)}{d} \sum_{n=1}^{\infty} \sum_{\substack{me^2=n \\ m_2 d = r^2}} \frac{a(m)}{me} \exp(-n\delta^4/Xd) \frac{1}{\phi(4N)}$$

$$\times \left\{ \frac{Y}{\delta^2} \frac{6}{\pi^2} \prod_{p|4Nm_2 d} \left(1 - \frac{1}{p^2}\right)^{-1} \frac{\phi(4Nm_2 d)}{4Nm_2 d} + \mathbf{O}\left(\frac{Y^{1/2}}{\delta} \mathrm{d}(4Nm_2 d)\right)\right\}.$$

The error term is

$$\ll Y^{1/2} \sum_{\delta \le X^{1/5}} \frac{1}{\delta} \sum_{d|\delta^2} \frac{\mathrm{d}(d)^2}{\sqrt{d}} \left\{ \sum_{m \le Xd/2\delta^4} \frac{\mathrm{d}(m)|a(m)|}{m} \log \frac{Xd}{m\delta^4} \right.$$

$$\left. + \mathbf{O}\left(\sum_{m \ge Xd/2\delta^4} \frac{\mathrm{d}(m)|a(m)|}{m} \exp\left(-\frac{m\delta^4}{Xd}\right)\right)\right\}.$$

Using Lemma 17 and partial summation, we see that

$$\sum_{m \le Xd/2\delta^4} \frac{\mathrm{d}(m)|a(m)|}{m} \log \frac{Xd}{m\delta^4} \ll \sqrt{\frac{Xd}{\delta^4}} \log X$$

and

$$\sum_{m \ge Xd/2\delta^4} \frac{\mathrm{d}(m)|a(m)|}{m} \exp\left(-\frac{m\delta^4}{Xd}\right) \ll \sqrt{\frac{Xd}{\delta^4}} \log X.$$

Therefore, the error term is

$$\ll Y^{1/2} \sum_{\delta \le X^{1/5}} \frac{1}{\delta} \sum_{d|\delta^2} \frac{\mathrm{d}(d)^2}{\sqrt{d}} \frac{X^{1/2} d^{1/2}}{\delta^2} \log X \ll X^{1/2} Y^{1/2} \log X.$$

Thus, the sum reduces to

$$\frac{6Y}{\pi^2 \phi(4N)} \sum_{\substack{\delta \le X^{1/5} \\ (\delta, 4N)=1}} \frac{1}{\delta^2} \sum_{d|\delta^2} \frac{\tilde{\mu}(d)}{d} \sum_{n=1}^{\infty} \sum_{\substack{me^2=n \\ m_2 d = r^2}} \frac{a(m)}{me} \exp(-n\delta^4/Xd)$$

$$\times \prod_{p|4Nm_2 d} \left(1 + \frac{1}{p}\right)^{-1} + \mathbf{O}(X^{1/2} Y^{1/2} \log X).$$

Now the sum over n is analysed as follows:

$$\sum_{n=1}^{\infty} \sum_{\substack{me^2=n \\ m_2 d = r^2}} \frac{a(m)}{me} \prod_{p|4Nm_2d} \left(1+\frac{1}{p}\right)^{-1} \exp(-n/X)$$

$$= \frac{1}{2\pi i} \int_{(2)} G_d(1+s)\zeta(1+2s)X^s\Gamma(s)\, ds$$

where

$$G_d(s) = \sum_{\substack{m=1 \\ m_2 d = r^2}}^{\infty} \frac{a(m)}{m^s} \prod_{p|4Nm_2d} \left(1+\frac{1}{p}\right)^{-1}.$$

By the argument used for $F_d(s)$, one finds an absolute constant $c > 1$ so that

$$|G_d(s)| \ll c^{\nu(d)} |L(2s, \mathrm{Sym}^2)\zeta(4s-2)^{-1}|,$$

for $\Re s > 3/4$ (say). Thus,

$$\sum_{n=1}^{\infty} \sum_{\substack{me^2=n \\ m_2 d = r^2}} \frac{a(m)}{me} \prod_{p|4Nm_2d} \left(1+\frac{1}{p}\right)^{-1} \exp(-n/X)$$

$$= \frac{1}{2} G_d(1)(\gamma + \log X) + \frac{1}{2} G_d'(1) + \mathbf{O}(c^{\nu(d)}),$$

as $X \to \infty$. Inserting the main term into our sum, we obtain

$$\frac{6Y}{\pi^2\phi(4N)} \sum_{\substack{\delta \le X^{1/5} \\ (\delta, 4N)=1}} \frac{1}{\delta^2} \sum_{d|\delta^2} \frac{\tilde{\mu}(d)}{d}$$

$$\times \left\{ \frac{1}{2}\left(\gamma + \log \frac{Xd}{\delta^4}\right) \sum_{m_2 d = r^2} \frac{a(m)}{m} \prod_{p|4Nm_2d} \left(1+\frac{1}{p}\right)^{-1} \right.$$

$$\left. - \frac{1}{2} \sum_{m_2 d = r^2} \frac{a(m)\log m}{m} \prod_{p|4Nm_2d} \left(1+\frac{1}{p}\right)^{-1} \right\}.$$

The second term gives an amount of $\mathbf{O}(Y)$. The contribution from the error term is also $\mathbf{O}(Y)$. The first term is

$$\frac{3Y}{\pi^2\phi(4N)} (\log X) \sum_{\substack{\delta \le X^{1/5} \\ (\delta, 4N)=1}} \frac{1}{\delta^2} \sum_{d|\delta^2} \frac{\tilde{\mu}(d)}{d} \sum_{m_2 d = r^2} \frac{a(m)}{m} \prod_{p|4Nr} \left(1+\frac{1}{p}\right)^{-1} + \mathbf{O}(Y).$$

Now using the definition of $\tilde{\mu}$, and making use of the fact that

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s}\left(\frac{\delta^2}{n}\right) = L(s) \sum_{d|\delta^2} \frac{\tilde{\mu}(d)}{d^s},$$

we obtain the identity

$$a(n)\left(\frac{\delta^2}{n}\right) = \sum_{\substack{d|\delta^2 \\ dm=n}} \tilde{\mu}(d)a(m).$$

Therefore, we find that the main term is

$$\frac{3Y(\log X)}{\pi^2\phi(4N)} \sum_{\substack{\delta \leq X^{1/5} \\ (\delta,4N)=1}} \frac{1}{\delta^2} \sum_{m_1} \frac{a(m_1)}{m_1} \sum_{\substack{r=1 \\ (r,2N)=1}}^{\infty} \frac{a(r^2)}{r^2}\left(\frac{\delta^2}{r^2}\right) \prod_{p|4Nr}\left(1+\frac{1}{p}\right)^{-1}.$$

In order to simplify this sum, consider the Dirichlet series

$$D_\delta(s) = \sum_{\substack{r=1 \\ (r,\delta)=1}}^{\infty} \frac{a(r^2)}{r^s} \prod_{p|4Nr}\left(1+\frac{1}{p}\right)^{-1}.$$

For $\mathrm{Re}(s) > 2$ it is an absolutely convergent series. Writing each r as $r = bu$ where $p|b \Rightarrow p|4N$ and $(u, 4N) = 1$, we see that

$$D_\delta(s) = \prod_{p|4N}\left(1+\frac{1}{p}\right)^{-1} \sum_{\substack{b \\ p|b \Rightarrow p|4N}} \frac{a(b^2)}{b^s} \sum_{\substack{u=1 \\ (u,4N\delta)=1}}^{\infty} \frac{a(u^2)}{u^s} \prod_{p|u}\left(1+\frac{1}{p}\right)^{-1}.$$

As in Section 1, write

$$B_p(s) = \sum_{j=0}^{\infty} \frac{a(p^{2j})}{p^{js}}.$$

Then the sum over u is equal to

$$\prod_{p \nmid 4N\delta}\left(1 + \left(1+\frac{1}{p}\right)^{-1}\left(B_p(s) - 1\right)\right)$$

and, factoring out B_p, we deduce that

$$D_\delta(s) = \prod_{p|4N}\left(1+\frac{1}{p}\right)^{-1}\left(\prod_{p\nmid\delta} B_p(s)\right) \prod_{p\nmid 4N\delta}\left(1 - \frac{1}{p+1}\left(1 - \frac{1}{B_p(s)}\right)\right).$$

Notice that the last product above converges absolutely for $\mathrm{Re}(s) > 1$ and at $s = 2$ can be bounded independently of δ. As for the second factor, we note that it is

$$\left(\sum_{n=1}^{\infty} \frac{a(n^2)}{n^s}\right) \prod_{p|\delta} B_p(s)^{-1}$$

and at $s = 2$ we have (on using the estimate $a(p^{2j}) \leq (2j+1)p^j$) that it is $\ll \delta/\phi(\delta)$.

Inserting this information into our sum, we see that it is

$$\frac{3Y(\log X)}{\pi^2\phi(4N)} \sum_{\substack{\delta \leq X^{1/5} \\ (\delta, 4N)=1}} \frac{1}{\delta^2} D_{2N\delta}(2) \sum_{n_1} \frac{a(n_1)}{n_1}$$

which is equal to

$$\frac{3Y(\log X)}{\pi^2\phi(4N)} \prod_{p|4N} \left(1 + \frac{1}{p}\right)^{-1} \sum_{n_1} \frac{a(n_1)}{n_1} \left(\sum_{\substack{n_2=1 \\ (n_2, 2N)=1}}^{\infty} \frac{a(n_2^2)}{n_2^2} \right)$$

$$\times \prod_{p \nmid 4N} \left(1 - \frac{1}{p+1}\left(1 - \frac{1}{B_p(2)}\right)\right)$$

$$\times \sum_{\substack{\delta \leq X^{1/5} \\ (\delta, 4N)=1}} \frac{1}{\delta^2} \left(\prod_{p|\delta} B_p(2)^{-1}\right) \prod_{p|\delta} \left(1 - \frac{1}{p+1}\left(1 - \frac{1}{B_p(2)}\right)\right)^{-1}.$$

By the estimate stated at the end of the previous paragraph, we may extend the sum over δ to infinity, thereby introducing an error of only $\mathbf{O}(YX^{-1/5+\varepsilon})$. Now let us simplify the sum over δ. As each summand is multiplicative in δ, we see that

$$\prod_{p \nmid 4N} \left(1 + \frac{1}{B_p(2)}\left(1 - \frac{1}{p+1}\left(1 - \frac{1}{B_p(2)}\right)\right)^{-1} \left(\frac{1}{p^2} + \frac{1}{p^4} + \cdots\right)\right)$$

$$= \prod_{p \nmid 4N} \left(1 + \frac{1}{B_p(2)(p^2-1)}\left(1 - \frac{1}{p+1}\left(1 - \frac{1}{B_p(2)}\right)\right)^{-1}\right).$$

Multiplying this by

$$\prod_{p \nmid 4N} \left(1 - \frac{1}{p+1}\left(1 - \frac{1}{B_p(2)}\right)\right),$$

we find that it becomes

$$\prod_{p \nmid 4N}\left(1-\frac{1}{p^2}\right)^{-1}\left(1+\frac{1}{p}\left(\frac{1}{B_p(2)}-1\right)\right).$$

We insert this calculation and the sum becomes

$$\frac{1}{2}\frac{1}{\phi(4N)}(Y\log X)\sum_{n_1}\frac{a(n_1)}{n_1}\left(\sum_{(n_2,2N)=1}\frac{a(n_2^2)}{n_2^2}\right)$$

$$\times\prod_{p\mid 4N}\left(1-\frac{1}{p}\right)\prod_{p\nmid 4N}\left(1+\frac{1}{p}\left(\frac{1}{B_p(2)}-1\right)\right).$$

In the notation of Remark 3 of Section 1,

$$\mathscr{P}(1)=\prod_{p\nmid 4N}\left(1+\frac{1}{p}\left(\frac{1}{B_p(2)}-1\right)\right).$$

The calculation in this remark shows that our sum is

$$\frac{1}{2}\frac{Y(\log X)}{\phi(4N)}\sum_{n_1}\frac{a(n_1)}{n_1}\sum_{\substack{n_2=1\\(n_2,2N)=1}}\frac{a(n_2^2)}{n_2^2}\prod_{p\mid 4Nn_2}\left(1-\frac{1}{p}\right).$$

Summarizing, we have proved that the sum is

$$\frac{1}{4}CY(\log X)+\mathbf{O}(Y)+\mathbf{O}(X^{1/2}Y^{1/2}\log X\log\log X).$$

All of our estimates were made under the assumption that

$$Y\log^{-B}Y\le X\le Y(\log Y)^{1+\nu}$$

where $0<\nu<\rho$. Combining the estimates of the sum and the integral, we deduce that if $X=Y/(\log Y)^{\lambda}$ for a small $\lambda>0$, then the sum of the sum and the integral is

$$\left(\frac{1}{\phi(4N)}\sum_{n_1}\frac{a(n_1)}{n_1}\sum_{n_2}\frac{a(n_2^2)}{n_2^2}\frac{\phi(4Nn_2)}{4Nn_2}\right)\left(\frac{1}{2}Y\log X-\frac{1}{2}Y\log\frac{X}{Y^2}\right)$$

$$+\mathbf{O}\left(Y(\log X)^{1-\nu}\right)$$

and this is

$$\frac{1}{2}CY(\log Y)+\mathbf{O}\left(Y(\log Y)^{1-\nu}\right),$$

which completes the proof of the theorem.

6. Concluding remarks

Our theorems can be generalised to include the case when D ranges over an arithmetic progression mod M. This gives rise to a more complicated constant. Though we have not developed it here, the method allows one to obtain asymptotic formulas of the form

$$\sum_{0<-D\leq Y} L_D^{(j)}(1) = C_j Y \log^j Y + \mathbf{O}(Y \log^{j-1} Y)$$

as $Y \to \infty$ for every $j \geq 1$. The results of these investigations will appear in a future work.

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References

[1] D. Bump, S. Friedberg, and J. Hoffstein, Non-vanishing theorems for L-functions of modular forms and their derivatives, preprint.

[2] A. S. Fainleib and O. Saparnijazov, Dispersion of real character sums and the moments of $L(1, \chi)$ (Russian), Izv. Akad. Nauk. USSR, Ser. fiz.-mat. Nauk. **19** Nr. 6 (1975), 24–29.

[3] M. Jutila, On the mean value of $L(1/2, \chi)$ for real characters, Analysis **1** (1981), 149–161.

[4] V. A. Kolyvagin, Finiteness of $E(\mathbf{Q})$ and $\text{III}_{E/\mathbf{Q}}$ for a subclass of Weil curves (Russian), Izv. Akad. Nauk. USSR, ser. Matem. **52** (1988).

[5] M. Ram Murty, Oscillations of Fourier coefficients of modular forms, Math. Ann. **262** (1983), 431–446.

[6] ______, Simple zeroes of certain L-series, Proc. Number Theory Conf. (ed. R. Mollin), Banff, Canada, 1987, pp. 427–439.

[7] M. Ram Murty and V. Kumar Murty, A variant of the Bombieri-Vinogradov theorem, in: *Number Theory*, ed. H. Kisilevsky and J. Labute, CMS Conf. Proc., **7** (1987), 243–272.

[8] V. Kumar Murty, On the Sato-Tate conjecture, in: *Number Theory Related to Fermat's Last Theorem*, ed. N. Koblitz, Birkhauser-Verlag, Boston, 1982, pp. 195–205.

[9] ______, Non-vanishing theorems for L-functions and their derivatives, *Automorphic Forms and Number Theory*, CRM, Montreal (1990), pp. 89–113.

[10] R. Rankin, Sums of powers of cusp form coefficients II, Math. Ann. **272** (1985), 593–600.

[11] K. Rubin, Tate-Shafarevic groups and L-functions of elliptic curves with complex multiplication, Inv. Math. **89** (1987), 527–559.

[12] F. Shahidi, On certain L-functions, Amer. J. Math. **103** (1981), 297–356.

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