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# GROSS-ZAGIER FORMULA FOR $\mathrm{GL}_2$*

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# Contents.



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# 1. Introduction and statements of results

In [20], Gross and Zagier proved a formula which relates the central derivatives of certain Rankin L-series and the heights of certain Heegner points on elliptic curves. Combined with Goldfeld's work on L-series [14], this formula gives a solution to Gauss' problem on class numbers; and combined with Kolyvagin's work on Euler system [16, 25], this formula gives the best evidence for the rank issue in the Birch and Swinnerton-Dyer conjecture. In [17], Gross has proposed a program to generalize this formula to totally real fields with anticyclotomic characters. In our previous paper [31], we have worked out the program when the character is trivial and the nonsplit level structure is small.

The present paper is devoted to working out the weight 2 case of the program. One immediate application is to generalize the results of Kolyvagin and Logachev [26], and Bertolini and Darmon [6] to obtain evidence toward the Birch and Swinnerton-Dyer conjecture in the rank 1 case for modular elliptic curves over totally real fields twisted by some anticyclotomic characters.

As a coproduct of the proof, we will also obtain a Gross-Zagier formula for the central values of certain Rankin L-series for forms with mixed holomorphic and Maass components at the archimedean places. There will be two applications of this Gross-Zagier formula. One is to generalize the recent work of Bertolini and Darmon [7, 8] to obtain evidence toward the Birch and Swinnerton-Dyer conjecture in the rank 0 case. The other one is to use the recent work [4] of Cogdell, Piateski-Shapiro, and Sarnak to prove the equidistribution of certain *toric orbits* of CM-points on quaternion Shimura varieties. This equidistribution statement generalizes a result of Duke [11] and is also recently announced by Cohen [5] using Duke's original method.

If we further assume that the work [4] of Cogdell, Piateski-Shapiro, and Sarnak can be extended to unramified anticyclotomic characters which is predicted by GRH and (which holds over $\mathbb{Q}$ by recent work of Kowalski, Michel, and Vanderkam [27]), then our Gross-Zagier formula will imply the equidistribution to certain Galois orbits of CM-points and thus gives some evidence toward the André-Oort conjecture concerning the Zariski topology of CM-points.

The applications to the Birch and Swinnerton-Dyer conjecture and the André-Oort conjecture will be treated in later papers.

In the following, we will describe the main results about the Gross-Zagier formula and proof.

### 1.1. Rankin-Selberg L-functions and kernels

Let $F$ be a totally real field of degree $g$ and discriminant $d$, with ring of adeles $\mathbb{A}$. Let $\phi$ be a Hilbert modular form of weight $(2, \dots, 2, 0, \cdots, 0)$ over $F$, which is a cuspidal newform of level $N$ and has trivial central character.

Let $K$ be a totally imaginary quadratic extension of $F$, and let $\omega$ be the nontrivial quadratic character of $\mathbb{A}^\times/F^\times \mathrm{N}\mathbb{A}_K^\times$. The conductor $c(\omega)$ is the relative discriminant of $K/F$. Let $\chi$ be a character of finite order of $\mathbb{A}_K^\times/K^\times\mathbb{A}^\times$. The conductor $c(\chi)$ is an ideal of $\mathcal{O}_F$, and we define the ideal $D = c(\chi)^2 c(\omega)$. The theory of theta series allows one to define a Hilbert modular form $\theta_\chi$ of weight $(1, \cdots, 1)$, whose L-function is equal to the Hecke L-series of $\chi$.

In this paper we will study the Rankin-Selberg convolution L-function

$$L(s, \phi, \theta_\chi) = L(s, \phi, \chi).$$

This is defined by an Euler product over primes $\wp$ of $F$, where the factors have degree $\leq 4$ in $\mathrm{N}\wp^{-s}$. This function has an analytic continuation to the entire complex plane, and satisfies a functional equation. We will assume the following

$$\text{(1.1.1)} \qquad \text{hypothesis:} \qquad \begin{cases} (c(\omega), c(\chi)) = 1, \\ \mathrm{ord}_\wp(D) \geq 1 \Longrightarrow \mathrm{ord}_\wp(N) \leq 1. \end{cases}$$

The functional equation is then

$$\text{(1.1.2)} \qquad L(2-s,\chi,\phi) = (-1)^{\#\Sigma} c(\chi,\phi)^{1-s} L(s,\chi,\phi)$$

where $c(\chi,\phi)$ is the conductor of the L-function $L(s,\chi,\phi)$,

$$c(\chi,\phi) = d^4 \mathrm{N}_{F/\mathbb{Q}}[N,D]^2 (N, c(\omega))$$

(here $[\cdot,\cdot]$ denotes the least common multiple, and $(\cdot,\cdot)$ denotes the greatest common divisor) and $\Sigma$ is the following set of places of $F$:

$$\text{(1.1.3)} \qquad \Sigma = \left\{ v \,\middle|\, \begin{array}{l} v \text{ is inifinite, and } \phi \text{ has weight 2 at } v, \text{ or} \\ v \text{ is finite, } v \nmid D, \text{ and } \omega_v(N) = -1, \text{ or} \\ v \text{ is finite, } v \mid (N, c(\omega)), \text{ and } a_v b_v = 1 \end{array} \right\}$$

where $a_v$ and $b_v$ are $v$-th Fourier coefficients of $\phi$ and $\theta_\chi$ respectively. If $v$ is in $\Sigma$ and unramified in $K$, $\chi_v = 1$. Furthermore, if $v$ is ramified in $K$, $\chi_v$ is unramified and $\chi_v^2 = 1$.

The general theory of Rankin-Selberg convolutions is due to Jacquet [22], but we will follow [20] in the case above, and will show that there is a form $\Theta(s,g)$ of level $[N,D]$ on $\mathrm{GL}_2(\mathbb{A})$ which is a kernel for the convolution. More precisely, we will show that for all new forms $\phi$ of level $N$:

$$\text{(1.1.4)} \qquad L(s,\chi,\phi) = (\phi, \Theta(s,g))_{[N,D]}$$

where $(\cdot,\cdot)_{[N,D]}$ is the Peterson product of level $[N,D]$.

We obtain the functional equation for $L(s,\chi,\phi)$ from that of $\Theta(s,g)$. Here our approach differs from [20], which computes $\mathrm{tr}_{[N,D]/N}(\Theta(s,g))$ as a kernel of level $N$. However, this trace is too difficult to compute in the general case (in [20], the authors were forced to assume that $D$ was square free, so $c(\omega)$ was odd and $c(\chi) = 1$).

Notice that the projection $\tilde{\Theta}(s,g)$ in the representation space $\Pi(\phi)$ is no longer a newform. But it is a multiple of a unique form $\phi_s^\sharp$ of level $[N,D]$ which is perpendicular to $\phi - \phi_s^\sharp$. The multiplier is then

$$\frac{L(s,\chi,\phi)}{(\phi_s^\sharp, \phi_s^\sharp)_{ND}}.$$

We call $\phi_s^\sharp$ a *quasi-newform* and will give $\phi_s^\sharp$ a direct definition in §3.1 in terms of characters $\chi_v$ for $v$ ramified in $K$.

### 1.2. Central derivatives

Our main formula expresses the central derivative $L'(1,\chi,\phi)$ in terms of the heights of CM-points on a Shimura curve, when $\phi$ is holomorphic and the sign of the functional equation (1.1.2) is $-1$, so $\#\Sigma$ is odd. Let $v$ be any real place of $F$, and let $B =_v B$ be the quaternion algebra over $F$ which ramified at the places in $\Sigma - \{v\}$. Let $G$ be the algebraic group over $F$, which is an inner form of $\mathrm{PGL}_2$, and has $G(F) = B^\times/F^\times$.

The group $G(F_v) \simeq \mathrm{PGL}_2(\mathbb{R})$ acts on $\mathcal{H}^\pm = \mathbb{C} - \mathbb{R}$. If $\widehat{F} = \mathbb{A}_f$ is the ring of finite adèles of $F$, and $U \subset G(\widehat{F})$ is open and compact, we get an analytic space

$$M_U(\mathbb{C}) = G(F)\backslash\mathcal{H}^\pm \times G(\widehat{F})/U.$$

Shimura proved these were the complex points of an algebraic curve $M_U$, which descends canonically to $F$ (embedded in $\mathbb{C}$, by the places $v$). The curve $M_U$ over $F$ is independent of the choice of $v$ in $\Sigma$.

To specify $M_U$, we must define $U \subset G(\widehat{F})$. To do this, we fix an embedding $K \longrightarrow B$, which exists, as all places in $\Sigma$ are either inert or ramified in $K$. Then $B = K + K\lambda$ with $\lambda \in B^\times$ satisfing $\alpha\lambda = \lambda\bar{\alpha}$ for $\alpha \in K$.

Let $\mathcal{O}_v \subset F_v$ be the local ring of integers, and let $\mathcal{O}_{K,v} \subset K_v$ be the integral closure of $\mathcal{O}_v$. For each finite place $v$ of $F$, let $A_v$ be an order of $B$ defined by

$$A_v = \mathcal{O}_{c(\chi_v)} + \mathcal{O}_{K,v} \cdot \lambda_v \cdot c(\chi_v)$$

Here $\mathcal{O}_{c(\chi_v)}$ is the order $\mathcal{O}_v + \mathcal{O}_{K,v}c(\chi_v)$ of $K_v$ and $\lambda_v$ is chosen integral over $\mathcal{O}_v$ whose norm $\mathrm{N}\lambda_v$ satisfies the following condition:

$$\mathrm{ord}_v(\mathrm{N}\lambda_v) = \mathrm{ord}_v(N/(N, D)).$$

Define an open compact subgroup $U_v$ of $G(F_v)$ by

$$U_v = A_v^\times/\mathcal{O}_v^\times. \tag{1.2.1}$$

Let $U = \prod_v U_v$. This defines the curve $M_U$ up to $F$-isomorphism. Let $X$ be its compactification over $F$, so $X = M_U$ unless $F = \mathbb{Q}$ and $\Sigma = \{\infty\}$, where $X$ is obtained by adding many cusps.

Notice that $X$ admits a natural action by

$$\Delta_T = \prod_{v|c(\chi)} T(\mathcal{O}_v) \cdot \prod_{v|c(\omega)} T(F_v)$$

via right multiplication on $G(\mathbb{A}_f)$, since $\Delta_T$ normalizes $U$ in $G(\mathbb{A}_f)$. Let $\Delta$ denote the subgroup of $G(\mathbb{A}_f)$ generated by $\Delta_T$ and $U$:

$$\Delta = U \cdot \Delta_T \tag{1.2.2}$$

and let $\chi_\Delta$ denote the character on $\Delta$ defined by

$$\chi_\Delta : \quad \Delta \longrightarrow \Delta_T \xrightarrow{\chi} \mathbb{C}^\times. \tag{1.2.3}$$

We will now construct points in $J$, the connected component of $\mathrm{Pic}(X)$, from CM-points on the curve $X$. The CM-points corresponding to $K$ on $M_U(\mathbb{C})$ form the set

$$G(F)_+\backslash G(F)_+ \cdot z \times G(\widehat{F})/U = T(F)\backslash G(\widehat{F})/U,$$

where $z \in \mathcal{H}^+$ is the unique fixed point of the torus points $K^\times/F^\times$. Let $\eta_\chi$ be a divisor on $X$ with complex coefficient defined by

$$\eta_\chi = \sum_{T(F)\backslash T(\mathbb{A}_f)/U_T} \chi^{-1}(t)[t]$$

where

$$U_T = T(\mathbb{A}_f) \cap U = \widehat{\mathcal{O}}^{\times}_{c(\chi)}/\widehat{\mathcal{O}}^{\times}.$$

If $\chi$ is not of form $\chi = \nu \cdot \mathrm{N}_{K/F}$ with $\nu$ a quadratic character of $F^\times\mathbb{A}^\times$, then $\eta_\chi$ has degree 0 on each fiber of $X$. Thus it defines a class $x_\chi$ in $\mathrm{Jac}(X)\otimes\mathbb{C}$. Otherwise we need a reference divisor to send $\eta_\chi$ to $\mathrm{Jac}(X)$. In the modular curve case, one uses cusps. In the general case, we use the *Hodge class* $\xi \in \mathrm{Pic}(X)\otimes\mathbb{Q}$: the unique class whose degree is 1 on each connected component and such that

$$\mathrm{T}_m\xi = \deg(\mathrm{T}_m)\xi$$

for all integral nonzero ideal $m$ of $\mathcal{O}_F$ prime to $ND$. The Heegner class we want now is the class difference

$$x_\chi := [\eta_\chi - \deg(\eta_\chi)\xi] \in \mathrm{Jac}(X)(K_\chi)\otimes\mathbb{C},$$

where $\deg(\eta_\chi)$ is the multi-degree of $\eta_\chi$ on geometric components, and $K_\chi$ is the abelian extention of $K$ corresponding to the group $T(F)\backslash T(\mathbb{A}_f)/U$.

Notice that this class has character $\chi_\Delta$ under the action by $\Delta$ on $\mathrm{Jac}(K_\chi)$. Let $y_\chi$ denote the $\phi$-typical component of $\eta_\chi$.

Our main theorem is now the following

THEOREM 1.2.1. *Let $\phi^\sharp$ be the quasi-newform as in §1 and §3.1. Then*

$$L'(1,\chi,\phi) = 2^{g+1} d_{K/F}^{-1/2}\cdot \|\phi^\sharp\|^2 \cdot \|y_\chi\|^2$$

*where*

- *$d_{K/F}$ is the relative discriminant of $K$ over $F$;*
- *$\|\phi^\sharp\|^2$ is the $L^2$-norm with respect to the Haar measure $dg$ which is the product of the the standard measure on $N(\mathbb{A})A(\mathbb{A})$, and the measure on the standard maximal compact group with*

$$\mathrm{vol}(\mathrm{SO}(F_\infty)U_0([N,D]) = 1;$$

- *$\|y_\chi\|$ is the Neron-Tate height of $y_\chi$.*

Gross and Zagier [20] originally proved Theorem 1.3.2 in the following special case:

$$\begin{cases} F = \mathbb{Q}, \\ \chi \text{ is unramified, } (D,2N)=1, \text{ and} \\ p \mid N \Longrightarrow p \text{ is split in } K. \end{cases}$$

The case treated in our previous paper [31] is when

$$\begin{cases} F \text{ is totally real,} \\ \chi \text{ is trivial, } (D,2N)=1, \text{ and} \\ \wp^2 \mid N \Longrightarrow \wp \text{ is split in } K. \end{cases}$$

One immediate application of our Gross-Zagier formula is to generalize the work of Kolyvagin-Logachev and Bertolini-Darmon [16, 25, 6] to obtain some evidence toward the Birch and Swinnerton-Dyer conjecture in rank 1 case. The details will be given in later papers. Here we just notice $y_\chi$ actually lives in some factor $A$ whose L-function is given by $\phi$ and its conjugates.

Let $\mathbb{Z}[\chi(\Delta)]$ be the subring of $\mathbb{C}$ generated by values $\chi(\Delta)$. and let $\mathbb{Z}[\phi]$ denote the subring generated by eigenvalues $a_\wp$ of $\mathrm{T}_\wp$ for all $\wp \nmid N$. Then we have

THEOREM 1.2.2. *There is a unique abelian subvariety of the Jacobian* $\mathrm{Jac}(X)$ *which is isogenous to* $\mathbb{Z}[\chi(\Delta)] \otimes_{\mathbb{Z}} A$ *(compatible with action by* $\Delta$*). Here "*$\otimes$*" means tensor product of abelian groups, and where* $A$ *is an abelian variety over* $F$ *of dimension equal to* $\operatorname{rank} \mathbb{Z}[\phi]$ *with an action by* $\mathbb{Z}[\phi]$ *such that*

$$L(s, A) \equiv \prod_{\sigma:\mathbb{Z}[\phi]\to\mathbb{C}} L(s, \phi^\sigma) \quad \text{mod } \textit{(factors at places dividing } N \cdot \infty\textit{)}$$

By Faltings' theorem, $A$ is uniquely determined by the above equality of $L$-functions up to isogenies.

## 1.3. Central values

We now return to the case where $\phi$ has possible nonholomorphic components, but we assume that the sign of the functional equation of $L(s, \chi, \phi)$ is $+1$, or equivalently, $\Sigma$ is even. In this case, we have an explicit formula for $L(1, \chi, \phi)$, which has an application to the distribution of CM-points on locally symmetric varieties covered by $(\mathcal{H}^+)^n$ where $n$ is the number of real places of $F$ where $\phi$ has weight 0.

More precisely, let $B$ be the quaternion algebra over $F$ ramified at $\Sigma$, and $G$ the algebraic group associated to $B^\times/F^\times$. Then

$$G(F \otimes \mathbb{R}) \simeq \mathrm{PGL}_2(\mathbb{R})^n \times \mathrm{SO}_3^{g-n}$$

acts on $(\mathcal{H}^\pm)^n$. The locally symmetric variety we will consider is

$$M_U = G(F)\backslash(\mathcal{H}^\pm)^n \times G(\widehat{F})/U,$$

where $U = \prod U_v$ was defined in the previous §. Then we have the following $q$-principle:

THEOREM 1.3.1 (§2.4). *There is a unique cuspidal function* $\phi_\chi$ *on* $M_U$ *with the following properties:*

1. $\phi_\chi$ *has character* $\chi_\Delta$ *under the action of* $\Delta$*;*
2. *for each finite place* $v$ *not dividing* $N \cdot D$*,* $\phi_\chi$ *is the eigenform for Hecke operators* $\mathrm{T}_v$ *with the same eigenvalues as* $\phi$*.*

The CM-points on $M_U$, associated to the embedding $K \longrightarrow B$, form the infinite set

$$G(F)_+\backslash G(F)_+ z \times G(\widehat{F})/U \simeq H\backslash G(\widehat{F})/U$$

where $z$ is a point in $\mathcal{H}^n$ fixed by $T$ and $H \subset G$ is the stabilizer of $z$ in $G$. Notice that $H$ is either isomorphic to $T$ if $n \neq 0$ or $H = G$ if $n = 0$. In any case there is a finite map

$$C_U := T(F)\backslash G(\mathbb{A}_f)/U \longrightarrow M_U.$$

The Gross-Zagier formula for central value we want to prove is the following:

THEOREM 1.3.2 (§4.4). *Let $\phi^\sharp$ be the form defined in §1, 1. Then*

$$L(1,\chi,\phi) = 2^{g+n} d_{K/F}^{-1/2} \cdot (\|\phi^\sharp\|/\|\phi_\chi\|)^2 \cdot |\ell_\chi(\phi_\chi)|^2$$

*where*

1. *$\ell_\chi(\phi_\chi)$ is the integral against $\chi(t^{-1})$ on $T(F)\backslash T(\mathbb{A}_f)$ with respect to the standard measure;*
2. *$\|\phi_\chi\|^2$ are $L^2$-norms with respect to the measure on $G(\mathbb{A})$ which is the product of the standard measure on $G(\mathbb{R})$ and the measure on $G(\mathbb{A}_f)$ such that* $\mathrm{vol}(\Delta) = 1$.

Notice that $\ell_\chi(\phi_\chi)$ is actually the evaluation of $\phi_\chi$ at the cycle $\eta_\chi$ defined in §1.2:

$$\ell_\chi(\phi_\chi) = \sum_{t \in T(F)\backslash T(\mathbb{A}_f)/U_T} \chi^{-1}(t)\phi_\chi(t).$$

There are two applications of this theorem. The first one is to generalize a recent work of Bertolini and Darmon [7, 8] to obtain some evidence about BSD-conjecture in rank 0 case. The second application is to use a recent work of Cogdell, Pieteski-Shapiro, and Sarnak [4] to obtain certain equidistribution statement of the toric orbits of CM-points. The details will be given in later papers.

### 1.4. Remarks on proof

The proof in this paper will be based on the following principle used in the original paper of Gross and Zagier [20]:

- The Fourier coefficients of a certain kernel form representing the derivative of the Rankin L-series should be given by the height pairing of CM-points.

But the techniques used in their proof are difficult to apply in the more general situation due to following fundamental obstructions:

- On a Shimura curve, there is no reference point such as a cusp, to send points on the curve to its Jacobian.
- On a Shimura curve, there is no reference modular form such as a Dedekind $\eta$-function to be used to compute the local self-intersection on CM-points.
- When an anticyclotomic character is ramified, since the trace computation is very massive, there is no workable expression of the kernel form to represent the derivative of the Rankin L-series,
- On a Shimura curve or even a modular curve, there is no explicit semistable model which can be used to compute the local intersection index of CM-points at supersingular points.

In our previous paper [31], we solved the first two problems by using multiplicity one for modular forms and Hodge index theory in Arakelov theory [12, 13]. The present paper is devoted to solve the remaining two issues with the following methods:

- We will work directly on kernel functions of high level but use *quasi-newform* projection instead of *newform* projection.
- We will not compute directly the local intersection at places where the Shimura curve has high level. Instead, we will obtain an asymptotic formula and show that this formula is sufficient by a *toric newform* theory.

Besides these technical improvements, we will also develop a notion of *geometric pairing* and prove a *local Gross-Zagier formula*. This formula replaces all mass combinatoric computations in the previous approaches and also provides a foundation for spectral decomposition used to prove the Gross-Zagier formula for central values of Maass forms.

**Acknowledgment.** Obviously, in this note, I am only trying to prove the simplest cases of the program outlined by B. Gross. I would like to express my indebtness to B. Gross for his deep insight concerning the arithmetic of the central values or derivatives of the Rankin's L-series which I have learned in his papers, manuscripts, and conversations.

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## 2. Automorphic forms on $\mathrm{GL}_2$

In this chapter, we want to review various facts about automorphic L-functions of $\mathrm{GL}_1$, $\mathrm{GL}_2$, and the Rankin-Selberg convolution of two forms on $\mathrm{GL}_2$. Our basic references are the papers of Tate [29], Jacquet and Langlands [23], and Jacquet [22].

Beside the general theory, we will also try to make computations using certain *newforms* with respect to the action of unipotent subgroups or the torus. The *unipotent newform theory*, or Atkin-Lehner theory, is discussed in Casselman's paper [3] in the adelic setting, while the *toric newform theory* is mainly due to Waldspurger [30].

### 2.1. L-functions for $\mathrm{GL}_1$

We first start with Tate's theory of L-functions for $\mathrm{GL}_1$.

**Nonarchimedean case.** Let $F$ be a nonarchimdean local field with a local parameter $\pi$. We normalize the absolute value on $F$ such that $q = |\pi|^{-1}$ is the cardinality of the residue field of $F$.

Let $\omega$ be a character of $F^\times$ with conductor $c(\omega) := \pi^{o(\omega)}\mathcal{O}_F$, that is the maximal ideal of $\mathcal{O}_F$ such that $\omega$ is trivial on $(1 + c(\omega))^\times$. The integer $o(\omega)$ is called the *order* of $\omega$. Then the L-function of $\omega$ is defined as follows:

$$L(s,\omega) = \begin{cases} (1-\omega(\pi)q^{-s})^{-1} & \text{if } \omega \text{ is unramified,} \\ 1 & \text{if } \omega \text{ is ramified.} \end{cases} \tag{2.1.1}$$

where $s \in \mathbb{C}$.

Let $\psi$ be a fixed nontrivial additive character of $F$. For a function $\Phi \in \mathcal{S}(F)$ (the space of compactly supported and locally constant functions) we define the Fourier transform by

$$\widehat{\Phi}(x) = \int_F \Phi(y)\psi(xy)dy \tag{2.1.2}$$

where $dx$ is a Haar measure on $F$ such that $\hat{\hat{\Phi}}(x) = \Phi(-x)$. If $\psi(x)$ is changed to $\psi_a(x) := \psi(ax)$ then $dx$ is changed to $|a|^{1/2}dx$ and $\hat{\Phi}(x)$ is changed to $|a|^{1/2}\hat{\Phi}(ax)$.

For example if $\psi$ has the *conductor* $c(\psi) := \pi^{o(\psi)}\mathcal{O}_F$, that is the maximal fractional ideal where $\psi$ is trivial, then $dx$ is such that the volume of $\mathcal{O}_F$ is $|\pi^{-o(\psi)}|^{1/2}$. The integer $o(\psi)$ is called the *order* of $\psi$.

For any $\Phi \in \mathcal{S}(F)$ we define the Mellin transform by

$$Z(s,\omega,\Phi) = \int_{F^\times} \Phi(x)\omega(x)|x|^s d^\times x \qquad (s \in \mathbb{C}, \quad \mathrm{Re}(s) >> 0) \tag{2.1.3}$$

where $d^\times x$ is a measure on $F^\times$ such that the volume of $\mathcal{O}_F^\times$ is 1. Then $Z(s,\omega,\Phi)$ is really a rational function of $q^s, q^{-s}$. One may show that the set of all $Z(s,\omega,\Phi)$ is a fractional ideal of $\mathbb{C}[q^s, q^{-s}]$ with $L(s,\omega)$ as a generator. The local functional equation shows the change when $s$ is replaced by $1-s$:

$$\frac{Z(1-s,\omega^{-1},\widehat{\Phi})}{L(1-s,\omega^{-1})} = \epsilon(s,\omega,\psi)\frac{Z(s,\omega,\Phi)}{L(s,\omega)} \tag{2.1.4}$$

where $\epsilon(s,\omega,\psi)$ is independent of $\Phi$ and is called the $\epsilon$-factor of $\omega$ with respect to $\psi$. If $\psi$ is changed to $\psi_a$ then $\epsilon(s,\omega,\psi)$ is changed to $\omega(a)|a|^{s-1/2}\epsilon(s,\omega,\psi)$.

If $\omega$ is unramified, and $\psi$ is of order 0, then we may use the characteristic function $\Phi_1$ on $\mathcal{O}_F$ to compute the $\epsilon$-factor:

$$Z(s,\Phi_1) = L(s,\omega), \qquad \epsilon(s,\omega,\psi) = 1. \tag{2.1.5}$$

If $\omega$ is ramified and $o(\psi) = 0$, we may compute the $\epsilon$-factor by using the restriction $\Phi_\omega$ of the function $\omega^{-1}$ on $\mathcal{O}_F^\times$:

$$\begin{aligned} Z(s,\Phi_\omega) &= L(s,\omega) = 1, \\ \epsilon(s,\omega,\psi) &= \epsilon(\omega,\psi)|\pi^{c(\omega)}|^{s-1/2}, \\ \epsilon(\omega,\psi) &= |a|^{1/2}\int_{\mathcal{O}_F^\times} \omega(xa)^{-1}\psi(xa)dx, \end{aligned} \tag{2.1.6}$$

where $a$ is a generator of $c(\omega)^{-1}$. Notice that $\epsilon(\omega,\psi)$ is a number of norm 1 if $\omega$ is unitary.

**Archimedean case.** First we consider the case where $F = \mathbb{R}$ with the usual absolute value. Then any nontrivial character will have the form

$$\psi(x) = e^{2\pi i\delta x}, \qquad (\delta \in \mathbb{R}^\times)$$

The self-dual measure $dx$ is $|\delta|^{1/2}$ times the usual measure on $\mathbb{R}$.

Let $\omega$ be a quasi-character of $\mathbb{R}^\times$ of the form

$$\omega(t) = |t|^r \mathrm{sgn}(t)^m, \qquad (r \in \mathbb{C}, \quad m = 0, 1).$$

Then we define

$$L(s,\omega) = \pi^{-(s+r+m)/2}\Gamma\left(\frac{s+r+m}{2}\right). \tag{2.1.7}$$

One may define the Mellin transform Zeta function as in the nonarchimedean case and show that $L(s,\omega)$ collecting all poles of these Zeta functions, and that the Zeta functions and L-function satisfy the same functional equation as in nonarchimedean case.

Again to compute the $\epsilon$-factor we may assume that $\delta = 1$. We can use the function

$$\Phi_\omega(x) = x^m e^{-\pi x^2}$$

to compute the $\epsilon$-factor:

$$\begin{aligned} Z(s, \Phi_\omega) &= L(s, \omega) \\ \epsilon(s, \omega, \psi) &= i^m. \end{aligned} \tag{2.1.8}$$

We now consider the case where $F = \mathbb{C}$ with normalized absolute value $|a|_{\mathbb{C}} = |z|^2$. Any nontrivial character of $\mathbb{C}$ has the form

$$\psi(z) = e^{4\pi i \mathrm{Re}(\delta z)}, \qquad (\delta \in \mathbb{C}^\times)$$

whose self-dual measure is $|\delta|_{\mathbb{C}}^{1/2}$ times $2dxdy$ $(z = x + yi)$.

Let $\omega$ be a quasi-character of $\mathbb{C}^\times$ with the form

$$\omega(z) = \begin{cases} |z|_{\mathbb{C}}^r z^m \\ \text{or} \\ |z|_{\mathbb{C}}^r \bar{z}^m, \end{cases} \qquad (r \in \mathbb{C}, \quad m \geq 0)$$

We define the L-function of $\omega$ to be

$$L(s, \omega) = 2(2\pi)^{-(s+r+m)}\Gamma(s + r + m). \tag{2.1.9}$$

Assume that $\delta = 1$. We may use the function

$$\Phi_\omega(z) = \begin{cases} e^{-\pi |z|^2} \bar{z}^m \\ \text{or} \\ e^{-\pi |z|^2} z^m \end{cases}$$

to compute the $\epsilon$-factor:

$$Z(s, \Phi_\omega) = L(s, \omega), \qquad \epsilon(s, \omega, \psi) = i^m. \tag{2.1.10}$$

**Global theory.** Let $F$ be now a global field and let $\mathbb{A}$ denote the ring of adeles of $F$. Let $\psi : F\backslash\mathbb{A} \to \mathbb{C}$ be a fixed nontrivial additive adele classes character of $F$.

Let $\omega : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ be an idele class quasi-character of $F$. Then we define the L-function $L(s, \omega)$ and $\epsilon$-factor by the product:

$$L(s, \omega) = \prod_v L(s, \omega_v) \tag{2.1.11}$$

$$\epsilon(s, \omega) = \prod_v \epsilon(s, \omega_v, \psi_v) \tag{2.1.12}$$

where $v$ runs through the set of all places of $F$, and $\omega_v$ and $\psi_v$ are components of $\omega$ and $\psi$ at the places $v$. One can show that these products are convergent for $\mathrm{Re}(s) >> 0$, and can be continued to a meromorphic functions on the whole complex plane, and that $L(s, \omega)$ satisfies a functional equation

$$L(s, \omega) = \epsilon(s, \omega)L(1 - s, \omega^{-1}). \tag{2.1.13}$$

This functional equation can be proved by combining the local functional equation with the global functional equation

$$\prod_v Z(s,\omega_v,\Phi_v)=\prod_v Z(1-s,\omega_v,\widehat{\Phi}_v)$$

for some Schwartz functions $\Phi_v$ which are the characteristic function of $\mathcal{O}_v$ for almost all places. This last functional equation is essentially a consequence of the Poisson summation formula.

## 2.2. L-functions for $\mathrm{GL}_2$

**Nonarchimedean case.** First we consider the case where $F$ is a nonarchimedean local field. Let $\psi$ be a fixed nontrivial additive character of $F$.

Let $\Pi$ be an irreducible, infinite dimensional, admissible representation of $\mathrm{GL}_2(F)$ with central character $\omega$, and with the L-function $L(s,\Pi)$ which has the form

$$L(s,\Pi)=\frac{1}{(1-\alpha|\pi|^s)(1-\beta|\pi|^s)}. \tag{2.2.1}$$

Then $\Pi$ can be realized in a Whittaker model $\mathcal{W}(\Pi,\psi)$, a space of locally constant functions $W$ on $\mathrm{GL}_2(F)$ such that

$$W\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)=\psi(x)W(g),\qquad \forall x\in F. \tag{2.2.2}$$

The L-function $L(s,\Pi)$ can be determined analytically by this model just as in $\mathrm{GL}_1$ case.

More precisely, for any $W\in\mathcal{W}(\Pi,\psi)$ define

$$\Psi(s,g,W)=\int_{F^\times} W\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right)|a|^{s-1/2}d^\times a \tag{2.2.3}$$

where $d^\times x$ is an invariant measure on $F^\times$ such that the volume of $\mathcal{O}_F^\times$ is 1. Then one may show that this integral is actually a rational function of $q^s, q^{-s}$, that $L(s,\Pi)$ collects all poles of these Mellin transforms, and that the Mellin transforms and the L-function satisfy the following functional equation

$$\frac{\Psi(1-s,wg,\widetilde{W})}{L(1-s,\widetilde{\Pi})}=\epsilon(s,\Pi,\psi)\omega^{-1}(\det g)\frac{\Psi(s,g,W)}{L(s,\Pi)}, \tag{2.2.4}$$

where

- $\widetilde{\Pi}$ is the *contragradient* of $\Pi$ which has the form
$$\widetilde{\Pi}=\Pi\otimes\omega^{-1};$$
- $\widetilde{W}(g)=W(g)\omega^{-1}(\det g)$ which is in $\mathcal{W}(\widetilde{\Pi},\psi)$;
- $w=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$;
- $\epsilon(s,\Pi,\psi)$ is independent of $\Phi$.

For $\delta \in F^\times$, if we change $\psi$, $W$, $\widetilde{W}$ respectively to

$$\psi_\delta, \qquad W_\delta(g) := W\left(\begin{pmatrix}\delta & 0\\ 0 & 1\end{pmatrix} g\right), \qquad \omega(\delta)\widetilde{W}_\delta,$$

then $\epsilon(s,\Pi,\psi)$ is changed to $\omega(\delta)|\delta|^{2s-1}\epsilon(s,\Pi,\psi)$. Thus in the computation of $\epsilon$-factors we may assume that the conductor of $\psi$ is 1. In this case, the $\epsilon$-factor has the form

$$\epsilon(s,\Pi,\psi) = |\pi^{o(\Pi)}|^{s-1/2}\epsilon(\Pi,\psi) \tag{2.2.5}$$

where $o(\Pi)$ is a nonnegative integer and is called the *order* of $\Pi$. The ideal $\pi^{o(\Pi)}\mathcal{O}_F$ is called the *conductor* of $\Pi$, and the complex number $\epsilon(\Pi,\psi)$ is called the *root number* of $\Pi$. Notice that the root number has norm 1 if $\omega$ is unitary.

**Archimedean case.** We now consider the real case $F = \mathbb{R}$ with additive character $\psi(x) = e^{2\pi i x}$. Then an irreducible, admissible, and infinite dimensional representation $\Pi$ of $\mathrm{GL}_2(\mathbb{R})$ is really a representation of $(\mathcal{G}, U)$ rather than a representation of $\mathrm{GL}_2(\mathbb{R})$ itself, where $\mathcal{G} = M_2(\mathbb{R})$ is the Lie algebra of $\mathrm{GL}_2(\mathbb{R})$, and $U = O_2(\mathbb{R})$. Such a representation can still be realized in a Whittaker model $\mathcal{W}(\Pi,\psi)$ of smooth functions on $\mathrm{GL}_2(\mathbb{R})$ with moderate growth where $(\mathcal{G}, U)$ acts by the right translation. One still can define the L-function $L(s,\Pi)$ which can then be determined (up to invertible functions) by analytic properties of $\mathcal{W}(\Pi,\psi)$.

**Principal series.** Let $F$ be a local field and let $\mu_1, \mu_2$ be two quasi-character of $F^\times$. Let $\mathcal{B}(\mu_1,\mu_2)$ denote the space *admissible* functions $f$ on $\mathrm{GL}_2(F)$ such that

$$f\left(\begin{pmatrix}a & x\\ 0 & b\end{pmatrix} g\right) = \mu_1(a)\mu_2(b)\left|\frac{a}{b}\right|^{1/2} f(g), \qquad \forall \begin{pmatrix}a & x\\ 0 & b\end{pmatrix} \in \mathrm{GL}_2(F)$$

where admissible means locally constant in the nonarchimedean case, and means smooth and $O_2(\mathbb{R})$-finite functions in the archimedean case. The $\mathcal{B}(\mu_1,\mu_2)$ admits an admissible representation by right translations. One may show that $\mathcal{B}(\mu_1,\mu_2)$ is isomorphic to $\mathcal{B}(\mu_2,\mu_1)$ when it is irreducible. To construct a Whittaker model for this representation, we notice that for any function $f$ in $\mathcal{B}(\mu_1,\mu_2)$, there is a Schwartz function $\Phi \in \mathcal{S}(F^2)$ such that

$$f = f_\Phi(g) := \mu_1(\det g)|\det(g)|^{1/2}\int_{F^\times}\Phi[(0,t)g]\mu_1\mu_2^{-1}(t)|t|d^\times t. \tag{2.2.6}$$

The Whittaker function corresponding to $f = f_\Phi$ is given by the following formula:

$$W_\Phi(g) = \mu_1(\det g)|\det(g)|^{1/2}\int_{F^\times}(\rho(g)\Phi)'[(t,t^{-1})]\mu_1\mu_2^{-1}(t)d^\times t \tag{2.2.7}$$

where $\rho(g)$ is the right translation, and $\Phi'$ is the inverse Fourier transform with respect to the second variable:

$$(\rho(g)\Phi)'(x,y) = \int_F \Phi[(x,u)g]\psi(-uy)du.$$

Let $\alpha_F$ denote the norm on $F$: $\alpha_F(x) = |x|$. If $\mu_1\mu_2^{-1} \neq \alpha_F^{\pm 1}$ the representation $\mathcal{B}(\mu_1,\mu_2)$ is irreducible and is denoted by $\Pi(\mu_1,\mu_2)$. We call this representation a *principal* representation. One has the following formula for the L-function and $\epsilon$-factors of $\Pi = \Pi(\mu_1, \mu_2)$ by the following formulas:

$$L(s, \Pi) = L(s, \mu_1)L(s, \mu_2), \tag{2.2.8}$$

$$\epsilon(s, \Pi, \psi) = \epsilon(s, \mu_1, \psi)\epsilon(s, \mu_2, \psi). \tag{2.2.9}$$

The central character of $\Pi$ is $\omega := \mu_1\mu_2$. The contragradient of $\Pi(\mu_1, \mu_2)$ is $\Pi(\mu_1^{-1}, \mu_2^{-1})$. If $F$ is nonarchimedean then the order of $\Pi$ is

$$o(\Pi) = o(\mu_1) + o(\mu_2). \tag{2.2.10}$$

If $F = \mathbb{R}$, we define the *weight* of $\Pi$ to be an integer $k = 0, 1$ such that $\omega(-1) = (-1)^k$.

If $\mu_1\mu_2^{-1} = \alpha_F$, then we may write $\mu_1 = \mu \cdot \alpha_F^{1/2}$, $\mu_2 = \mu \cdot \alpha_F^{-1/2}$ with $\mu$ a quasi-character of $F^\times$. Then $\mathcal{B}(\mu_1, \mu_2)$ contains a unique irreducible representation of codimension 1. We call this representation a *special representation* with twist $\mu$, and denote it as $\sigma(\mu)$. We define the L-function and $\epsilon$-factor of $\Pi = \sigma(\mu)$ by

$$L(s, \Pi) = L(s, \mu_1), \tag{2.2.11}$$

$$\epsilon(s, \Pi, \psi) = \epsilon(s, \mu_1, \psi)\epsilon(s, \mu_2, \psi)\frac{L(1 - s, \mu_1^{-1})}{L(s, \mu_2)}. \tag{2.2.12}$$

The central character of $\Pi$ is $\omega = \mu_1\mu_2 = \mu^2$. The contragradient of $\Pi(\mu)$ is $\Pi(\mu^{-1})$. If $F$ is nonarchimdean, then the order of $\Pi$ is 1 if $\mu$ is unramified, and $2o(\mu)$ if $\mu$ is ramified. If $F = \mathbb{R}$, then the *weight* of $\Pi$ is defined to be 2.

One case we will use is when $F$ is nonarchimedean and $\mu$ is unramified. In this case $\sigma(\mu)$ has $\epsilon$-factor $-\mu(\pi)$ by taking limit $s \longrightarrow 1/2$ in the above formula.

If $F$ is nonarchimdean, a representation is called *supercuspidal* if it is not principal or special.

**Weil representation.** Let $K$ be a quadratic extension of $F$. Let $\eta$ be a character on $F^\times$ corresponding to the extension $K/F$. Let $\chi$ be a quasi-character of $K^\times$. Then there is a unique irreducible and admissible representation $\Pi = \Pi(\chi)$ of $\mathrm{GL}_2(F)$ such that

$$L(s, \Pi) = L_K(s, \chi), \tag{2.2.13}$$

$$\begin{aligned} \epsilon(s, \Pi, \psi) &= \epsilon(s, \omega, \psi)\epsilon_K(s, \chi, \psi_K), \\ \epsilon(\Pi, \psi) &= \epsilon(\omega, \psi)\epsilon_K(\chi, \psi_K), \end{aligned} \tag{2.2.14}$$

where $\psi_K = \psi \circ \mathrm{tr}_{K/F}$. The central character of $\Pi(\chi)$ is $\omega = \eta \cdot \chi|_{F^\times}$. If the residue character of $F$ is not 2, every irreducible, admissible, infinite dimensional representation of $\mathrm{GL}_2(F)$ is either principal, special, or isomorphic to $\Pi(\chi)$.

If $K/F$ is nonarchimedean, and $\chi$ is of the form $\mu \cdot \mathrm{N}_{K/F}$, then

$$\Pi(\chi) = \Pi(\mu, \mu \cdot \eta) \tag{2.2.15}$$

where $\mu$ is an unramified character of $F^\times$.

If $K/F$ is nonarchimedean, and $\chi$ is not of the form as above, then $\Pi(\chi)$ is supercuspidal in the sense that $L(s, \Pi(\chi) \otimes \mu) = 1$ for any character $\mu$ of $F^\times$.

If $K = \mathbb{C}$, and $\chi$ has a form

$$\chi(z) = |z|_{\mathbb{C}}^r z^m, \qquad (m \geq 0),$$

then $\Pi(\chi)$ is *discrete* of weight $m + 1$. This means that $\Pi(\chi)$ appears in $L^2(\mathrm{GL}_2(\mathbb{R}))$ as discrete spectrum. More precisely, we may take this discrete spectrum generated by a holomorphic modular form of weight $m + 1$.

**Jacquet-Langlands local correspondence.** One may also construct representations by using a definite quaternion algebra $B$. By *Jacquet-Langlands correspondence*, there is a 1-1 correspondence between irreducible, admissible, and discrete representations of $\mathrm{GL}_2$ and irreducible representations of $B^\times$. In this correspondence, one dimensional characters $\mu \cdot \det$ of $B^\times$ will give special representations $\sigma(\mu)$.

**Langlands local correspondence.** First lets consider the case $F$ is nonarchimedean. Let $W_F$ denote the Weil group: the subgroup of $\mathrm{Gal}(\bar{F}/F)$ of elements whose images in the residue group $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ are integral powers of the Frobenius. Then Langlands correspondence gives a 1-1 correspondence between irreducible two dimensional representations of $W_F$ and supercuspidal representations of $\mathrm{GL}_2(F)$ which is compatible with twists by characters and the formalism of L-functions and $\epsilon$-facts. For example if $\Pi = \Pi(\chi)$ with $\chi$ a character of $K^\times$, here $K$ is a quadratic extension of $F$, then we may consider $\chi$ as a character of the Weil group $W_K$ via local class field theory. The representation of $W_F$ corresponding to $\Pi(\chi)$ is the induced representation $\mathrm{Ind}_{W_K}^{W_F}(\chi)$.

We now consider the case where $F = \mathbb{R}$. Then the Weil group $W_\mathbb{R}$ is generated by $\mathbb{C}^\times$ and $j$ such that

$$j^2 = -1, \qquad jx = \bar{x}j, \qquad \forall x \in \mathbb{C}^\times.$$

One has obvious homomorphisms

$$W_\mathbb{R}^{\mathrm{ab}} \simeq \mathbb{R}^\times \longrightarrow \mathbb{R}^\times/\mathbb{R}_+^\times \simeq \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

The Langlands correspondence gives a 1-1 correspondence between irreducible representation of $W_\mathbb{R}$ and discrete series of $\mathrm{GL}_2(\mathbb{R})$ which has the same properties as in the nonarchimedean case.

### 2.3. Theories on newforms

We now continue to work on representations of $\mathrm{GL}_2(F)$ for $F$ a local field.

**Atkin-Lehner theory.** Just as in the $\mathrm{GL}_1$ case, the conductor or the order of $\Pi$ will measure the ramification of $\Pi$. For any $c \geq 0$, lets define

$$U_0(\pi^c) = \left\{ \gamma \in \mathrm{GL}_2(\mathcal{O}_F) : \quad \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \pi^c \right\}. \tag{2.3.1}$$

$$U_1(\pi^c) = \left\{ \gamma \in \mathrm{GL}_2(\mathcal{O}_F) : \quad \gamma \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod \pi^c \right\}. \tag{2.3.2}$$

We say that a function $W$ in $\mathcal{W}(\Pi, \psi)$ has *level* $\pi^c$ if it is invariant under $U_1(\pi^c)$. Then we have the following:

PROPOSITION 2.3.1 ([3]). *The order $o(\Pi)$ is the minimal nonnegative integer $c$ such that $\mathcal{W}(\Pi, \psi)$ has a nonzero function of level $\pi^c$. Moreover,*

1. *If $c = o(\Pi)$, then the space $\mathcal{W}(\Pi, \psi)$ has a unique element $W_\Pi$ of level $\pi^c$ and takes value 1 at the unit element $e$ in $\mathrm{GL}_2(F)$.*
2. *If $c \geq o(\Pi)$ then the space of functions in $\mathcal{W}(\Pi, \psi)$ of level $\pi^c$ has dimension $c - o(\Pi) + 1$ and is generated by*

$$W_{\Pi,i}(g) := W\left(g \begin{pmatrix} \pi^{-i} & 0 \\ 0 & 1 \end{pmatrix}\right) \qquad (i = 0, 1, \cdots, c - o(\Pi)).$$

The function $W_\Pi(g)$ is called a Whittaker *newform* with respect to character $\psi$. With this function and definition in (2.2.3), one has the following

$$L(s,\Pi) = \Psi(s, W_\Pi), \tag{2.3.3}$$

$$W_\Pi(gh) = \epsilon(\Pi,\psi)W_{\widetilde{\Pi}}(g)\omega(\det g)^{-1}, \tag{2.3.4}$$

where

$$h := \begin{pmatrix} 0 & 1 \\ -\pi^c & 0 \end{pmatrix}, \qquad c = o(\Pi) \tag{2.3.5}$$

is the *Atkin-Lehner operator* of order $c$.

In this paper we will use a modified notion of newforms. To define it, we assume that $\Pi$ is unitary. Then there is a hermitian and positive pairing

$$(\cdot,\cdot): \quad \mathcal{W}(\Pi,\psi)\times\mathcal{W}(\Pi,\psi) \longrightarrow \mathbb{C}$$

such that

$$(\rho(g)W_1, \rho(g)W_2) = (W_1, W_2).$$

We say a vector $W \in \mathcal{W}(\Pi,\psi)$ is *quasi-new*, if $W$ is nonzero, and

$$(W, W_\Pi - W) = 0.$$

Let $V$ be a space of forms in $\mathcal{W}(\Pi,\psi)$ containing the newvector $W_\Pi$. Then the correspondence

$$v \longrightarrow \{w \in V, \quad (v,w) = 0\}$$

gives a one-one correspondence between the quasi-newvector in $V$ and hyperplane not containing $W_\Pi$.

For example, let $c \geq o(\Pi)$ be a fixed integer, then we may take $V$ to be the space of forms of level $\pi^c$. Then there is unique quasi-new vector perpendicular to

$$W\left(\begin{pmatrix} \pi^{-i} & 0 \\ 0 & 1 \end{pmatrix}\right), \qquad c - o(\Pi) \geq i \geq 1.$$

**Weights.** The analogue of the *order* of a representation in the archimedean case is *weight*: we say a form $W \in \mathcal{W}(\Pi,\psi)$ has *weight* $m$ if

$$W\left(g\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = W(g)e^{2\pi i m\theta}, \qquad \forall\theta \in \mathbb{R}/\mathbb{Z}. \tag{2.3.6}$$

One can show that the weight $k$ of a representation $\Pi$ is the minimal nonnegative integer such that $\Pi$ has a nonzero vector of weight $k$. Moreover for any integer $n$, the space of forms in $\mathcal{W}(\Pi,\psi)$ is one dimensional if $|n| \geq k$, $n \equiv k \pmod 2$. Otherwise it is 0.

If $\Pi$ is not of the form $\Pi = \Pi(\alpha^{r_1}\mathrm{sgn}, \alpha^{r_2}\mathrm{sgn})$, then with definition in (2.2.3), there is a unique and Whittaker functions $W_\Pi$ of weight $k$ such that

$$L(s,\Pi) = \Psi(s, W_\Pi), \tag{2.3.7}$$

$$W_\Pi(gw) = \epsilon(\Pi,\psi)W_{\widetilde{\Pi}}(g)\omega^{-1}(\det g). \tag{2.3.8}$$

Again, we call $W_\Pi$ the new vector for $\Pi$ with respect to the additive character $\psi$.

In case $\Pi = \Pi(\alpha^{r_1}\mathrm{sgn}, \alpha^{r_2}\mathrm{sgn})$, we call a Whittaker function $W(g)$ of weight 0 a *newform* if $W(g)\mathrm{sgn}(\det g)$ is a newform for $\Pi(\alpha^{r_1}, \alpha^{r_2})$. Notice that $\Psi(s, W) = 0$ as $W\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ is odd in $a \in \mathbb{R}^\times$.

**Waldspurger theory.** Let $F$ be a nonarchimedean local field. Let $K$ be a quadratic extension of $F$ (which is either split $K = F \oplus F$ or a field) embedded into $M_2(F)$. Let $T$ denote the torus $K^\times/F^\times$ in $G = \mathrm{PGL}_2(F)$.

Let $G' = B^\times/F^\times$ where $B$ is a quaternion division algebra over $F$. We also embed $K$ into $B$ if $K$ is nonsplit and also denote $T$, the torus $K^\times/F^\times$ in $G'$.

Let $\Pi$ be an irreducible, admissible, and infinite dimensional representation of $G$. If $\Pi$ is $L^2$, let $\Pi'$ denote the corresponding representation of $G'$ by the Jacquet-Langlands correspondence.

Let $\mathcal{U}(\Pi, T)$ (resp. $\mathcal{U}(\Pi', T)$) denote the space of linear maps from $\Pi$ (resp. $\Pi'$) to the space of continuous functions on $T\backslash G$ (resp. $T'\backslash G'$) with compatible $G$. (resp. $G'$) action. Set $\mathcal{U}(\Pi', T)$ to be zero if it is can't be defined as above.

If $T$ is not split, let $\Pi^T$ (resp. $(\Pi')^T$) denote the subspace of $\Pi$ (resp. $\Pi'$) invariant under $T$. Then we have the following fundamental criterion for the existence of $T$-invariant vectors in $\Pi$ or $\Pi'$.

THEOREM 2.3.2 ([30], Proposition 1, Lemma 1, Theorem 2). *With notation as above, one has that*

$$\dim \mathcal{U}(\Pi, T) + \dim(\Pi', T) = 1$$

*and that if $T$ is not split then,*

$$\dim \mathcal{U}(\Pi, T) = \dim \Pi^T, \qquad \dim \mathcal{U}(\Pi', T) = \dim(\Pi')^T.$$

*Moreover,*

1. *If $T$ is split or $\Pi$ is principal, then $\mathcal{U}(\Pi, T) \neq 0$.*
2. *If $T$ is not split and $\Pi = \sigma(\mu)$ ($\mu^2 = 1$) is special, then*

$$\mathcal{U}(\Pi, T) \neq 0 \iff \mu \circ \mathrm{N}_{K/F} \neq 1,$$
$$\mathcal{U}(\Pi', T) \neq 0 \iff \mu \circ \mathrm{N}_{K/F} = 1.$$

3. *If $T$ is nonsplit and $K/F$ is unramified, then*

$$\mathcal{U}(\Pi, T) \neq 0 \iff o(\Pi) \quad \textit{is even},$$
$$\mathcal{U}(\Pi', T) \neq 0 \iff o(\Pi) \quad \textit{is odd}.$$

**Toric newforms and Gross-Prasad's theory.** In [19], Gross and Prasad studied the invariant vector from a different point of view, i.e., by analyzing the subspace $\Pi^\Gamma$ (resp. $(\Pi')^\Gamma$) of vectors invariant under $\Gamma = R^\times$ where $R$ is an order of $M_2(F)$ or $B$ of discriminant $c(\Pi)$ containing $\mathcal{O}_K$.

THEOREM 2.3.3 ([19], see also [31]). *Assume either $K/F$ is unramified, or $\Pi$ is principal, or $\Pi$ is special with prime conductor. Then*

$$\dim \Pi^\Gamma = \dim \mathcal{U}(\Pi, T), \qquad \dim(\Pi')^\Gamma = \dim \mathcal{U}(\Pi', T).$$

Everything is proved in [32] except the case where $K/F$ is ramified and $\Pi = \sigma(\mu)$.

LEMMA 2.3.4 ([19]). *If $K/F$ is ramified and $\Pi = \Pi(\mu)$ is special of prime conductor, then*

1. *$\Pi^\Gamma$ is one dimensional and stable under $T$ with a unramified character which sends $\pi_K$ to $-\mu(\pi)$.*

2. $(\Pi')^\Gamma$ *is one dimensional and stable under* $T$ *with a unramified character which sends* $\pi_K$ *to* $\mu(\pi)$.

*Proof.* Indeed, there is nothing to prove in the case of a one dimensional representation. In case of $\Pi$, $\Gamma$ is isomorphic to $U_0(\pi)$ while $\pi_K$ acts like the Atkin-Lehner operator. Thus the lemma follows from the functional equation of the Whittaker newform and computation of the epsilon-factor $\epsilon(\Pi, \psi) = -\eta(\pi)$. $\square$

Let $\chi$ now be a character of $K^\times/F^\times$. We want to study invariant vectors under the action of $T$ with character $\chi$ under the same conditions as in Theorem 2.3.3. When $\chi$ is unramified, then either $\chi = 1$ in the situation of Theorem 2.3.3, or in the situation of Lemma 2.3.4 with described character. Thus we need only treat the case where $\chi$ is ramified.

We assume that $o(\Pi) \le 1$. Lets assume that $\mathcal{O}_K$ is embedded into $M_2(\mathcal{O}_F)$ and let

$$\Gamma = (\mathcal{O}_K + c(\chi)M_2(\mathcal{O}_F))^\times. \tag{2.3.9}$$

Now $\chi$ can be extended to a character of $\Gamma$ in the obvious way. We are concerned the existence of a nonzero subspace $\Pi^\chi$ of vectors $v$ in $\Pi$ such that

$$\gamma v = \chi(\gamma) v, \qquad \gamma \in \Gamma. \tag{2.3.10}$$

THEOREM 2.3.5. *Assume that* $K/F$ *is unramified, that* $\chi$ *is ramified, and that* $o(\Pi) \le 1$. *Then* $\dim \Pi^\chi = 1$.

*Proof.* Our assumption implies $\Pi$ is included in the space $\mathcal{B}(\mu, \mu^{-1})$ of locally constant functions on $\mathrm{GL}_2(F)$ such that

$$f\left(\begin{pmatrix} a & x \\ 0 & b\end{pmatrix} g\right) = \mu(a/b)|a/b|^{1/2} f(g),$$

where $\mu$ is an unramified character of $F^\times$. It suffices to show the theorem for this space because in the case $\Pi = \Pi(\mu)$, $\mu^2 = \alpha_F$, the one-dimensional subquotient of $\mathcal{B}(\mu, \mu^{-1})$ is isomorphic to $\mu \cdot \det g$ which does not have $\chi$-eigen vectors.

The $\chi$-eigen subspace of $\mathcal{B}(\mu, \mu^{-1})$ for $\Gamma$ is the space of functions $f$ on $\mathrm{GL}_2(\mathcal{O}_F)$ such that

$$f\left(\begin{pmatrix} a & x \\ 0 & b\end{pmatrix} g\gamma\right) = \mu(a/b) f(g)\chi(\gamma)$$

for all $\gamma$ in $\Gamma$.

First we treat the case where $K$ is a field. Let $u$ be a trace-free element of $\mathcal{O}_K^\times$. Then we have an embedding $K \to M_2(F)$ given by

$$a + bu \longrightarrow \begin{pmatrix} a & b \\ bu^2 & a\end{pmatrix}.$$

With this embedding one has the decomposition $\mathrm{GL}_2(F) = B_1(F)T(F)$ where $B_1(F)$ is the set of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix}$. Since $\chi$ is trivial on $F^\times$, the $\chi$-eigen subspace for $\Gamma$ is included in the $\chi$-eigen subspace for $T$. But it is easy to see that the $\chi$-eigen subspace of $T$ is one-dimensional and is generated by

$$f_0\left(\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix} t\right) = |a|^{1/2}\mu(a)\chi(t).$$

To show the theorem for this case, we need only show that $f_0$ is in the $\chi$-eigen subspace of $\Gamma$. In other words we want to show for any $g \in \mathrm{GL}_2(F)$, $\gamma \in \Gamma$, that

$$f_0(g\gamma) = f_0(g)\chi(\gamma).$$

Since $T$ normalizes $\Gamma$ and fixes the character $\chi$, one has the decomposition

$$\begin{aligned} g &= \beta \cdot t, \qquad (\beta \in B_1(F), t \in T(F)),\\ t\gamma t^{-1} &= \alpha \cdot \tau \qquad (\alpha \in B_1(\mathcal{O}_F), \tau \in T(\mathcal{O}_F). \end{aligned}$$

Thus $g\gamma$ has the decomposition $\beta\alpha \cdot \tau t$. The above equation follows easily.

It remains to consider the case where $K = F \oplus F$ and $\chi = (\mu, \mu^{-1})$. Let $K$ be embedded into $M_2(F)$ diagonally. Then $\Gamma$ consists of matrices congruent to elements in $T(\mathcal{O}_F)$ modulo $\pi^n$. It is not difficult to show that $B(F)\backslash G(F)/\Gamma$ is represented by the following elements

$$e := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$
$$\beta_m := \begin{pmatrix} 1 & 0 \\ \pi^m & 1 \end{pmatrix}, \quad \gamma_m := \begin{pmatrix} 0 & 1 \\ -1 & \pi^m \end{pmatrix} \quad (0 < m < n).$$

One can verify explicitly that the $\chi$-eigen subspace of $V$ is one dimensional and is generated by the following function supported on $B(F)\alpha\Gamma$:

$$f_0\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \alpha\gamma\right) = |a/b|^{1/2}\mu(a/b)\chi(\gamma).$$

$\square$

We call the space $\Pi^\chi$ the space of *toric newvectors* with a prescribed character $\chi$. Notice when $\chi$ is ramified, our treatment is slightly different than [19], where Gross-Prasad obtained the same result about invariants under $\widetilde{R}^\times$ with $\widetilde{R}$ an order of $B$ containing $\mathcal{O}_{c(\chi)}$ *optimally.*

### 2.4. Automorphic forms on $\mathrm{GL}_2$

**Automorphic forms and cusp forms.** Let $F$ be a number field. Let $\mathbb{A}$ denote the adeles of $F$. Let $\omega$ be a quasi-character of $F^\times\backslash\mathbb{A}^\times$. Let $\mathcal{A}(\omega)$ denote the space of *automorphic forms* on $\mathrm{GL}_2(\mathbb{A})$ which are the smooth functions with moderate growth on $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})$, and with character $\omega$ under the translation by the center $Z(\mathbb{A}) = \mathbb{A}^\times$. The space $\mathcal{A}(\omega)$ admits a representation $\rho$ of $\mathrm{GL}_2(\mathbb{A})$:

$$(\rho(g)f)(x) = f(xg) \tag{2.4.1}$$

For each place $v$ of $F$ let $\Pi_v$ be a representation of $F_v$ such that for all but finite many $v$, $\Pi_v$ is unramified with a fixed newvector $v_\wp$. Then we can define the representation $\Pi := \otimes_v \Pi_v$ of $\mathrm{GL}_2(\mathbb{A})$ as a direct limit

$$\Pi_S := \otimes_{v\in S}\Pi_v$$

over finite subsets $S$ of $F$ such that for two $S \subset S'$ containing all archimedean places and ramified places of $\Pi_v$, the structure map $\Pi_S \longrightarrow \Pi_{S'}$ is given by tensoring with

$$\otimes_{S'\backslash S} v_\wp.$$

We say $\Pi$ is *automorphic*, if $\Pi$ is isomorphic to a subrepresentation of $\mathcal{A}(\omega)$.

Let $\psi$ be a fixed nontrivial character of $F\backslash\mathbb{A}$. Then for $\phi \in \mathcal{A}(\omega)$ one has the following Fourier expansion

$$\phi(g) = C_\phi(g) + \sum_{\xi \in F^\times} W_\phi\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} g\right) \tag{2.4.2}$$

where $C_\phi(g)$ is the constant coefficient, and and $W_\phi(g)$ is the Whittaker coefficient of $\phi$:

$$C_\phi(g) := \int_{F\backslash\mathbb{A}} \phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) dx \tag{2.4.3}$$

$$W_\phi(g) := \int_{F\backslash\mathbb{A}} \phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) \psi(-x) dx. \tag{2.4.4}$$

Here $dx$ is the associated self-dual measure on $F\backslash\mathbb{A}$ which is actually the unique Haar measure of volume 1.

A form $\phi \in \mathcal{A}(\omega)$ is called *cuspidal* if $C_\phi(g) = 0$ for all $g \in \mathrm{GL}_2(\mathbb{A})$. Let $\mathcal{A}_0(\omega)$ denote the space of cusp forms in $\mathcal{A}(\omega)$ which is stable under the action by $\mathrm{GL}_2(\mathbb{A})$. An irreducible, admissible, and infinite dimensional representation $\Pi$ of $\mathrm{GL}_2(\mathbb{A})$ of central character $\omega$ is called *cuspidal* if it appears in $\mathcal{A}_0(\omega)$. It is well known that if $\Pi$ is cuspidal then the multiplicity of $\Pi$ in $\rho$ is 1:

THEOREM 2.4.1 (Strong multiplicity one, [3]). *Let $\Pi = \otimes \Pi_v$ and $\Pi' = \otimes \Pi'_v$ be two cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$ such that $\Pi_v \simeq \Pi'_v$ for all but finitely many places $v$ of $F$. Then $\Pi \simeq \Pi'$.*

For a cuspidal representation $\Pi$, we let $\mathcal{A}(\Pi)$ denote the space of cuspidal forms. Then for any collection of Whittaker functions in $W_v \in \mathcal{W}(\Pi_v.\psi_v)$ with almost all $W_v$ are newform, one may form a global Whittaker function $W = \otimes_v W_v$, and a cusp form

$$\phi(g) = \sum_{\xi \in F^\times} W\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} g\right). \tag{2.4.5}$$

**L-functions.** Let $\Pi = \otimes \Pi_v$ be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$. Let $L(s,\Pi)$ denote the product of $L(s,\Pi_v)$ and let $\epsilon(s,\Pi)$ denote the product of $\epsilon(s,\Pi_v,\psi_v)$ which is convergent for $\mathrm{Re}(s) >> 0$. Then we have

THEOREM 2.4.2. *The function $L(s,\Pi)$ ($\mathrm{Re}(s) >> 0$) can be continued to a holomorphic function on the whole complex plane and satisfies the functional equation*

$$L(s,\Pi) = \epsilon(s,\Pi) L(1-s,\Pi) \tag{2.4.6}$$

*Proof.* Indeed, for any place $v$, one may find a Whittaker function $W_v$ such that $\Psi(s,e,W_v) \neq 0$, and that for almost all finite $v$, $W_v$ equals the standard spherical function. Let $\phi$ be a form with Whittaker function $W := \prod W_v$. Then one has

$$\prod_v \Psi(s,e,W_v) = \int_{F^\times\backslash\mathbb{A}^\times} \phi\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} |a|^{s-1/2} d^\times a,$$

and

$$\prod_v \Psi(1-s,w,\widetilde{W}_v)=\int_{F^\times\backslash\mathbb{A}^\times}\phi\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}w\right)\omega^{-1}(a)|a|^{1/2-s}d^\times a.$$

These two quantities are equal since

$$\phi\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}w\right)=\phi\left(w^{-1}\begin{pmatrix}a&0\\0&1\end{pmatrix}w\right)=\omega(a)\phi\begin{pmatrix}a^{-1}&0\\0&1\end{pmatrix}.$$

□

Let $\delta\in\mathbb{A}^\times$ be such that every local additive character

$$\psi_v^0(x):=\psi_v(\delta_v^{-1}x)$$

has conductor 1. Let $W_v^0(g)$ be the newform for $\Pi_v$. Then we may define a Whittaker function $W(g)=\otimes W_v(g)$ for $\psi^0:=\otimes\psi_v^0$ and a *newform* $\phi_\Pi$ by

$$\phi_\Pi(g)=\sum_{\xi\in F^\times}W\left(\begin{pmatrix}\xi\delta&0\\0&1\end{pmatrix}g\right) \tag{2.4.7}$$

With this newform, since $|\delta|=d_F^{-1}$, one has

$$\begin{aligned}\int_{F^\times\backslash\mathbb{A}^\times}\phi\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}\right)|a|^{s-1/2}d^\times a=&\int_{\mathbb{A}^\times}W\begin{pmatrix}a\delta&0\\0&1\end{pmatrix}|a|^{s-1/2}d^\times a\\ =&d_F^{s-1/2}L(s,\Pi).\end{aligned}$$

**Hecke operators.** Assume that $\omega=1$ and let $\phi$ be a fixed form in $\mathcal{A}_0(\omega)$. Let $S$ be a finite subset of places such that if $v\notin S$, then $v$ is a nonarchimedean place and $\phi$ is invariant under $\mathrm{GL}_2(\mathcal{O}_v)$. For a nonzero $a\in\widehat{\mathcal{O}}^S$ an integral finite $S$- idele, let $\mathrm{T}_a$ be the Hecke operator corresponding to the characteristic function on the set

$$H(a)=\left\{g\in M_2(\widehat{\mathcal{O}}_F^S),\qquad \det g\cdot\widehat{\mathcal{O}}_F^S=a\cdot\widehat{\mathcal{O}}_F^S\right\}. \tag{2.4.8}$$

Then $H(a)$ has a disjoint decomposition:

$$H(a)=\coprod_{\alpha,\beta,\gamma}\begin{pmatrix}\alpha&\beta\\0&\gamma\end{pmatrix}\mathrm{GL}_2(\widehat{\mathcal{O}}_F^S) \tag{2.4.9}$$

where $\alpha,\gamma$ are integral ideles modulo $\widehat{\mathcal{O}}_F^{S,\times}$ such that $\alpha\gamma=a$, and $\beta$ is an integral adele modulo $\alpha$.

It follows that for $g\in\mathrm{GL}_2(\mathbb{A}_S)$ and $y\in\mathbb{A}^S$,

$$\begin{aligned}\mathrm{T}_aW_\phi\left(g\begin{pmatrix}y&0\\0&1\end{pmatrix}\right)=&\sum_{\alpha,\beta,\gamma}W_\phi\left(g\begin{pmatrix}y\alpha/\gamma&y\beta/\gamma\\0&1\end{pmatrix}\right)\\ =&\sum_{\gamma|y_f\delta\alpha}W_\phi\left(g\begin{pmatrix}y\alpha/\gamma&0\\0&1\end{pmatrix}\right)\sum_{\beta\mod\alpha}\psi(y\beta/\gamma)\\ =&\sum_{a,d|y_f\delta}W_\phi\left(g\begin{pmatrix}y\alpha/\gamma&0\\0&1\end{pmatrix}\right)|\alpha|^{-1}.\end{aligned}$$

Thus we have the formula

$$\mathrm{T}_a W_\phi \left( g \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = \sum_{\gamma | (y_f \delta, a)} W_\phi \left( g \begin{pmatrix} ya/\gamma^2 & 0 \\ 0 & 1 \end{pmatrix} \right) |\gamma/a| \tag{2.4.10}$$

It follows that if $a$ is prime to $y_f\delta$,

$$W_\phi \left( g \begin{pmatrix} ya & 0 \\ 0 & 1 \end{pmatrix} \right) = |a| \mathrm{T}_a W_\phi \left( g \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right). \tag{2.4.11}$$

If $\phi$ belongs to an irreducible and cuspidal representation $\Pi$, then we have $|a|\mathrm{T}_a\phi = \widehat{\Pi}(a)\phi$ where

$$\widehat{\Pi}(a) = W^S \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \tag{2.4.12}$$

It follows that

$$\sum_a \widehat{\Pi}(a)|a|^{s-1/2} = \prod_{\wp \notin S} L(s, \Pi_\wp). \tag{2.4.13}$$

For $\wp \notin S$, $\Pi_\wp$ is unramified, thus is uniquely determined by $L(s, \Pi_\wp)$ and then by $a_\wp$.

**Jacquet-Langlands correspondence.** Let $B$ be a quaternion algebra over $F$ and let $G = B^\times$ as an algebraic group over $F$. Then we have the same notions of automorphic forms, automorphic representations, and the multiplicity one or strong multiplicity one.

Let $\Pi' = \otimes \Pi'_v$ be an irreducible and admissible representation of $G(\mathbb{A})$ and let $\Pi = \otimes \Pi_v$ be an irreducible and admissible representation of $\mathrm{GL}_2(\mathbb{A})$ obtained by applying Jacquet-Langlands correspondence componentwise. Then $\Pi'$ is automorphic and cuspidal if and only if $\Pi$ is automorphic and cuspidal.

**Proof of Theorem 1.2.2 and 1.3.1.** We now return to the situation of Introduction where a form $\phi$ over a totally real field $F$ and a character $\chi$ of $\mathbb{A}_K^\times / K^\times \mathbb{A}^\times$ are given such that the hypothesis (1.1.1) is satisfied, where $K$ is an imaginary quadratic extension of $F$. The functional equation of $L(s, \chi, \phi)$ has sign $(-1)^{\#\Sigma}$ where $\Sigma$ is a finite set of places defined in (1.1.3).

Let $S$ be a finite set of archimedean places of $F$ such that

- $S \cup \Sigma$ contains all archimedean places of $F$,
- $\Sigma - S$ has even cardinality.

Let $B$ be a quaternion algebra over $F$ which is ramified exactly at places in $\Sigma - S$ and let $G$ be the inner form of $\mathrm{PGL}_{2,F}$ associate to $B^\times / F^\times$. Let $\Delta$ be an open compact subgroup of $G(\mathbb{A}_f)$ defined in (1.2.2) and $\chi_\Delta$ a character on $\Delta$ defined in (1.2.3).

THEOREM 2.4.3. *There is a unique cusp form $\phi_\chi$ on $G(\mathbb{A})$ with the following properties:*

1. *$\phi_\chi$ has the same weight as $\phi$ at places in $S$, and has weight $0$ at other infinite places;*
2. *$\phi_\chi$ has character $\chi_\Delta$ under the action of $\Delta$;*
3. *for each finite place $v$ not dividing $N \cdot D$, $\phi_\chi$ is the eigenform for Hecke operators $\mathrm{T}_v$ with the same eigenvalues as $\phi$.*

*Proof.* Let $\Pi$ be the irreducible and cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ generated by $\phi$. For each place $v$ in $\Sigma$, $\Pi_v$ is nonprincipal. This is clear for $v \mid \infty$; for $v$ finite, we just need to notice that any principal representation has even order. Thus $\Pi$ will have a Jacquet-Langlands correspondence $\Pi'$ of $G(\mathbb{A})$. The existence and uniqueness of $\phi_\chi$ now is determined by the local representation $\Pi'_v$ and follows from the results in the last section §2.3. □

To prove Theorem 1.2.2, we take $S$ to be the set $\{\tau\}$. Then the form $\phi_\chi$ in Theorem 2.4.3 is on the Shimura curve $X$ defined in §1.2. Theorem 1.2.2 now follows from the standard Eichler-Shimura theory.

To prove Theorem 1.3.1, we take $S = \emptyset$.

**$q$-expansion principle.** Let $\Pi_\infty = (\Pi_v, v \mid \infty)$ be a fixed representation of $\mathrm{GL}_2(F_\infty) = \prod_{v|\infty} \mathrm{GL}_2(F_v)$ at the archimedean place with trivial central character. Let $N$ be an ideal of $\mathcal{O}_F$. For each representation $\Pi$ with conductor $N$ and infinite component $\Pi_\infty$, fix one quasi-newform $\phi_\Pi$. Let $\mathcal{A}^\sharp(\Pi_\infty, N)$ denote the space of cusp-forms generated by $\phi_\Pi$. Notice that $\mathcal{A}^\sharp(\Pi_\infty, N)$ is a finite dimension space with an action by Hecke operators $\mathrm{T}_a$ for $(a, N) = 1$.

Let $\ell$ be a unique linear functional on $\mathcal{A}^\sharp(\Pi_\infty, N)$ such that

$$\ell(\phi_\Pi) = 1.$$

THEOREM 2.4.4 (q-expansion principle). *Let $S$ be a set of places containing infinite places and places dividing $N$. Let $\mathbb{T}^\sharp = \mathbb{T}^\sharp(\Pi_\infty, N)$ denote the ring of endomorphism of $\mathcal{A}^\sharp = \mathcal{A}^\sharp(\Pi_\infty, N)$ generated by $\mathrm{T}_a$ for $a$ prime to $S$. Then the pairing*

$$\mathbb{T}^\sharp \times \mathcal{A}^\sharp \longrightarrow \mathbb{C}, \qquad \langle t, \phi\rangle = \ell(t\phi)$$

*is nondegenerate in both variables.*

*Proof.* The space $\mathcal{A}^\sharp$ is a direct sum of one dimensional space $\mathbb{C}\phi_\Pi$. The action of $\mathbb{T}^\sharp$ is given by a character $t \longrightarrow a_\Pi(t)$. The (strong) multiplicity one implies that the characters $t \longrightarrow a_\Pi(t)$ are all different. The assertion now follows from the linear independence of the characters $a_\Pi(t)$. □

### 2.5. Rankin-Selberg convolution

In the rest of this chapter, we will review Jacquet's theory [22] of Rankin-Selberg convolutions of L-functions for $\mathrm{GL}_2$. For our purpose, we only consider the convolutions which can be written as a *single* Mellin-transform of Whittaker functions. First, lets consider the nonarchimedean case.

**Nonarchimedean case.** Let $F$ be a nonarchimdean field. Let $\Pi_i$ $(i = 1, 2)$ be two admissible representations of $\mathrm{GL}_2(F)$ with central characters $\omega_i$. Then the convolution L-function $L(s, \Pi_1 \times \Pi_2)$ is the inverse of a polynomial of $q^{-s}$ which is the common denominator of all the following Mellin transforms:

$$\Psi(s, W_1, W_2, \Phi) = \int_{Z(F)N(F)\backslash G(F)} W_1(g)W_2(\epsilon g)f_\Phi(s, \omega, g)dg. \tag{2.5.1}$$

where $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $W_i \in \mathcal{W}(\Pi_i, \psi)$, $\Phi \in \mathcal{S}(F^2)$, $\omega = \omega_1 \cdot \omega_2$, and

$$f_\Phi(s, \omega, g) = |\det g|^s \int_{F^\times} \Phi[(0, t)g]|t|^{2s}\omega(t)d^\times t. \tag{2.5.2}$$

Again one has a functional equation:

$$\frac{\Psi(1-s,\widetilde{W}_1,\widetilde{W}_2,\widetilde{\Phi})}{L(1-s,\widetilde{\Pi}_1\times\widetilde{\Pi}_2)}=\epsilon(s,\Pi_1\times\Pi_2,\psi)\frac{\Psi(s,W_1,W_2,\Phi)}{L(s,\Pi_1\times\Pi_2)} \tag{2.5.3}$$

where

$$\widetilde{W}_i(g)=W_i(g)\omega_i(\det(g))^{-1} \tag{2.5.4}$$

$$\widetilde{\Phi}(x,y)=\int\Phi(u,v)\psi(yu-xv)dudv. \tag{2.5.5}$$

The L-function $L(s,\Pi_1\times\Pi_2)$ can also be defined by algebraic means. If one of $\Pi_i$ is principal, say $\Pi_1=\Pi(\mu_1,\mu_2)$, then

$$L(s,\Pi_1\times\Pi_2)=L(s,\mu_1\otimes\Pi_2)\cdot L(s,\mu_2\otimes\Pi_2) \tag{2.5.6}$$

$$\epsilon(s,\Pi_1\times\Pi_2,\psi)=\epsilon(s,\mu_1\otimes\Pi_2,\psi)\cdot\epsilon(s,\mu_2\otimes\Pi_2,\psi). \tag{2.5.7}$$

If one of $\Pi_i$ is special, say $\Pi_1=\sigma(\mu)$, then

$$L(s,\Pi_1\times\Pi_2)=L(s,\mu\alpha_F^{1/2}\otimes\Pi_2), \tag{2.5.8}$$

$$\begin{aligned}\epsilon(s,\Pi_1\times\Pi_2,\psi)&=\epsilon(s,\mu\alpha_F^{1/2}\otimes\Pi_2,\psi)\epsilon(s,\mu\alpha_F^{-1/2}\otimes\Pi_2,\psi)\\&\quad\cdot\frac{L(1-s,\widetilde{\Pi}_2\otimes\alpha^{-1/2}\mu^{-1})}{L(s,\Pi_2\otimes\alpha^{-1/2}\mu)}.\end{aligned} \tag{2.5.9}$$

Assume now that both $\Pi_i$ are supercuspidal. Then each $\Pi_i$ corresponds to some irreducible two dimensional representation $\rho_i$ of the Weil group $W_F$. Then we have:

$$L(s,\Pi_1\times\Pi_2)=L(s,\rho_1\times\rho_2). \tag{2.5.10}$$

$$\epsilon(s,\Pi_1\times\Pi_2,\psi)=\epsilon(s,\rho_1\times\rho_2,\psi). \tag{2.5.11}$$

In general, $L(s,\Pi_1\times\Pi_2)$ is some combination of $\Psi(s,W_1,W_2,\Phi)$. But it will have a nice expression as a single Mellin transform under the following hypothesis:

- *One of* $\Pi_i$ *is either unramified or special with an unramified twist.*

In this case, if we write

$$L(s,\Pi_1)=\prod_{i=1}^{2}(1-\alpha_i|\pi|^s)^{-1},\qquad L(s,\Pi_2)=\prod_{j=1}^{2}(1-\beta_j|\pi|^s)^{-1},$$

then one can show that the Rankin-Selberg convolution L-function is given by:

$$L(s,\Pi_1\times\Pi_2)=\prod_{i,j=1}^{2}(1-\alpha_i\beta_j|\pi|^s)^{-1}. \tag{2.5.12}$$

Without loss of generality, we assume that $\Pi_1$ satisfies the above hypotheses and

$$c_1:=\mathrm{ord}(\Pi_1)\le c_2:=\mathrm{ord}(\Pi_2), \tag{2.5.13}$$

and that the additive character $\psi$ has order 0. In the following we want to show that

$$L(s,\Pi_1\times\Pi_2)=\Psi(s,W_1,W_2,\Phi),$$

where $W_i$ be the Whittaker newfunction for $\Pi_i$, and

$$\Phi(x,y) = \begin{cases} 1 & \text{if } o(\omega)=0,\ |x|\le|\pi|^{c_2},\ |y|\le 1, \\ \omega^{-1}(y) & \text{if } o(\omega)>0,\ |x|\le|\pi|^{c_2},\ |y|=1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.14}$$

Notice that an invariant measure $dg$ on $Z(F)N(F)\backslash \mathrm{GL}_2(F)$ has decomposition $dg = |a|^{-1}dadk$ with respect to the decomposition $G(F) = Z(F)N(F)A(F)U$ where $da$ corresponds to the Haar measure on $F^\times$ such that $\mathcal{O}_F^\times$ has volume 1 and $dk$ is a measure on $\mathrm{GL}_2(\mathcal{O}_F)$. We normalize the measure such that the volume of $U_0(\pi^{c_2})$ is 1.

PROPOSITION 2.5.1. *Assume that either $\Pi_1$ or $\Pi_2$ is not special of prime conductor. For each $j$ between $0$ and $c_2-c_1$, one has*

$$\Psi\left(s, \rho\begin{pmatrix}\pi^{-j} & 0\\ 0 & 1\end{pmatrix} W_1, W_2, \Phi\right) = |\pi|^{j(s-1/2)}\alpha_j L(s, \Pi_1\times\Pi_2)$$

*where $\alpha_n$ is defined by*

$$L(s,\Pi_2) = \sum_n \alpha_n |\pi|^{ns}.$$

*Proof.* Using the decomposition $G(F) = Z(F)N(F)A(F)U$, we may write

$$\begin{aligned}
&\Psi\left(s, \rho\begin{pmatrix}\pi^{-j} & 0\\ 0 & 1\end{pmatrix} W_1, W_2, \Phi\right)\\
&= \int_{F^\times\times U} W_1\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} k \begin{pmatrix}\pi^{-j} & 0\\ 0 & 1\end{pmatrix}\right)\cdot\\
&\quad\cdot W_2\left(\begin{pmatrix}-a & 0\\ 0 & 1\end{pmatrix} k\right) f(s,\omega,k,\Phi)|a|^{s-1}dkd^\times a
\end{aligned}$$

If $c_2 = c_1 = j = 0$, then by definition of $\Phi$, one can show that for $k\in U$,

$$f(s,\omega,k,\Phi) = L(2s,\omega).$$

It follows that

$$\Psi(s,W_1,W_2,\Phi) = L(2s,\omega)\int_{F^\times} W_1\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right) W_2\left(\begin{pmatrix}-a & 0\\ 0 & 1\end{pmatrix}\right)|a|^{s-1}d^\times a.$$

The proposition now follows from the formula

$$\sum_n W\begin{pmatrix}\pi^n & 0\\ 0 & 1\end{pmatrix}|\pi|^{n(s-1/2)} = L(s,\Pi_i).$$

If $c_2 > c_1 + j$ then $\Phi = \Phi_1 + \Phi_2$ where $\Phi_1$ is the restriction of $\omega^{-1}$ on $\pi^{c_2}\mathcal{O}_F\times\mathcal{O}_F^\times$ while $\Phi_2$ is either zero or the characteristic function of $\pi^{c_2}\mathcal{O}_F\times\pi\mathcal{O}_F$. It is easy to see that $\Phi_2$ is invariant under $U_1(\pi^{c_2-1})$. Thus in the above formula, we may replace $\Phi$ by $\Phi_1$ since $\rho\begin{pmatrix}\pi^{-j} & 0\\ 0 & 1\end{pmatrix} W_1$ is invariant under $U_1(\pi^{c_2-1})$ while $\Pi_2$ has conductor $c_2$. Now for $k\in U$,

$$f(s,\omega,k,\Phi_1) = \begin{cases}\omega(k)^{-1} & \text{if } k\in U_0(\pi^{c_2}),\\ 0 & \text{otherwise,}\end{cases}$$

where for $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(\pi^{c_2})$ with $c_2 > 0$, $\omega(k)$ is defined to be $\omega(d)$.

It follows that

$$\begin{aligned}
&\Psi\left(s, \rho\begin{pmatrix} \pi^{-j} & 0 \\ 0 & 1 \end{pmatrix} W_1, W_2, \Phi\right) \\
=&\int_{F^\times} W_1\left(\begin{pmatrix} a\pi^{-j} & 0 \\ 0 & 1 \end{pmatrix}\right) W_2\left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s-1} d^\times a \\
=&|\pi|^{j(s-1)} \int_{F^\times} W_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) W_2\left(\begin{pmatrix} -a\pi^j & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s-1} d^\times a \\
=&|\pi|^{j(s-1/2)}\alpha_j \int_{F^\times} W_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) W_2\left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s-1} d^\times a \\
=&|\pi|^{j(s-1/2)}\alpha_j L(s, \Pi_1 \times \Pi_2).
\end{aligned}$$

Here we have used the fact that since $\Pi_2$ is ramified, $L(s, \Pi_2)$ is of degree 1 or 0 and $\alpha_i\alpha_j = \alpha_{i+j}$. It follows that for $a$ integral,

$$W_2\begin{pmatrix} -a\pi^j & 0 \\ 0 & 1 \end{pmatrix} = |\pi|^{j/2}\alpha_j W_2\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}.$$

It remains to treat the case where $c_2 = c_1 + j > 0$. If $\omega$ is ramified, then we may use the same method to compute as above. If $\omega$ is unramified then using the facts that

$$\begin{aligned}
W_1\left(g\begin{pmatrix} \pi^{-j} & 0 \\ 0 & 1 \end{pmatrix}\right) &= \mathrm{const} \cdot W_1\left(g\begin{pmatrix} 0 & 1 \\ -\pi^{c_2} & 0 \end{pmatrix}\right)\omega_1(\det g), \\
W_2\left(g\begin{pmatrix} 0 & 1 \\ -\pi^{c_2} & 0 \end{pmatrix}\right) &= \mathrm{const} \cdot \widetilde{W}_2(g)\omega_2(\det g),
\end{aligned}$$

where $\widetilde{W}_i$ is the standard Whittaker function for $\widetilde{\Pi}_i$, we have

$$\Psi\left(s, \rho\begin{pmatrix} \pi^{-j} & 0 \\ 0 & 1 \end{pmatrix} W_1, W_2, \Phi\right) = \mathrm{cont} \cdot \Psi\left(s, \widetilde{W}_1, \widetilde{W}_2, \rho\begin{pmatrix} 0 & 1 \\ -\pi^{c_2} & 0 \end{pmatrix}\Phi\right).$$

Since $\rho\begin{pmatrix} 0 & 1 \\ -\pi^{c_2} & 0 \end{pmatrix}\Phi$ is invariant under $\mathrm{GL}_2(\mathcal{O}_F)$, the integral must be zero by using the decomposition $G(F) = Z(F)N(F)A(F)U$. □

**Archimedean case.** Let $\Pi_i$ $(i = 1, 2)$ be two irreducible, admissible and infinite dimensional representations of $\mathrm{GL}_2(\mathbb{R})$. Then we can define the Rankin-Selberg convolution in the same manner as in the nonarchimdean case. In particular if one of $\Pi_i$ is principal, say $\Pi_2 \simeq \Pi(\mu_1, \mu_2)$ then one can show that

$$L(s, \Pi_1 \times \Pi_2) = L(s, \Pi_1 \otimes \mu_1)L(s, \Pi_1 \otimes \mu_2), \tag{2.5.15}$$

$$\epsilon(s, \Pi_1 \times \Pi_2) = \epsilon(s, \Pi_1 \otimes \mu_1, \psi)L(s, \Pi_1 \otimes \mu_2, \psi). \tag{2.5.16}$$

If both $\Pi_i$ are discrete, say $\Pi_i = \Pi(\chi_i)$, then one can show that

$$L(s, \Pi_1 \times \Pi_2) = L_{\mathbb{C}}(s, \chi_1 \otimes \chi_2)L_{\mathbb{C}}(s, \chi_1 \otimes \bar{\chi}_2) \tag{2.5.17}$$

$$\epsilon(s, \Pi_1 \times \Pi_2, \psi) = \epsilon_{\mathbb{C}}(s, \chi_1 \otimes \chi_2, \psi_{\mathbb{C}})\epsilon_{\mathbb{C}}(s, \chi_1 \otimes \bar{\chi}_2, \psi_{\mathbb{C}}). \tag{2.5.18}$$

Indeed, $\Pi_i$ corresponds to two representations of the Weil group $W_{\mathbb{R}}$: $\mathrm{Ind}(\chi_1)$ and $\mathrm{Ind}(\chi_2)$ by Langlands local correspondence. Thus $\Pi_1 \otimes \Pi_2$ corresponds to $\mathrm{Ind}(\chi_1) \otimes \mathrm{Ind}(\chi_2)$. The conclusion now follows from the fact that

$$\mathrm{Ind}(\chi_1) \otimes \mathrm{Ind}(\chi_2) \simeq \mathrm{Ind}\left(\chi_1 \otimes \mathrm{Ind}(\chi_2)|_{\mathbb{C}^\times}\right) \simeq \mathrm{Ind}(\chi_1 \otimes \chi_2 \oplus \chi_1 \otimes \bar{\chi}_2).$$

As in the nonarchimedean case, we want to express $L(s, \Pi_1 \times \Pi_2)$ as a canonical Mellin transform $\Psi(s, W_1, W_2, \Phi)$. For this, we assume the following

- *For each $i$, $\Pi_i$ is either discrete, or principal of type $\Pi(\alpha^{r_1}, \alpha^{r_2})$.*

Without loss of generality, we assume further that their weights $k_i$ satisfies $k_1 \geq k_2$. Then we have Whittaker functions $W_i$ of $\Pi_i$ of weights $k_1, -k_2$ such that

$$\int_{\mathbb{R}^\times} W_i \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} d^\times a = L(s, \Pi_i).$$

Moreover our assumption implies that the function

$$a \longrightarrow \phi_i(a) := W_i \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

is either even or supported on one connected component of $\mathbb{R}^\times$.

We fix a measure $dg$ on $N(\mathbb{R})Z(\mathbb{R})\backslash \mathrm{GL}_2(\mathbb{R})$ which is a product $|a|^{-1} dadk$ with respect to the decomposition $\mathrm{GL}_2(\mathbb{R}) = Z(\mathbb{R})N(\mathbb{R})A(\mathbb{R})\mathrm{SO}_2(\mathbb{R})$, where $da$ is induced by a usual measure on $\mathbb{R}^\times$, and $dk$ is such that $\mathrm{SO}_2(\mathbb{R})$ has volume 1.

PROPOSITION 2.5.2. *Assume that the conductor of $\psi$ is 1. Let $\Phi$ be the function in $\mathcal{S}(\mathbb{R}^2)$ defined by*

$$\Phi(x, y) = c(ix + y)^{n_1 - n_2} e^{-\pi(x^2 + y^2)},$$

*where $c$ is a positive constant:*

$$c = \begin{cases} 1 & \textit{if } \Pi_2 \textit{ are principal,} \\ 2^{k_2 - 1} & \textit{if } \Pi_2 \textit{ is discrete.} \end{cases}$$

*Then*

$$\Psi(s, W_1, W_2, \Phi) = L(s, \Pi_1 \times \Pi_2).$$

*Proof.* First we use the decomposition $G(F) = Z(F)N(F)A(F)U$ and the fact that $f_\Phi$ has weight $k_2 - k_1$. We may write

$$\Psi(s, W_1, W_2, \Phi) = c_0 \int_0^\infty \phi_1(a)\phi_2(-a) f_\Phi(s, \omega, e)|a|^{s-1} d^\times a$$

where $c_0 = 1$ unless both $\Pi_i$ are of weight 0. Otherwise $c_0 = 1$. Write $\omega(x) = |x|^r \mathrm{sgn}(x)^{k_1 - k_2}$, then

$$f_\Phi(s, \omega, e) = cG_2(2s + t + k_1 - k_2).$$

We need to compute the integral here. Write

$$G_1(s) = \pi^{-s/2}\Gamma(s/2), \qquad G_2(s) = 2(2\pi)^{-s}\Gamma(s) = G_1(s)G_1(s+1).$$

Then we have $\sigma_i$, $\tau_i$ such that

$$\int_0^\infty \phi_1(a)|a|^{s-1/2}d^\times a = c_1 L(s,\Pi_1) = c_1 G_1(s+\sigma_1)G_1(s+\sigma_2),$$

$$\int_0^\infty \phi_2(a)|a|^{s-1/2}d^\times a = c_2 L(s,\Pi_2) = c_2 G_1(s+\tau_1)G_1(s+\tau_2),$$

where $c_i = 1$ unless $\Pi_i$ is of weight 0. Otherwise $c_i = 1/2$. Now by Barnes lemma,

$$\int_0^\infty \phi_1(a)\phi_2(-a)|a|^{s-1}d^\times a = 2\frac{c_1c_2\prod_{i,j}G_1(s+\sigma_i+\tau_j)}{G_1(2s+\sigma_1+\sigma_2+\tau_1+\tau_2)}.$$

In summary we have

$$\Psi(s,W_1,W_2,\Phi) = 2cc_0c_1c_2\frac{G_1(2s+t+k_1-k_2)}{G_1(2s+\sigma_1+\sigma_2+\tau_1+\tau_2)}\prod_{i,j}G_1(s+\sigma_i+\tau_j).$$

We now want to check if the right hand side equals $L(s,\Pi_1\otimes\Pi_2)$.

First case: both $\Pi_i$ are principal of weight 0. We write

$$\Pi_1 = \Pi(\alpha_{\mathbb{R}}^{\sigma_1},\alpha_{\mathbb{R}}^{\sigma_2}),\qquad \Pi_2 = \Pi(\alpha_{\mathbb{R}}^{\tau_1},\alpha_{\mathbb{R}}^{\tau_2}),$$

then

$$t = \sigma_1+\sigma_2+\tau_1+\tau_2,\qquad L(s,\Pi_1\otimes\Pi_2) = \prod_{i,j}G_1(s+\sigma_i+\tau_j).$$

The identity follows.

Second case: $\Pi_1$ is discrete and $\Pi_2$ is principal. Then we may write

$$\Pi_1 = \Pi(\chi),\quad \chi(z) = |z|_{\mathbb{C}}^r z^m,\qquad \Pi_2 = \Pi(\alpha^{\tau_1},\alpha^{\tau_2}).$$

In this case

$$\sigma_1 = r+m,\quad \sigma_2 = r+m+1,\quad k_1 = m+1,\quad k_2 = 0,$$
$$t = 2r+m+\tau_1+\tau_2\quad L(s,\Pi_1\otimes\Pi_2) = \prod_{i,j}G_1(s+\sigma_i+\tau_j).$$

Again, the identity follows also.

Last case: both $\Pi_i$ are discrete. We write

$$\Pi_i = \Pi(\chi_i),\qquad \chi_i(z) = |z|_{\mathbb{C}}^{r_i}z_i^m,\quad (m_1\geq m_2).$$

Then

$$k_i = m_i+1,\quad \sigma_1 = r_1+m_1,\quad \sigma_2 = r_1+m_1+1,$$
$$\tau_1 = r_2+m_2,\quad \tau_2 = r_2+m_2+1,\quad t = 2r_1+2r_2+m_1+m_2,$$
$$L(s,\Pi_1\times\Pi_2) = G_2(s+r_1+r_2+m_1+m_2)G_2(s+r_1+r_2+m_1).$$

Equality now follows as we express everything in terms of $G_1(s+u)$ using the formula

$$G_1(2s) = 2^{s-1}G_2(s) = 2^{s-1}G_1(s)G_1(s+1)$$

□

If $\Pi_1$ is discrete and $\Pi_2 = \mathrm{sgn}\cdot\Pi_2'$ where $\Pi_2' = (\alpha^{r_1},\alpha^{r_2})$, then

$$L(s,\Pi_1\otimes\Pi_2) = L(s,\Pi_1\otimes\Pi_2')$$

Thus the proposition still works in this case.

Similarly, we may treat the case $\Pi_i = \mathrm{sgn}\cdot\Pi_i'$ of the above type.

**Global case.** Let $F$ be a totally real field with nontrivial character $\psi = \otimes\psi_v$ of $\mathbb{A}_F/F$. For $i = 1, 2$, let $\Pi_i$ be an irreducible and cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Then we can define the global Rankin-Selberg convolution L-function and $\epsilon$-factors:

$$L(s, \Pi_1 \times \Pi_2) = \prod_v L(s, \Pi_{1,v} \times \Pi_{2,v}), \tag{2.5.19}$$

$$\epsilon(s, \Pi_1 \times \Pi_2) = \prod_v \epsilon(s, \Pi_{1v} \times \Pi_{2v}, \psi_v). \tag{2.5.20}$$

Of course the definition of $\epsilon(s, \Pi_1 \times \Pi_2)$ does not depend on the choice of $\psi$ even if the local components $\epsilon(s, \Pi_{1v} \times \Pi_{2v}, \psi_v)$ do. One may show that the above product is absolutely convergent for $\mathrm{Re}(s) >> 0$ and $L(s, \Pi_1 \times \Pi_2)$ can be continued to a holomorphic function to the whole complex plane. Moreover, $L(s, \Pi_1 \times \Pi_2)$ satisfies an obvious functional equation:

$$L(s, \Pi_1 \times \Pi_2) = \epsilon(s, \Pi_1 \times \Pi_2) L(1 - s, \Pi_1 \times \Pi_2). \tag{2.5.21}$$

To prove the functional equation, one takes Whittaker functions $W_i(g) = \otimes_v W_{i,v}(g_v)$ for $\Pi_i$ with respect to $\psi$ and a function $\Phi = \otimes\Phi_v$ in $\mathcal{S}(\mathbb{A}^2)$ such that $\Psi(s, W_{1v}, W_{2v}, \Phi_v) \neq 0$ for every $v$. Let $\phi_i$ now be automorphic functions with Whittaker functions $W_i(g)$. Let $f_\Phi(s, g)$ denote a function on $\mathbb{C} \times G(\mathbb{A})$ defined by

$$f_\Phi(s, g) = \prod_v f_{\Phi_v}(s, g_v). \tag{2.5.22}$$

Then $f_\Phi(s, g)$ is invariant under the left multiplication by $B(F)$ and with character $\omega^{-1}$ under the action by the center $Z(\mathbb{A})$. Let $E(s, g)$ be an Eisenstein series defined by the following formula:

$$E_\Phi(s, g) = \sum_{\gamma \in B(F)\backslash G(F)} f_\Phi(s, \gamma g). \tag{2.5.23}$$

Then

$$\begin{aligned}
&\int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} \phi_1(g)\phi_2(g) E_\Phi(s, g) dg \\
=&\int_{Z(\mathbb{A})B(F)\backslash \mathrm{GL}_2(\mathbb{A})} \phi_1(g)\phi_2\left(\epsilon g\right) f_\Phi(s, g) dg \\
=&\int_{Z(\mathbb{A})N(F)\backslash \mathrm{GL}_2(\mathbb{A})} \phi_1(g) W_2\left(\epsilon g\right) f_\Phi(s, g) dg \\
=&\int_{Z(\mathbb{A})N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} W_1(g) W_2(\epsilon g) f_\Phi(s, g) dg \\
=&\Psi(s, W_1, W_2, \Phi),
\end{aligned}$$

where the measures $dg$ on $\mathrm{PGL}_2(\mathbb{A})$ and $N(\mathbb{A})Z(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})$ are chosen such that their "ratio" on $N(F)\backslash N(\mathbb{A})$ has volume 1. The functional equation now follows from the local equations and the functional equation for Eisenstein series:

$$E_\Phi(s, g, \omega) = \omega(\det g) E_{\widetilde{\Phi}}(1 - s, g, \omega^{-1}). \tag{2.5.24}$$

Let $\delta \in \mathbb{A}^\times$ such that the character $\psi_v^0(x) := \psi(\delta_v^{-1}x)$ of $F_v$ has conductor 1 at every place. Assume that with respect to $\psi^0$ there are Whittaker functions $W_i^0$ of $\Pi_i$ and a function $\Phi \in \mathcal{S}(\mathbb{A}^2)$ such that

$$L(s, \Pi_1 \times \Pi_2) = \Psi(s, W_1^0, W_2^0, \Phi),$$

such as the selected cases we have treated in last two sections. Then if we define

$$W_i(g) = W_i^0 \left( \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

It follows that

$$\Psi(s, W_1, W_2, \Phi) = |\delta|^{1/2-s} \Psi(s, W_1^0, W_2^0, \Phi).$$

In other words, if we take $\phi_i$ with Whittaker functions $W_i$, then we have the simple expression for the Rankin L-function:

$$L(s, \Pi_1 \times \Pi_2) = |\delta|^{s-1/2} \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \phi_1(g)\phi_2(g)E_\Phi(s, g)dg. \tag{2.5.25}$$

# 3. Kernel functions

In this chapter we will study the kernel function for certain Rankin-Selberg convolutions. More precisely, we will first construct a kernel $\Theta(s, g)$ as described in the end of §1.1. This kernel depends only on the character $\chi$ and the type of $\phi$ but is not be unique. We choose the simplest one so that *a functional equation holds*. Then we compute the central value, the central derivative, and the holomophic projection. These procedures are quite close to those used by Gross and Zagier [20].

The important difference is that we will not take the trace to the same level as $\phi$. Actually some experimental computation shows that the trace is so complicated that there is no way to compare with the geometric pairing. Of course, there will be some problems created by high levels if we don't take trace. But this can be taken care of by our new notion of *quasi-newforms* in §2.3. On the other hand, since no trace needed, this method has better flexibility than [20]. For example even in the classical case $F = \mathbb{Q}$, $\chi = 1$, our method works for even discriminant $D$.

## 3.1. Kernel functions

We now start with our basic setting as in §1.1. Let $F$ be a totally real field. Let $\Pi$ be an irreducible and cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ with trivial central character, and conductor $N$. Assume that at each archimedean place $\Pi$ is either principal, or discrete of weight 2.

Let $K/F$ be a totally imaginary quadratic extension. Let $\omega$ denote the associated quadratic character of $\mathbb{A}^\times$ with conductor $c(\omega)$. Let $\chi$ be a finite character of $\mathbb{A}_K^\times/\mathbb{A}^\times K^\times$ whose conductor $c(\chi)$ is prime to $c(\omega)$. Let $\Pi(\chi)$ be the induced irreducible representation of $\mathrm{GL}_2(\mathbb{A})$. Then $\Pi(\chi)$ has weight $(1, \cdots, 1)$, level $D = c(\chi)^2 c(\omega)$, and central character $\omega$.

**Epsilon-factors.** Assume that for $\wp \mid D$, $\mathrm{ord}_\wp(N) \le 1$. (For applications we need only assume this after both $\Pi_\wp$ and $\Pi(\chi)_\wp$ are twisted by quadratic characters at $\wp$.) Let $\psi = \otimes\psi_v$ be a nontrivial character of $\mathbb{A}_F/F$. The $\epsilon$-factor is given as follows:

$$\epsilon_v(\Pi_1 \times \Pi_2, \psi) = \begin{cases} -\omega_v(-1) & \text{if } v \in \Sigma, \\ \omega_v(-1) & \text{if } v \notin \Sigma, \end{cases} \tag{3.1.1}$$

where

$$\Sigma : \quad \begin{cases} \text{infinite places where } \Pi \text{ has weight 2,} \\ \text{finite places } \wp \nmid D \text{ such that } \omega_\wp(N) = -1, \\ \text{finite places } \wp \mid (N, c(\omega)) \text{ such that } \mu_\wp\nu_\wp(\pi) = 1, \\ \qquad \text{where } \Pi_\wp = \Pi_\wp(\mu_\wp),\ \chi_\wp = \nu_\wp \circ \mathrm{N}_{K_\wp/F_\wp}. \end{cases} \tag{3.1.2}$$

Notice that in the last case of the above list, $\mu_\wp(\pi)$ and $\nu_\wp(\pi)$ are actually the parameters of the local L-functions of $\Pi_\wp$ and $\chi_\wp$:

$$L(s, \Pi_\wp) = \frac{1}{1 - \mu_\wp(\pi)|\pi|^s}, \qquad L(s, \chi_\wp) = \frac{1}{1 - \nu_\wp(\pi)|\pi|^s}.$$

**Kernel $\Theta_T$.** We now want to apply §2.5 to $\Pi_1 = \Pi$, $\Pi_2 = \Pi(\chi)$. We write

$$L(s, \Pi_1 \times \Pi_2) =: L(s, \Pi \otimes \chi). \tag{3.1.3}$$

Let $\phi$ be the newform for $\Pi$, and let $\theta_\chi$ be the newform for $\Pi(\chi)$ defined in (2.4.7). Then

$$L(s, \Pi \otimes \chi) = |\delta|^{s-1/2} \int \phi(g)\theta_\chi(g)E(s, g)dg \tag{3.1.4}$$

where $\delta \in \mathbb{A}^\times$ is the conductor of any fixed additive character. Thus $|\delta|^{-1}$ is the discriminant $d$ of $F$.

Let $S$ be the set of places dividing $c(\omega)$. For each $v \in S$, fix a uniformizer $\pi_v$ such that $\omega_v(\pi_v) = 1$. For each subset $T$ of $S$, let $h_T$ denote the Atkin-Lehner operator of level $c(\omega)$: an element in $G(\mathbb{A})$ which has component 1 outside of $T$, and has component

$$h_v := \begin{pmatrix} 0 & 1 \\ -\pi_v^{o(\omega_v)} & 0 \end{pmatrix} \tag{3.1.5}$$

at $v \in T$, and let $\pi_T^*$ denote the idele which has component 1 out side $T$ and has elements

$$\pi_v^* := \pi_v^{o(\omega_v) - o(\Pi_v)} \tag{3.1.6}$$

at $v$. Also we define

$$\gamma_T(s) = \prod_{v \in T} \gamma_v(s), \tag{3.1.7}$$

$$\gamma_v(s) = \nu_v(\pi_v)^{o(\omega_v)}|\pi_v^*|^{1/2-s} \cdot (-1)^{\#\{v\}\cap\Sigma}.$$

Set

$$\Theta_T(s,g)=\gamma_T(s)\theta_\chi(gh_T^{-1})E(s,gh_T^{-1}). \tag{3.1.8}$$

LEMMA 3.1.1. *For any integral idele $a$ dividing $[N,D]/N=D/(N,D)$, one has*

$$\begin{aligned}&|\delta|^{s-1/2}\int\phi\left(g\begin{pmatrix}a^{-1}&0\\0&1\end{pmatrix}\right)\Theta_T(s,g)dg\\=&|a_{S-T}/a_T|^{s-1/2}\nu^*(a)L(s,\Pi\otimes\chi),\end{aligned}$$

*where*

$$\nu^*(a)=\begin{cases}\nu(a) & \text{if } a|c(\omega),\\ 0 & \text{otherwise.}\end{cases}$$

*Proof.* Indeed, let $h_T^0$ be the Atkin-Lehner operator of level $c(\Pi)$ over places over $T$, then by Proposition 2.5.1,

$$\begin{aligned}&\int\phi\left(g\begin{pmatrix}a^{-1}&0\\0&1\end{pmatrix}\right)\theta_\chi(gh_T)E(s,gh_T)dg\\=&\int\phi\left(gh_T^{-1}\begin{pmatrix}a^{-1}&0\\0&1\end{pmatrix}\right)\theta_\chi(g)E(s,g)dg\\=&\epsilon(1/2,\Pi_T)\int\phi\left(gh_T^{-1}\begin{pmatrix}a^{-1}&0\\0&1\end{pmatrix}h_T^0\right)\theta_\chi(g)E(s,g)dg\\=&\epsilon(1/2,\Pi_T)\nu(\pi_T^*)|\pi_T^*\cdot a_{S-T}/a_T|^{s-1/2}c_T(a)L(s,\Pi\otimes\chi),\end{aligned}$$

The conclusion now follows from the fact that

$$\epsilon(1/2,\Pi_v)=\begin{cases}1 & \text{if } \Pi_v \text{ is unramified,}\\ -\mu_v(\pi_v) & \text{if } \pi_v=\sigma(\mu_v).\end{cases}$$

□

**Kernel $\Theta$.** We define a *kernel function* by

$$\Theta(s,g)=2^{-|S|}|\delta|^{s-1/2}\sum_{T\subset S}\Theta_T(s,g). \tag{3.1.9}$$

Then

$$L(s,\Pi\otimes\chi)=\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})}\phi(g)\Theta(s,g)dg. \tag{3.1.10}$$

Notice that $\phi$ has level $N$ but $\Theta$ has level $[N,D]$. By Lemma 3.1.1, we have:

LEMMA 3.1.2. *For any integral idele $a$ dividing $[N,D]/N=D/(N,D)$, one has*

$$\begin{aligned}&\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})}\phi\left(g\begin{pmatrix}a^{-1}&0\\0&1\end{pmatrix}\right)\Theta(s,g)dg\\=&\prod_{v|S}\frac{|a|_v^{s-1/2}+|a|_v^{1/2-s}}{2}\cdot\nu^*(a)L(s,\Pi\otimes\chi).\end{aligned}$$

The advantage of using $\Theta$ instead of $\Theta_T$ is that it has more symmetry. Actually from Lemma 3.1.2, one sees that the projection of $\Theta(s,g)$ on the space $\Pi$ should have the same functional equation as $L(s,\Pi\otimes\chi)$. We will show this functional equation in the next section.

But now let us give an important definition to describe this projection.

DEFINITION 3.1.3. *The quasi-new form $\phi_s^\sharp$ is defined to be the unique* quasi-newform *of level $[N,D]$ propotional to the projection of $\bar{\Theta}(s,g)$. In other words, $\phi_s^\sharp$ is perpendicular to the following hyperplane which is the orthogonal complement of $\bar{\Theta}(s,g)$ on the subspace of forms in $\Pi$ of level $[N,D]$:*

$$\left\{\sum_{a\left|\frac{D}{(N,D)}\right.} c_a\phi\left(g\begin{pmatrix}a^{-1} & 0\\ 0 & 1\end{pmatrix}\right): \qquad \sum c_a\nu_s^*(a)=0,\right\}$$

*where*

$$\nu^*(a)_s=\prod_{v|S}\frac{|a|_v^{s-1/2}+|a|_v^{1/2-s}}{2}\begin{cases}\nu(a) & \text{if } a|c(\omega),\\ 0 & \text{otherwise.}\end{cases}$$

*Write $\phi^\sharp=\phi^\sharp_{1/2}$.*

By Lemma 3.1.2, we have

PROPOSITION 3.1.4. *The projection of $\bar{\Theta}(s,g)$ on $\Pi$ is given by*

$$\frac{L(s,\Pi\otimes\chi)}{(\phi_s^\sharp,\phi_s^\sharp)}\cdot\phi_s^\sharp.$$

### 3.2. Functional equation

In this section we want to show the functional equation of the kernel function constructed in the last section:

THEOREM 3.2.1.

$$\Theta(s,g)=\epsilon(s,\Pi\otimes\chi)\Theta(1-s,g),$$

*where*

$$\epsilon(s,\Pi\otimes\chi)=(-1)^{\#\Sigma}|c(\psi)^{-4}c(\Pi\otimes\chi)|^{s-1/2}$$

*and*

$$c(\Pi\otimes\chi)=[N,D]^2(N,c(\omega)).$$

By Lemma 3.1.2, this gives a new proof of the following functional equation of Rankin-Selberg L-functions without using the local equations.

THEOREM 3.2.2.

$$L(s,\Pi\otimes\chi)=\epsilon(s,\Pi\otimes\chi)L(1-s,\Pi\otimes\chi).$$

The main idea of the proof is to use the functional equation of the Eisenstein series

$$E_\Phi(s,g) = \omega(\det g)E_{\widetilde{\Phi}}(1-s,g) \tag{3.2.1}$$

and a precise computation of $\widetilde{\Phi}$. Notice that $\Phi$ is a product of local $\Phi_v$ in $\mathcal{S}(F_v^2)$. Thus we will compute $\widetilde{\Phi}_v$ case by case.

LEMMA 3.2.3. *Let $\delta_v \in F_v^\times$ such that $\psi_v^0(x) := \psi(\delta_v^{-1}x)$ is of order $0$.*

1. *For a finite place $v$,*

$$\widetilde{\Phi}_v(x,y) = |\delta_v\pi_v^{c_v}|\Phi\left[(x,y)\delta_v\pi_v^{c_v}\right],$$

*if $\omega_v$ is unramified, and*

$$\widetilde{\Phi}_v(x,y) = |\delta_v\pi_v^{3c_v/2}|\epsilon(\omega,\psi^0)\Phi_v\left[-\pi^{c_v}\delta_v(x,y)h_v\right]$$

*if $\omega_v$ is ramified, where $\pi_v$ is a fixed local parameter such that $\omega_v(\pi_v)=1$, and $c_v = \mathrm{ord}_v([N,D])$, and*

$$h_v = \begin{pmatrix} 0 & 1 \\ -\pi_v^{c_v} & 0\end{pmatrix},$$

2. *For an archimedean place $v$,*

$$\widetilde{\Phi}_v(x,y) = -|\delta_v|\Phi(x\delta_v, y\delta_v).$$

*Proof.* Let $\widetilde{\Phi}_v^0$ denote the Fourier transform with respect to $\psi_v^0$. Then

$$\widetilde{\Phi}_v(x,y) = |\delta_v|\widetilde{\Phi}_v^0(x\delta_v, y\delta_v).$$

Let assume that $v$ is nonarchimedean first. For each character $\mu$ of $F_v^\times$ define

$$\Phi_\mu(x) = \begin{cases} 1 & \text{if } c(\mu)=0,\ |x|\le 1,\\ \mu^{-1}(x) & \text{if } c(\mu)>0,\ |x|=1,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\Phi_v(x,y) = \Phi_{1_v}(x\pi_v^{-c_v})\Phi_{\omega_v}(y)$$

where $1_v$ denote the trivial character of $F_v^\times$. It follows that

$$\widetilde{\Phi}_v^0(x,y) = |\pi_v|^{c_v}\widehat{\Phi}_{1_v}^0(y\pi_v^{c_v})\widehat{\Phi}_\omega^0(-x).$$

Notice that for a general character $\mu$ of $F^\times$,

$$\widehat{\Phi}_\mu^0(x) = |\pi_v|^{o(\mu)/2}\mu^{-1}(\pi_v^{o(\mu)})\epsilon(\mu,\psi^0)\Phi_{\mu^{-1}}(x\pi^{o(\mu)}).$$

It follows that

$$\begin{aligned}\widetilde{\Phi}_v(x,y) &= |\delta_v|\widetilde{\Phi}_v^0(x\delta_v,y\delta_v)\\ &= |\delta_v\pi_v^{c_v}|\widehat{\Phi}_{1_v}^0(y\pi_v^{c_v}\delta_v)\widehat{\Phi}_\omega^0(-x\delta_v)\\ &= |\delta_v\pi_v^{c_v+o(\omega)/2}|\omega(\pi_v^{o(\omega)})\epsilon(\omega,\psi^0)\Phi_{1_v}(y\pi_v^{c_v}\delta_v)\Phi_\omega(-x\pi^{o(\omega)}\delta_v).\end{aligned}$$

If $\omega_v$ is unramified, then

$$\begin{aligned}\widetilde{\Phi}_v(x,y) =&|\delta_v\pi_v^{c_v}|\Phi_{1_v}(y\pi_v^{c_v}\delta_v)\Phi_{1_v}(x\delta_v)\\ =&|\delta_v\pi_v^{c_v}|\Phi\left[(x,y)\delta_v\pi_v^{c_v}\right].\end{aligned}$$

If $\omega_v$ is ramified, then $c_v = o(\omega_v)$, and

$$\begin{aligned}\widetilde{\Phi}_v(x,y) &= |\delta_v\pi_v^{3c_v/2}|\omega(\pi_v^{c_v})\epsilon(\omega,\psi^0)\Phi_v(y\pi_v^{2c_v}\delta_v, -x\pi^{c_v}\delta_v)\\ &= |\delta_v\pi_v^{3c_v/2}|\omega(\pi_v^{c_v})\epsilon(\omega,\psi^0)\Phi_v\left[-\pi^{c_v}\delta_v(x,y)h_v\right].\end{aligned}$$

It remains to consider the case where $v$ is archimedean. In this case

$$\begin{aligned}\Phi_v(x,y) &= (ix+y)e^{-\pi(x^2+y^2)},\\ \widetilde{\Phi}_v^0(x,y) &= -\Phi_v(x,y).\end{aligned}$$

□

Lets now find a functional equation for $f_\Phi(s,g)$. By definition,

$$f_{\widetilde{\Phi}_v}(1-s,g) = |\det g|^{1-s}\int_{F_v^\times}\widetilde{\Phi}_v[(0,t)g]|t|^{2(1-s)}\omega(t)d^\times t.$$

Lemma 3.2.4. *Write $f_v(s,g)$ (resp. $\widetilde{f}_v(s,g)$) for $f_{\Phi_v}(s,g)$ (resp. $f_{\widetilde{\Phi}_v}(s,g)$). Let $\beta_v(s)$ denote the function*

$$\beta_v(s) = \begin{cases}|\pi_v^{c_v}|^{2s-1}\omega_v(\pi_v^{c_v}) & \text{if } v\nmid\infty,\ o(\omega_v)=0,\\ |\pi_v^{3c_v}|^{s-1/2}\epsilon(\omega_v,\psi^0) & \text{if } v\nmid\infty,\ o(\omega_v)>0,\\ 1 & \text{if } v\mid\infty.\end{cases}$$

*Then:*

$$\widetilde{f}_v(1-s,g) = \begin{cases}|\delta_v|^{2s-1}\omega(-\delta_v)\beta_v(s)f_v(1-s,gh_v). & \text{if } v\nmid\infty,\ o(\omega_v)>0,\\ |\delta|^{2s-1}\omega_v(-\delta_v)\beta_v(s)f_v(1-s,g) & \text{otherwise.}\end{cases}$$

**Proof of Theorem 3.2.1.** Lets write $S$ (resp. $\bar{S}$) for finite places where $\omega_v$ is ramified (resp. unramified). Let $\beta(s)$ be the product of $\beta_v(s)$, then

$$f_{\widetilde{\Phi}}(1-s,g) = |\delta|^{2s-1}\omega(\delta)\beta(s)f(1-s,gh_S),$$

$$E(s,g) = \omega(\det g)E_{\widetilde{\Phi}}(1-s,g) = |\delta|^{2s-1}\omega(\delta\det g)\beta(s)E(1-s,gh_S).$$

It follows that

$$\begin{aligned}\Theta(s,g) =&2^{-\#S}|\delta|^{s-1/2}\sum_{T\subset S}\gamma_T(s)\theta_\chi(gh_T^{-1})E(s,gh_T^{-1})\\ =&2^{-\#S}|\delta|^{3s-3/2}\omega(\delta\det g))\sum_{T\subset S}\gamma_T'(s)\theta_\chi(gh_T^{-1})E(1-s,gh_T^{-1}h_S)\\ =&2^{-\#S}|\delta|^{3s-3/2}\omega(\delta\det g)\sum_{T\subset S}\gamma_{\bar{T}}'(s)\theta_\chi(gh_Th_S^{-1})E(1-s,gh_T),\end{aligned}$$

where $\bar{T} = S \setminus T$,

$$\gamma'_T(s) = \beta(s)\gamma_T(s).$$

Recall that $\theta_\chi$ is a form whose Whittaker function is

$$W_\chi(g) := W_\chi^0\left(\begin{pmatrix}\delta & 0\\ 0 & 1\end{pmatrix} g\right),$$

where

$$W_\chi^0(g) = \prod_v W_{\chi,v}(g_v),$$

with $W_{\chi,v}$ a newform in $\mathcal{W}(\Pi(\chi_v), \psi_v^0)$ unless $v$ is infinite and $\Pi_v$ is of weight 2. If $v$ is infinite and $\Pi_v$ is of weight 2, $\rho(\epsilon)W_{\chi,v}^0(g)$ is a newform of $\Pi_v$ where $\epsilon = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$.

LEMMA 3.2.5. *Let $\infty^+$ (resp. $\infty^-$ denote the archimedean places where $\Pi_v$ is weight 2 (resp. 0). Then*

$$W_{\chi,v}^0(g) = W_{\chi,v}(g)^0\omega_v(\det g) \qquad \forall v \in \bar{S} \cup \infty^-,$$
$$W_{\chi,v}^0(gh_v^{-1}) = \omega_v(-\pi_v^{c_v}\det g)\nu_v(\pi_v^{c_v})\epsilon(\omega_v, \psi_v^0)W_{\chi,v}^0(g) \qquad \forall v \in S,$$
$$W_{\chi,v}^0(g) = W_{\chi,v}^0(g)\omega_v(-\det g), \qquad \forall v \in \infty^+.$$

*Proof.* The first equality is true because both sides are newforms for $\Pi_v = \Pi_v \otimes \omega_v$. The second one follows from our Atkin-Lehner theory in §2.3 and the fact that

$$\epsilon(\Pi(\chi)_v, \psi_v^0) = \epsilon(\nu_v, \psi_v^0)\epsilon(\nu_v \cdot \omega_v, \psi_v^0) = \nu_v(\pi_v^{c_v})\epsilon(\omega_v, \psi_v^0).$$

The last one is true because both sides are newforms after $g$ is replaced by $g\epsilon$. □

By this lemma, we have the following functional equation of theta series:

$$\theta_\chi(gh_S^{-1}) = \theta_\chi(g)\cdot\omega(\delta\det g)(-1)^{\#\infty^-}\alpha,$$

where

$$\alpha = \prod_{v\in S}\alpha_v, \qquad \alpha_v = \nu_v(\pi_v^{c_v})\epsilon(\omega_v, \psi_v^0).$$

It follows that

$$\Theta(s,g) = 2^{-\#S}|\delta|^{3s-3/2}\sum_{T\subset S}\gamma_T^*(s)\theta_\chi(gh_T)E(1-s, gh_T),$$

where

$$\begin{aligned}\gamma_T^*(s) =&\beta(s)\cdot\gamma_{\bar{T}}(s)\cdot(-1)^{\#\infty^-}\cdot\alpha\\ =&\prod_{v\in\bar{S}}|\pi_v^{c_v}|^{2s-1}\cdot\prod_{v\in T}|\pi_v^{3c_v}|^{s-1/2}\nu_v(\pi_v^{c_v})(-1)^{\#T\cap\Sigma}\\ &\cdot\prod_{v\in\bar{T}}|\pi_v^{2c_v+o(\Pi_v)}|^{s-1/2}\cdot(-1)^{\#\Sigma}\\ =&(-1)^{\#\Sigma}\prod_{v\in\bar{S}}|\pi_v^{2c_v}|^{s-1/2}\cdot\prod_{v\in S}|\pi_v^{2c_v+o(\Pi_v)}|^{s-1/2}\gamma_T(1-s).\end{aligned}$$

Theorem 3.2.1 now follows easily.

### 3.3. Fourier expansion

In this paper we will study the Fourier expansion of $\Theta$ in great detail, i.e., the constant term $C_\chi(s,g)$ and the Whittaker function $W_\chi(s,g)$. Let

$$\theta_\chi(g) = C_\chi(g) + \sum_{\eta\in F^\times} W_\chi\left(\begin{pmatrix}\eta & 0\\ 0 & 1\end{pmatrix} g\right), \tag{3.3.1}$$

$$E(s,g) = C(s,g) + \sum_{\xi\in F^\times} W\left(s, \begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} g\right), \tag{3.3.2}$$

be Fourier expansions of $\theta_\chi$ and $E(s,g)$ respectively. Then $\Theta(s,g)$ will have Fourier expansion

$$\Theta(s,g) = C_\chi(s,g) + \sum_{\alpha\in F^\times} W_\chi\left(s, \begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} g\right), \tag{3.3.3}$$

with

$$C_\chi(s,g) = \sum_{\substack{\xi+\eta=0\\ \xi,\eta\in F^\times}} W(s,\xi,\eta,g) + C_\chi^*(s,g), \tag{3.3.4}$$

$$W_\chi(s,g) = \sum_{\substack{\xi+\eta=1\\ \xi,\eta\in F^\times}} W(s,\xi,\eta,g) + W_\chi^*(s,g), \tag{3.3.5}$$

where

$$\begin{aligned} W(s,\xi,\eta,g) = 2^{-\#S}|\delta|^{s-1/2}\sum_{T\subset S}\gamma_T(s) W_\chi\left(\begin{pmatrix}\eta & 0\\ 0 & 1\end{pmatrix} gh_T^{-1}\right)\\ \cdot W\left(s, \begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} gh_T^{-1}\right), \end{aligned} \tag{3.3.6}$$

$$C_\chi^*(s,g) = 2^{-\#S}|\delta|^{s-1/2}\sum_{T\subset S}\gamma_T(s) C_\chi(gh_T^{-1}) C(s, gh_T^{-1}), \tag{3.3.7}$$

$$\begin{aligned} W_\chi^*(s,g) = 2^{-\#S}|\delta|^{s-1/2}\sum_{T\subset S}\gamma_T(s)\big(C_\chi(gh_T^{-1}) W(s, gh_T^{-1})\\ + W_\chi(g) C(s, gh_T^{-1})\big). \end{aligned} \tag{3.3.8}$$

Notice that $W(s,\xi,\eta,g)$, $C_\chi^*(s,g)$, and $W_\chi^*(s,g)$ share the same function equation as $L(s,\Pi\otimes\chi)$ by the same argument as above, since the functional equation of $E(s,g)$ and $\theta_\chi(g)$ will give the same functional equations to each term of their Fourier expansions.

In the following we want to compute the Fourier expansion explicitly for $g$ of the form $\begin{pmatrix}a\delta^{-1} & 0\\ 0 & 1\end{pmatrix}$. But first we need to compute them for $E(s, gh_T^{-1})$ and $\theta_\chi(gh_T^{-1})$.

**Fourier expansion of Eisenstein series.** Lets first compute the constant term $C(s,g)$ Using decomposition

$$\mathrm{GL}_2(F) = P(F)\coprod P(F)wN(F), \qquad w = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \tag{3.3.9}$$

one may compute the Fourier expansion with respect to $\psi$ to obtain

$$(3.3.10)\qquad \begin{aligned} C(s,g) &= \int_{F\backslash\mathbb{A}} E\left(s, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx = f(g) + \int_{\mathbb{A}} f\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx \\ &= f(s,g) + \omega(\det g)\widetilde{f}(s,g) \end{aligned}$$

Since $f(s,g)$ and $\widetilde{f}(1-s,g)$ are in the space of principal series $\mathcal{B}(\alpha^{s-1/2}, \alpha^{1/2-s}\omega)$, we have the following

LEMMA 3.3.1.

$$C\left(\begin{pmatrix} a\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_T^{-1}\right) = |a|^s f_T(s) + |a|^{1-s}\omega(a)\widetilde{f}_T(s),$$

*where*

$$f_T(s) = |\delta|^{s-1/2} f(s, h_T^{-1}), \qquad \widetilde{f}_T(s) = |\delta|^{1/2-s}\omega(\delta)\widetilde{f}(s, h_T^{-1}).$$

Lets now compute the Whittaker function.

$$\begin{aligned} W(s,g) &= \int_{F\backslash\mathbb{A}} E\left(s, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \psi(-x)dx = \int_{\mathbb{A}} f\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-x)dx \\ &= |\det g|^s \int_{\mathbb{A}^\times} |t|^{2s}\omega(t)dt^\times \int_{\mathbb{A}} \Phi[(-t, -tx)g]\psi(-x)dx \\ &= |\det g|^s \int_{\mathbb{A}^\times} (\rho(g)\Phi)'(t, t^{-1})|t|^{2s-1}\omega(t)dt^\times, \end{aligned}$$

where $\Phi'$ is the partial inverse Fourier transform:

$$\Phi'(x,y) = \int_{F_v} \Phi(x,u)\psi_v(-uy)du.$$

For each place $v$, write $W_v(s,g)$ using the same formula in local integrals. In the following we want to compute $f_v(g)$ case by case for $g = \begin{pmatrix} a\delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ or $g = \begin{pmatrix} a\delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_v^{-1}$.

LEMMA 3.3.2. *Assume that $v$ is finite and $\omega_v$ is unramified. For $a \in F_v^\times$, $W\left(s, \begin{pmatrix} a\delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \neq 0$ only if $|a| \le |\pi_v^{c_v}|$. In this case it is given by*

$$|a|^{1/2}|\pi_v^{c_v}\delta_v|^{s-1/2}\omega_v(\delta_v)\frac{|a\pi_v^{1-c_v}|^{s-1/2} - |a\pi_v^{1-c_v}|^{1/2-s}\omega(a\pi_v^{1-c_v})}{|\pi_v|^{s-1/2} - |\pi|^{1/2-s}\omega(\pi_v)}.$$

*Proof.* Recall that $\Phi_v(x,y)$ is given by

$$\Phi_v(x,y) = \Phi_{1_v}(\pi_v^{-c_v}x)\Phi_{1_v}(y)$$

where $1_v$ denotes the trivial character of $F_v^\times$.

$$\begin{aligned}
W\left(s, \begin{pmatrix} a\delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) =& |a\delta_v^{-1}|^s \int_{F_v^\times} \Phi'(a\delta_v^{-1}t, t^{-1})|t|^{2s-1}\omega(t)d^\times t \\
=& |a|^s \int_{F_v^\times} |\delta_v|^{1/2-s}\Phi_{1_v}(a\delta_v^{-1}t\pi_v^{-c_v})\Phi_{1_v}(t^{-1}\delta_v)|t|^{2s-1}\omega(t)d^\times t \\
=& |a|^s|\delta_v|^{1/2-s}\int_{|\pi_v^{c_v}a^{-1}\delta_v|\geq|t|\geq|\delta_v|} |t|^{2s-1}\omega(t)d^\times t \\
=& |a|^s|\delta_v|^{s-1/2}\omega(\delta_v)\sum_{i=0}^{\mathrm{ord}_v(a)-c_v} |\pi^i|^{1-2s}\omega(\pi^i),
\end{aligned}$$

where the sum is zero if $c_v > \mathrm{ord}_v(a)$. □

LEMMA 3.3.3. *Assume that $v$ is finite and $\omega_v$ is ramified.*

1. *For $a \in F_v^\times$, $W\left(s, \begin{pmatrix} a\delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \neq 0$ only if $|a| \leq 1$. In this case it is given by*

$$|a|^s|\delta_v|^{s-1/2}\omega(-\delta_v)|\pi_v^{c_v}|^{2s-1/2}\epsilon(\omega_v, \psi_v^0).$$

2. *For $a \in F_v^\times$, $W\left(s, \begin{pmatrix} a\delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_v^{-1}\right) \neq 0$ only if $|a| \leq 1$. In this case it is given by*

$$|a|^{1-s}|\delta_v|^{s-1/2}\omega(-a\delta_v)|\pi_v^{c_v}|^s.$$

*Proof.* Again, we know that

$$\Phi_v(x, y) = \Phi_{1_v}(\pi_v^{-c_v}x)\Phi_{\omega_v}(y).$$

The Fourier transform of $\Phi_{\omega_v}$ with respect to the unramified character $\psi_v^0$ is given by

$$|\pi_v^{c_v}|^{1/2}\epsilon(\omega_v, \psi^0)\Phi_{\omega_v}(x\pi_v^{c_v}).$$

It follows that

$$\begin{aligned}
& W\left(s, \begin{pmatrix} a\delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \\
=& |a\delta_v^{-1}|^s \int_{F_v^\times} \Phi'(a\delta_v^{-1}t, t^{-1})|t|^{2s-1}\omega(t)d^\times t \\
=& |a|^s \int_{F_v^\times} |\delta_v|^{1/2-s}|\pi_v|^{c_v/2}\epsilon(\omega_v, \psi_v^0)\Phi_{1_v}(a\delta_v^{-1}t\pi_v^{-c_v})\Phi_{\omega_v}(-t^{-1}\delta_v\pi_v^{c_v})|t|^{2s-1}\omega(t)d^\times t \\
=& |\delta_v|^{s-1/2}\omega(-\delta_v)|\pi_v^{c_v}|^{2s-1/2}\epsilon(\omega_v, \psi_v^0)\Phi_{1_v}(a)|a|^s.
\end{aligned}$$

This proves the first part of the lemma. For the second part, we notice that

$$\Phi\left[(x, y)\begin{pmatrix} a\delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_v^{-1}\right] = \Phi(y, -a\delta_v^{-1}\pi_v^{-c_v}x).$$

It follows that

$$\begin{aligned}
&W\left(s, \begin{pmatrix} a\delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_v^{-1}\right) \\
=&|a\delta_v^{-1}\pi_v^{-c_v}|^s \int_{F_v^\times} \Phi'(a\delta_v^{-1}t, t^{-1})|t|^{2s-1}\omega(t)d^\times t \\
=&|a|^s \int_{F_v^\times} |\delta_v|^{1/2-s}|\pi_v^{c_v}|^{1-s}\Phi_1(\pi_v^{c_v}\delta_v t^{-1})\Phi_{\omega_v}(-a\delta_v^{-1}\pi_v^{-c_v}t)|t|^{2s-1}\omega(t)d^\times t \\
=&|a|^{1-s}|\delta_v|^{s-1/2}\omega(-a\delta_v)|\pi_v^{c_v}|^s\Phi_{1_v}(a).
\end{aligned}$$

□

Lemma 3.3.4. *Assume that $F_v = \mathbb{R}$, then*

$$\begin{aligned}
W\left(s, \begin{pmatrix} a\delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) =&|\delta_v|^{s-1/2}\omega(-a\delta_v)\frac{\Gamma(s+1/2)}{\pi^{s+1/2}} \\
&\cdot |a|^{1-s} \cdot \int_{\mathbb{R}} \frac{e^{2\pi i a x}}{(i+x)(1+x^2)^{s-1/2}}dx.
\end{aligned}$$

*Proof.* In this case,

$$\Phi_v(x,y) = (ix+y)e^{-\pi(x^2+y^2)}.$$

First change the order of the Fourier transform and Mellin transform:

$$\begin{aligned}
&W\left(s, \begin{pmatrix} a\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \\
=&|a\delta^{-1}|^s \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \Phi(a\delta^{-1}t, x)e^{-2\pi i t^{-1}\delta_v x}dx|t|^{2s-1}\omega(t)d^\times t \\
=&|a|^s|\delta_v|^{s-1/2}\omega(\delta_v)\int_{\mathbb{R}^\times}\int_{\mathbb{R}} \Phi(at, -tx)e^{2\pi i x}dx|t|^{2s}\omega(t)d^\times t \\
=&|a|^s|\delta_v|^{s-1/2}\omega(\delta_v)\int_{\mathbb{R}} e^{2\pi i x}dx\int_{\mathbb{R}^\times} \Phi(at, -xt)|t|^{2s}\mathrm{sgn}(t)d^\times t,
\end{aligned}$$

The integral over $\mathbb{R}^\times$ is

$$\begin{aligned}
&2(ia-x)\int_0^\infty t^{1+2s}e^{-\pi t^2(a^2+x^2)}d^\times t \\
=&(ia-x)\int_0^\infty \left(\frac{t}{\pi(a^2+x^2)}\right)^{s+1/2} e^{-t}d^\times t \\
=&(ia-x)\frac{\Gamma(s+1/2)}{(\pi(a^2+x^2))^{s+1/2}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&W\left(s, \begin{pmatrix} a\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \\
=&|\delta|^{s-1/2}\omega_v(\delta_v)\frac{\Gamma(s+1/2)}{\pi^{s+1/2}} \cdot |a|^s \cdot \int_{\mathbb{R}} \frac{(ia-x)e^{2\pi i x}}{(a^2+x^2)^{s+1/2}}dx \\
=&|\delta|^{s-1/2}\omega_v(-a\delta_v)\frac{\Gamma(s+1/2)}{\pi^{s+1/2}} \cdot |a|^{1-s} \cdot \int_{\mathbb{R}} \frac{e^{2\pi a x}}{(i+x)(1+x^2)^{s-1/2}}dx.
\end{aligned}$$

□

**Fourier expansion of theta series.** Recall that the series $\theta_\chi$ is a form in the space of the representation $\Pi(\chi)$ which has Whittaker function

$$W_\chi^0\left(\begin{pmatrix}\delta & 0\\ 0 & 1\end{pmatrix} g\right),$$

where $W_\chi^0(g) = \prod W_{\chi,v}^0(g_v)$ and $W_v^0(g_v)$ are new vectors unless $v$ is archimedean where $\Pi_v$ has weight 2. If $\Pi_v$ is of weight 2, then $W_{\chi,v}^0\left(g\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)$ is a new vector. In the following lets compute the Fourier expansion of $\theta_\chi$. Again we will start with the constant term.

LEMMA 3.3.5. *The constant term $C_\chi(g)$ is nonzero only if $\chi$ is of the form $\nu \cdot \mathrm{N}_{K/F}$ with $\nu$ a quadratic character on $F^\times\backslash\mathbb{A}^\times$. In this case we have*

$$C_\chi\left(\begin{pmatrix}a\delta^{-1} & 0\\ 0 & 1\end{pmatrix} h_T^{-1}\right) = \nu(a)|a|^{1/2}c_{\chi,T} + \nu\omega(a)|a|^{1/2}\widetilde{c}_{\chi,T}$$

*where $c_{\chi,T}$, $\widetilde{c}_{\chi,T}$ are constants independent of $a$.*

*Proof.* The representation $\Pi(\chi)$ is non-cuspidal only if $\chi = \nu \cdot \mathrm{N}_{K/F}$. In this case, it is the principle series $\Pi(\nu, \nu\omega)$. Thus there is a $\Phi \in \mathcal{S}(\mathbb{A}^2)$ such that the constant term is given by

$$f_\Phi(g, \nu, \nu\omega) + f_{\widetilde{\Phi}}(g, \nu\omega, \nu)$$

where for two characters $\mu_1, \mu_2$,

$$f_\Phi(g, \mu_1, \mu_2) = \mu_1(\det g)|\det g|^{1/2}\int_{\mathbb{A}^\times} \Phi[(0,t)g]\mu_1\mu_2^{-1}(t)\omega(t)|t|d^\times t.$$

The conclusion of the lemma now follows easily. □

LEMMA 3.3.6. *Assume that $v$ is nonarchimdean.*

1. *If $K_v = F_v \oplus F_v$, $\chi_v = (\mu_v, \mu_v^{-1})$, then*

$$W_\chi^0\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} = |a|^{1/2}\begin{cases}\frac{\mu(a\pi)-\mu^{-1}\omega(a\pi)}{\mu(\pi)-\mu^{-1}\omega(\pi)} & \text{if } |a| \le 1,\ o(\mu) = 0,\\ 1 & \text{if } |a| = 1,\\ 0 & \text{otherwise.}\end{cases}$$

2. *If $K_v/F_v$ is unramified field extension, then*

$$W_\chi^0\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} = \begin{cases}|a|^{1/2} & \text{if } \chi_v = 1,\ \mathrm{ord}(a) \in 2\mathbb{Z}_{\ge 0},\\ 1 & \text{if } |a| = 1,\\ 0 & \text{otherwise.}\end{cases}$$

3. *If $K_v/F_v$ is a ramified field extension, $\chi = \nu \circ \mathrm{N}_{K_v/F_v}$, then*

$$W_{\chi,v}^0\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} = \begin{cases}|a|^{1/2}\nu(a) & \text{if } |a| \le 1,\\ 0 & \text{otherwise.}\end{cases}$$

*Proof.* All the conclusions follow from the identity

$$\int_{F^\times} W^0_{\chi,v}\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}|a|^{s-1/2}d^\times a = L(s,\Pi(\chi_v)) = L(s,\chi_v),$$

and the fact that the value $W\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}$ depends only on $|a|$. □

LEMMA 3.3.7. *Assume that $F_v = \mathbb{R}$.*
1. *If $\Pi$ is of weight 0, then*

$$W^0_{\chi,v}\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} = \begin{cases} 2|a|^{1/2}e^{-2\pi a} & \text{if } a>0,\\ 0 & \text{otherwise.}\end{cases}$$

2. *If $\Pi$ is of weight 2, then*

$$W^0_{\chi,v}\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} = \begin{cases} 2|a|^{1/2}e^{2\pi a} & \text{if } a<0,\\ 0 & \text{otherwise.}\end{cases}$$

*Proof.* It is sufficient to show the first part. Notice that in this case, the values of $W$ at $\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}$ determine the values of $W(g)$ as it has weight 1. One only needs now to show that this $W(g)$ gives the right L-function when twisted with characters of $\mathbb{R}^\times$. □

**Fourier expansion of $\Theta(s,g)$.** Lets start with $W(s,\xi,\eta,g)$ for $g=\begin{pmatrix} a\delta^{-1} & 0\\ 0 & 1\end{pmatrix}$. From our definition, it is actually a product of $W_v(s,\xi,\eta,g_v)$ where

$$\begin{aligned} &W_v(s,\xi,\eta,g)\\ =&\frac{1}{2}|\delta|^{s-1/2}W_{\chi,v}\left(\begin{pmatrix} \eta & 0\\ 0 & 1\end{pmatrix}g\right)W_v\left(s,\begin{pmatrix} \xi & 0\\ 0 & 1\end{pmatrix}g\right)\\ &+\frac{1}{2}|\delta|^{s-1/2}\gamma_v(s)W_{\chi,v}\left(\begin{pmatrix} \eta & 0\\ 0 & 1\end{pmatrix}gh_v^{-1}\right)W_v\left(s,\begin{pmatrix} \xi & 0\\ 0 & 1\end{pmatrix}gh_v^{-1}\right),\end{aligned}\tag{3.3.11}$$

if $\omega_v$ is ramified; otherwise

$$W_v(s,\xi,\eta,g) = |\delta|^{s-1/2}W_{\chi,v}\left(\begin{pmatrix} \eta & 0\\ 0 & 1\end{pmatrix}g\right)W_v\left(s,\begin{pmatrix} \xi & 0\\ 0 & 1\end{pmatrix}g\right).\tag{3.3.12}$$

Thus, the value of $W(s,\xi,\eta,g)$ has been computed in the previous lemmas. When $\omega_v$ is ramified, we have the following simplification:

LEMMA 3.3.8. *Assume that $\omega_v$ is ramified. Then $W(s,\xi,\eta,e)$ is nonzero only if both $|\xi\leq 1$, and $|\eta|\leq 1$. In this case*

$$\begin{aligned} W_v\left(s,\xi,\eta,\begin{pmatrix} \delta^{-1} & 0\\ 0 & 1\end{pmatrix}\right) =&\frac{1}{2}|\delta_v|^{2s-1}\omega(-\delta_v)\epsilon(\omega,\psi_v)\cdot|\eta\xi\pi^{c_v}|^{1/2}\nu(\eta)|\pi_v^{*+}|^{s-1/2}\\ &\cdot\left[|\pi_v^{*-}\xi|^{s-1/2}+(-1)^{\{v\}\cap\Sigma}\omega_v(-\eta\xi)|\pi_v^{*-}\xi|^{1/2-s}\right]\end{aligned}$$

*where*

$$\pi_v^{*+} = \pi_v^{o(\omega_v)+o(\Pi)/2},\qquad \pi_v^{*+} = \pi_v^{o(\omega_v)-o(\Pi)/2}.$$

*Proof.* Apply Lemma 3.3.3, 3.3.6, 3.2.5. □

Also it is not difficult to check the following

LEMMA 3.3.9. *For each place $v$ of $F$, $\xi, \eta \in F_v^\times$,*

$$W_v(s, \xi, \eta, g) = \omega_v(-\xi\eta)\epsilon(s, \Pi \otimes \chi, \psi)W(1-s, \xi, \eta, g).$$

It remains to treat $C_\chi^*$ and $W_\chi^*$.

LEMMA 3.3.10. *The function $C_\chi^*(s, g) = 0$ unless $\chi$ is of form $\nu \cdot \mathrm{N}_{K/F}$. It is a linear combination of functions in*

$$f_1(s, g) \in \mathcal{B}(\alpha^s\nu, \alpha^{-s}\nu), \qquad f_2(s, g) \in \mathcal{B}(\alpha^s\nu\omega, \alpha^{-s}\nu\omega),$$
$$f_3(s, g) \in \mathcal{B}(\alpha^{1-s}\nu\omega, \alpha^{s-1}\nu\omega), \qquad f_4(s, g) \in \mathcal{B}(\alpha^{1-s}\nu, \alpha^{s-1}\nu),$$

*which are holomorphic in $s$, of opposite weight as $\Pi$, and invariant under $U_0([N, D])$.*

*Proof.* It is clear that the function $C_\chi^*(s, g) = 0$ unless $\chi$ is of form $\nu \cdot \mathrm{N}_{K/F}$. In this case it is a linear combination of constant terms of products of a form in $\Pi(\nu, \nu\omega)$ and a form in $\Pi(\alpha^{s-1/2}, \alpha^{1/2-s}\omega)$ with coefficients holomorphic in $s$.

Notice that the constant term of a form $E_f \in \Pi(\mu_1, \mu_2)$ has constant term

$$f(g) + \widetilde{f}(g), \qquad f \in \mathcal{B}(\mu_1, \mu_2), \quad \widetilde{f} \in \mathcal{B}(\mu_2, \mu_1).$$

Since the product of two principal in $\mathcal{B}(\mu_1, \mu_2)$, $\mathcal{B}(\nu_1, \nu_2)$ will be in

$$\mathcal{B}(\mu_1\nu_1\alpha^{1/2}, \mu_2\nu_2\alpha^{-1/2}),$$

we see that $C_\chi^*(s, g)$ is a linear combination of functions in

$$f_1(s, g) \in \mathcal{B}(\alpha^s\nu, \alpha^{-s}\nu), \qquad f_2(s, g) \in \mathcal{B}(\alpha^s\nu\omega, \alpha^{-s}\nu\omega),$$
$$f_3(s, g) \in \mathcal{B}(\alpha^{1-s}\nu\omega, \alpha^{s-1}\nu\omega), \qquad f_4(s, g) \in \mathcal{B}(\alpha^{1-s}\nu, \alpha^{s-1}\nu).$$

□

LEMMA 3.3.11. *Let $g$ denote $\begin{pmatrix} a\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. If $\chi$ is not of the form $\nu \circ \mathrm{N}_{K/F}$, the function $W_\chi^*(s, g)$ ($a \in \mathbb{A}^\times$) is a sum of $W_\chi^\pm(s, g)$, where*

$$W_\chi^+(s, -) \in \mathcal{W}(\Pi(\chi) \otimes \alpha^s, \psi), \qquad W_\chi^-(s, -) \in \mathcal{W}(\Pi(\chi) \otimes \alpha^{1-s}, \psi).$$

*If $\chi = \nu \circ \mathrm{N}_{K/F}$, then $W_\chi^*(s, g)$ is a sum of the above two terms and two more terms $W_\nu^\pm(s, g)$, where*

$$W_\nu^+(s, -) \in \mathcal{W}(\Pi(\alpha^s\nu, \alpha^{1-s}\nu\omega), \psi), \qquad W_\nu^-(s, -) \in \mathcal{W}(\Pi(\alpha^{1-s}\nu, \alpha^s\nu\omega), \psi).$$

*Moreover, $W_\chi^\pm$ (resp. $W_\nu^\pm$) are invariant under $U_1([N, D])$ and holomorphic in $s$, and has opposite weight as $\theta_\chi$ (resp. $E(s, -)$).*

*Proof.* This follows from the definition and Lemma 3.3.1, 3.3.5, and the fact that every function $f(s, g)$ in $\mathcal{B}(\alpha^{s-1/2}, \alpha^{1/2-s}\omega)$ is holomorphic in $s$. □

### 3.4. Central values and derivatives

Depending on whether $\Sigma$ is even or odd, in this section we want to compute the values or derivatives of the Fourier coefficients of $\Theta(s, g)$ at $s = 1/2$.

**Central values.** Assume that $\Sigma$ is even. We want to compute the Fourier coefficients of $\Theta(1/2, g)$ for $g = \begin{pmatrix} a\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. The *degenerate* terms are easily deduced from Lemma 3.3.10, 3.3.11. We now treat compute $W_v(1/2, \xi, \eta, g)$. First assume that $F$ is non-archimedean and $\omega$ is unramified. In this case,

$$W_v(1/2, \xi, \eta, g) = W_{\chi,v}\left(\begin{pmatrix} \eta a\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) W_v\left(\frac{1}{2}, \begin{pmatrix} \xi a\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

If $\chi_v$ is unramified then by Lemma 3.3.6,

$$W_{\chi,v}\left(\begin{pmatrix} a\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) = W^0_{\chi,v}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \neq 0$$

only if $a \in \mathrm{N}(\mathcal{O}_{K,v})$. In this case

$$\begin{aligned} & W_{\chi,v}\left(\begin{pmatrix} a\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \\ =& |a|^{1/2} \begin{cases} 1 & \text{if } K_v \text{ is a field,} \\ \sum_{i+j=\mathrm{ord}(a)} \mu(\pi^{i-j}) & \text{if } K_v = F_v \oplus F_v,\ \chi_v = (\mu, \mu^{-1}). \end{cases} \end{aligned}$$

Similarly by Lemma 3.3.2,

$$W_v\left(1/2, \begin{pmatrix} a\delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \neq 0$$

only if $a\pi_v^{-c_v} \in \mathrm{N}(\mathcal{O}_K)$. In this case,

$$W_v\left(\frac{1}{2}, \begin{pmatrix} a\delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) = |a|^{1/2}\omega_v(\delta_v) \begin{cases} 1 & \text{if } K_v \text{ is a field,} \\ \mathrm{ord}(a\pi_v^{-c_v}) + 1 & \text{if } K_v = F_v \oplus F_v, \end{cases}$$

where $c_v = \mathrm{ord}_v([N, D])$. We assume further that either $c_v$ or $\mathrm{ord}_v(a)$ is zero. Then we have the following:

Lemma 3.4.1. *Assume that both $\chi$ and $\omega$ are unramified. The value*

$$W_v(1/2, \xi, \eta, g) \neq 0$$

*only if both $\eta a$ and $\xi a\pi^{-c}$ are in $\mathrm{N}(\mathcal{O}_K)$. In this case it is given by*

$$W_v(1/2, \xi, \eta, g) = \omega_v(\delta_v)|\eta\xi|^{1/2}|a|$$

*if $K$ is a field, and*

$$W_v(1/2, \xi, \eta, g) = \omega(\delta_v)|\eta\xi|^{1/2}|a| \frac{\mu(\eta a\pi) - \mu^{-1}(\eta a\pi)}{\mu(\pi) - \mu^{-1}(\pi)} \cdot \mathrm{ord}(\xi a\pi^{1-c})$$

*if $K = F \oplus F$ and $\chi = (\mu, \mu^{-1})$.*

If $\chi_v$ is ramified, then

$$W_{\chi,v}\left(\begin{pmatrix} a\delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \neq 0$$

only if $a$ is invertible. In this case

$$W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = 1.$$

Lemma 3.4.2. *Assume that $\chi_v$ is ramified. Let $g_0 = \begin{pmatrix} \delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. Then the value $W_v(1/2, \xi, \eta, g_0) \neq 0$ only if both $|\eta|_v = 1$ and $\xi\pi^{-c}$ are in $\mathrm{N}(\mathcal{O}_{K,v})$. In this case it is given by*

$$W_v(1/2, \xi, \eta, g) = \omega_v(\delta_v)|\eta\xi|_v^{1/2}$$

*if $K_v$ is a field, and*

$$W_v(1/2, \xi, \eta, g) = \omega_v(\delta_v)|\eta\xi|_v^{1/2} \cdot \mathrm{ord}(\xi\pi^{1-c_v})$$

*if $K_v = F_v \oplus F_v$.*

Lets now treat the case where $\omega_v$ is ramified. In this case $\chi_v = \nu \cdot \mathrm{N}$ with $\nu$ a quadratic character of $F_v^\times$.

Lemma 3.4.3. *Assume that $\omega_v$ is ramified. Then $W_v(1/2, \xi, \eta, g_0) \neq 0$ only if*

$$|\xi|_v \leq 1, \quad \omega_v(-\eta\xi) = (-1)^{\#\{v\}\cap\Sigma}.$$

*In this case,*

$$W_v(1/2, \xi, \eta, g_0) = \epsilon(\omega, \psi_v)^{-1}|\eta\xi\pi^{c_v}|_v^{1/2}\nu(\eta).$$

It remains to treat the archimedean case $F_v = \mathbb{R}$. By Lemma 3.3.4, 3.3.7, the kernel function $W_v(1/2, g)$ with $g = \begin{pmatrix} a\delta^{-1} & 0 \\ o & 1 \end{pmatrix}$ is a product of two functions

$$W_{\chi,v}(g) = \begin{cases} 2|a|^{1/2}e^{-2\pi a} & \text{if } a > 0,\ v \in \infty^-, \\ 2|a|^{1/2}e^{2\pi a} & \text{if } a < 0,\ v \in \infty^+, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} W_v(1/2, g) =& \omega(-a\delta_v)\pi^{-1}|a|^{1/2}\int_{\mathbb{R}^\times} \frac{e^{2\pi i a x}}{i + x}dx \\ =& \begin{cases} 0 & \text{if } a > 0, \\ -2i\omega_v(\delta_v)|a|^{1/2}e^{2\pi a} & \text{otherwise.} \end{cases} \end{aligned}$$

Thus we have

Lemma 3.4.4. *Assume that $F = \mathbb{R}$. Then*

$$W(1/2, \xi, \eta, g) = \begin{cases} -4i\omega_v(\delta_v)|\eta\xi|^{1/2}|a|e^{2\pi a(\xi-\eta)} & \textit{if } a\eta > 0 \textit{ and } a\xi < 0 \\ 0 & \textit{Otherwise} \end{cases}$$

*if* $\Pi_v$ *is of weight* 0, *and*

$$W_v(1/2,\xi,\eta,g)=\begin{cases}-4i\omega_v(\delta_v)|\eta\xi|^{1/2}|a|e^{2\pi a(\xi+\eta)} & \text{if } a\eta<0,\ a\xi<0\\ 0 & \text{otherwise}\end{cases}$$

*if* $\Pi_v$ *of weight* 2

The lemma actually implies that the complex conjugation of $\Theta(1/2,g)$ is holomorphic of weight 2 (resp. nonholomorphic of weight 0) at infinite places where $\Pi$ is of weight 2 (resp. non-holomorphic of weight 0).

**Central derivatives.** Assume that $\Sigma$ is odd. Then by Theorem 3.2.1, $\Theta(1/2,g)=0$. We want to compute its derivative $\Theta'(1/2,g)$ at $s=1/2$. Again the degenerate term can be easily deduced from Lemma 3.3.10, 3.3.11. Lets now compute the central derivative for $W(s,\xi,\eta,g)$ for $g$ of the form $\begin{pmatrix}a\delta^{-1} & 0\\ 0 & 1\end{pmatrix}$. Recall that $W(s,\xi,\eta,g)$ is a product of $W_v(s,\xi,\eta,g)$, and that $W_v(s,\xi,\eta,g)$ satisfies the functional equation

$$W_v(s,\xi,\eta,g)=\omega_v(-\xi\eta)\epsilon(s,\Pi_v\otimes\chi_v)W_v(1-s,\xi,\eta,g), \tag{3.4.1}$$

$$\epsilon(1/2,\Pi_v\otimes\chi_v)=(-1)^{\#\Sigma\cap\{v\}}. \tag{3.4.2}$$

It follows that

$$W'(1/2,\xi,\eta,g)=\sum_v W^v(1/2,\eta,\xi,g^v)\cdot W_v'(1/2,\eta.\xi,g) \tag{3.4.3}$$

where $W^v$ is the product of $W_\ell$ over places $\ell\neq v$, and $W_v'$ is the derivative for the variable $s$, and $v$ runs through the places with

$$\omega_v(-\xi\eta)=(-1)^{1+\#\Sigma\cap\{v\}},\qquad \omega_\ell(-\xi\eta)=(-1)^{\#\Sigma\cap\{\ell\}},\quad \forall\ell\neq v.$$

In particular we need only consider the finite places which are not split in $K$. In the following we want to compute $W_v'(1/2,\xi,\eta,g)$ such that

$$\omega_v(-\xi\eta)=(-1)^{1+\#\Sigma\cap\{v\}}=\begin{cases}1 & \text{if } v\in\Sigma,\\ -1 & \text{if } v\notin\Sigma.\end{cases} \tag{3.4.4}$$

First, let's consider the case where $v$ is a place of $F$ which is inert and unramified for the extension $K/F$, and such that $\chi_v$ is unramified. In this case

$$W_v(s,\xi,\eta,g)=W_{\chi,v}\begin{pmatrix}\eta a\delta_v^{-1} & 0\\ 0 & 1\end{pmatrix}W_v\left(s,\begin{pmatrix}\xi a\delta_v^{-1} & 0\\ 0 & 1\end{pmatrix}\right).$$

Then by Lemma 3.3.2, 3.3.6, the $W_{\chi,v}$ term is nonzero only if $\mathrm{ord}(\eta a)$ is even and nonnegative in which case the value is given by $|\eta a|^{1/2}$. Then the $W_v(s,-)$ term is zero at $s=1/2$ and has nonzero derivative only if $\mathrm{ord}(\xi a\pi^{-c})$ is odd and nonnegative in which case the derivative is given by

$$\omega(\delta_v)|\xi a|^{1/2}\log|\xi a\pi^{1-c}|.$$

LEMMA 3.4.5. *Let $v$ be a finite place of $F$ which is inert and unramified in $K$ such that $\chi_v$ is unramified. Then the only non trivial contribution is when* $\mathrm{ord}(\eta a)$ *is even and nonnegative, and* $\mathrm{ord}(\xi a\pi^{-c})$ *is odd and positive. In this case, we have*

$$W_v'(1/2,\xi,\eta,g)=\omega_v(\delta_v)|\eta\xi|_v^{1/2}\cdot|a|_v\cdot\log|\xi a\pi^{1-c}|_v.$$

We now consider the case where $v$ is inert in $K$ and $\chi_v$ is ramified. Assume that $a = \delta_v^{-1}$. The $W_{\chi,v}$ term is nonzero only if $\eta$ is invertible. In this case its value is 1.

LEMMA 3.4.6. *Let $v$ be a finite place of $F$ which is inert and unramified in $K$ such that $\chi_v$ is ramified. Then the only non trivial contribution is when* $\mathrm{ord}(\eta) = 1$ *and* $\mathrm{ord}(\xi\pi^{-c})$ *is odd and positive. In this case, we have*

$$W_v'(1/2, \xi, \eta, g_0) = \omega_v(\delta_v)|\xi|_v^{1/2} \cdot \log|\xi\pi^{1-c}|_v.$$

Lets now treat the case where $\omega$ is ramified. In this case, by Lemma 3.3.8, $\chi_v = \nu \cdot \mathrm{N}_{K/F}$ and for $|\eta| \le 1$,

$$\begin{aligned} W(s, \xi, \eta, g_0) =& \frac{1}{2}\epsilon(\omega, \psi_v)^{-1}|\eta\xi\pi^{c_v}|^{1/2}\nu(\eta)|\pi^{*+}|^{s-1/2} \cdot \\ &\cdot \left(|\pi\xi^{*-}|^{s-1/2} - |\pi\xi^{*-}|^{1/2-s}\right). \end{aligned}$$

LEMMA 3.4.7. *Assume that $\omega_v$ is ramified and $\chi_v = \nu \circ \mathrm{N}_{K/F}$. Then the only case with nontrivial contribution is when both $\xi$ and $\eta$ are integral and*

$$\omega_v(-\eta\xi) = \begin{cases} 1 & \text{if } v \in \Sigma, \\ -1 & \text{if } v \notin \Sigma. \end{cases}$$

*In this case*

$$W'(1/2, \xi, \eta, g_0) = \epsilon(\omega_v, \psi)^{-1}|\eta\xi\pi^{c_v}|_v^{1/2}\nu(\eta)\log|\xi\pi^{*-}|_v.$$

Finally we treat the archimedean place. The nontrivial case is when $\eta a < 0$ (resp. $\eta a > 0$) when $v \in \infty^+$ (resp. $v \in \infty^-$) and $\xi a > 0$. In this case, $W'(1/2, \xi, \eta, g)$ is the product of

$$W\left(\begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix} g\right) = 2|\eta a|^{1/2}e^{-2\pi|\eta a|},$$

and

$$\begin{aligned} W_0'\left(1/2, \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g\right) =& \omega(-a\xi\delta_v)\pi^{-1}|\xi a|^{1/2}\frac{\partial}{\partial s}\bigg|_{s=1/2}\int_{\mathbb{R}} \frac{e^{2\pi i\xi a x}dx}{(i+x)(1+x^2)^{s-1/2}} \\ =& 2i\omega_v(\delta_v)|\xi a|^{1/2}q_0(4\pi\xi a)e^{2\pi\xi a}, \end{aligned}$$

where

$$q_0(t) = \int_0^1 e^{-t/x}d^\times x.$$

Thus finally we have

LEMMA 3.4.8. *The only trivial contribution is when $\eta a < 0$ (resp. $\eta a > 0$) if $v \in \infty^+$ (resp. $v \in \infty^-$) and $\xi a > 0$. In this case,*

$$W'(1/2, \xi, \eta, g) = 4i\omega_v(\delta_v)|\eta\xi|^{1/2} \cdot |a| \cdot q_0(4\pi\xi a) \cdot e^{2\pi(\xi a - |\eta a|)}.$$

### 3.5. Holomorphic projection

In this section we assume that $\Sigma$ is odd, and that at every infinite place $\Pi$ is discrete of weight 2. We want to find the holomorphic projection of $\bar{\Theta}'(1/2, g)$. That is a holomorphic form $\Phi$ of weight 2 such that $\bar{\Theta}'(1/2, g) - \Phi$ is perpendicular to any holomorphic form. Here a form $\phi$ of weight 2 is called *holomorphic* if its Whittaker function satisfies

$$W_\phi \begin{pmatrix} ay_\infty\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \widehat{\phi}(a) W_\infty \begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix} \tag{3.5.1}$$

where $\widehat{\phi}$ is a function of integral ideles $a$, and $W_\infty = \prod_{v\nmid\infty} W_v$ is the Whittaker function for weight 2 such that:

$$W_v \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = \begin{cases} 2ae^{-2\pi a} & \text{if } a > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5.2}$$

The number $\widehat{\phi}(a)$ is called the *$a$-th Fourier coefficient* of $\phi$.

Lets first state a formula for holomorphic projection. For any Whittaker function $W$ on $\mathrm{GL}_2(\mathbb{A})$ of weight 2, any integral idele $a \in \mathbb{A}_f^\times$, and any complex number $\mathrm{Re}(\sigma) > 0$, let's define

$$W_\sigma(a) = (2\pi)^g \int_{F_\infty^+} W \left( \begin{pmatrix} y \cdot a\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) e^{-2\pi y} y^\sigma d^\times y \tag{3.5.3}$$

provided the integral converges.

Lemma 3.5.1. *Let $\widetilde{\phi}$ be an automorphic form for $\mathrm{PGL}_2(\mathbb{A})$ which has asymptotic behavior $O(|a|^{1-\epsilon})$ near each cusp. Then $W_{\widetilde{\phi},\sigma}(a)$ is holomorphic at $\sigma = 0$ and the holomorphic projection $\phi$ of $\widetilde{\phi}$ has Fourier coefficients given by the following formula:*

$$\widehat{\phi}(a) = \lim_{\sigma \to 0} W_{\widetilde{\phi},\sigma}(a).$$

*Proof.* For a fixed subgroup $U_0([D, N])$ as before and a finite idele $a$ lets define $H_{a,\sigma}(g)$ to be a Whittaker function on $\mathrm{GL}_2(\mathbb{A})$ of weight 2, invariant under $Z(\mathbb{A})U_0([N, D])$, supported on $Z(\mathbb{A})A(\mathbb{A})U_0([D, N])$, and such that

$$H_{a,\sigma} \left( \begin{pmatrix} y\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) = \begin{cases} |y|^\sigma W_\infty \left( \begin{pmatrix} y_\infty & 0 \\ o & 1 \end{pmatrix} \right) & \text{if } y_f \in a\widehat{\mathcal{O}}_F^\times, \\ 0 & \text{otherwise,} \end{cases}$$

where $\sigma$ is a complex number. Let $P_{a,\sigma}(g)$ denote the following Poincareé series

$$P_{a,\sigma}(g) = \sum_{\gamma \in Z(F)N(F)\backslash \mathrm{GL}_2(F)} H_{a,\sigma}(\gamma g).$$

Then $P_{a,\sigma}$ is absolutely convergent for $\mathrm{Re}(\sigma) > 0$ and defines a nonholomorphic form of weight 2 for $U_0([N,D])$. For any cuspform $\widetilde{\phi}$ for $U_0([N,D])$ of weight 2, we have

$$\begin{aligned}(\widetilde{\phi}, P_{a,\sigma}) &= \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \widetilde{\phi}(g)\bar{P}_{a,\sigma}(g)dg = \int_{Z(\mathbb{A})N(F)\backslash G(\mathbb{A})} \widetilde{\phi}(g)\bar{H}_{a,\sigma}(g)dg \\ &= \int_{Z(\mathbb{A})N(\mathbb{A})\backslash G(\mathbb{A})} (W_{\widetilde{\phi}}\bar{H}_{a,\sigma})(g)dg = \int_{\mathbb{A}^\times} (W_{\widetilde{\phi}}\bar{H}_{a,\sigma})\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)\frac{d^\times\alpha}{|\alpha|} \\ &= |\delta||a|^\sigma \int_{F_\infty^+} W_{\widetilde{\phi}}\left(\begin{pmatrix} y\cdot a\delta^{-1} & 0\\ 0 & 1\end{pmatrix}\right) e^{-2\pi y}y^\sigma d^\times y.\end{aligned}$$

If $\phi$ is the holomorphic projection of $\widetilde{\phi}$ then $(\phi, P_a) = (\widetilde{\phi}, P_a)$. As $W_{\widetilde{\phi}}(g) = W_\infty(g_\infty)W_\phi(g_f)$, we have

$$\begin{aligned}(\widetilde{\phi}, P_a) =(\phi, P_a) &= 2^g|\delta||a|^\sigma W_\phi\left(\begin{pmatrix} a\delta_f^{-1} & 0\\ 0 & 1\end{pmatrix}\right)\int_{F_\infty^+} e^{-4\pi y}y^{1+\sigma}d^\times y \\ &= |\delta||a|^\sigma W_\phi\left(\begin{pmatrix} a\delta_f^{-1} & 0\\ 0 & 1\end{pmatrix}\right)\cdot \frac{2^g\Gamma(1+\sigma)^g}{(4\pi)^{g(1+\sigma)}}.\end{aligned}$$

Taking the limit $\sigma \to 0$, the lemma follows. □

We want to apply Lemma 3.5.1 to $\bar{\Theta}'$. First of all lets study the asymptotic behavior at a cusp.

LEMMA 3.5.2. *There is an automorphic form $E'(g)$ on $\mathrm{PGL}_2(\mathbb{A})$ which is a sum of Eisenstein series or their derivatives such that for any $g \in \mathrm{GL}_2(\mathbb{A})$, $a \in \mathbb{A}^\times$, as $a \to \infty$,*

$$\bar{\Theta}'\left(1/2, \begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} g\right) = E'\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} g\right) + O_g(|a|^{1-\epsilon}).$$

*More precisely, $E'(g) \neq 0$ only if $\chi = \nu \cdot \mathrm{N}_{K/F}$. In this case it is a sum*

$$E'(g) = E_1'(1/2, g) + E_2'(1/2, g)$$

*where $E_1(s,g)$ and $E_2(s,g)$ are Eisenstein series formed by functions*

$$f_1(s,g) \in \mathcal{B}(\nu\alpha^s, \nu\alpha^{-s}), \qquad f_2(s,g) \in \mathcal{B}(\nu\omega\alpha^s, \nu\omega\alpha^{-s})$$

*which are holomorphic in s near s=1/2, of weight 2, and invariant under $U_0([N,D])$.*

*Proof.* The constant term of an automorphic form the is always invariant from left under $B(F)$. Thus we can form Eisenstein series using the constant term of $\bar{\Theta}'(1/s, g)$. To get informations on the asymptotic behavior, we want to study this constant more precisely. From the Fourier expansion, one easily sees that for any $g \in \mathrm{GL}_2(\mathbb{A})$, $a \in \mathbb{A}^\times$,

$$\bar{\Theta}'\left(1/2, \begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} g\right) = \bar{C}_\chi^{*\prime}\left(1/2, \begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} g\right) + O_g(|a|^{1-\epsilon})$$

as $a \to \infty$. By Lemma 3.3.10, the function $\bar{C}_\chi^*(s,g) \neq 0$ only if $\chi = \nu \cdot \mathrm{N}_{K/F}$. In this case it is a sum

$$\sum_{i=1}^4 f_i(s,g)$$

as in Lemma 3.3.10. When taking the derivative at $s = 1/2$, we may assume $f_3 = f_4 = 0$ as $f_4(1-s, g)$ will be in the first space, and $f_3(1-s, g)$ will be in the second space.

Let $E_1(s, g)$, $E_2(s, g)$ be Eisenstein series formed by $f_1$ and $f_2$. We define $E$ to be the derivative at $1/2$ of $E_1 + E_2$. Then $E$ has constant term $f_1' + f_2' + \widetilde{f}_1' + \widetilde{f}_2'$ where

$$\widetilde{f}_1 \in \mathcal{B}(\alpha^{-s}\nu, \alpha^s\nu), \qquad \widetilde{f}_2 \in (\alpha^{-s}\nu\omega, \alpha^s\nu\omega).$$

Thus $f_1'$ and $f_2'$ has the bound $O(\log|a|)$ at the cusp. Thus, we have the right asymptotic behavior given in the lemma. $\square$

Let us apply this lemma for the form

$$\widetilde{\Phi}(g) := \bar{\Theta}'(1/2, g) - E'(g) \tag{3.5.4}$$

which has the same holomorphic projection as $\bar{\Theta}'(1/2, g)$. Let $\Phi$ denote its holomorphic projection. With respect to the additive character $\psi$, the Whittaker function of $\widetilde{\Phi}$ is a sum of following Whittaker functions:

$$W(v, \xi, \eta, g) := \bar{W}^v(1/2, \xi, \eta, \epsilon g)\bar{W}_v'(1/2, \xi, \eta, \epsilon g), \tag{3.5.5}$$

$$A(g) := \bar{W}_\chi^{*'}(1/2, \epsilon g), \qquad B(g) := \bar{W}'(\epsilon g), \tag{3.5.6}$$

where $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $W'(g)$ is the Whittaker function of $E'(g)$, $\xi, \eta \in F^\times$ and $v$ is a place of $F$ such that

$$\xi + \eta = 1, \qquad \omega_\ell(-\xi\eta) = (-1)^{\#\{\ell\}\cap\Sigma}, \quad \forall \ell \neq v.$$

Let $W_\sigma(v, \xi, \eta, a)$, $A_\sigma(a)$ and $B_\sigma(a)$ denote the integrals defined at the beginning of the section for these Whittaker functions. Then by Lemma 3.5.1, the Fourier coefficient of $\Phi$ is given by

$$\widehat{\Phi}(a) = \lim_{\sigma \to 0} \left( \sum_{\xi, \eta, v} W_\sigma(v, \xi, \eta, a) + A_\sigma(a) + B_\sigma(a) \right). \tag{3.5.7}$$

Lets describe the contributions of the last two terms first. We need some notation.

DEFINITION 3.5.3. *Let $\mathbb{N}_F$ denote the semigroup of nonzero ideals of $\mathcal{O}_F$. For each $a \in \mathbb{N}_F$, let $|a|$ denote the* inverse norm *of $a$. For a fixed ideal $M$, let $\mathbb{N}_F(M)$ denote the sub-semigroup of ideals prime to $M$.*

*A function $f$ on $\mathbb{N}_F(M)$ is called* quasi-multiplicative *if*

$$f(a_1a_2) = f(a_1) \cdot f(a_2)$$

*for all coprime $a_1, a_2 \in \mathbb{N}_F(M)$. For a quasi-multiplicative function $f$, let $\mathcal{D}(f)$ denote the set of all $f$-*derivations*, that is the set of all a linear combinations*

$$g = cf + h$$

*where $c$ is a constant, and $h$ satisfies*

$$h(a_1a_2) = h(a_1)f(a_2) + h(a_2)f(a_1)$$

*for all* $a_1, a_2 \in \mathbb{N}_F(M)$ *with* $(a_1, a_2) = 1$.

*For a representation* $\Pi$, *the Fourier coefficients* $\widehat{\Pi}(a)$ *is defined to be*

$$\widehat{\Pi}(a) := W_{\Pi,f}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},$$

*where* $W_{\Pi,f}$ *is the product of Whittaker newvectors at finite places. In other words,* $\widehat{\Pi}(a)$ *is defined such that the finite part of L-series has expansion*

$$L_f(s, \Pi) = \sum \widehat{\Pi}(a)|a|^{s-1/2}.$$

*Then* $\widehat{\Pi}(a)$ *is quasi-multiplicative.*

*Let* $f_\sigma(a)$ *be a function on* $\mathbb{N}_F(M)$ *which is meromorphically depends on* $\sigma \in \mathbb{C}$, $\mathrm{Re}(\sigma) > 0$ *with at most a simply pole at* $\sigma = 0$, *then we denote the* quasi-limit

$${}'\lim_{\sigma \to 0} f_\sigma(a)$$

*the constant term in the Laurent expansion:*

$${}'\lim_{\sigma \to 0} f_\sigma(a) = \lim_{\sigma \to 0} \left(f_\sigma(a) - residue \cdot \sigma^{-1}\right).$$

LEMMA 3.5.4. *The function* $f_\sigma$ *is holomorphic at* $\sigma = 0$ *with the constant term*

$$A := \lim_{\sigma \to 0} A_\sigma \in \mathcal{D}(\widehat{\Pi}(\chi) \otimes \alpha^{1/2}).$$

*The function* $B_\sigma$ *is meromorphic at* $\sigma = 0$ *with a simple pole with constant*

$$B :=' \lim_{\sigma \to 0} B_\sigma \in \mathcal{D}(\Pi(\alpha^{1/2}\nu, \alpha^{-1/2}\nu)) + \mathcal{D}(\widehat{\Pi}(\alpha^{1/2}\nu\omega, \alpha^{-1/2}\nu\omega)).$$

*Proof.* Let's study $A_\sigma(a)$ first for $a \in \mathbb{N}_F(ND)$. By Lemma 3.3.11, for $g = \begin{pmatrix} ay_\infty\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, the Whittaker function $W_\chi^*(s, g)$ is a sum of four Whittaker functions

$$W_\chi^\pm(s, g), \qquad W_\nu^\pm(s, g),$$

where $W_\nu^\pm \neq 0$ only if $\chi = \nu \circ \mathrm{N}_{K/F}$. We want to study the contribution of $W_\chi^\pm$. The argument for $W_\nu^\pm$ is similar. Due to the symmetry $s \longrightarrow 1 - s$, when we compute $W_\chi^{*'}(1/2, g)$, we may forget $W_\chi^-$. Since $W_\chi^+$ is invariant under $\Gamma_1([N, D])$, it has spherical decomposition

$$W_\chi^+(s, g) = W_\chi^0\left(s, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \cdot W_\chi^\infty\left(s, \begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix}\right) \cdot W_\chi^+(s),$$

where $W_\chi^0$ is the product of the newvectors in the space of Whittaker functions for the representation $\Pi(\chi) \otimes \alpha^s$ over places prime to $ND$, and where $W_\chi^\infty(s, -)$ is the Whittaker function at $\infty$ with weight $-1$. It follows that the contribution to $A_\sigma(a)$ from $W_\chi^+$ is the derivative at $s = 1/2$ of the sum of the following integrals

$$W_{\chi,\sigma}(s, a) := \bar{W}_\chi^0\left(s, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \cdot \bar{W}_\chi^+(s) \cdot I_\chi(s, \sigma),$$

where

$$I\chi(s,\sigma) = (2\pi)^g \int_{F_\infty^+} \bar{W}_\chi^\infty \left(s, \begin{pmatrix} -y & 0 \\ 0 & 1 \end{pmatrix}\right) e^{-2\pi y} y^\sigma d^\times.$$

By explicit computation, one may show that $I_\chi(s,\sigma)$ is holomorphic at $(s,\sigma) = (1/2, 0)$. It follows that the contribution of $W_\chi^\pm$ part is the derivative of $W_{\chi,0}(s,a)$ at $s = 1/2$. It is indeed in $\mathcal{D}(\widehat{\Pi}(\chi) \otimes \alpha^{1/2})$.

The computation for $B_\sigma(a)$ is similar. The only difference is that when computing the above integral with respect to the Whittaker function of $\Pi(\alpha^s, \alpha^{-s})$ of $\mathrm{GL}_2(\mathbb{R})$ of weight 2, one gets singularity near $(s,\sigma) = (1/2, 0)$ of the form

$$\text{const} \cdot \frac{\sigma}{\sigma + s - 1/2}.$$

Thus its value at $s = 1/2$ has no singularity at $\sigma = 0$ but its derivative at $s = 1/2$ has a simple pole at $\sigma = 0$. $\square$

It remains to compute

$$W_\sigma(v,\xi,\eta,a) := (2\pi)^g \int_{F_\infty^+} \bar{W}\left(v,\xi,\eta, \begin{pmatrix} -ay\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) e^{-2\pi y} y^\sigma d^\times y. \tag{3.5.8}$$

If $v$ is finite, it is equal to the product

$$\bar{W}_f^v\left(1/2,\xi,\eta, \begin{pmatrix} -a\delta_f^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \bar{W}_v'\left(1/2,\xi,\eta, \begin{pmatrix} -a\delta_f^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right),$$

and

$$(2\pi)^g \int_{F_\infty^+} \bar{W}_\infty\left(1/2,\xi,\eta, \begin{pmatrix} -y & 0 \\ 0 & 1 \end{pmatrix}\right) e^{-2\pi y} y^\sigma d^\times y$$

which is nonzero only if $\xi a$ and $\eta a$ are both integral. By Lemma 3.4.4, the last term is nonzero only if $0 < \xi < 1$. In this case, it is given by

$$\omega_\infty(\delta_\infty)(2i)^g |\eta\xi|_\infty^{1/2} \frac{\Gamma(1+\sigma)^g}{(4\pi)^{g\sigma}}.$$

It follows that for a fixed $a$ there are only finitely many triples $\xi, \eta, v$ such that $W_\sigma(v,\xi,\eta,a) \neq 0$. Thus, in the contribution from finite $v$, we may simply take special values.

We now assume that $v$ is an infinite place. Then $\bar{W}_\sigma(v,\xi,\eta,a)$ is the product

$$\bar{W}_f\left(1/2,\xi,\eta, \begin{pmatrix} -a\delta_f^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right),$$

and

$$I_{\infty,\sigma}^v(\xi,\eta) := (2\pi)^{g-1} \int_{F_{\infty-\{v\}}^+} \bar{W}_{\infty-\{v\}}\left(1/2,\xi,\eta, \begin{pmatrix} -y & 0 \\ 0 & 1 \end{pmatrix}\right) e^{-2\pi y} d^\times y,$$

and

$$I_{v,\sigma}(\xi,\eta) := (2\pi) \int_0^\infty \bar{W}_v'\left(1/2,\xi,\eta, \begin{pmatrix} -y & 0 \\ 0 & 1 \end{pmatrix}\right) e^{-2\pi y} y^s d^\times y.$$

By 3.4.4, $I^v_{\infty,\sigma}(\xi,\eta)\neq 0$ only if $0<\xi_{\infty-\{v\}}<1$ and the value is given by

$$\prod_{\ell\in\infty-\{v\}}\omega_\ell(\delta_\ell)(2i)|\eta\xi|_\infty^{1/2}\frac{\Gamma(1+\sigma)}{(4\pi)^\sigma}.$$

By 3.4.8, $I_{v,\sigma}(\xi,\eta)\neq 0$ only if $\xi_v<0$, and $W'$-term is equal to

$$4i\omega_v(\delta_v)|\eta\xi|_v^{1/2}|y|q_0(4\pi|\xi|_v y)e^{-2\pi y}.$$

Thus the integral is equal to

$$\begin{aligned}&-8i\omega_v(\delta_v)\pi|\xi\eta|^{1/2}\int_0^\infty q_0(4\pi|\xi|_v y)e^{-4\pi y}y^{1+\sigma}d^\times y\\ =&-2i\omega_v(\delta_v)|\xi\eta|^{1/2}\frac{\Gamma(1+\sigma)}{(4\pi)^\sigma}\int_1^\infty\frac{dx}{x(1+|\xi|_v x)^{1+\sigma}}.\end{aligned}$$

Proposition 3.5.5. *With respect to the standard Whittaker function for holomorphic weight 2 forms, the a-the Fourier coefficients $\widehat{\Phi}(a)$ of the holomorphic projection $\Phi$ of $\bar{\Theta}'(1/2,g)$ is a sum*

$$\widehat{\Phi}(a)=A(a)+B(a)+\sum_v\widehat{\Phi}_v(a)$$

*where $A$, $B$ are given in Lemma 3.5.4, and the sum is over places of $F$ which are not split in $K$, with $\widehat{\Phi}_v(a)$ given by the following formulas:*

1. *if $v$ is a finite place then $\widehat{\Phi}_v(a)$ is a sum over $\xi\in F$ with $0<\xi<1$ of the following terms:*

$$(2i)^g|\eta\xi|_\infty^{1/2}\cdot\bar{W}_f^v\left(1/2,\xi,\eta,\begin{pmatrix}a\delta_f^{-1}&0\\0&1\end{pmatrix}\right)\cdot\bar{W}'_v\left(1/2,\xi,\eta,\begin{pmatrix}a\delta_f^{-1}&0\\0&1\end{pmatrix}\right)$$

2. *if $v$ is an infinite place, then $\widehat{\Phi}_v(a)$ is the constant term at $s=0$ of a sum over $\xi\in F$ such that $0<\xi_w<1$ for all infinite place $w\neq v$ and $\xi_v<0$ of the following terms:*

$$(2i)^g|\xi\eta|_\infty^{1/2}\cdot\bar{W}_f\left(1/2,\xi,\eta,\begin{pmatrix}a\delta_f^{-1}&0\\0&1\end{pmatrix}\right)\cdot\int_1^\infty\frac{-dx}{x(1+|\xi|_v x)^{1+s}}.$$

## 4. Geometric pairing of CM-cycles

In this chapter, we will study the local term of the so called *geometric pairing* of CM-cycles induced by a fixed *multiplicity function.* The height pairing of CM-points on Shimura curves will be the sums of various geometric pairings by choosing different quaternion algebras and multiplicity functions (or Green's functions). These algebras are the ${}_vB$ of the distance 1 from the odd set $\Sigma$, which admit an embedding $K\longrightarrow_v B$. Our main result is the *local Gross-Zagier formula* which relates the *linking number* of the pairing to some local components of the Fourier coefficients of the kernel functions and is given in the last chapter. This formula actually replaces all the combinatoric computation in the original approaches of Gross and Zagier. As an immediate application, we prove a Gross-Zagier formula for the central values of Rankin L-functions by spectral decomposition of the geometric pairing when the the multiplicity function is some Whittaker function.

### 4.1. Geometric pairing of CM-cycles

**CM-cycles.** Let $G$ be an inner form of $\mathrm{PGL}_2$ over $F$. This means that $G = B^\times/F^\times$ with $B$ a quaternion algebra over $F$. Let $K$ be a totally imaginary quadratic extension of $F$ which is embedded into $B$. Let $T$ denote the subgroup of $G$ given by $K^\times/F^\times$. Then the set

$$C := T(F)\backslash G(\mathbb{A}_f) \tag{4.1.1}$$

is called the set of CM-points. This set admits a natural action by $T(\mathbb{A}_f)$ (resp. $G(\mathbb{A}_f)$) by left (resp. right) multiplications.

There is a map from $C$ to the Shimura variety defined by $G$

$$G(F)_+\backslash\mathcal{H}^n \times G(\mathbb{A}_f)$$

as in §1.3 which sending the class of $g \in G(\mathbb{A}_f)$ to the class of $(z, g)$ where $z \in \mathcal{H}^n$ is fixed by $T$. This map is an embedding if $G$ is not totally definite. In our later study of local intersection, there is a situtation where $G$ is definite but $\mathcal{H}^n$ is replaced by the formal neighborhood $Y$ of a supersiggular point of a Shiumra variety. Thus in this case, one has an embedding of CM-points into a *formal Shimura variety.*

The set of CM-points has a topology induced from $G(\mathbb{A}_f)$ and has a unique $G(\mathbb{A}_f)$-invariant measure $dx$ up to constants such that every open and compact subset has finite and positive measure. Lets fix one measure on $T(\mathbb{A}_f)$ such that the volume of $T(\widehat{\mathcal{O}}_F) = \prod \mathcal{O}_{K,v}^\times F_v^\times/F_v^\times$ is 1. Then $dx$ is uniquely determined by its quotient on $T(\mathbb{A}_f)\backslash G(\mathbb{A}_f)$ which we may define as a product of the measure on $T(F_v)\backslash G(F_v)$ over all finite places $v$ of $F$. In practice, we will insist that $\mathrm{vol}(T(F_v)\backslash T(F_v)\cdot U_v) = 1$ for some compact and open subgroup of $G(F_v)$.

The set

$$\mathcal{S}(C) = \mathcal{S}(T(F)\backslash G(\mathbb{A}_f)) \tag{4.1.2}$$

of locally constant functions with compact support is called the set of CM-cycles which admits a natural action by $T(\mathbb{A}_f)\times G(\mathbb{A}_f)$. The $L^2$-norm induces a hermitian structure on $\mathcal{S}(C)$ such that the action of $T(\mathbb{A}_f)\times G(\mathbb{A}_f)$ is unitary. Since $T(F)\backslash T(\mathbb{A}_f)$ is compact, one has a natural orthogonal decomposition

$$\mathcal{S}(C) = \oplus_\chi \mathcal{S}(\chi, C) \tag{4.1.3}$$

where the sum is over characters of $T(F)\backslash T(\mathbb{A}_f)$.

There is also a local decomposition for each character $\chi$:

$$\mathcal{S}(\chi, C) = \otimes_v \mathcal{S}(\chi_v, G(F_v)) \tag{4.1.4}$$

where tensor product is a limit tensor product over the set of all finite places of $F$ and $\mathcal{S}(\chi_v, G(F_v))$ is the set of locally constant functions on $G(F_v)$ with character $\chi_v$ under the left multiplication by $T(F_v)$ and with compact support modulo $T(F_v)$. Fix a maximal order $\mathcal{O}_B$ of $B$. Thus any element $\phi$ in $\mathcal{S}(\chi, C)$ will have a decomposition

$$\phi = \phi_S \otimes_{v\notin S} \phi_v^0$$

where $S$ is a finite set of finite places which contains all places over which $\chi_p$ is ramified, $\phi_\wp^0$ supported on $T(F_v)\cdot G(\mathcal{O}_v)$ and takes value 1 on $G(\mathcal{O}_v)$, where $G(\mathcal{O}_v) = \mathcal{O}_{B,v}^\times \cdot F_v^\times/F_v^\times$. The hermitian structure on $\mathcal{S}(\chi, C)$ is the product of a hermitian structure on $\mathcal{S}(\chi_p, G(F_v))$.

**Geometric pairing.** In the following we will define a class of pairings on CM-cycles which are *geometric*. To do this, lets write CM-points in a slightly different way,

$$C = G(F)\backslash(G(F)/T(F)) \times G(\mathbb{A}_f), \tag{4.1.5}$$

then the topology and measure of $C$ is still induced by those of $G(\mathbb{A}_f)$ and the *discrete* ones of $G(F)/T(F)$.

Let $m$ be a *real valued* function on $G(F)$ which is $T(F)$-invariant and such that $m(\gamma) = m(\gamma^{-1})$. Then $m$ can be extended to $G(F)/T(F) \times G(\mathbb{A}_f)$ such that

$$m(\gamma, g_f) = \begin{cases} m(\gamma) & \text{if } g_f = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.6}$$

We now have a kernel function

$$k(x, y) = \sum_{\gamma \in G(F)} m(x^{-1}\gamma y) \tag{4.1.7}$$

on $C \times C$. Then we can define a pairing on $\mathcal{S}(C)$ by

$$\begin{aligned} \langle \phi, \psi \rangle &= \int_{C^2} \phi(x) k(x, y) \bar{\psi}(y) dx dy \\ &:= \lim_{U \to 1} \int_{C^2} \phi(x) k_U(x, y) \bar{\psi}(y) dx dy \end{aligned} \tag{4.1.8}$$

where $U$ runs through the open subgroup of $G(\mathbb{A}_f)$ and

$$k_U(x, y) = \mathrm{vol}(U)^{-2} \int_{U^2} k(xu, yv) du dv.$$

This pairing is called a *geometric pairing with multiplicity function $m$*. For two function $\psi$ and $\phi$ in $\mathcal{S}(\chi, T(\mathbb{A}_f)\backslash G(\mathbb{A}_f)$, one has

$$\begin{aligned} \langle \phi, \psi \rangle &= \int_{[T(F)\backslash G(\mathbb{A}_f)]^2} \phi(x) \sum_{\gamma \in G(F)} m(x^{-1}\gamma y) \bar{\psi}(y) dx dy \\ &= \sum_{\gamma \in T(F)\backslash G(F)/T(F)} m(\gamma) \langle \phi, \psi \rangle_\gamma \end{aligned} \tag{4.1.9}$$

where

$$\begin{aligned} \langle \phi, \psi \rangle_\gamma &= \int_{T(F)\backslash G(\mathbb{A}_f)} \sum_{\delta \in T(F)\backslash T(F)\gamma T(F)} \phi(\delta y) \bar{\psi}(y) dy \\ &= \int_{T_\gamma(F)\backslash G(\mathbb{A}_f)} \phi(\gamma y) \bar{\psi}(y) dy \end{aligned} \tag{4.1.10}$$

and where

$$T_\gamma := \gamma^{-1} T \gamma \cap T = \begin{cases} T & \text{if } \gamma \in N_T, \\ 1 & \text{otherwise.} \end{cases} \tag{4.1.11}$$

where $N_T$ is the normalizer of $T$ in $G$. The integral $\langle \phi, \psi \rangle_\gamma$ is called the *linking number* of $\phi$ and $\psi$ at $\gamma$.

**Local linking numbers.** Let $\chi$ be a character of $T(F)\backslash T(\mathbb{A}_f)$. In the following we want to compute the linking number of the pairing on the space $\mathcal{S}(\chi, C)$. In this case

$$\langle\phi,\psi\rangle_\gamma = \int_{T(\mathbb{A}_f)\backslash G(\mathbb{A}_f)} \widetilde{\phi}(\gamma, y)\bar{\psi}(y)dy \tag{4.1.12}$$

where

$$\widetilde{\phi}(\gamma, y) = \int_{T_\gamma(F)\backslash T(\mathbb{A}_f)} \phi(t^{-1}\gamma ty)dt. \tag{4.1.13}$$

If $\gamma \in N_T$, then

$$\widetilde{\phi}(\gamma, y) = \mathrm{vol}(T(F)\backslash T(\mathbb{A}_f)) \cdot \phi(\gamma y) \begin{cases} 1 & \text{if } \gamma \in T \text{ or } \chi^2 = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.14}$$

If $\gamma \notin N_T$ and $\phi = \otimes\phi_v$, then we have decomposition

$$\widetilde{\phi}(\gamma, y) = \prod \widetilde{\phi}_v(\gamma, y_v), \qquad \widetilde{\phi}_v(\gamma, y_v) = \int_{G(F_v)} \phi(t^{-1}\gamma ty)dt. \tag{4.1.15}$$

Notice that when $\gamma \notin N_T$, $\widetilde{\phi}_v(\gamma, y_v)$ depends on the choice of $\gamma$ in its class in $T(F)\backslash G(F)/T(F)$ while their product $\widetilde{\phi}(\gamma, y)$ does not. This problem can be solved by taking $\gamma$ to be a *trace free element* in its class which is unique up to conjugation by $T(F)$. This can be seen for example by writing $B = K + K\epsilon$ where $\epsilon \in B$ is an element such that $\epsilon^2 \in F^\times$ and $\epsilon x = \bar{x}\epsilon$. Notice that the function

$$\xi(a + b\epsilon) = \frac{\mathrm{N}(b\epsilon)}{\mathrm{N}(a + b\epsilon)} \tag{4.1.16}$$

defines an embedding

$$T(F)\backslash G(F)/T(F) \longrightarrow F \tag{4.1.17}$$

such that $\xi(\gamma) = 0$ (resp. 1) iff $\xi \in T$ (resp. $\xi \in N_T - T$). The image of $G(F) \setminus N_T$ is the set of $\xi \in F$ such that $\xi \neq 0, 1$ and where for any place $v$ of $F$,

$$1 - \xi^{-1} \in \begin{cases} \mathrm{N}(K^\times) & \text{if } B_v \text{ is split,} \\ F^\times \setminus \mathrm{N}(K^\times) & \text{if } B_v \text{ is not split,} \end{cases}$$

or equivalently,

$$\omega_v(-\xi\eta) = (-1)^{\delta(B_v)}, \tag{4.1.18}$$

where $\eta = 1 - \xi$, and $\delta(B_v) = 0$ if $B_v$ is split and $\delta(B_v) = 1$ if $B_v$ is nonsplit. Then we may write $\gamma(\xi)$ for a trace free element $\gamma \in G(F)$ with $\xi(\gamma) = \xi$. We may write $m(\xi)$ for $m(\gamma(\xi))$ and $\widetilde{\phi}(\xi, y)$ for $\widetilde{\phi}(\gamma(\xi), y)$. We extend $m(\xi)$ to all $F$ by setting $m(\xi) = 0$ if $\xi$ is not in the image of (4.1.17).

In the following computation, we will fix one order $R$ of $B$ such that

$$R_v = \mathcal{O}_{K,v} + \mathcal{O}_{K,v}\lambda_v \tag{4.1.19}$$

where

- $\lambda_v \in B_v^\times$ such that $\lambda_v x = \bar{x}\lambda_v$ for all $x \in K$,
- $\lambda_v^2 \in F_v^\times$, and $\lambda_v$ is divisible by $c(\chi_v)$.

Let $\Delta$ be a subgroup of $G(\mathbb{A}_f)$ generated by images of $\widehat{R}^\times$ and $K_v^\times$ for $v$ ramified in $K$:

$$\Delta = \prod_{v \nmid c(\omega_v)} R_v^\times F_v^\times / F_v^\times \cdot \prod_{v | c(\omega_v)} R_v^\times K_v^\times / F_v^\times$$

and take an $a \in \mathbb{A}_f^\times$, such that $\mathrm{ord}_v(a) = 0$ if $R_v$ is not maximal. Then we set the measure on $C$ such that the quotient measure on $T(\mathbb{A}_f)\backslash G(\mathbb{A}_f)$ has volume 1 on $T(\mathbb{A}_f)\backslash T(\mathbb{A}_f)\Delta$. Now the character can be naturally extended to a character of $\Delta$. We will compute the geometric pairing for

$$\phi = \mathrm{T}_a\phi_\Delta, \qquad \psi = \phi_\Delta, \qquad \phi_\Delta = \prod \phi_{\Delta_v} \tag{4.1.20}$$

with $\phi_{\Delta_v}$ supported on $T(F_v)\cdot\Delta_v$ and such that

$$\phi_\Delta(tu) = \chi(t)\chi(u), \qquad u \in \Delta. \tag{4.1.21}$$

The Hecke operator here is defined as

$$\mathrm{T}_a\phi(x) = \prod_v \mathrm{T}(a_v)\phi_v, \qquad \mathrm{T}_{a_v}\phi_v = \int_{H(a_v)} \phi_v(xg)dg, \tag{4.1.22}$$

where

$$H(a_v) := \{g \in M_2(\mathcal{O}_v) : \quad |\det g| = |a_v|\}, \tag{4.1.23}$$

and $dg$ is a measure such that $\mathrm{GL}_2(\mathcal{O}_v)$ has volume 1. Then we have

$$\begin{aligned} &\langle \mathrm{T}_a\phi_\Delta, \phi_\Delta\rangle \\ =&\mathrm{vol}(T(F)\backslash T(\mathbb{A}_f))\left(m(0)\mathrm{T}_a\phi_\Delta(e) + m(1)\mathrm{T}_a\phi_\Delta(\epsilon)\delta_{\chi^2=1}\right) \\ &+ \sum_{\xi\neq 0,1} m(\xi)\prod_v \ell_v(\mathrm{ord}_v(a_v),\xi) \end{aligned} \tag{4.1.24}$$

where $\epsilon \in N_T \setminus T$, and

$$\ell_v(n,\xi) = \int_{T(F_v)} T(\pi_v^n)\phi_{\Delta_v}(t^{-1}\gamma(\xi)t)dt. \tag{4.1.25}$$

## 4.2. Linking numbers

In this section we want to compute the local linking numbers defined at the end of the last section. Thus, we change the notation to let $F$ denote a nonarchimedean local field. Let $B$ denote a quaternion algebra over $F$, and let $G$ denote the algebraic group $B^\times/F^\times$.

Let $K/F$ be a quadratic extension of $F$ embedded into $B$. Let $R$ be an order of $B$ of the type

$$R = \mathcal{O}_K + \mathcal{O}_K\lambda, \qquad \lambda = \pi_K^m\epsilon \tag{4.2.1}$$

where

- $\epsilon \in N_T \setminus T$ such that $\mathrm{ord}(\epsilon^2) = 0$ unless $B$ is nonsplit and $K/F$ is unramified where $\mathrm{ord}(\epsilon^2) = 1$;
- $\pi_K \in K^\times$ is a local parameter if $K$ is nonsplit; otherwise it is the local parameter of one component of $K = F \oplus F$;
- $m \geq \mathrm{ord}_K(\chi)$.

Let $T = K^\times/F^\times$ denote the subgroup of $G$. Let $\chi$ be a character of $T(F)$ and let $\Delta$ denote $R^\times$ if $K/F$ is unramified, and $R^\times \cdot T(F)$ if $K/F$ is ramified. Then the character $\chi$ can be extended to $R^\times$. Let $\phi$ be a function on $G(F)$ supported on $T(F) \cdot \Delta$ such that

$$\phi(tu) = \chi(t)\chi(u) \qquad t \in T(F), \quad u \in \Delta. \tag{4.2.2}$$

Let $n$ be a nonnegative integer such that $n = 0$ if $\Delta$ is not maximal. Then we want to compute the following *degenerate terms*

$$\ell(n, 0) := \mathrm{T}_{\pi^n}\phi(e), \qquad \ell(n, 1) := \mathrm{T}_{\pi^n}\phi(\epsilon) \tag{4.2.3}$$

where $\epsilon \in N_T \setminus T$ and local linking number is

$$\ell(n, \xi) := \int_{T(F)} \mathrm{T}_{\pi^n}\phi_\Delta(t^{-1}\gamma(\xi)t)dt \tag{4.2.4}$$

where the $dx$ is a Haar measure on $T(F)$ normalized such that the volume of $T(\mathcal{O}_v)$ is one if $v$ is split, and the volume of $T(F_v)$ is one if $v$ is nonsplit. Here $\xi \in F$ such that $\xi \neq 0, 1$ and such that

$$1 - \xi^{-1} \in \begin{cases} \mathrm{N}(K^\times) & \text{if } B \text{ is split,} \\ F^\times \setminus \mathrm{N}(K^\times) & \text{if } B \text{ is non-split,} \end{cases} \tag{4.2.5}$$

and $\gamma(\xi) \in B^\times$ is a trace free element such that $\xi(\gamma) = \xi$. We extend this definition to all $\xi \in F$ by insisting that $\ell(n, \xi) = 0$ if $\xi$ does not satisfy the above condition. Lets start with the degenerate terms.

**Degenerate terms.**

LEMMA 4.2.1. *If $n = 0$, then*

$$\ell(0, 0) = 1, \qquad \ell(0, \epsilon) = \begin{cases} 1 & \textit{if } \chi^2 = 1,\ m = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

*If $n > 0$, $K/F$ is nonsplit, then*

$$\ell(n, 0) = \ell(n, 1) = \begin{cases} 1 & \textit{if } n \equiv 0 \mod 2, \\ 0 & \textit{otherwise.} \end{cases}$$

*If $n > 0$ and $K = F \oplus F$ with $\chi = (\mu, \mu^{-1})$, then*

$$\ell(n, 0) = \ell(n, 1) = \sum_{i+j=n} \mu(\pi)^{i-j}.$$

*Proof.* The case of $n = 0$ is clear.

If $n > 0$, then $B$ is split and $\Delta$ is maximal so we may fix one isomorphism $G \simeq \mathrm{PGL}_2$ such that $\Delta = \mathrm{PGL}_2(\mathcal{O}_F)$. We are thus reduced to the computation of the local integrals

$$\int_{H(\pi^n)} \phi(g)dg = \int_{H(\pi^n)} \phi(\epsilon g)dg,$$

where

$$H(\pi^n) = \{g \in M_2(\mathcal{O}_v) : \quad \mathrm{ord}_v(\det g) = n\}.$$

Lets evaluate this integral in two cases.

Case 1: $K$ is an unramified field extension of $F$. Then we may write

$$M_2(F_v) = K_v + K_v\epsilon,$$

where $\epsilon^2 = 1$ such that $\epsilon x = \bar{x}\epsilon$ for all $x \in \mathcal{O}_{K,v}$. Now $H(\pi^n)$ is a sum of $H_i$ $(i = 0, \cdots, [n/2])$ where

$$H_i = \left\{\pi^i(a + b\epsilon) \in H(\pi^n) : \quad (a, b) = 1\right\}.$$

Notice that $H_i$ is not disjoint with $K^\times$ if and only if $i = n/2$. It follows that

$$\int_{H(\pi^n)} \phi(g)dg = \begin{cases} 1 & \text{if } n \equiv 0 \mod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Case 2: $K = F \oplus F$, and $\chi = (\mu, \mu^{-1})$. Then $H(\pi^n)$ has the following representatives modulo $\mathrm{GL}_2(\mathcal{O}_v)$:

$$\begin{pmatrix} \pi^i & x \\ 0 & \pi^j \end{pmatrix} \qquad i + j = n, x \mod \pi^i.$$

The term with $x \neq 0$ has trivial contribution to the integral. Thus we have

$$\int_{H(\pi^n)} \phi(g)dg = \sum_{i+j=n} \mu(\pi)^{i-j}.$$

□

**Unramified case.** We now assume that both $K/F$ and $\chi$ are unramified, that $B = M_2(F)$ is split, and $\Delta = \mathrm{PGL}_2(\mathcal{O}_F)$ is maximal. We want to compute $\ell(n, \xi)$.

LEMMA 4.2.2. *Assume that $K$ is a field. Let $\eta = 1 - \xi$. Then $\ell(n, \xi) \neq 0$ only if both $\mathrm{ord}(\xi\pi^n)$ and $\mathrm{ord}(\eta\pi^n)$ are even and nonnegative. In this case,*

$$\ell(n, \xi) = 1.$$

*Proof.* By definition $\ell(n, \xi) \neq 0$ only if $\mathrm{ord}(1 - \xi^{-1}) = \mathrm{ord}(\eta\xi^{-1})$ is even or equivalently, $\xi = \xi(\gamma)$ for some trace free $\gamma \in M_2(F)$.

Under our assumption, $\chi = 1$ and $\phi(g) \neq 0$ only if $g \in \pi^n\Delta$ for some $n$. In this case it is 1. It follows that $\ell(n, \xi) \neq 0$ only if $\mathrm{ord}_\pi(\det \gamma) + n$ is an even number, say $2m$. It follows that

$$\ell(n, \xi) = \int_{T(F)} \int_{H(\pi^n)} \phi(\pi^{-m}t^{-1}\gamma tg)dgdt.$$

Without loss of generality we assume that $\gamma$ is given by $u(1+\alpha\epsilon)$ where $u$ is a trace-free unit of $\mathcal{O}_K$, and $\epsilon^2 = -1$. Let $|\pi|^w = |\det\gamma| = |1-\alpha\bar{\alpha}|$. Then $2m = n+w$.

Now $H(\pi^n)$ is the union of $H_i$ $(0 \le i \le [n/2])$:

$$H_i = \pi^i \left\{a + b\epsilon \in H(\pi^{n-2i}), \quad (a,b) = 1\right\}.$$

Thus

$$\ell(n,\xi) = \sum_{i\ge 0} \ell_i(n,\xi),$$

where

$$\ell_i(n,\xi) = \int_{T(F)}\int_{H_i} \phi(\pi^{-m}t^{-1}\gamma tg)dgdt.$$

If $i = n/2$, then $H(1) = \Delta$ and

$$\ell_i(n,\xi) = \int_{T(F)} \phi\left(\pi^{i-m}t^{-1}\gamma t\right) dt.$$

This is nonzero only if $\gamma \in \pi^{w/2}\Delta$ and is given by

$$\mathrm{vol}(T(F)) = 1.$$

Notice that the condition $\gamma \in \pi^{w/2}\Delta$ is equivalent to $w \le 0$.

If $i < n/2$, as $\det(a + b\epsilon) = \bar{a}a - \pi^{2c}\bar{b}b$, one even has $|a| = |b| = 1$ for every $a + b\epsilon$ in $H_i$. Thus there is a finite subset $B_i$ of $b \in \mathcal{O}_K$ such that $|\mathrm{N}(b) - 1| = |\pi|^{n-2i}$ such that

$$H_i = \bigcup_{b\in B_i} (1 + b\epsilon)\pi^i\Delta.$$

To give a nice description of $B_i$, we notice that for $b, b' \in \mathcal{O}_K$ with

$$|\bar{b}b - 1| = |\bar{b}b - 1| = |\pi|^{n-2i},$$

we have

$$(\pi^c + b\epsilon)(\pi^c + b'\epsilon)^{-1} \in \Delta,$$

if and only if $b \equiv b' \pmod{\pi^{n-2i}}$. Thus the projection $\mathcal{O}_K^\times \longrightarrow (\mathcal{O}_K/\pi^{n-2i})^\times$ is injective on $B_i$. The image of $B_i$ is exactly the set $(\mathcal{O}_K/\pi^{n-2i})^{\mathrm{N}=1}$ of elements of norm 1, since every element $b \in (\mathcal{O}_K/\pi^{n-2i}\mathcal{O}_K)^\times$ with norm 1 can be lifted to an element $\widehat{b}$ of $\mathcal{O}_K^\times$ such that $|\mathrm{N}(\widehat{b}) - 1| = |\pi|^{n-2i}$.

The contribution from $H_i$ is given by

$$\sum_{b\in B_i}\int_{|t|=1} \phi\left(\pi^{i-m}(1 + \alpha t\epsilon)(1 + b\epsilon)\right) dt.$$

The matrix inside the integral is

$$\pi^{i-m}\left[(1 + \alpha\bar{b}t) + (b + \alpha t)\epsilon\right].$$

If we first sum over $b$ and then compute the integral, then the integral simply counts the number of $b$ such that this matrix is integral. Write $j = m - i = (n - 2i + w)/2$ then $2j - w = 2n - i > 0$ and the contribution is

$$\ell_i(\eta, n) = \#\left\{ \begin{array}{c} b \mod \pi^{2j-w} \\ |b\bar{b} - 1| = |\pi|^{2j-w} \end{array} \middle| \begin{array}{c} |\bar{b}^{-1} + \alpha| \leq |\pi|^j \\ |b + \alpha| \leq |\pi|^j \end{array} \right\}.$$

Recall that $|1 - \bar{\alpha}\alpha| = |\pi|^w$. If $|\alpha| < 1$, then $w = 0$ and the last equation gives $j \leq 0 \leq w/2$. This is a contradiction. If $|\alpha| > 1$, then $w < 0$ and $|\alpha| = |\pi|^{w/2}$. The last equation implies that $w/2 \geq j$, which is again a contradiction. Thus we must have $|\alpha| = 1$ and $w \geq 0$.

The last two equations imply that $|\bar{b}^{-1} - b| \leq |\pi|^j$ (resp. $|\bar{\alpha}^{-1} - \alpha| \leq |\pi|^j$) or equivalently, $|\bar{b}b - 1| \leq |\pi|^j$ (resp. $|\alpha\bar{\alpha} - 1| \leq |\pi|^j$). By the first equation (resp. definition of $w$) we have $2j - w \geq j$ (resp. $w \geq j$). Thus we have $j = w > 0$. Notice that in this case the system has a unique solution.

In summary, we obtain that $\ell(n, \xi) \neq 0$ only if $n - w$ is even and nonnegative. In this case,

$$\ell(n, \xi) = 1.$$

The lemma now follows since

$$\xi = \frac{-\mathrm{N}(\alpha)}{1 - \mathrm{N}(\alpha)}.$$

□

Lemma 4.2.3. *Assume that $K = F \oplus F$ is split, and $\chi = (\mu, \mu^{-1})$. Then $\ell(n, \xi)$ is nonzero only if $|\xi\pi^n| \leq 1$. In this case,*

$$\ell(n, \xi) = \frac{\mu(\eta\pi^{n+1}) - \mu^{-1}(\eta\pi^{n+1})}{\mu(\pi) - \mu^{-1}(\pi)} \cdot \mathrm{ord}(\xi\pi^{n+1})$$

*where $\eta = 1 - \xi$.*

*Proof.* In this case, we identify $T$ with the group of matrices $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ in $\mathrm{PGL}_2$, and set

$$\epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} -1 & \alpha \\ 1 & 1 \end{pmatrix}, \qquad \alpha = \frac{\xi}{1 - \xi}.$$

Now $H(\pi^n)$ is the union

$$H(\pi^n) = \bigcup_{\substack{i+j=n \\ x \in (\mathcal{O}_F/\pi^k)^\times}} \begin{pmatrix} \pi^i & 0 \\ 0 & \pi^j \end{pmatrix} \begin{pmatrix} \pi^k & x \\ 0 & 1 \end{pmatrix} \Delta.$$

The function $\phi(g)$ is nonzero if and only if this matrix is in $\begin{pmatrix} \pi^u & 0 \\ 0 & \pi^v \end{pmatrix} \Delta$ for some $u, v$ such that $|\pi|^{u+v} = |\det g|$. In this case the value of $\phi$ is given by $\chi(\pi^{u-v})$. Thus

$$\ell(n, \xi) = \sum_{u+v=n+w} \mu(\pi)^{u-v} \ell(n, \xi, u, v),$$

where $u, v$ are integers, $|\pi|^w = |\det\gamma| = |1+\alpha|$, and

$$\ell(n,\xi,u,v) = \sum_{\substack{i+j=n\\ x\in\mathcal{O}_F/\pi^i}} \int_{F^\times} \phi\left(\begin{pmatrix}\pi^{-u} & 0\\ 0 & \pi^{-v}\end{pmatrix}\begin{pmatrix}-1 & \alpha t^{-1}\\ t & 1\end{pmatrix}\begin{pmatrix}\pi^i & x\\ 0 & \pi^j\end{pmatrix}\right).$$

The product of these 3 matrices is

$$\begin{pmatrix}-\pi^{i-u} & -x\pi^{-u}+\alpha\pi^{j-u}t^{-1}\\ t\pi^{i-v} & t\pi^{-v}x+\pi^{j-v}\end{pmatrix}.$$

Change variable $t \to t\pi^{v-i}$ we obtain

$$\begin{pmatrix}-\pi^{i-u} & \pi^{-u}(-x+\alpha\pi^{u-w}t^{-1})\\ t & \pi^{-i}(tx+\pi^{u-w})\end{pmatrix}.$$

Notice that the value

$$\sum_{x\in\mathcal{O}_F/\pi^i} \phi\left(\begin{pmatrix}-\pi^{i-u} & \pi^{-u}(-x+\alpha\pi^{u-w}t^{-1})\\ t & \pi^{-i}(tx+\pi^{u-w})\end{pmatrix}\right)$$

depends only on $|t|$. It follows that

$$\ell(n,\xi,u,v) = \#\left\{\begin{array}{c} 0\le i\le n,\\ k\ge 0\\ x \mod \pi^i\end{array}\;\middle|\;\begin{array}{c} i\ge u\\ |x-\alpha\pi^{u-w-k}|\le|\pi|^u\\ |x+\pi^{u-w-k}|\le|\pi|^{i-k}\end{array}\right\}.$$

First we assume that $w>0$. Let $\beta=(\alpha+1)\pi^{-w}\in\mathcal{O}_F^\times$ then

$$\ell(n,\xi,u,v) = \#\left\{\begin{array}{c} 0\le i\le n,\\ k\ge 0\\ x \mod \pi^i\end{array}\;\middle|\;\begin{array}{c} i\ge u\\ |x+\pi^{u-w-k}-\beta\pi^{u-k}|\le|\pi|^u\\ |x+\pi^{u-w-k}|\le|\pi|^{i-k}\end{array}\right\}.$$

If $u-w-k<0$ then the third condition implies that $u-w-k\ge i-k$ or $u\ge w+i$ which contradicts to the first condition. Thus the quantity is nonzero only if $n\ge u\ge w$; in this case, we may replace $x$ by $\pi^{u-w-k}+\pi^{i-k}y$ for $y\in\mathcal{O}_F/\pi^k$. The equation becomes

$$\ell(n,\xi,u,v) = \#\left\{\begin{array}{c} u\le i\le n,\\ u-w\ge k\ge 0\\ y \mod \pi^k\end{array}\;\middle|\;|y-\beta\pi^{u-i}|\le|\pi|^{u+k-i}\right\}.$$

If $u<i$, then the condition implies $u-i\ge u+k-i$ or simply $k=0$ and $y=0$. The contribution in this case is $n-u$. If $u=i$, then the equation has a unique solution in $y$. Thus the contribution is $u-w+1$. Thus we have

$$w>0 \Longrightarrow \ell(n,\xi,u,v) = \begin{cases} n-w+1 & \text{if } n\ge u\ge w,\\ 0 & \text{otherwise.}\end{cases}$$

We now consider the case $w=0$. Write $\alpha=\beta\pi^t$ with $t\ge 0$ and $\beta\in\mathcal{O}_F$. Then

$$\ell(n,\xi,u,v) = \#\left\{\begin{array}{c} 0\le i\le n,\\ k\ge 0\\ x \mod \pi^i\end{array}\;\middle|\;\begin{array}{c} i\ge u\\ |x-\beta\pi^{u+t-k}|\le|\pi|^u\\ |x+\pi^{u-k}|\le|\pi|^{i-k}\end{array}\right\}.$$

If $u-k<0$, then the last equation gives $u-k \geq i-k$ or $u \geq i$. Combining with the first equation we have $u=i$. The second equation is solvable only if $u+t-k \geq 0$. In this case it has the unique solution $x=\beta u^{u+t-k}$, which also satisfies the third equation. Thus this case has nontrivial solution only if $0 \leq u \leq n$; and the contribution is $t$ ( $=$ the number of $k$'s such that $u+t \geq k > u$).

Assume now that $u \geq k$. Then $i \geq k$ and we may replace $x$ by $-\pi^{u-k}+\pi^{i-k}y$ with $y \in \mathcal{O}_F/\pi^k$. The contribution is

$$\#\left\{ \begin{array}{c} u \leq i \leq n, \\ u \geq k \geq 0 \\ y \mod \pi^k \end{array} \,\middle|\, |y-(1+\alpha)\pi^{u-i}| \leq |\pi|^{u+k-i} \right\}.$$

If $u<i$, the condition implies that $u-i \geq u-i+k$. Thus $k=0$. The equation has a unique solution and the contribution in this case is $n-u$ ( $=$ the number of $i$'s such that $u<i \leq n$). If $u=i$, then still the equation has a unique solution and the contribution is $u+1$. Thus we obtain

$$w=0 \Longrightarrow \ell(n,\xi,u,v)=\begin{cases} n+t+1 & \text{if } n \geq u \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

It remains to treat the case where $w<0$. Let $\beta=(\alpha+1)\pi^{-w} \in \mathcal{O}_F^\times$ then

$$\ell(n,\xi,u,v)=\#\left\{ \begin{array}{c} 0 \leq i \leq n, \\ k \geq 0 \\ x \mod \pi^i \end{array} \,\middle|\, \begin{array}{c} i \geq u \\ |x+\pi^{u-w-k}-\beta\pi^{u-k}| \leq |\pi|^u \\ |x+\pi^{u-w-k}| \leq |\pi|^{i-k} \end{array} \right\}.$$

It $u-k<0$ then the second equation implies that $u-k \geq u$. Thus $k=0$ and $u<0$; in this case the second equation trivially holds for all $x$. The last equation is solvable only if $u-w \geq 0$ then it has a unique solution $x=-\pi^{u-w}$. The contribution in the case $u<k$ is nonzero only if $0>u \geq w$. Then it is given by $n+1$ ( $=$ number of $i$'s).

If $u-k \geq 0$, then we may replace $x$ by

$$-\pi^{u-w-k}+\beta\pi^{u-k}+\pi^u y, \qquad y \in \mathcal{O}_F/\pi^{i-u}.$$

The contribution is then

$$\#\left\{ \begin{array}{c} u \leq i \leq n, \\ u \geq k \geq 0 \\ y \mod \pi^{i-u} \end{array} \,\middle|\, |\beta\pi^{-k}+y| \leq |\pi|^{i-k-u} \right\}.$$

If $k>0$, then the equation implies that $-k \geq i-k-u$. Thus $u=i$. The contribution is $u$ (number of $k$'s). If $k=0$, then the equation still has a unique solution. The contribution is $n-u+1$ ( $=$ number of $i$'s). Thus the contribution in the case $u \geq k$ is nonzero only if $n \geq u \geq 0$, and then it is given by $n+1$. Thus we have

$$w<0 \Longrightarrow \ell(n,\xi,u,v)=\begin{cases} n+1 & \text{if } w \leq u \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

We now apply the following formula for integers $n \geq c$,

$$\sum_{\substack{a+b=n+c \\ c \leq a \leq n}} x^a y^b = \sum_{\substack{a+b=n-c \\ 0 \leq a \leq n-c}} x^{a+c}y^{b+c} = (xy)^c \cdot \frac{x^{n-c+1}-y^{n-c+1}}{x-y}.$$

We obtain that $\ell(n,\xi)$ is nonzero only if $w \le n$. In this case,

$$\ell(n,\xi) = \frac{\mu(\pi)^{n-w+1} - \mu^{-1}(\pi)^{n-w+1}}{\mu(\pi) - \mu^{-1}(\pi)} \cdot (n+t-w+1)$$

where $\gamma = \begin{pmatrix} -1 & \alpha \\ 1 & 1 \end{pmatrix}$, $|\pi|^w = |\det\gamma|$, $|\pi|^t = |\alpha|$. The lemma now follows since

$$\eta = \frac{1}{1+\alpha}, \qquad \xi = \frac{\alpha}{1+\alpha}.$$

□

**Ramified case.** It remains to compute $\ell(0,\xi)$ in the ramified case where $\Delta$ is not maximal.

LEMMA 4.2.4. *Assume that $K$ is split, that $\chi = (\mu,\mu^{-1})$. Then $\ell(0,\xi)$ is nonzero only if $|\xi| \le |\det\lambda|$. In this case,*

$$\ell(0,\xi) = \mu(-1)\mathrm{ord}(\xi\pi\det\lambda^{-1}).$$

*Proof.* We now embed $K$ into the diagonal of $M_2(F)$ such that $\pi_K$ is sent to $\begin{pmatrix} 0 & 0 \\ 0 & \pi \end{pmatrix}$. Then $R$ is the order of matrices $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F)$ with $|z| \le |\pi|^m$. As before we may take $\gamma = \begin{pmatrix} -1 & \alpha \\ 1 & 1 \end{pmatrix}$ with $\alpha = \xi/(1+\xi)$. The integral,

$$\ell(0,\xi) = \int_{F^\times} \phi\left(\begin{pmatrix} -1 & \alpha t^{-1} \\ t & 1 \end{pmatrix}\right) dt,$$

is nonzero only if there are some elements $u, v, t \in F^\times$ such that

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\begin{pmatrix} -1 & \alpha t^{-1} \\ t & 1 \end{pmatrix} = \begin{pmatrix} -u & u\alpha t^{-1} \\ vt & v \end{pmatrix} \in \Delta.$$

This implies that $u, v \in \mathcal{O}_F^\times$, that $|\pi|^{\mathrm{ord}(\alpha)} \le |t| \le |\pi|^m$, and that $\mathrm{ord}(\alpha) \ge m$. Conversely, if $\mathrm{ord}(\alpha) \ge m$, then

$$\begin{aligned}\ell(0,\xi) &= \int_{|\pi|^{\mathrm{ord}(\alpha)} \le |t| \le |\pi|^m} \phi\left(\begin{pmatrix} -1 & \alpha t^{-1} \\ t & 1 \end{pmatrix}\right) dt \\ &= \mu(-1)(\mathrm{ord}(\alpha) - m + 1).\end{aligned}$$

The lemma follows. □

LEMMA 4.2.5. *Assume that $K$ is an unramified extension of $F$. Then $\ell(0,\xi)$ is nonzero only if $\xi\det\lambda^{-1}$ is even and non-negative. In this case*

$$\ell(0,\xi) = \chi(u).$$

*Proof.* Write $\mathrm{ord}(\det\lambda) = \delta(B) + 2m$ where $\delta(B) = 0$ if $B$ is split and $\delta(B) = 1$ if $B$ is nonsplit, and where $u$ is any trace free unit of $\mathcal{O}_K$.

By definition, $\ell(0,\xi) \ne 0$ only if $\mathrm{ord}_v(\eta\xi^{-1}\pi_v^{\delta(B)})$ is even or equivalently, $\xi = \xi(\gamma)$ for some trace free element in $B^\times$.

In this way, we may write $\gamma = u(1+\alpha\epsilon)$ with $u$ a trace free unit of $\mathcal{O}_K$. Now

$$\ell(0,\xi) = \int_{T(\mathcal{O}_F)} \phi(t^{-1}\gamma t)dt.$$

The integral is nonzero only if $\gamma \in T\Delta$. This is equivalent to the fact that the number $w = \mathrm{ord}(\det\gamma)$ is even and $\leq 0$, and that $|\alpha| \leq |\pi|^{m-w/2}$. This in turn is equivalent to $|\alpha| \leq |\pi|^m$ (then $w = 0$). In this case the integral equals $\chi(u)$. Since $\xi = \mathrm{N}(\alpha)\pi^{\delta(B)}/(1+\mathrm{N}(\alpha)\pi^{\delta(B)})$, the lemma follows. □

Lemma 4.2.6. *Assume that $K/F$ is ramified, that $\chi$ is unramified with the form $\chi = \nu \circ \mathrm{N}$. Then $\ell(0,\xi)$ is nonzero only if*

$$|\xi| \leq |\det\lambda|, \qquad \omega_v(-\xi\eta) = (-1)^{\delta(B)}$$

*where $\eta = 1-\xi$, and $\delta(B) = 0$ if $B$ is split and $\delta(B) = 1$ if $B$ is nonsplit. In this case,*

$$\ell(0,\xi) = \nu(\eta\pi)$$

*where $\nu$ is a (quadratic) character of $F^\times$ such that $\chi = \nu\circ\mathrm{N}$.*

*Proof.* By definition, $\ell_v(0,\xi)$ is nonzero only if

$$\omega_v(1-\xi^{-1}) = \omega_v(-\xi\eta) = (-1)^{\delta(B)}$$

or equivalently, $\xi = \xi(\gamma)$ for some trace free element $\gamma \in B^\times$. In this case, the integral is a sum

$$\begin{aligned}
\ell(0,\xi) &= \int_{T(\mathcal{O}_F)} \phi(t^{-1}\gamma t)dt + \int_{\pi_K T(\mathcal{O}_F)} \phi(t^{-1}\gamma t)dt \\
&= \int_{T(\mathcal{O}_F)} \left(\phi(t^{-1}\gamma t) + \phi(t^{-1}\pi_K^{-1}\gamma\pi_K t)\right) dt \\
&= \phi(\gamma).
\end{aligned}$$

In the last step, we have used the fact that $\pi_K$ normalizes $\Delta$.

Now $\phi(\gamma)$ is nonzero only if $\gamma \in T\Delta$, or equivalently, $|\xi| \leq |\pi|^m$. In this case $\phi(\gamma) = \nu(\det\gamma)$. We may choose $\gamma$ of the form $\pi_K(1+\alpha\epsilon)$ with $\pi_K^2 = \pi$ to be a parameter of $F$. Then

$$\phi(\gamma) = \nu(-\pi(1-\epsilon^2\mathrm{N}(\alpha)) = \nu(-\pi\eta^{-1}) = \nu(\pi\eta).$$

□

### 4.3. Local Gross-Zagier formula

We now go back to the global setting in §3.1 and §4.1 with even $\Sigma$ and the quaternion algebra $B$ ramified exactly at places in $\Sigma$ and the elements $\lambda_v \in B^\times$ given by the following formula:

$$\mathrm{ord}(\det\lambda_v) = \begin{cases} \mathrm{ord}_v([D,N]) & \text{if } v \text{ is unramified in } K, \\ 0 & \text{if } v \text{ is ramified in } K. \end{cases} \tag{4.3.1}$$

In this section we want to prove a *local Gross-Zagier formula* by comparing the local Fourier coefficients $W_v(1/2, \xi, \eta, g)$ computed in §3.4 and the linking numbers $\ell_v(n, \xi)$ computed in §4.2. Then we apply this to the global case to get some *pre-Gross-Zagier formula* with arbitrary multiplicity function.

Let $v$ be a fixed finite place of $F$. We have extended the definition to all $\xi \in F \setminus \{0, 1\}$ by insisting that $\ell_v(n, \xi) = 0$ when $\xi$ is not in the image of (4.1.17).

LEMMA 4.3.1 (Local Gross-Zagier formula). *Let $\eta = 1 - \xi$ and $g = \begin{pmatrix} \pi_v^n \delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ such that $n = 0$ if $\Delta_v$ is not maximal. Then*

$$\bar{W}_v \left( \frac{1}{2}, \xi, \eta, \epsilon g \right) = |c(\omega_v)|^{1/2} \cdot \epsilon(\omega_v, \psi_v) \chi_v(u) \cdot |\eta\xi|_v^{1/2} |\pi_v^n| \cdot \ell_v(n, \xi)$$

*where $u$ is any trace free element in $K^\times$.*

*Proof.* First lets consider the unramified case: $c_v = 0$. This case follows easily from Lemma 3.4.1, Lemma 4.2.2, and 4.2.3.

Lets consider now the ramified case: $c_v > 0$ but $\omega_v$ is unramified. The formula follows from Lemma 3.4.1, 3.4.2, 4.2.4, 4.2.5.

The case where $\omega_v$ is ramified follows from Lemma 3.4.3 and 4.2.6. □

COROLLARY 4.3.2 (pre-GZF). *Let $\langle \cdot, \cdot \rangle$ be the geometric pairing on the CM-cycle with multiplicity function $m$ on $F$ such that $m(\xi) = 0$ if $\xi$ is not in the image of (4.1.17). Assume that $\delta_v = 1$ for $v \mid \infty$. Then there are constants $c_1$, $c_2$ such that for an integral idele $a$ prime to $ND$,*

$$\begin{aligned} |c(\omega)|^{1/2} |a| \langle \mathrm{T}_a \phi_\Delta, \phi_\Delta \rangle =& (c_1 m(0) + c_1 m(1)) |a|^{1/2} W_{\chi, f}(g) \\ &+ i^{[F:\mathbb{Q}]} \sum_{\xi \in F \setminus \{0,1\}} |\xi\eta|_\infty^{1/2} \bar{W}_f(1/2, \xi, \eta, g) m(\xi), \end{aligned}$$

*where $g = \begin{pmatrix} a\delta_f^{-1} & 0 \\ 0 & 1 \end{pmatrix}$.*

*Proof.* This follows from the above theory and the fact that $\prod_v \epsilon(\omega_v, \psi_v) = 1$ and that for $v \mid \infty$, $\epsilon(\omega_v, \psi_v) = i$. □

This pre-GZF will be used for odd $\Sigma$ with $\Sigma$ replaced by ${}_v\Sigma$ for each place $v$, where

$$(4.3.2) \qquad {}_v\Sigma := \begin{cases} \Sigma \setminus \{v\} & \text{if } v \in \Sigma, \\ \Sigma \cup \{v\} & \text{if } v \notin \Sigma. \end{cases}$$

Let ${}_vB$ denote the quaternion algebra ramified at ${}_v\Sigma$.

### 4.4. Gross-Zagier formula for special values

We now want to apply the pre-Gross-Zagier formula for multiplicity function to be the product of the Whittaker function on $\mathrm{GL}_2(F_v)$ $(v \mid \infty)$:

$$(4.4.1) \qquad m(\xi, g_\infty) = |\eta\xi|_\infty^{-1/2} i^{-[F:\mathbb{Q}]} \bar{W}_\infty(1/2, \xi, \eta, \epsilon g_\infty)$$

where $g_\infty \in \mathrm{GL}_2(F_\infty)$ is viewed as a parameter. We set

$$m(0, g_\infty) = m(1, g_\infty) = 0.$$

By corollary 4.3.2, one obtains:

LEMMA 4.4.1. *The complex conjugate of the kernel function $\bar{\Theta}$ has Whittaker function:*

$$\bar{W}\left(1/2, \epsilon g_\infty \cdot \begin{pmatrix} a\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) = |c(\omega)|^{1/2}|a|\langle \mathrm{T}_a\phi, \phi\rangle(g_\infty)$$

*where $a$ is a finite integral idele which has component 1 at those places where either $\chi$, $\Pi$, or $K/F$ is ramified.*

Let $\infty^+$ (resp. $\infty^-$) be the infinite places of $F$ where $\Pi$ is discrete (resp. principal). Then $m(\xi, g_\infty)$ is a product of $m_v(\xi, g)$ where $m_v(\xi, g)$ has weight 2 (resp. 0) if $v \in \infty^+$ (resp. $v \in \infty^-$). By Lemma 3.4.4, its value at $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ is given as follows:

$$m_v\left(\xi, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} 4|a|e^{-2\pi a} & \text{if } 1 \geq \xi \geq 0,\ a > 0,\ v \in \infty^+, \\ 4|a|e^{2\pi a(\xi-\eta)} & \text{if } a\xi \leq 0,\ a\eta \geq 0,\ v \in \infty^-, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4.2}$$

**Spectral decomposition.** Let $U_f = \prod_v U_v$ be an open and compact subgroup of $G(\mathbb{A}_f)$ defined in §4.1 with $\lambda_v$ given in §4.3, and let $U$ be the subgroup $U_\infty U_f$ of $G(\mathbb{A})$ where $U_\infty$ is the unique maximal connected compact subgroup of $G(\mathbb{R})$ containing $T(\mathbb{R})$. Take a measure on $G(F)\backslash G(\mathbb{A})$ induced by a standard measure on $G(\mathbb{R})$ and such that $\mathrm{vol}(U) = 1$. We now consider $m$ as a function on $G(\mathbb{R})$ for a fixed $g_\infty \in \mathrm{GL}_2(\mathbb{R})$. Let $k(x, y)$ to be the kernel function

$$k(x,y) = \sum_{\gamma \in G(F)} m_U(x^{-1}\gamma y) \tag{4.4.3}$$

where

$$m_U(x) = \int_U m_U(xu)du. \tag{4.4.4}$$

In this section we want to decompose $k(x, y)$ into the eigenfunctions in $x, y$.

LEMMA 4.4.2. *As Whittaker functions on $\mathrm{GL}_2(F_\infty)$,*

$$k(x,y)(g_\infty) = 2^{[F:\mathbb{Q}]+n} \sum_{\phi_i} W_i(g_\infty) \cdot \phi_i(x)\bar{\phi}_i(y) + \textit{continuous contribution}$$

*where $n = \#\infty^-$, and the sum is over all cuspidal eigenforms $\phi$ of Laplacian and Hecke operators on $G(F)\backslash G(\mathbb{A})/U$. Here "continous contribution" means a sum of integrations of Eisenstein series. Thus for a cuspidal eigenform $\phi$,*

$$\int_{G(F)\backslash G(\mathbb{A})} k(x,y)\bar{\phi}(y)dy = 2^{[F:\mathbb{Q}]+n} W_\phi(g_\infty)\phi(x).$$

*Proof.* Notice that for a function $\phi$ on $G(F)\backslash G(\mathbb{A})/U$ one has the identity

$$\begin{aligned} \int_{G(F)\backslash G(\mathbb{A})} k(x,y)\phi(y)dy &= \int_{G(\mathbb{A})} m_U(y)\phi(xy)dy \\ &= \int_{G(\mathbb{R})} m(y)\phi(xy)dy \\ &=: \rho(m)(\phi)(x). \end{aligned}$$

Thus, it suffices to study the action defined by $m_v$ on the space of functions on

$$G(F_v)/U_v = \mathcal{H}^{\pm}$$

for $v \in \infty^-$.

It is well known that the action of $\rho(m_v)$ commutes with the action of the product of Laplacians

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

and that the induced action of $\rho(m_v)$ on each eigenspace of the Laplacian with fixed eigenvalue is constant. Thus if $\phi$ an eigenform for $\Delta$ with eigenvalue $\frac{1}{4}+t^2$ with $t \in \mathbb{C}$, then $\phi$ is also an eigenfunction of $\rho(m_v)$:

$$\rho(m_v)\phi := \int_{\mathcal{H}^{\pm}} m_v(x)\phi(xy)dy = \Lambda\phi(x)$$

where $\Lambda$ is a number depending only on $t$.

For example, one may compute $\Lambda$ by choosing a function $\phi$ of weight 0 supported on $\mathrm{GL}_2(\mathbb{R})_+$ such that

$$\phi\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right) = y^{1/2+it}.$$

Then

$$\Lambda = (\rho(m_v)\phi)(e) = \int m_v\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right) y^{-1/2+it}dxd^\times y.$$

Using coordinates

$$\xi\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right) = -\frac{(y-1)^2+x^2}{4y},$$

one obtains

$$\begin{aligned}\Lambda =& \int_{ay>0} 4|a|\exp\left[-2\pi a\frac{x^2+y^2+1}{2y}\right]|y|^{-3/2+it}dxdy\\ =&4|a|^{1/2}\int_0^\infty \exp\left[-\pi|a|(y+y^{-1})\right]y^{it}d^\times y\\ =&4W_t\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right),\end{aligned}$$

where $W_t$ is the Whittaker newfunction for the representation $\Pi(\alpha^{it}, \alpha^{-it})$ □

It follows that the pairing $\langle\cdot,\cdot\rangle$ on functions on $G(F)\backslash G(\mathbb{A})$ with compact support is automorphic. More precisely, for any two functions $\phi$ and $\psi$ on $G(F)\backslash G(\mathbb{A})/U$, let $\alpha_U(\phi,\psi)$ denote the form of $\mathrm{PGL}_2(\mathbb{A})$ of weight 2 (resp. 0) at places of $\infty^+$ (resp. $\infty^-$) by the following formula:

(4.4.5) $$\alpha(\phi,\psi)(z) = \sum_i \phi_i(z)^\sharp(\phi_i,\phi)\overline{(\phi_i,\psi)} + \text{continuous contribution}$$

where $\phi_i^\sharp$ is a *quasi-newform* of weight $(2,\cdots,2,0,\cdots,0)$ in the representation $\Pi_i$ of $\mathrm{PGL}_2(\mathbb{A})$ corresponding to the representation $\Pi_i'$ of $G(\mathbb{A})$ generated by $\phi_i$ via Jacquet-Langlands theory. We now have

LEMMA 4.4.3.

$$|a|\langle \mathrm{T}_a\phi,\psi\rangle(g_\infty) = 2^{[F:\mathbb{Q}]+n}W_{\alpha(\phi,\psi)}\left(g_\infty\cdot\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\right).$$

*Proof.*

$$\begin{aligned}
|a|\langle \mathrm{T}_a\phi,\psi\rangle(g_\infty) =& |a|(\mathrm{T}_a\phi\otimes\psi, k(x,y)(g_\infty))\\
=&2^{[F:\mathbb{Q}]+n}\sum_i W_i(g_\infty)(\phi_i,|a|\mathrm{T}_a\phi)\overline{(\phi_i,\psi)}\\
=&2^{[F:\mathbb{Q}]+n}\sum_i W_i(g_\infty)(|a|\mathrm{T}_a\phi_i,\phi)\overline{(\phi_i,\psi)}\\
=&2^{[F:\mathbb{Q}]+n}\sum_i W_i\left(g_\infty\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\right)(\phi_i,\phi)\overline{(\phi_i,\psi)}\\
=&2^{[F:\mathbb{Q}]+n}W_{\alpha(\phi,\psi)}\left(g_\infty\cdot\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\right). \qquad \square
\end{aligned}$$

**Gross-Zagier formula for central values.** Fix a character $\chi$ of $T(F)\backslash T(\mathbb{A}_f)$. We have defined a certain function $\phi=\phi_\Delta$ on $\mathcal{S}(\chi, T(\mathbb{A}_f)\backslash G(\mathbb{A}_f))$ in §4.1. Let $\Psi$ denote the form $2^{[F:\mathbb{Q}]+n}|c(\omega)|^{1/2}\alpha(\phi,\phi)$ which has the form

$$\Psi(z) = 2^{[F:\mathbb{Q}]+n}|c(\omega)|^{1/2}\sum_i \phi_i^\sharp(z)\left|\ell_\chi(\phi_{i,\chi})\right|^2 + \text{continuous contribution} \tag{4.4.6}$$

where

$$\ell_\chi(\phi_{i,\chi}) = \int_{T(F)\backslash T(\mathbb{A}_f)}\phi_{i,\chi}(t)\chi(t^{-1})d^\times t \tag{4.4.7}$$

where $d^\times t$ is a Haar measure such that $\widehat{\mathcal{O}}_K^\times$ has volume 1, and where $\phi_{i,\chi}$ is a *toric newform* in $\Pi'$ which satisfies the following conditions:

$\phi_{i,\chi}$ has character $\chi$ under action by $\Delta$.

LEMMA 4.4.4. *The forms* $\bar\Theta(1/2,-)$ *and* $\Psi$ *have the same projection to quasi-newforms.*

*Proof.* Then by Lemma 4.4.1, 4.4.3, 4.3.2, 3.3.11, for fixed $g_\infty$, we have shown that the form

$$\bar\Theta(1/2,-) - 2^{[F:\mathbb{Q}]+n}|c(\omega)|^{1/2}\Psi$$

has Whittaker function in $\mathcal{W}(\Pi(\chi)\otimes\alpha^{1/2})$ for

$$g=\begin{pmatrix} a\delta_f^{-1} & 0\\ 0 & 1\end{pmatrix}g_\infty$$

with $a$ integral and prime to $ND$. It must be zero as $\Pi(\chi)\otimes\alpha^{1/2}$ has nontrivial central character. Thus $\bar\Theta$ and $\Psi$ must have the same projection to quasi-newforms. $\square$

**Proof of theorem 1.4.1.** Let $\Pi$ be an irreducible and automorphic representation of weight $(2,\cdots,0,\cdots)$ of $\mathrm{PGL}_2(\mathbb{A})$, and let $\phi^\sharp$ and $\phi_\chi$ be the corresponding forms for $\Pi$ and $\Pi'$, then

$$L(1/2,\Pi\otimes\chi)=(\phi^\sharp,\bar{\Theta})=(\phi^\sharp,\Psi)=2^{[F:\mathbb{Q}]+n}|c(\omega)|^{1/2}\cdot\frac{\|\phi^\sharp\|^2}{\|\phi_\chi\|^2}|\ell_\chi(\phi_\chi)|^2 \tag{4.4.8}$$

where $\phi_\chi$ is a toric newform with character $\chi$ under $\Delta$ via Jacquet-Langlands. Notice that the measures on $\mathrm{PGL}_2(\mathbb{A})$ and $G(\mathbb{A})$ are taken by taking a standard measure at archimedean place such that $U_0([D,N])$ and $\widehat{R}^\times$ both have volume 1. Theorem 1.4.1 now follows easily.

# 5. Shimura curves and CM-points

In this chapter we want to review the theory of Shimura curves, following Shimura, Deligne[9], Caroyal[2], and the author's earlier work [31]. We will start with some canonical local system on the Shimura curves which is an analogue of the elliptic curve on modular curves. For example, CM-points now become the points with nontrivial endomorphisms. These system will be used to construct the integral models, and to study the reduction of CM-points. Finally we will study the local intersection index of distinct CM-point on the generic fiber on the model when the Shimura curve has minimal level structure. This is basically a consequence of Gross' theory [15] of canonical and quasi-canonical liftings.

For high level structure, the local intersection numbers are difficult to compute as one has no explicit semistable model for Shimura curves. But the local index formula for minimal level will give an asymptotic formula for the index of high level. Thanks to the *toric* newform theory in §2.3, this asymptotic formula is sufficient for our computation in the next chapter. It may not be a bad idea in the future to recover the index formula for high level structure from the Gross-Zagier formula proved in the next chapter.

### 5.1. Some local systems

Lets fix a totally real field $F$ and a quaternion algebra $B$ of $F$ indefinite at one place $\tau=\tau_1$ of $F$ and definite at other real places. In applications, $B$ will be the algebra ${}_vB$, with $v=\tau$, associated to an odd set $\Sigma$ containing all real places. In this chapter, we will let $G$ denote the algebraic group $B^\times$ rather than $B^\times/F^\times$.

Let $h_0$ denote an embedding $\mathbb{S}\longrightarrow G_\mathbb{R}$ of algebraic groups over $\mathbb{R}$ with trivial coordinates at $\tau_i$ $(i\geq 2)$, where $\mathbb{S}=\mathbb{C}^\times$ as an algebraic group over $\mathbb{R}$. Now for any compact open compact subgroup $U$ of $G(\mathbb{A}_f)$ we have the Shimura curve

$$M_U=G(F)\backslash\mathcal{H}^\pm\times G(\mathbb{A}_f)/U. \tag{5.1.1}$$

where $\mathcal{H}^\pm$ is the conjugacy class of $h_0$ under $G(\mathbb{R})$ which is isomorphic to $\mathbb{C}-\mathbb{R}$.

Write $V_0$ for $B$ as a left $B$-module. Then the right multiplication of $G$ on $V_0$ gives an identification $G=\mathrm{GL}_B(V_0)$. The embedding $h_0:\mathbb{S}\longrightarrow G_\mathbb{R}$ now defines a Hodge structure on $V_{0,\mathbb{R}}$.

By the strong approximation theorem, the set of canonical component of $M_U$ is identified with

$$G(F)\backslash\{\pm 1\} \times G(\mathbb{A}_f)/U \simeq G(F)_+\backslash G(\mathbb{A}_f)/U \overset{\det}{\simeq} F^\times\backslash\mathbb{A}_f^\times / \det(U) =: Z_{\det U}. \tag{5.1.2}$$

**Moduli interpretation of $M_U$.** We want to show that $M_U$ parameterizes the pairs of a Hodge structure and an $U$-level structure on $V_0$ (see Deligne [9]). More precisely, $M_U$ parameterizes the set of the isomorphism classes of the following objects $(V, h, \bar{\kappa})$ where

1. $V$ is a free $B$-module of rank 1;
2. $h$ is an embedding from $\mathbb{S} \longrightarrow \mathrm{GL}_B(V_\mathbb{R})$ which has trivial component at $\tau_i$ for $i > 1$;
3. $\bar{\kappa}$ is a class in $\mathrm{Isom}(V_0, V)/U$;

where an isomorphism of two objects $(V, h, \bar{\kappa})$ and $(V', h', \bar{\kappa}')$ is an isomorphism $\iota : V \longrightarrow V'$ of $B$-modules satisfying the following conditions:

- $h' = \iota \circ h \circ \iota^{-1}$;
- $\bar{\kappa}' = \iota \circ \bar{\kappa}$.

Indeed, for any object $(V, h, \bar{\kappa})$ as above we may fix an isomorphism $\iota : V \longrightarrow V_0$ of $B$-modules. Then $h_\iota := \iota \circ h \circ \iota^0$ is an embedding of $\mathbb{S}$ into $G(\mathbb{R})$ with trivial components at $\tau_i$ for $i > 1$. Thus $h_\iota$ is conjugate to $h_0$. It follows that $h_\iota$ defines an element in $\mathcal{H}^\pm$. Also $\iota\bar{\kappa}$ defines an element in $G(\mathbb{A}_f)/U$. Thus the object $(V, h, \bar{\kappa})$ defines an element in $M_U$. Conversely, for a given point $x$ in $M_U$ represented by $(h, g)$ one may define an object $(V_0, h, \bar{\kappa})$ where $\bar{\kappa}$ is the class of multiplication by $g$ on $\widehat{V}_0$.

**Moduli interpretation of $Z_D$.** One may also show that for a compact open subgroup $D$ of $\mathbb{A}_F^\times$, the set

$$Z_D := F_+^\times\backslash\mathbb{A}_F^\times/D$$

parameterizes the objects $(L, \bar{\epsilon}, \bar{\lambda})$ where $F_+^\times$ denotes the set of totally positive elements in $F$, and

1. $L$ is a free $F$-module of rank 1;
2. $\epsilon$ is an *orientation* of $L$: $\epsilon \in F_+^\times\backslash\mathrm{Isom}(L, F)$;
3. $\bar{\lambda}$ is a $D$-level structure: $\bar{\lambda} \in \mathrm{Isom}(\widehat{F}, \widehat{L})/D$.

Indeed, the correspondence is given by

$$(L, \bar{\epsilon}, \bar{\lambda}) \longrightarrow F_+^\times \cdot (\epsilon \circ \lambda(1)) \cdot U \in Z_D.$$

**Moduli interpretation of** $\det : M_U \longrightarrow Z_D$**.** Let $D = \det(U)$. For any object $(V, h, \bar{\kappa})$ parameterized by $M_U$, one may define an object $(L_V, \bar{\epsilon}_h, \bar{\lambda}_\kappa)$ as follows:

1. $L_D$ is the $F$-vector space $\det(V)$ generated by symbols $\langle x, y\rangle$ modulo relations such that the pairing is symmetric, $F$-bilinear, and $B$-hermitian in the following sense:

   $$\langle bx, y\rangle = \langle x, \bar{b}y\rangle, \qquad b \in B.$$

   It can be showed that $\det(V)$ is one dimensional.
2. let $\iota : V \longrightarrow B = V_0$ be a $B$-linear isomorphism such that $\iota \circ h \circ \iota^{-1}$ is in the connected component as $h_0$. Then $\bar{\epsilon}_h$ is the class of

   $$\det(\iota) : F = \det(B) \longrightarrow \det(V),$$

   where $F = \det(B)$ is identified by sending 1 to $\langle 1, 1\rangle$.

3. $\bar{\lambda}$ is the class of

$$\det(\kappa) : \det(\widehat{V}) \longrightarrow \det(\widehat{B}) = \widehat{F}.$$

Then it can be shown that the map $M_U \longrightarrow Z_{\det U}$ is given by this correspondence of objects.

**Universal objects.** When $U$ is sufficiently small, the universal object $(V_U, h_U, \bar{\kappa}_U)$ does exist in the sense that $V_U$ is a local system of invertible $B$ modules on $M_U$ with a Hodge structure $h_U$ which makes $V_U^1 = V_U \otimes_\tau \mathbb{R}$ an algebraic vector bundle on $M_U$ of rank 2 with one action by $B$ whose trace is the standard one on $B$, and $\bar{\kappa}_M$ is a level structure $\kappa_x : \widehat{V}_0 \longrightarrow \widehat{V}_x$ for each geometric point $x \in M_U$. Here for an abelian group $M$, $\widehat{M}$ denotes $M \otimes \widehat{\mathbb{Z}}$. Physically, one has the following identification:

$$V_U = G(F)\backslash V_0 \times \mathcal{H}^\pm \times G(\mathbb{A}_f)/U, \tag{5.1.3}$$

$$V_U^1 = G(F)\backslash V_0^1 \times \mathcal{H}^\pm \times G(\mathbb{A}_f)/U \tag{5.1.4}$$

where $V_0^1 = V_0 \otimes_{\tau_1} \mathbb{R}$ such that $U$ has trivial action on $V_0$ and such that $\gamma \in G(F)$ acts on $V_0$ by right multiplication by $\gamma^{-1}$. It follows that

$$\widehat{V}_U = G(F)\backslash \mathcal{H}^\pm \times G(\mathbb{A}_f) \times \widehat{V}_0/U \tag{5.1.5}$$

where the action of $G(F)$ on $\widehat{V}_0$ is trivial and the action of $U$ on $\widehat{V}_0$ is given by right multiplication. The map $V_U \longrightarrow \widehat{V}_U$ is given by

$$(v, z, g) \longrightarrow (z, g, vg)$$

and the level structure $\bar{\kappa}$ is given by the class of the identity map.

Similarly, $Z_D$ has a universal object

$$L_D = F_+^\times \backslash F \times \widehat{F}^\times / D, \tag{5.1.6}$$

$$\widehat{L}_D = F_+^\times \backslash \widehat{F}^\times \times \widehat{F}/D. \tag{5.1.7}$$

Here the action of $(a, b) \in F_+^\times \times D$ sends $(x, y) \in F \times \widehat{F}$ to $(xa^{-1}, yb)$. The map $L_D \longrightarrow \widehat{L}_D$ is given by $(x, y) \in F \times \widehat{F}^\times$ to $(y, xy) \in \widehat{F}^\times \times \widehat{F}$. The pairing $V_U \times V_U \longrightarrow L_D$ and $\widehat{V}_U \times \widehat{V}_U \longrightarrow \widehat{L}_D$ are given in the obvious manner.

**Galois actions.** By Shimura's theory, $M_U$ is defined over $F$ with Galois action on the set $Z_D$ of connected components given by class field theory

$$\nu : \mathrm{Gal}(\bar{F}/F) \longrightarrow F_+^\times \backslash \mathbb{A}_f^\times / D.$$

One may show that with this canonical structure, the vector bundle $V_U^1$ is defined over $F$. Thus for one object $x = (V, h, \bar{\kappa}) \in X(\bar{F})$ and $\sigma \in \mathrm{Gal}(\bar{F}/F)$, the $\mathbb{C}$-vector spaces $V_x^1$ and $V_{x^\sigma}^1$ both have some $\bar{F}$-structure $V_{x,\bar{F}}^1$ and $V_{x^\sigma,\bar{F}}^1$ and $\sigma$ induces an $\sigma$ -linear isomorphism (which is still denoted as $\sigma$) from $V_{x,\bar{F}}^1$ to $V_{x^\sigma,\bar{F}}^1$.

Similarly, the local system $\widehat{V}$ is also defined over $F$. Thus for one object $x = (V, h, \bar{\kappa})$ and one $\sigma \in \mathrm{Gal}(\bar{F}/F)$, there is a morphism which is still denoted as $\sigma$ from $\widehat{V}_x \longrightarrow \widehat{V}_{x^\sigma}$ such that $\bar{\kappa}_{x^\sigma} = \bar{\kappa} \circ \sigma$. The determinant of this map induces a similar map on the local systems on $Z_D$. More precisely, if $x = (L, \bar{\epsilon}, \bar{\lambda})$ is one object parameterized by $Z_D$, then $x^\sigma = (L, \bar{\epsilon}, \bar{\sigma} \cdot \nu(\sigma))$ and the morphism $\sigma : \widehat{L} \longrightarrow \widehat{L}$ is just the multiplication by $\nu(\sigma)$.

**Integral structure.** To get an integral structure of local systems, we may take a maximal order $\mathcal{O}_B$ such that $U \subset \widehat{\mathcal{O}}_B^\times$. Let $V_{0,\mathbb{Z}}$ be the lattice in $V_0$ corresponding to $\mathcal{O}_B$. Then the lattice $\kappa_x(V_{0,\mathbb{Z}}) =: V_{x,\mathbb{Z}}$ in $V_x$ is independent of the choice of $\kappa_x \in \bar{\kappa}_M$. Thus $M_U$ also parameterizes integral objects $(V_\mathbb{Z}, h, \bar{\kappa})$ where $V_\mathbb{Z}$ is an invertible $\mathcal{O}_B$-module, $h$ is Hodge structure on $V_\mathbb{R}$ as before, and $\bar{\kappa}$ is an $U$-class of isomorphism $\widehat{\mathcal{O}}_B \longrightarrow \widehat{V}_\mathbb{Z}$.

Similarly, for any fixed $\mathcal{O}_F$-fractional ideal $L_0$, $Z_D$ parameterizes objects $(L_\mathbb{Z}, \bar{\epsilon}, \bar{\lambda})$ where $L_\mathbb{Z}$ is an invertible $\mathcal{O}_F$-module, and $\bar{\epsilon}$ is an orientation of $L := L_\mathbb{Z} \otimes \mathbb{Q}$, and $\bar{\lambda}$ is a $D$-class of isomorphism $\lambda : \widehat{L}_{0,\mathbb{Z}} \longrightarrow \widehat{L}_\mathbb{Z}$.

For the morphism $M_U \longrightarrow Z_D$, we take $L_{0,\mathbb{Z}}$ to be the $\mathcal{O}_F$-submodule $\det(\mathcal{O}_B)$ of $F$ generated by $\langle x, y\rangle$ for $x, y \in B_\mathbb{Z}$. Then the image of an object $(V_\mathbb{Z}, h, \bar{\kappa})$ will be $(L_{V,\mathbb{Z}}, \bar{\epsilon}_h, \bar{\lambda}_\kappa)$ with $L_{V,\mathbb{Z}} = \det(V_\mathbb{Z})$. Thus $\widetilde{V}_U = V_U/V_{U,\mathbb{Z}}$ and $\widetilde{L}_D = L_D/L_{D,\mathbb{Z}}$ form systems of divisible groups on $M_U$ and $Z_D$.

For any fixed positive integral idele $n$, one has a *Weil pairing*

$$\langle x, y\rangle_n := n\langle x', y'\rangle$$

on $\widetilde{V}_U[n]$ with values in $\widetilde{L}_D[n]$, where $x, y \in \widetilde{V}_U$ represented by $x, y \in V_U$. If $U$ contains $U(n) := (1 + n\widehat{B})^\times$, then the level $U$ structure can be described as a class of isomorphism

$$\widetilde{V}_{U,0}[n] \longrightarrow \widetilde{V}_U[n]$$

modulo $U$.

If $B = M_2(\mathbb{Q})$, then $M_U$ parameterizes objects of elliptic curves with level structure with a universal object $(\mathcal{E}, \kappa_\mathcal{E})$. Then $V_U^1 = \mathrm{Lie}(\mathcal{E})^2$ with a natural action by $B$, and $\widetilde{V}_U = \mathcal{E}_{tor}^2$.

## 5.2. Homomorphisms and CM-points

For any two objects $x = (V, h, \bar{\kappa})$ and $x' = (V', h', \bar{\kappa})$ of $M_U$, one can define the $F$-module $\mathrm{Hom}^0(x, x')$ of homomorphisms $\alpha \in \mathrm{Hom}_B(V, V')$ such that for any $z \in \mathbb{C}^\times$,

$$h'(z) \circ \alpha_\mathbb{R} = \alpha_\mathbb{R} \circ h(z).$$

Write $\mathrm{End}^0(x)$ for $\mathrm{Hom}^0(x, x)$. Then $\mathrm{End}^0(x)$ is either $F$ or a totally imaginary quadratic extension $K$ of $F$. In the second case, we call $x$ a CM-point by $K$. The induced action of $K$ on the complex space $V^1 = V \otimes_\tau \mathbb{R}$ is given by a complex embedding of $K$ which we still denote as $\tau$.

For two points $x, x'$, the $F$-vector space $\mathrm{Hom}^0(x, x')$ has rank $\leq 2$. If this space is nonzero, then we say $x$ and $x'$ are isogenous and any nonzero element in this vector space is called a quasi-isogeny. It is easy to show that $\mathrm{Hom}^0(x, x')$ has dimension 2 if and only if both $x$ and $x'$ has CM by isomorphic imaginary quadratic extensions $K$ and $K'$. We may further fix an isomorphisms $K \simeq K'$ with respect to the embeddings into $\mathbb{C}$ defined in the previous paragraph.

For a fixed imaginary quadratic extension $K$ of $F$ and an embedding $\tau : K \subset \mathbb{C}$ extending that of $F$, the set $C_U$ of CM-points on $M_U$ by $K$ is not empty. Indeed, we may fix an embedding $\alpha : K \longrightarrow B$ which induces a Hodge structure $h_0$ on $V_0 = B$ with trivial component at places $\tau_i$ for $i \neq 1$ and equal to $\alpha \otimes_{\tau_1} \mathbb{R}$ at $\tau_1$. We now may take $x_0$ to be a point on $M_U$ corresponding to the object $(V_0, h_0, \bar{\kappa})$ where $\kappa$ is the identity map $\widehat{B} \longrightarrow \widehat{V}$. For the identification

$$M_U = G(F)\backslash\mathcal{H}^{\pm} \times G(\mathbb{A}_f)/U, \tag{5.2.1}$$

$x_0$ corresponds to the point represented by $(h_0, 1)$ where $h_0 \in X$ is one of two fixed points by $K^\times \subset G(F)$. (The other one is the complex conjugation of $h_0$). All CM-points by $K$ with fixed $\tau_1$ are then given by

$$C_U = G(F)\backslash G(F)h_0 \times G(\mathbb{A}_f)/U = T(F)\backslash G(\mathbb{A}_f)/U \tag{5.2.2}$$

where $T = K^\times$ is the torus in $G$.

By Shimura's theory, the Galois action of $\mathrm{Gal}(\bar{F}/K)$ is given by class field theory and multiplication of $T(\mathbb{A}_f)$ from left hand side. More precisely, if $\sigma \in \mathrm{Gal}(\bar{F}/K)$, $x = (V, h, \bar{\kappa})$, then $x^\sigma = (V, h, \nu(\sigma)\cdot\bar{\kappa})$ and the action on local system is given by right multiplication by $\nu(\sigma)$, where $\nu$ is the reciprocity map $\mathrm{Gal}(\bar{K}/K) \longrightarrow K^\times\backslash\mathbb{A}_{K,f}^\times$.

## 5.3. Canonical models

**Integral model.** It is well known that $M_U$ has a canonical integral model $\mathcal{M}_U$ over $\mathcal{O}_F$ which is regular if $U$ is sufficiently small. Let $\mathcal{O}_U$ be the ring of the abelian extension $F_U$ of $F$ corresponding to the class

$$F_+^\times\backslash\widehat{F}^\times/\det(U).$$

Then $Z_U$ has a model $\mathcal{Z}_U$ over $F$ and is isomorphic to $\mathrm{Spec}F_U$. The map $M_U \longrightarrow Z_U$ induces a map

$$\mathcal{M}_U \longrightarrow \mathcal{Z}_U := \mathrm{Spec}\mathcal{O}_U. \tag{5.3.1}$$

The local system $V_U^1$ and $\widetilde{V}_U$ can be extended to $\mathcal{M}_U$ to a vector bundle and a local system of divisible groups such that $\mathrm{Lie}(\widetilde{V}_U) = V_U^1$.

Let $\bar{F}$ denote the algebraic closure of $F$ in $\mathbb{C}$. We want to study the reduction of points on $M_U \otimes \bar{F}$. Notice that $Z_U \otimes \bar{F}$ is naturally isomorphic to $Z_U(\bar{F}) \times \mathrm{Spec}\bar{F}$. Thus $Z_U$ has an integral model

$$\overline{\mathcal{Z}}_U := Z_U(\bar{F}) \times \mathrm{Spec}\bar{\mathcal{O}}_F := \coprod_{Z_U(\bar{F})} \mathrm{Spec}\bar{\mathcal{O}}_F \tag{5.3.2}$$

Notice that this scheme has a natural map to $\mathcal{Z}_U$. Let $\overline{\mathcal{M}}_U$ be the tensor product of $\overline{\mathcal{Z}}_U$ and $\mathcal{M}_U$ over $\mathcal{Z}_U$.

**Formal modules.** We now fix one prime $\wp$ of $F$ and let $\bar{\wp}$ be an extension of $\wp$ to $\bar{F}$. We assume that $U$ is a product $U = U^\wp \cdot U_\wp$ and want to study reduction of $\overline{\mathcal{M}}_U$ at $\overline{\wp}$, following the method of Carayol [2] where $\wp$ was assumed to be unramified in $B$. See also Katz-Mazur [24] for the case of modular curves. Lets write $\mathcal{O}_\wp$ for $\mathcal{O}_{F,\wp}$, $\overline{\mathcal{O}}_\wp$ for $\mathcal{O}_{\bar{F},\bar{\wp}}$, $\mathcal{M}_{U,\wp}$ for $\mathcal{M}_U \otimes_{\mathcal{O}_F} \mathcal{O}_\wp$, and $\overline{\mathcal{M}}_{U,\wp}$ for $\overline{\mathcal{M}}_U \otimes_{\overline{\mathcal{O}}_F} \overline{\mathcal{O}}_\wp$.

Then over $\mathcal{M}_{U,\wp}$ , the prime to $p$-part of $(\widetilde{V}_U, \bar{\kappa})$ can be extended to an etale system on $\mathcal{M}_{U,\wp}$ but $\wp$-part

$$(\widetilde{V}_U \otimes \mathcal{O}_\wp, \bar{\kappa} \otimes \mathcal{O}_\wp)$$

can only be extended to a system of *special* formal $\mathcal{O}_{B,\wp}$-module with a *Drinfeld* level structure,

$$(\mathcal{V}, \bar{\alpha}).$$

Here special modules and Drinfeld level structure are defined as follows:

- $\mathcal{V}$ is *special* means that $\mathrm{Lie}(\mathcal{V})$ is a locally free sheaf over $\mathcal{O}_{\mathcal{M}_U} \otimes \mathcal{O}_E$ of rank 1 where $\mathcal{O}_E$ is an unramified quadratic extension $\mathcal{O}_\wp$ in $\mathcal{O}_{B,\wp}$.
- A *Drinfeld level structure* means an $U$-class of morphisms

$$\alpha : \wp^{-n}\mathcal{O}_B/\mathcal{O}_B \longrightarrow \mathcal{V}[\wp^n]$$

such that cycles of the latter space is generated by the image.

When $U^\wp$ is sufficiently small, $\mathcal{M}_{U,\wp}$ is regular and is locally a universal deformation of $\mathcal{V}$ with its level structure in the special fiber. We write $\mathcal{V}^0$ for the isogeny class of $\mathcal{V}$.

Similarly the local system $(\widetilde{L}, \bar{\lambda})$ will also extend to a divisible group over $\mathcal{Z}_{U,\wp} = \mathrm{Spec}(\mathcal{O}_{U,\wp})$ whose prime to $\wp$-part is etale, and its $\wp$-part is a formal $\mathcal{O}_\wp$-module with a level structure

$$(\mathcal{L}, \bar{\beta})$$

such that the induced action of $\mathcal{O}_\wp$ on $\mathcal{L}$ is the usual multiplication of $\mathcal{O}_\wp$ inside $\mathcal{O}_{U,\wp}$. The level structure again is also defined by a $\det(U)$-class of surjective morphism

$$\beta : \wp^{-n}\mathcal{L}_0/\mathcal{L}_0 \longrightarrow \mathcal{L}[\wp^{-n}]$$

where $\mathcal{L}_0 = \det(\mathcal{O}_B)$ is the pairing module of $\mathcal{O}_B$. For any generator $t \in \mathcal{O}_F$ of order 1 at $\wp$, the level structure is compatible with the pairing:

$$\langle \alpha(x), \alpha(y) \rangle_t = \beta(\langle x, y \rangle_t).$$

The map $\overline{\mathcal{Z}}_{U,\wp} \longrightarrow \mathcal{Z}_{U,\wp}$ then classifies the lifting $(\mathcal{L}, \bar{\beta})$ to the geometric generic fiber.

**Homomorphism.** Let $x$ and $x'$ be two geometric points in the special fiber of $\mathcal{M}_{U,\wp}$. Then we define $\mathrm{Hom}^0(x, x')$ to be the subgroup in

$$\mathrm{Hom}((\mathcal{V}_x^0, \widehat{V}_x^\wp), (\mathcal{V}_{x'}^0, \widehat{V}_{x'}))$$

generated by $\mathrm{Hom}^0(y, y')$ for all liftings $y, y'$ of $x, x'$ to the geometric points of $\mathcal{M}_{U,\wp}$. We say $x$ and $x'$ are *isogenous* if $\mathrm{Hom}^0(x, x') \neq 0$ and any nontrivial element in this group is called a *quasi-isogeny*.

### 5.4. Reductions

In this section we want to study the reduction of the integral model of a Shimura curve for a fixed prime $\wp$ of $F$. More precisely, we will study the set of irreducible components in the fiber over $\wp$, and the set of three classes of closed geometric points in the special fiber: ordinary points, supersingular points, and super special points. We will also identify the reduction of CM-points in each fiber.

**Case of unramified prime.** First let's consider the case where $\wp$ is unramified in $B$. We want to study the smoothness of the special fiber of $\mathcal{M}_\wp$. Let $U_0$ denote $U^\wp \cdot \mathcal{O}_{B,\wp}$. Then one can show that $\mathcal{M}_{U_0,\wp}$ has good reduction when $U^\wp$ is sufficiently small, see Carayol [2] when $\wp$ is unramified in $B$, and Katz-Mazur [24] when $B = M_2(\mathbb{Q})$.

To study the general case, lets fix one isomorphism $\mathcal{O}_{B,\wp} = M_2(\mathcal{O}_\wp)$. Then every $\mathcal{O}_{B,\wp}$ module $M$ can be uniquely written as

$$M = N \oplus N \qquad N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M \tag{5.4.1}$$

as $\mathcal{O}_F$-modules such that the action of $\mathcal{O}_{B,\wp}$ is given by left multiplication on $N^2$. One symmetric pairing $M \times M \longrightarrow P$ is equivalent to an alternative pairing $N \times N \longrightarrow P$. By this convention, the formal $\mathcal{O}_{B,\wp}$-module $\mathcal{V}$ is then given by two copies of one formal module $\mathcal{E}$ of dimension 1 with a usual Weil pairing with values in $\mathcal{L}$. The level structure is then a usual level structure

$$(\wp^{-n}\mathcal{O}_\wp/\mathcal{O}_\wp)^2 \longrightarrow \mathcal{E}[\wp^n].$$

A geometric point $x$ at the special fiber of $\mathcal{M}_{U,\wp}$ is called *supersingular*, if $\mathcal{E}_x$ is connected. Otherwise, it is called *ordinary*. If it is ordinary, then the level structure has a kernel of rank 1 and thus defines an element in $\lambda \in \mathbb{P}^1(F_\wp)/U_\wp$. One may show that for any given $\lambda \in \mathbb{P}^1(F_\wp)/U_\wp$, and a fixed connected component $\mathcal{M}^0_{U,\wp}$, the points in the special fiber which are either supersingular or ordinary corresponding to $\lambda$ actually form an irreducible component $I_\lambda$ of the special fiber. Thus, the supersingular points are only singular points in the special fiber. These $I_\lambda$'s are called the *Igusa curves*. The nature map $\mathcal{M}_{U,\wp} \longrightarrow \mathcal{M}_{U_0,\wp}$ induces an isomorphism between each $I_\lambda$ and the special fiber of the $\mathcal{M}^0_{U_0,\wp}$.

Let $\mathbb{F}$ be the algebraic closure of the residue field of $F_\wp$ and let $\mathcal{M}_{U,\mathbb{F}}$ (resp. $\overline{\mathcal{M}}_{U,\mathbb{F}}$) be the geometric special fiber of $\mathcal{M}_{U,\wp}$ (resp. $\overline{\mathcal{M}}_{U,\wp}$). Since $\mathcal{O}_U$ is totally ramified over $\mathcal{O}_{U_0}$, the set of connected component of $\mathcal{M}_{U,\mathbb{F}}$ is the same as that of $\mathcal{M}_{U_0,\mathbb{F}}$ thus the same as $M_{U_0}$. It follows that the set of irreducible component of $\mathcal{M}_{U,\mathbb{F}}$ is given by

$$G(F)_+\backslash G(\mathbb{A}_f)/U_0 \times \mathbb{P}(F_\wp)/U_\wp.$$

From this one easily obtains the following:

LEMMA 5.4.1. *If $\wp$ is split in $B$, then the set of irreducible components of $\overline{\mathcal{M}}_{U,\mathbb{F}}$ is given by*

$$G(F)_+\backslash G(\mathbb{A}_f)/U \times \mathbb{P}(F_\wp)/U_\wp.$$

**Ordinary points.** Let $x$ be a fixed ordinary point on $\mathcal{M}_{U,\mathbb{F}}$. Then it can be shown that $K := \mathrm{End}^0(x)$ is a totally imaginary quadratic extension of $F$ which is split at $\wp$. We may fix one splitting $K_\wp = F_\wp^2$, such that the divisible group $\mathcal{E}_x$ is isogenous to a direct sum $\mathcal{E}_x^{et} \oplus \mathcal{E}_x^0$ compatible with the action of $K$, where $\mathcal{E}_x^0$ is a formal group of dimension 1 and $\mathcal{E}_x^{et}$ is etale. In this way, one obtains the diagonal embedding $K \longrightarrow B$ such that at $\wp$, it is given by the diagonal embedding. Let $\wp^e, \wp^0$ denote two induced primes of $K$. It also can be shown that two ordinary points $x$ and $x'$ are isogenous if and only if they have isomorphic endomorphism rings. We may fix such isomorphisms such that they induce the same action on tangent spaces of the associated formal groups.

Let $K$ be a fixed totally imaginary quadratic extension of $F$ with a fixed splitting $K_\wp \simeq F_\wp^2$. Then the set of geometric ordinary points on $\mathcal{M}_{U,\mathbb{F}}$ with endomorphisms by $K$ and with given splitting can be identified with

$$K_0^\times \backslash \left( \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \backslash \mathrm{GL}_2(\mathcal{O}_\wp) \right) \times G(\mathbb{A}_f^\wp)/U,$$

where $K_0^\times$ denotes the subgroup of $K^\times$ of elements with order 0 at two places of $K$ over $\wp$.

Indeed, let $x_0 = (\mathcal{E}_0, \widetilde{V}_0^\wp, \bar{\kappa}_0)$ be a fixed ordinary point with CM by $K$. Using one $\kappa \in \bar{\kappa}$, we may identify $\mathcal{E}_0^{et}$ with $F_\wp/\mathcal{O}_\wp$, and $\widetilde{V}_0^\wp$ with $\widehat{B}^\wp/\widehat{\mathcal{O}}_B^\wp$. Then for any ordinary point $x = (\mathcal{E}, \widetilde{V}^\wp, \bar{\kappa})$ with CM by $K$, there is an isogeny $\alpha : x \longrightarrow x_0$ which induces an isomorphism on divisible groups at $\wp$. Such an $\alpha$ is unique up to multiplication by elements in $K_0^\times$. Such an isogeny now induces an element

$$(z, g) \in \mathrm{Hom}^*(\mathcal{O}_\wp^2, \mathcal{O}_\wp) \times G(\mathbb{A}_f^\wp)$$

such that the surjective map $\alpha \circ \kappa = (z, g)$, where $\mathrm{Hom}^*$ means the set of *surjective homomorphisms.* In this way we may identify the set of ordinary points with CM by $K$ with

$$K_0^\times \backslash \mathrm{Hom}^*(\mathcal{O}_\wp^2, \mathcal{O}_\wp) \times G(\mathbb{A}_f^\wp)/U.$$

Our assertion now follows from the identity

$$\mathrm{Hom}^*(\mathcal{O}_\wp^2, \mathcal{O}_\wp) = \mathrm{pr}_1 \cdot \mathrm{GL}_2(\mathcal{O}_\wp) = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \Big\backslash \mathrm{GL}_2(\mathcal{O}_\wp)$$

where $\mathrm{pr}_1$ denote the projection of $\mathcal{O}_\wp$ onto the first factor.

The maps from CM-points by $K$ over $\bar{F}_\wp$, to CM-points by $K$ over $\mathbb{F}$, and to irreducible components over $\mathbb{F}$ are given by the obvious ones, via the identity

$$\mathbb{P}^1(F_\wp) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \Big\backslash \mathrm{GL}_2(F_\wp).$$

We now want to study the ordinary points on $\overline{\mathcal{M}}_{U,\mathbb{F}}$ which are exactly ordinary points on $\mathcal{M}_U$ with an lifting of determinant level structure to the geometric generic fiber. In the above setting, for a given isogeny $\alpha : \mathcal{E} \longrightarrow \mathcal{E}_0$, we will have a triple $(z, g, a)$ with $a \in \mathcal{O}_F^\times$. The set of ordinary points on $\overline{\mathcal{M}}_{U,\mathbb{F}}$ is then identified with

$$K_0^\times \backslash \mathrm{Hom}^*(\mathcal{O}_\wp^2, \mathcal{O}_\wp) \times \mathcal{O}_F^\times \times G(\mathbb{A}_f^\wp)/U$$

where $K_0^\times$ and $U_\wp$ acts on $\mathcal{O}_F^\times$ by determinants. It is easy to show that the map $g \longrightarrow (\mathrm{pr}_1 \cdot g, \det g)$ induces a bijection:

$$\mathrm{Hom}^*(\mathcal{O}_\wp^2, \mathcal{O}_\wp) \times \mathcal{O}_F^\times = N(\mathcal{O}_\wp) \backslash \mathrm{GL}_2(\mathcal{O}_\wp)$$

with compatible action by $K_0^\times$ and $U_\wp$. Thus we have shown that the set of ordinary points on $\overline{\mathcal{M}}_{U,\wp}$ is identified with

$$K_0^\times \backslash \left(N(\mathcal{O}_\wp) \backslash \mathrm{GL}_2(\mathcal{O}_\wp)\right) \times G(\mathbb{A}_f^\wp)/U.$$

Using the decomposition

$$\mathrm{GL}_2(F_\wp) = K_\wp^\times \cdot N(F_\wp) \cdot \mathrm{GL}_2(\mathcal{O}_\wp)$$

we then obtain the following

LEMMA 5.4.2. *The set of ordinary points on $\overline{\mathcal{M}}_{U,\mathbb{F}}$ with CM by $K$ is identified with*

$$K^\times \backslash \left(N(F_\wp)\backslash \mathrm{GL}_2(F_\wp)\right) \times G(\mathbb{A}_f^\wp)/U.$$

*The reduction maps from the CM-points on $M_U \otimes_F \bar{F}$ to ordinary points and irreducible components on $\overline{\mathcal{M}}_{U,\mathbb{F}}$ are given by the following obvious ones:*

$$\begin{aligned} K^\times\backslash G(\mathbb{A}_f)/U &\longrightarrow K^\times \backslash \left(N(F_\wp)\backslash \mathrm{GL}_2(F_\wp)\right) \times G(\mathbb{A}_f^\wp)/U \\ &\longrightarrow F_+^\times \backslash \mathbb{A}_f^\times / \det(U) \times \mathbb{P}(F_\wp)/U_\wp \end{aligned}$$

*where the second map sends the class of* $(g_\wp, g^\wp)$ *to the class of* $(\det(g_\wp g^\wp), g_\wp)$.

**Supersingular points.** We now want to give a description of the set of supersingular points on $\mathcal{M}_{U,\mathbb{F}}$ which is the same as on $\mathcal{M}_{U_0,\mathbb{F}}$, where $U_0 = \mathrm{GL}_2(\mathcal{O}_\wp)U^\wp$. It can be shown that all supersingular points are isogenous to each other, and for a fixed supersingular point $x_0 = (\mathcal{V}_0, \widetilde{V}_0^\wp, \bar{\kappa}_0)$, the endomorphism ring $B' := \mathrm{End}^0(x_0)$ is a quaternion algebra which can be obtained from $B$ by changing invariants at $\tau$ and $\wp$. In other words, in our notation $B =_\tau B$ and $B' =_\wp B$ with ramification set ${}_v\Sigma$ and ${}_\wp\Sigma$ defined at (4.3.2) respectively. Let $G'$ denote the algebraic group $(B')^\times$ over $F$. Fix one $\kappa_0 \in \bar{\kappa}_0$. We may embed $B$ into $G(\mathbb{A}_f^\wp)$ and identify $\widehat{V}_0^\wp$ with $\widehat{B}^\wp$. Then for any supersingular point $x = (\mathcal{V}, \widetilde{V}^\wp, \bar{\kappa})$, we have an isogeny $\alpha : x \longrightarrow x_0$ of degree prime to $p$ which is unique up to composition with elements of $G'(F)_0$ of order 0 at $\wp$. The level structures now induce one element $g \in G(\mathbb{A}_f^\wp)$ such that

$$g := \alpha \circ \kappa \in G(\mathbb{A}_f^\wp).$$

By this way we may show that the set of supersingular points on $\mathcal{M}_{U,\mathbb{F}}$ can be identified with

$$G'(F)_0\backslash G(\mathbb{A}_f^\wp)/U^\wp = G'(F)\backslash G'(\mathbb{A}_f)/U'$$

where $U' = \mathcal{O}_{B',\wp}^\times \cdot U^\wp$. The morphism from supersingular points to the set of connected components

$$\det : B \longrightarrow F_+^\times,$$

and the map from CM-points by $K$ to the set of supersingular points is given by

$$\begin{aligned} T(F)\backslash G(\mathbb{A}_f)/U &\longrightarrow G'(F)\backslash G'(\mathbb{A}_f)/U' \\ [g] &\longrightarrow [g'_\wp \cdot g^\wp], \end{aligned}$$

where $g'_\wp \in B(\wp)_\wp$ is any element with norm $\det g_\wp$. Similarly, one can show the following.

LEMMA 5.4.3. *The set of supersingular points on $\overline{\mathcal{M}}_{U,\mathbb{F}}$ is identified with*

$$G'(F)_0\backslash \mathcal{O}_\wp^\times \times G(\mathbb{A}_f^\wp)/U = G'(F)\backslash F_\wp^\times \times G(\mathbb{A}_f^\wp)/U$$

*where $G'(F)$ and $U_\wp$ act on $F_\wp^\times$ by determinant. The maps from CM-points on $M_U \otimes_F \bar{F}$ to supersingular points and to the set of connected components on $\overline{\mathcal{M}}_{U,\mathbb{F}}$ are given by the following obvious ones:*

$$\begin{aligned} K^\times \backslash G(\mathbb{A}_f)/U &\longrightarrow G'(F)\backslash F_\wp^\times \times G(\mathbb{A}_f^\wp)/U \\ &\longrightarrow F_+^\times \backslash \mathbb{A}_f^\times / \det(U) \end{aligned}$$

*where the first map sends the class of $g$ to the class of $(\det g_\wp, g^\wp)$ and the second map sends the class of $(x, g^\wp)$ to the class of $x \det g^\wp$.*

**Case of ramified primes.** It remains to study the reduction of $\mathcal{M}_{U,\wp}$ in the case that $B$ is not split at $\wp$. In this case, the group $\widetilde{V}$ is a connected formal group. It follows that the map

$$\mathcal{M}_{U,\wp} \to \mathcal{M}_{U_0,\wp}$$

is purely inseparable at the fiber over $\wp$. So the set of irreducible components of $\mathcal{M}_{U,\mathbb{F}}$ over $\wp$ is the same as that of $\mathcal{M}_{U_0,\mathbb{F}}$.

In this nonsplit case, one can show that all points in the special fiber are $F$-isogenous, and the $F$-endomorphism ring is a quaternion algebra $B'$ over $F$ obtained by changing the invariants of $B$ at $\tau$ and $\wp$. Again, we let $G'$ denote the algebraic group $(B')^\times$ over $F$.

To study the irreducible components of $\mathcal{M}_{U_0,\mathbb{F}}$ over $\wp$ we can use the uniformization theorem of Cerednik – Drinfeld [1, 10]. We need some notations to state this theorem. Let $\widehat{\mathcal{M}}_{U_0}$ denote the formal completion of $\mathcal{M}_{U_0}$ along its special fiber over $\wp$. Fix an isomorphism:

$$\widehat{B}' \simeq M_2(F_\wp) \cdot \widehat{B}^\wp$$

where the superscript $\wp$ means that the component at the place $\wp$ is removed. Let $\widehat{\Omega}$ denote Deligne's formal scheme over $\mathcal{O}_\wp$ obtained by blowing-up $\mathbb{P}^1$ along its rational points in the special fiber over the residue field $k$ of $\mathcal{O}_\wp$ successively. So the generic fiber $\Omega$ of $\widehat{\Omega}$ is a rigid analytic space over $F_\wp$ whose $\bar{F}_\wp$ points are given by $\mathbb{P}^1(\bar{F}_\wp) - \mathbb{P}^1(F_\wp)$. The group $\mathrm{GL}_2(F_\wp)$ has a natural action on $\widehat{\Omega}$. The theorem of Cerednik-Drinfeld gives a natural isomorphism

$$\widehat{\mathcal{M}}_{U_0} \simeq G'(F)\backslash \widehat{\Omega} \widehat{\otimes} \mathcal{O}_\wp^{\mathrm{ur}} \times \widehat{B}^{\times,\wp}/U^\wp \tag{5.4.2}$$

where $\mathcal{O}_\wp^{\mathrm{ur}}$ denote the completion of the maximal unramified extension of $\mathcal{O}_\wp$ with an action by $G'(F)$ given by

$$g \in B(\wp)^\times \longrightarrow \mathrm{Fr}^{-\mathrm{ord}_\wp \det g}.$$

Since $\Omega$ is connected, the set of geometric components of $\mathcal{M}_{U,\mathbb{F}}$ is identified with

$$G'(F)\backslash \mathbb{Z} \times \widehat{B}^{\wp,\times}/U^\wp = G'(F)_0 \backslash \widehat{B}^{\wp,\times}/U^\wp,$$

where $G'(F)_0$ means elements of $B'$ of order 0 at $\wp$. Taking det, this set is then identified with

$$F_+^\times \backslash \widehat{F}^\times / U_0.$$

To obtain a description of the special fiber of $\widehat{\mathcal{M}}_{U_0}$, we notice that the irreducible components of special fiber of $\widehat{\Omega}$ correspond one-to-one to the classes modulo $F^\times$ of $\mathcal{O}_\wp$ lattices in $F_\wp^2$. Consequently, one has the following.

LEMMA 5.4.4. *The set of geometric irreducible geometric components of $\widehat{\mathcal{M}}_{U_0}$ over $\wp$ is indexed by the set*

$$\begin{aligned}&G'(F)\backslash \mathrm{GL}_2(F_\wp)/F_\wp^\times \mathrm{GL}_2(\mathcal{O}_\wp)\times\mathbb{Z}\times\widehat{B}^{\times,\wp}/U^\wp\\ \simeq &G'(F)_e\backslash G'(\mathbb{A}_f)/\mathrm{GL}_2(\mathcal{O}_\wp)U^\wp,\end{aligned}$$

*where $G'(F)_e$ means the set of elements in $B'$ with even order at $\wp$.*

**Superspecial points.** A point $x$ in the special fiber of $\mathcal{M}_{U_0,\wp}$ is called *superspecial* if the corresponding formal $\mathcal{O}_{B,\wp}$-module $\mathcal{V}_{\mathbb{F}}$ is a direct sum of two formal $\mathcal{O}_\wp$-module of dimension 1 and height 2. Let $\Omega_{\mathbb{F}}$ be a fixed formal $\mathcal{O}_\wp$-module over $\mathbb{F}$ of dimension 1 and height 2 which is unique up to isomorphism. Let $\mathcal{O}_{B,\wp}\simeq \mathrm{End}_{\mathcal{O}_\wp}(\Omega)$ be a fixed isomorphism which is unique up to conjugation. Then there is an isomorphism

$$\mathcal{V}_{\mathbb{F}}\simeq\Omega_{\mathbb{F}}\oplus\Omega_{\mathbb{F}}$$

which is unique up to conjugation by $\mathrm{GL}_2(\mathcal{O}_{B,\wp})$. The action of $\mathcal{O}_{B,\wp}$ on $\mathcal{V}$ is given by an embedding

$$\iota:\quad \mathcal{O}_{B,\wp}\longrightarrow M_2(\mathcal{O}_{B,\wp}).$$

It is easy to see that the set of isomorphism classes of superspecial $\mathcal{V}_{\mathbb{F}}$ is in 1-1 correspondence with the set of conjugacy classes of $\iota$. For a fixed $\iota$, let $R_\iota$ denote the centralizer of the image of $\iota$.

Fix one superspecial point $x_0=(\mathcal{V}_0,\widehat{V}_0^\wp,\kappa^\wp)$ of conjugacy class $[\iota]$. Via $\kappa_0^\wp$, one may identify $\widehat{V}^\wp$ with $\widehat{B}^\wp$, and $\widehat{B}'^\wp$ with $\widehat{B}^\wp$. Then for any superspecial point $x=(\mathcal{V}_x,\widehat{V}_x^\wp,\kappa_x^\wp)$ we may find a quasi-isogeny $\alpha: x\longrightarrow x_0$ which induces an isomorphism between $\mathcal{V}_x$ and $\mathcal{V}_0$. Such an $\iota$ is unique up to multiplication by elements of $G'(F)_0$ of elements whose components at $\wp$ is in $R_\iota^\times$. The level structure $\kappa_x^\wp$ now induces one $g^\wp\in G'(\mathbb{A}_f^\wp)$. Thus we have the following:

LEMMA 5.4.5. *The set of superspecial points of class $[\iota]$ is identified with*

$$G'(F)_0\backslash G'(\mathbb{A}_f^\wp)/U^\wp,$$

*where $G'(F)_0$ denotes the elements in $G'(F)$ with images in $R_\iota^\times$.*

Now, let $K$ be a totally imaginary quadratic extension embedded in $B$. We want to study the reduction $C_U$ of CM points by $K$. We will only consider *special* points in $C_U$, i.e., those points whose endomorphism has maximal component at $\wp$. We want to show that the special CM-points have superspecial reduction. First, let's construct some special formal $\mathcal{O}_{B,\wp}$-module over $\overline{\mathcal{O}}_\wp$.

Then $\mathcal{O}_{B,\wp}$ can be written as

$$\mathcal{O}_{B,\wp}=\mathcal{O}_{K,\wp}+\mathcal{O}_{K,\wp}\epsilon \tag{5.4.3}$$

where $\epsilon\in B_\wp^\times$ such that $x\epsilon=\epsilon\bar{x}$ for any $x\in K_\wp$, and that $\epsilon^2\in F_\wp^\times$ with order

$$\mathrm{ord}_\wp(\epsilon^2)=\begin{cases}1 & \text{if } K_\wp/F_\wp \text{ is unramified,}\\ 0 & \text{if } K_\wp/F_\wp \text{ is ramified.}\end{cases} \tag{5.4.4}$$

Let $\Omega$ be a formal $\mathcal{O}_{K,\wp}$-module of height 1 and dimension 1 over $\bar{\mathcal{O}}_\wp$. A $K_\wp$-special module over $\bar{\mathcal{O}}_\wp$ is the following module:

$$\mathcal{V} \simeq \Omega \oplus \Omega,$$

such that for $x, y \in \Omega, \alpha \in \mathcal{O}_{K,\wp}$,

$$\epsilon(x,y) = (\epsilon^2 y, x), \qquad \alpha(x,y) = (\alpha x, \bar{\alpha} y).$$

In this case all $K_\wp$-special points have superspecial reduction with the same conjugacy class and the corresponding ring $R_\wp := R_\iota$ is given by the following

$$R_\wp = \mathcal{O}_K + \epsilon^2 \mathcal{O}_K \epsilon', \qquad \epsilon' = \begin{pmatrix} 0 & \epsilon^{-1} \\ \epsilon & 0 \end{pmatrix}.$$

LEMMA 5.4.6. *All $K_\wp$-special points have $K_\wp$-special module at $\wp$. Moreover the set of special CM -points by $K$ is given by*

$$T(F)_0 \backslash G'(\mathbb{A}_f^\wp)/U^\wp$$

*where $T(F)_0$ denotes the set of elements in $T(F)$ whose components at $\wp$ has order 0. Moreover the map from special CM-points by $K$ to the set of superspecial points and to the set of irreducible components are given by the following natural projection:*

$$\begin{aligned} T(F)_0 \backslash G'(\mathbb{A}_f^\wp)/U^\wp &\longrightarrow G'(F)_0 \backslash G'(\mathbb{A}_f^\wp)/U^\wp \\ &\longrightarrow G'(F)_e \backslash G'(\mathbb{A}_f)/\mathrm{GL}_2(\mathcal{O}_\wp)U^\wp \end{aligned}$$

*Proof.* For any special CM-point $x = (V, h, \kappa)$, it suffices to show that the Tate module $\mathbb{T}_\wp := \mathbb{T}_\wp(\widetilde{V})$ is isomorphic to $\mathcal{O}_{B,\wp}$ with action by $\mathcal{O}_{B,\wp}$ by left multiplication and with action by $\mathcal{O}_{K,\wp}$ by right multiplication.

First, we consider the case where $K_\wp$ is unramified. As

$$\mathcal{O}_{K,\wp} \otimes \mathcal{O}_{K,\wp} = \mathcal{O}_{K,\wp}^2,$$

any $\mathcal{O}_{B,\wp} \otimes \mathcal{O}_{K,\wp}$-module is a direct sum with an action by $\epsilon$. The conclusion follows easily.

We now consider the case where $K_\wp$ over $F_\wp$ is ramified. Then any $\mathcal{O}_{B,\wp} \otimes \mathcal{O}_{K,\wp}$ module is a module $M$ over the discrete valuation ring $A := \mathcal{O}_{K,\wp}[\epsilon]$ with an action

$$\alpha : \quad \mathcal{O}_{K,\wp} \longrightarrow \mathrm{End}_{\mathcal{O}_{K,\wp}}(M),$$

such that $\alpha(a)\epsilon = \epsilon\alpha(\bar{a})$ for any $a \in \mathcal{O}_{K,\wp}$. The $\mathcal{O}_{B,\wp} \otimes \mathcal{O}_{K,\wp}$-module $T_\wp := T_\wp(\mathcal{V}_\wp)$ now has rank 1 over $A$, thus is free of rank 1. Lets fix one isomorphism

$$\phi : \quad \mathbb{T}_\wp \simeq A,$$

and let $\eta \in \mathrm{End}(T_\wp)$ be the endomorphism over $\mathcal{O}_{K,\wp}$ given by the conjugation of $A/\mathcal{O}_{K,\wp}$. Then for any $a \in \mathcal{O}_{K,\wp}$ which is trace free, $\alpha(a)\eta$ commutes with $\epsilon$. Thus it must be given by

$$\alpha(a) = \phi^{-1} \circ \eta a x \circ \phi$$

where $x \in A^\times$. Since $\alpha(a^2) = a^2$ we have that $x\bar{x} = 1$. Thus, there is an $y \in R^\times$, $x = y/\bar{y}$. By replacing $\phi$ by $y \circ \phi$, we may assume that $x = 1$. The conclusion follows easily. ∎

### 5.5. Local CM-intersections

In this section we are going to compute the local intersection index of CM-points at their reduction. When the level structure is minimal, the formula can be proved using Gross' theory of canonical and quasi-canonical lifting. When the level structure is not maximal then there are some fundamental obstructions to computing the local index, since no explicit semistable model is known. We will prove asymptotic formulas which are apparently sufficient for the applications in the next chapter.

**Ordinary case.** First lets consider a prime $\wp$ of $F$ which is split in $K$. Let $\pi$ be a fixed local parameter of $F_\wp$. Then all CM-points in $C_U$ will have ordinary reduction over $\mathbb{F}$. In particular all these reductions are smooth points in the special fiber. If $U$ is sufficiently small so that $\overline{\mathcal{M}}_{U,\wp}$ is representable, the *geometric intersection index* $(x,y)_{U,\wp}$ of two distinct CM points $x$ and $y$ in $\overline{M}_U$ can be defined to be the maximal rational number $t$ such that

$$\bar{x} = \bar{y} \mod \pi^t$$

where $\bar{x}$ and $\bar{y}$ are closures of $x$ and $y$ in $\overline{\mathcal{M}}_U$. This definition can be extended to divsiors with disjoint support in $C_U$. For general $U$, we take $U'$ a subgroup of $U$ such that $\overline{\mathcal{M}}_{U',\wp}$ is representable and then define

$$(x,y)_{U,\wp} = [U:U']^{-1}(\alpha^*x, \alpha^*y)_{U',\wp}$$

where $\alpha^*$ denote the pull-back map of divisors induced by the projection $\overline{\mathcal{M}}_{U',\wp} \longrightarrow \overline{\mathcal{M}}_{U,\wp}$.

We have shown that the reduction of ordinary CM-points on $\overline{\mathcal{M}}_U$ is given by the following projection:

$$(5.5.1) \qquad T(F)\backslash G(\mathbb{A}_f)/U \longrightarrow T(F)\backslash [N(F_\wp)\backslash G(F_\wp)] \times G(\mathbb{A}_f^\wp)/U.$$

Thus the intersection of CM-points is taken in the set $N(F_\wp)$. More precisely, let $x, y$ be two CM-points with the same ordinary reduction. Then $x$ and $y$ can be represented by elements $g, h \in G(\mathbb{A}_f)$ such that

$$h^\wp = g^\wp, \qquad h_\wp = ng_\wp$$

with $n \in N(F_\wp)$. Then the intersection of $x$ and $y$ depends only $n$ when $U^\wp$ is sufficiently small. In order to describe intersection precisely, lets give a modular interpretation of $N(F_\wp)$.

Let $\mathcal{L}$ be the unique formal $\mathcal{O}_\wp$-module over $\overline{\mathcal{O}}_\wp$ of dimension 1 and height 1 with a fixed base $\zeta$ of $T_\wp(\mathcal{L})$. By a *polarization* on an $\mathcal{O}_\wp$-module $\mathcal{E}$ over $\overline{\mathcal{O}}_\wp$, we mean a system of Weil pairings of group schemes

$$\langle \cdot, \cdot \rangle_n \qquad \mathcal{E}[\wp^n] \times \mathcal{E}[\wp^n] \longrightarrow \mathcal{L}[\wp^n]$$

with respect to a uniformizer $\pi$ of $\mathcal{O}_\wp$. This pairing thus induces a pairing on $T_\wp(\mathcal{E})$.

Let $\mathcal{X}$ be the set of isomorphism classes of objects $(\mathcal{E}, \alpha, \beta)$ where

1. $\mathcal{E}$ is a polarized divisible $\mathcal{O}_\wp$-module over $\overline{\mathcal{O}}_F$ of height 2;
2. $\alpha$ is an isomorphism from $\mathcal{E}_\mathbb{F}$ to $F_\wp/\mathcal{O}_\wp \oplus \mathcal{L}_\mathbb{F}$;
3. $\beta$ is an isomorphism from $\mathcal{O}_\wp^2 \to T_\wp(\mathcal{E})$;

such that the following two conditions are verified:

- $\det\beta$ is of determinant 1 in sense that when composing with the Weil pairing, $\det\beta$ as a level structure of $\mathcal{L}$ is given by the base $\zeta$;
- the morphism

$$T_\wp(\alpha)\circ\beta: \qquad \mathcal{O}_\wp^2 \longrightarrow T_\wp(F_\wp/\mathcal{O}_\wp)$$

is given by the first projection and the base $\xi = \lim_n \pi^{-n}$ of of $T_\wp(F_\wp/\mathcal{O}_\wp)$.

Then $\mathcal{X}_\wp$ may be identified with $N(F_\wp)$. Indeed, let $\mathcal{E}_0$ be the divisible group $\mathcal{L}\oplus F_\wp/\mathcal{O}_\wp$ with a canonical polarization, a canonical deformation $\alpha_0$, and a canonical level structure

$$\beta_0 : F_\wp^2 \longrightarrow V_\wp(\mathcal{E}_0), \qquad \beta_0(a,b) = a\xi + b\zeta.$$

Then for any object $(\mathcal{E},\alpha,\beta)$ there is a unique isogeny $\phi:\mathcal{E}\to\mathcal{E}_0$ so that $\phi$ respects the reduction maps $\alpha$'s. Now $\phi$ and $\beta$'s induce an element $g\in N(F_\wp)$ which acts on $F_\wp^2$ by right multiplications on row vectors.

For any $x\in F_\wp$ let $(E_x,\alpha_x,\beta_x)$ be the object corresponding to

$$n(x) := \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} \in N(F_\wp)$$

in the above correspondence. For $n$ a positive integer, let $m(n,x)$ be the maximal rational number $t$ such that modulo $\pi^t$, the $(E_x,\alpha_x)$ is isomorphic to $(E_0,\alpha_0)$ and that $\beta$ and $\beta_0$ induces the same level structure modulo $\wp^n$.

Lemma 5.5.1. *Assume that $n\geq \mathrm{ord}(x)+1$. Then*

$$m(n,x) = \frac{1}{q^{n-\mathrm{ord}(x)-1}(q-1)}.$$

*Proof.* Under the quasi-isogeny $\phi: E_x\to E_0$ with respect to the reduction morphism $\alpha$'s , the image $T_\wp(E_x)$ is the following lattice of $T_\wp(E_0)=\mathcal{O}_\wp^2$:

$$\mathcal{O}_\wp^2\cdot\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}\xi\\ \zeta\end{pmatrix} = \mathcal{O}_\wp(x\zeta+\xi)+\mathcal{O}_\wp\zeta,$$

with the level structure

$$\kappa(a,b) = a(x\zeta+\xi)+b\zeta.$$

We first consider the case where $x\in\mathcal{O}_\wp$. Then $\phi$ is an isomorphism of divisible modules. We may take $E_x = E_0$ with the above level structure. Modulo $\wp^n$ this level structure gives two generators

$$x\zeta_{\wp^n}+\xi_{\wp^n}, \qquad \zeta_{\wp^n}.$$

Thus for $n\geq\mathrm{ord}(x)+1$,

$$m(n,x) = \mathrm{ord}(x\zeta_{\wp^n}) = \mathrm{ord}(\zeta_{\wp^{n-\mathrm{ord}(x)}}) = \frac{1}{q^{\mathrm{ord}(x)-n-1}(q-1)}.$$

Here we have used the fact that $\mathcal{O}_\wp^{\mathrm{ur}}(\zeta_{\wp^n})$ is a totally ramified Galois extension of $\mathcal{O}_\wp^{\mathrm{ur}}$ with group

$$(\mathcal{O}_\wp/\wp^n)^\times.$$

It remains to treat the case where $x \notin \mathcal{O}_\wp$. Let $x = x'\pi^{-s}$ with $x' \in \mathcal{O}_\wp^\times$. Let $u : E_0 \to E_0$ be an isogeny inducing the map

$$a\xi + b\zeta \longrightarrow x^{-1}a\xi + b\zeta$$

on $V_\wp(E_0) \simeq F_\wp^2$. Then there is an isogeny $v : E_0 \longrightarrow E$ such that $\phi \circ v = u$. For $\phi$ to be an isomorphism over some $\overline{\mathcal{O}}_F$-scheme $S$ if and only if the isogeny $u_S$ and $v_S$ have the same kernel. By construction,

$$\ker(u) = \frac{\mathcal{O}_\wp x\xi + \mathcal{O}_\wp\zeta}{\mathcal{O}_\wp\xi + \mathcal{O}_\wp\zeta} = \mathcal{O}_\wp\xi_{\wp^s}$$
$$\ker(v) = \frac{\mathcal{O}_\wp(x\xi + x\zeta) + \mathcal{O}_\wp\zeta}{\mathcal{O}_\wp\xi + \mathcal{O}_\wp\zeta} = \mathcal{O}_\wp(\zeta_{\wp^s} + \xi_{\wp^s}).$$

Thus if $\phi$ is an isomorphism of formal groups over some $\mathcal{O}_\wp^{\mathrm{ur}}$-scheme $S$, then one must have $\zeta_{\wp^s} = 0$ on $S$. Assume now this is the case. Then $\phi$ is an isomorphism which transform the level structure $\kappa$ modulo $\wp^n$ on $E_0$ to the level structure

$$(a, b) \longrightarrow a(\xi_{\wp^n} + x'\zeta_{\wp^{n+s}}) + b\zeta_{\wp^n}.$$

(Notice that $\zeta_{\wp^{n+s}} \in E_0(S)[\wp^n]$ as $\zeta_{\wp^s} = 0$) The condition $\beta = \beta_0$ modulo $\wp^n$ is equivalent to $\zeta_{\wp^{n+s}} = 0$. Thus

$$m(n, x) = \mathrm{ord}(\zeta_{\wp^{n+s}}) = \frac{1}{q^{n+s-1}(q-1)}.$$

This completes the proof of the proposition. □

**Supersingular case.** We now consider a prime $\wp$ of $F$ which is nonsplit in $K$ but split in $B =_\tau B$. As usual, let $B' =_\wp B$ and let $G$ and $G'$ denote the algebraic groups over $F$ associated to $B^\times$ and $(B')^\times$. Let $\bar{F}_\wp$ be an algebraic closure of $K_\wp$ with algebraically closed residue field $\mathbb{F}$. Then all points in $C_U$ have supersingular reductions at $\mathbb{F}$ and the reduction is given by the following map

$$T(F)\backslash G(\mathbb{A}_f)/U \longrightarrow G'(F)\backslash F_\wp^\times \times G(\mathbb{A}_f^\wp)/U. \tag{5.5.2}$$

If we write CM-points as

$$G'(F)\backslash \left(G(F) \times_{T(F)} G(F_\wp)\right) \times G(\mathbb{A}_f^\wp)/U, \tag{5.5.3}$$

then this reduction map is given by

$$\begin{aligned} G'(F) \times_{T(F)} G(F_\wp)/U_\wp &\longrightarrow F_\wp, \\ g_1 \times g_2 &\longrightarrow \det(g_1) \cdot \det(g_2). \end{aligned} \tag{5.5.4}$$

It follows that the local intersection of CM-points is given by a distribution on

$$\left\{ (g_1, g_2) \in G'(F_\wp) \times_{T(F_\wp)} G(F_\wp) \,\middle|\, \det(g_1)\det(g_2) \in \det(U_\wp) \right\}.$$

More precisely, let $x$ and $y$ be two CM-points with the same reduction. Then $x$ and $y$ can be represented by $g, h \in G(\mathbb{A}_f)$ such that

$$h^\wp = \gamma g^\wp, \qquad \det(h_\wp) = \det(\gamma) \cdot \det(g_\wp)$$

for an $\gamma \in G'(F)$. The intersection of $x$ and $y$ depends only on $(\gamma, g_\wp \cdot h_\wp^{-1})$ in the above set when $U^\wp$ is sufficiently small.

To describe the local intersection moreprecisely we need a description of this set in terms of formal $\mathcal{O}_\wp$-modules. Let $\bar{\mathcal{E}}$ be a polarized formal $\mathcal{O}_\wp$-module of height 2 over $\mathbb{F}$ with an endomorphism given by $B'_\wp$ which is unique up to isomorphism. Let $\mathcal{X}_\wp$ denote the set of isomorphism classes of objects $(\mathcal{E}, \alpha, \beta)$ where

1. $\mathcal{E}$ is a polarized formal $\mathcal{O}_\wp$-module of height 2 over $\overline{\mathcal{O}}_\wp$ with endomorphism by some order in $K_\wp$;
2. $\alpha : \mathcal{E}_{\mathbb{F}} \to \bar{\mathcal{E}}$ is an isomorphism of formal $\mathcal{O}_\wp$-modules with degree 1 (with respect to the polarizations);
3. $\beta : F_\wp^2 \longrightarrow V_\wp(\mathcal{E}_F)$ is an isomorphism of degree 1.

Then we have an identification

$$\mathcal{X}_\wp = \left\{(g_1, g_2) \in G'(F_\wp) \times_{T(F_\wp)} G(F_\wp) : \quad \det(g_1) \cdot \det(g_2) = 1\right\}.$$

To see this let $\mathcal{E}_0$ be the canonical deformation of $\bar{\mathcal{E}}$ with respect to the embedding $K_\wp \to B_\wp$ with the canonical rigidification $\alpha_0$ and a fixed $U_\wp$-level structure $\beta_0$. Then for any object $(\mathcal{E}, \alpha, \beta)$, we have an isogeny $\phi : \mathcal{E} \longrightarrow \mathcal{E}_0$ with compatible action by elements in $K$. The isogeny $\phi$ induces element $(g_1, g_2) \in G'(F_\wp) \times G(F_\wp)$:

$$g_1 : \quad \bar{\mathcal{E}} \xrightarrow{\phi_F^{-1}} \mathcal{E}_{\mathbb{F}} \xrightarrow{\alpha} \bar{\mathcal{E}}$$

$$g_2 : \quad F_\wp^2 \xrightarrow{\beta} V_\wp(\mathcal{E}_F) \xrightarrow{V_\wp(\phi)} V_\wp(\mathcal{E}_{0,F}) \xrightarrow{\beta_0^{-1}} F_\wp^2.$$

It is easy to see that the class of $(g_1, g_2)$ in $G'(F_\wp) \times_{T(F_\wp)} G(F_\wp)$ is independent of choice of $\phi$.

Conversely, for any pair $[(g_1, g_2)]$ as above, there is an isogeny $\phi : \mathcal{E} \to \mathcal{E}_0$ and an $U$-level structure $\beta$ such that $g_2$ is given by the above formula. The isogeny $\phi$ induces an isogeny $\phi_{\mathbb{F}} : \mathcal{E}_{\mathbb{F}} \to \bar{\mathcal{E}}$. There is a unique isogeny $\alpha : \mathcal{E}_{\mathbb{F}} \to \mathcal{E}$ such that $g_1$ is given by the above formula.

The intersection theory on $\mathcal{X}_\wp$ is difficult to describe because the universal deformation ring of supersingular points with level structure is singular in general. But for the minimal level structure, the intersection theory can be formulated by Gross' theory of canonical and quasi-canonical liftings. In the following lets describe the intersection for the minimal level structure: $U_\wp = \mathrm{GL}_2(\mathcal{O}_\wp)$.

Fix one element $(g_1, g_2)$ of $\mathcal{X}_\wp$. Modulo $\mathrm{GL}_2(\mathcal{O}_\wp)$, we may assume that $g_2 = \begin{pmatrix} 1 & 0 \\ 0 & \pi^c \end{pmatrix}$ for some $c \geq 0$. Indeed, write $\mathcal{O}_K = \mathcal{O}_\wp + \mathcal{O}_\wp \delta$ and take an embedding of $T$ into $\mathrm{GL}_2$ by the obvious isomorphism $\mathcal{O}_\wp^2 \simeq \mathcal{O}_K$. Then by multiplying $g_2$ by some element of $K_\wp^\times$, we may assume that $\mathcal{O}_K \subset g_2(\mathcal{O}_K)$ and $g_2(\mathcal{O}_K)/g_2(\mathcal{O}_K)$ is cyclic and is generated by the image of $\pi^{-c}$. This implies that

$$g_2(\mathcal{O}_K) = \mathcal{O}_\wp \wp^{-c} + \mathcal{O}_\wp \delta = \begin{pmatrix} \pi^{-c} & 0 \\ 0 & 1 \end{pmatrix} \mathcal{O}_K$$

for some $c \in \mathbb{N}$. Consequently, $g_2 \in \begin{pmatrix} \pi^{-c} & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_\wp)$. We call $c$ the *conductor* of $(g_1, g_2)$. Let $(\mathcal{E}_c, \alpha_c, \beta_c)$ be the object of conductor $c$ and let $m(g_1, g_2)$ denote the maximal rational number $t$ such that this object is isomorphic to $(\mathcal{E}_0, \alpha_0, \beta_0)$ modulo $\pi^t$.

Let $\epsilon$ be a trace free element in $B'_\wp$ such that $x\epsilon = \epsilon\bar{x}$ for any $x \in K_\wp$. Then $B'_\wp = K_\wp + K_\wp\epsilon$. For any $g = a + b\epsilon \in (B'_\wp)^\times$ define

$$\xi(g) = \frac{\det b\epsilon}{\det g} = \frac{-\epsilon^2 \mathrm{N}(b)}{\mathrm{N}(a) - \epsilon^2\mathrm{N}(b)}. \tag{5.5.5}$$

LEMMA 5.5.2. *If $c = 0$, then*

$$m(g_1, g_2) = \frac{1}{2}\mathrm{ord}(\pi\xi(g_1)).$$

*If $c > 0$, then*

$$m(g_1, g_2) = \begin{cases} \frac{1}{q^{2c-2}(q^2+1)} & \text{if } K_\wp/F_\wp \text{ is unramified,} \\ \frac{1}{2q^c} & \text{if } K_\wp/F_\wp \text{ is ramified.} \end{cases}$$

*Proof.* By construction, $g_2^{-1}$ is integral thus $\phi^{-1}$ gives an isogeny $\psi : \mathcal{E}_0 \longrightarrow \mathcal{E}_c$ with kernel generated by a $\zeta \in \mathcal{E}_0[\wp^c]$. Now $g_1$ is given by

$$g_1 = \alpha \cdot \psi_{\mathbb{F}}.$$

The number $m(g_1, g_2)$ is the maximal rational number $t$ such that $\alpha$ can be extended to isomorphism modulo $\pi^t$. Thus $m(g_1, g_2)$ is also the maximal rational number $t$ such that $g_1$ can be extended to an endomorphism of $\mathcal{E}_0$ modulo $\pi^t$, and such that $g_1$ kills kernels of $\psi$, or equivalently, $g_1$ kills $\zeta$.

First we assume that $c = 0$. Then $g_1$ and $\psi$ are isomorphisms and Gross' theorem shows that $m(g_1, g_2)e(K_\wp/F_\wp)$ is the maximal integer $m$ such that $g_1 \in \mathcal{O}_{K,\wp} + \pi_K^{m-1}\mathcal{O}_B$. We may choose a decomposition $B_\wp = K_\wp + K_\wp\epsilon$ such that $\epsilon x = \bar{x}\epsilon$ for any $x \in K_\wp$, and $\epsilon^2 \in F_\wp^\times$ with order given by

$$\mathrm{ord}(\epsilon^2) = \begin{cases} 1 & \text{if } K_\wp/F_\wp \text{ is unramified,} \\ 0 & \text{if } K_\wp/F_\wp \text{ is ramified.} \end{cases}$$

Write $g_1 = a + \pi_K^{m-1}b\epsilon$ with $a \in \mathcal{O}_K, b \in \mathcal{O}_K^\times$, then

$$\xi(g_1) = -\pi_K^{2m-2}\epsilon^2 b^2 \det(g_1)^{-1}.$$

We now assume that $c > 0$. Then over $\mathcal{O}_{K,\wp}^{\mathrm{ur}}$, all cyclic submodules $D_i$ of $\mathcal{E}_0[\wp^c]$ are conjugate to each other. The total intersection is 1. Thus

$$(e(K_\wp/F_\wp)m(g_1, g_2))^{-1} = \#(\mathcal{O}_{K,\wp}/\pi^c)^\times/(\mathcal{O}_\wp/\pi^c)^\times.$$

□

We want to treat now the case where $U_\wp$ is not maximal where $\mathcal{M}_U$ need to be replaced by some resolution of singularities after a base change. We will only consider so called *special CM-points* $C_U^0$ which are represented by $g \in G(\mathbb{A}_f)$ whose component at $\wp$ is in $T(F_\wp) \cdot U_\wp$. Thus we have identification:

$$C_U^0 = T(F)_0 \backslash G(\mathbb{A}_f^\wp)/U^\wp$$

where $T(F)_0$ denotes the elements in $T(F)$ whose image in $T(F_\wp)$ is in $U_\wp$. Let $G'(F)_0$ denote the elements in $G'(F)$ whose image in $G'(F_\wp)$ has determinant in $\det(U_\wp)$.

Lemma 5.5.3. *Let $L$ be a finite extension of $\mathcal{O}_F^{\mathrm{ur}}$ over which all points in $C_U^0$ are rational. Let $\mathcal{M}'_{U,\wp}$ be the minimal resolution of singularities of $\mathcal{M}_U \otimes \mathcal{O}_L$. Then the reduction of $C_U^0$ is given by*

$$T(F)_0 \backslash G(\mathbb{A}_f^\wp)/U^\wp \longrightarrow G'(F)_0 \backslash G'(\mathbb{A}_f)/U'$$

*where $U' = U'_\wp \cdot U^\wp$ with $U'_\wp$ an open compact subgroup of $G'(F_\wp)_0$ containing $T(F_\wp)_0$.*

*Proof.* Then the reduction on $\mathcal{M}_{U,\wp}$ is given by

$$T(F)_0 \backslash G(\mathbb{A}_f^\wp)/U^\wp \longrightarrow G'(F)_0 \backslash G(\mathbb{A}_f^\wp)/U^\wp.$$

Let $X$ be the formal neighborhood of a supersingular point in $\mathcal{M}_{U,\wp}$ structure when $U^\wp$ is sufficiently small. Then $X$ is isomorphic to the universal defomation scheme of a formal $\mathcal{O}_\wp$-module of height 2 with level $U_\wp$-structure. It is wellknown that $X$ is regular and has an action by $G'(F_\wp)_0$. Let $X'$ be the inverse image of $X$ in $\mathcal{M}'_{U,\wp}$ which is also the minimal resolution of sigularities of $X \otimes \mathcal{O}_L$. By functoriality, $X'$ has an action by $G'(F_\wp)_0$. It induces an action on the special fiber $X'_{\mathbb{F}}$ of $X'$. By continuity, it is factored by an open subgroup $U'_\wp$ of $G'(F_\wp)_0$. Thus reduction of CM-points which is given locally by

$$G'(F_\wp)_0/T(F_\wp)_0 \longrightarrow X'_{\mathbb{F}}$$

has a finite image $Y$. The reduction of CM-points in the minimal regular model $M_U \otimes L$ is given by

$$T(F)_0 \backslash G(\mathbb{A}_f^\wp)/U^\wp \longrightarrow G'(F)_0 \backslash Y \times G(\mathbb{A}_f^\wp)/U^\wp.$$

□

Since we may rewrite $C_U^0$ in the form

$$C_U^0 = G'(F)_0 \backslash G'(F)_0/T(F)_0 \times G'(\mathbb{A}_f^\wp)/U^\wp,$$

thus the reduction of CM-points is induced by the map

$$G(F_\wp)_0/T(F_\wp)_0 \longrightarrow G'(F_\wp)/U_\wp.$$

The intersection theory is given by some function $m(g)$ on $G'(F_\wp)_0/T(F_\wp)_0$ in the following sense when $U^\wp$ is sufficiently small. Let $x, y$ be two special CM-points in $C_U^0$ represented by $g, h \in G(\mathbb{A}_f^\wp)$. Then $x$ and $y$ have the same reduction only if there is a $\gamma \in G'(F)$ such that $h = \gamma g$. Then the local intersection of $x$ and $y$ is given by $m(\gamma)$.

Lemma 5.5.4. *The local intersection of CM-points with respect to $U$-level structure is given by a function on $G'(F_\wp)_0/T(F_\wp)_0$ such that*

$$m(g) = m_0(g) + m'(g), \qquad g \notin T(F_\wp)$$

*where $m_0(g)$ is supported on $U'_\wp$ and is the restriction of $\frac{1}{2}\mathrm{ord}\xi(g)$ and $m'(g)$ is a locally constant function on $G'(F_\wp)$.*

*Proof.* Let $X_0$ denote $X$ in the proof of the previous lemma corresponding to the maximal group $\mathrm{GL}_2(\mathcal{O}_\wp)$. Let $X'_0$ denote the base change $X_0 \otimes \mathcal{O}_L$. Then $X'_0$ is smooth and the map $X' \longrightarrow X'_0$ is generically etale. Let $y$ be a point in $Y$. Then the local ring of $y$ at $Y$ is isomorphic to $\mathcal{O}_L[[T]]$, so is the local ring of $x$ in $X'_L$. Thus the map $X' \longrightarrow X'_L$ is given by a power series $f(T) = \sum_i a_i T^i \in \mathcal{O}_L[[T]]$ with $a_1 \neq 0$. It follows that $\mathrm{ord}(f(T)/T)$ is locally constant. □

**Superspecial case.** It remains to treat the case where $\wp$ is a prime of $F$ which is not split in $B$. First, let's consider the case where $U_\wp$ is maximal. The reduction from the special CM-points to superspecial points takes the form

$$T(F)_0\backslash G(\mathbb{A}_f^\wp)/U^\wp \longrightarrow G'(F)_0\backslash G'(\mathbb{A}_f^\wp)/U^\wp, \tag{5.5.6}$$

where $G'(F)_0$ is the subgroup of elements on $G'(F)$ whose components at $\wp$ are in $R_\wp^\times$, where $R_\wp$ is constructed in the last section which takes the form

$$R_\wp = \mathcal{O}_K + \epsilon^2\mathcal{O}_K\epsilon', \qquad \epsilon' = \begin{pmatrix} 0 & \epsilon^{-1} \\ \epsilon & 0 \end{pmatrix}.$$

Thus the local intersection occurs in $G'(F)_0/T(F)_0$. More precisley if $x$ and $y$ are two special CM-points with the same reduction. Then $x$ and $y$ can be represented by $g, h \in G(\mathbb{A}_f^\wp)$ such that

$$h = \gamma g$$

with a $\gamma \in G'(F)_0$. When $U^\wp$ is sufficiently small, the intersection of $x$ and $y$ depends only on $\gamma$. As in previous cases, we need a modular interpretation in the formal $\mathcal{O}_{B,\wp}$-module level.

Let $\bar{\mathcal{V}}$ be a superspecial $\mathcal{O}_{B,\wp}$-module over $\mathbb{F}$. Consider the set $\mathcal{X}_\wp$ of the following objects $(\mathcal{E}, \alpha)$ where

1. $\mathcal{V}$ is a formal $\mathcal{O}_{B,\wp} \otimes \mathcal{O}_{K,\wp}$-module;
2. $\alpha : \bar{\mathcal{V}} \longrightarrow \mathcal{V}_{\mathbb{F}}$ is an isomorphism.

It is easy to see that this set is identified with

$$R_\wp^\times/\mathcal{O}_{K,\wp}^\times.$$

More precisely, let $(\mathcal{V}_0, \alpha_0)$ be a fixed object. We identify $\bar{\mathcal{V}}$ with $\mathcal{V}_{0,\mathcal{F}}$ via $\alpha_0$. Then for any object $(\mathcal{V}, \alpha)$ there is an isomorphism $\phi : \mathcal{V} \longrightarrow \mathcal{V}_0$ of $\mathcal{O}_{B,\wp} \otimes \mathcal{O}_{K,\wp}$ -modules which is unique up to action by $\mathcal{O}_{K,\wp}^\times$. There is an element $g \in R_\wp^\times = \mathrm{Aut}(\bar{\mathcal{V}}_0)$ such that $\phi_{\mathbb{F}} \circ \alpha = g$.

LEMMA 5.5.5. *Let $(\mathcal{V}, \alpha)$ be an object corresponding to an object $g \in R_\wp^\times$. Then the maximal rational number $t$ such that $(\mathcal{V}, \alpha)$ and $(\mathcal{V}_0, \alpha_0)$ are isomorphic modulo $\pi^t$ is given by*

$$m(g) = \begin{cases} \frac{1}{2}\mathrm{ord}\xi(g) & \text{if } K_\wp/F_\wp \text{ is unramified} \\ \frac{1}{2}\mathrm{ord}\pi\xi(g) & \text{if } K_\wp/F_\wp \text{ is ramified} \end{cases}$$

*Proof.* Let $\mathcal{V}_{0,m}$ denote $\mathcal{V}_0 \otimes \mathcal{O}_{K,\wp}^{\mathrm{ur}}/\pi_K^m$ and $\mathcal{V}_m$ denote $\mathcal{V} \otimes \mathcal{O}_{K,\wp}^{\mathrm{ur}}/\pi_K^m$. Then the intersection number times $e(K_\wp/F_\wp)$ is the maximal integer $m$ such that $\alpha : \mathcal{V}_{0,\mathbb{F}} \longrightarrow \mathcal{V}_{\mathbb{F}}$ can be extended to an isomorphism from $\mathcal{V}_{0,m}$ to $\mathcal{V}_m$, or the maximal integer $m$ such that

$$g = \phi_{\mathbb{F}} \circ \alpha \in R_m := \mathrm{End}_{\mathcal{O}_{B,\wp}}(\mathcal{V}_{0,m}).$$

By Lemma 5.4.6, we may decompose $\mathcal{V}_0$ as a direct sum $\mathcal{V}_0 = \Omega \oplus \Omega$ where $\Omega$ is a $\mathcal{O}_{K,\wp}$-module of dimension 1 and height 1 over $\mathcal{O}_\wp^{\mathrm{ur}}$ with standard action by

$$\mathcal{O}_{B,\wp} = \mathcal{O}_{K,\wp} + \mathcal{O}_{K,\wp}\epsilon$$

given as follows. For $x, y \in \Omega$, $\alpha \in \mathcal{O}_{K,\wp}$,

$$\alpha(x,y) = (\alpha x, \bar{\alpha} y), \qquad \epsilon(x,y) = (\epsilon^2 y, x).$$

Let $\Omega_m$ denote the reduction of $\Omega$ modulo $\pi_K^m$. Now $R_m$ is the centralizer of $\mathcal{O}_{B,\wp}$ in $\mathrm{End}(\mathcal{V}_{0,m})$ and, therefore,

$$R_m = R_\wp \cap M_2(\mathrm{End}(\Omega_{0,m})).$$

By Gross' theorem,

$$\mathrm{End}(\Omega_m) = \mathcal{O}_{K,\wp} + \pi_K^{m-1}\mathcal{O}_{K,\wp}\epsilon.$$

It is easy now to see that

$$R_m = \mathcal{O}_K + \epsilon^2 \pi_K^{m-1}\mathcal{O}_K \epsilon'.$$

For $g = a + \epsilon^2 b \pi_K^{m-1}\epsilon' \in R_\wp^\times$ with $b \in \mathcal{O}_{K,\wp}^\times$, then

$$\mathrm{ord}_{\pi_K}(\xi(g)) = \mathrm{ord}_{\pi_K}(\mathrm{N}(\epsilon^2\pi_K^{m-1})) = \begin{cases} 2m & \text{if } K_\wp/F_\wp \text{ is unramified,} \\ 2(m-1) & \text{if } K_\wp/F_\wp \text{ is ramified.} \end{cases}$$

□

We consider now the general case of $U_\wp$. The same proof of Lemma 5.5.3, 5.5.4 gives the following:

LEMMA 5.5.6. *Let $C_U^0$ denote the set of special points with level $U$ structure. Let $L$ be a finite extension of $\mathcal{O}_F^{\mathrm{ur}}$ over which all points in $C_U^0$ are rational. Let $\mathcal{M}'_{U,\wp}$ be the minimal resolution of singularities of $\mathcal{M}_U \otimes \mathcal{O}_L$. Then the reduction of $C_U^0$ is given by*

$$T(F)_0 \backslash G(\mathbb{A}_f^\wp)/U^\wp \longrightarrow G'(F)_0 \backslash G'(\mathbb{A}_f)/U'$$

*where $U' = U'_\wp \cdot U^\wp$ with $U'_\wp$ an open compact subgroup of $G'(F_\wp)$ containing $T(F_\wp)_0$. Moreover the local intersection of CM-points with respect to $U$-level structure is given by a distribution on $G'(F_\wp)_0/T(F_\wp)_0$ such that*

$$m(g) = m_0(g) + m'(g), \qquad g \notin T(F_\wp)$$

*where $m_0(g)$ is supported on $U'_\wp$ and is the restriction of $\frac{1}{2}\mathrm{ord}\xi(g)$ and $m'(g)$ is a locally constant function on $G'(F_\wp)$.*

# 6. Gross-Zagier formula

In this chapter, we are going to compute the height pairing and finish the proof of the Gross-Zagier formula. We will start with a review of Arakelov theory on an arithmetic surface, and the arithmetic Hodge index theorem which will express height pairings as a sum of Green's functions over places of number fields with respect to a fixed *arithmetic polarization.* Then, we apply this theory to Shimura curves polarized by the *Hodge class*, and compute the Green's functions of *distinct CM-points* on Shimura curves. Strictly speaking, we can only compute the height pairing of CM-points modulo (1) the contributions from intersections of CM-points with Eisenstein class, (2) self-intersections of CM-points, and (3) the coefficients of some forms on compact quaternion algebras. Finally, we will show that all these non-computable contributions are *negligible.*

### 6.1. Calculus on arithmetic surfaces

In this section we will reviewing the Arakelov theory on arithmetic surfaces and arithmetic Hodge index theory. The basic references are [12, 13, 28, 32]. The only new concept is the Green's function over nonarchimedean places.

**Arithmetic divisors and hermitian line bundles.** Let $F$ be a number field. By an arithmetic surface over $\mathrm{Spec}\mathcal{O}_F$, we mean a projective and flat morphism $\mathcal{X} \longrightarrow \mathrm{Spec}\mathcal{O}_F$ such that that $\mathcal{X}$ is a regular scheme of dimension 2. Let $\widehat{\mathrm{Div}}(\mathcal{X})$ denote the group of *arithmetic divisors* on $\mathcal{X}$. Recall that an arithmetic divisor on $\mathcal{X}$ is a pair $\widehat{D} := (D, g)$ where $D$ is a divisor on $\mathcal{X}$ and $g$ is a function on

$$X(\mathbb{C}) = \coprod X_\tau(\mathbb{C})$$

with some logarithmic singularities on $|D|$ such that for each archimedean place $\tau$ of $F$, and each point $x_0 \in X_\tau(\mathbb{C})$ with local coordinate $t$, the function

$$x \longrightarrow g(x) + \mathrm{ord}_{x_0}(D_\tau) \log |t(x)|$$

can be extended to a smooth function in a neighborhood of $x_0$. The form $-\frac{\partial\bar{\partial}}{\pi i} g$ on $X(\mathbb{C}) \setminus |D|$ can be extended to a smooth form $c_1(\widehat{D})$ on $X(\mathbb{C})$ which is called the *curvature* of the divisor $\widehat{D}$. If $f$ is a nonzero rational function on $\mathcal{X}$ then we can define the corresponding *principal arithmetic divisor* by

$$\widehat{\mathrm{div}} f = (\mathrm{div} f, -\log |f|). \tag{6.1.1}$$

An arithmetic divisor $(D, g)$ is called *vertical* (resp. *horizontal*) if $D$ is supported in the special fibers (resp. $D$ does not have component supported in the special fiber).

The group of arithmetic divisors is denoted by $\widehat{\mathrm{Div}}(\mathcal{X})$ while the subgroup of principal divisor is denoted by $\widehat{\mathrm{Pr}}(\mathcal{X})$. The quotient $\widehat{\mathrm{Cl}}(\mathcal{X})$ of these two groups is called the *arithmetic divisor class group* which is actually isomorphic to the group $\widehat{\mathrm{Pic}}(\mathcal{X})$ of hermitian line bundles on $\mathcal{X}$. Recall that a hermitian line bundle on $\mathcal{X}$ is a pair $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$, where $\mathcal{L}$ is a line bundle on $\mathcal{X}$ and $\|\cdot\|$ is hermitian metric on $\mathcal{L}(\mathbb{C})$ over $X(\mathbb{C})$. For a rational section $\ell$ of $\mathcal{L}$, we can define the corresponding divisor by

$$\widehat{\mathrm{div}}(\ell) = (\mathrm{div}\ell, -\log \|\ell\|). \tag{6.1.2}$$

It is easy to see that the divisor class of $\widehat{\mathrm{div}}(\ell)$ does not depend on the choice of $\ell$. Thus one has a well defined map from $\widehat{\mathrm{Pic}}(\mathcal{X})$ to $\widehat{\mathrm{Cl}}(\mathcal{X})$. This map is an isomorphism with converse defined by assigning an arithmetic divisor $\widehat{D} = (D, g)$ to an arithmetic line bundle $\mathcal{O}(\widehat{D}) = (\mathcal{O}(D), \|\cdot\|)$ such that the canonical section $\ell$ of $\mathcal{O}(D)$ has metric

$$\|\ell\|(x) = e^{-g(x)}.$$

One may show that the curvature of an arithmetic divisor depends only on its class and thus can be defined on $\widehat{\mathrm{Pic}}(\mathcal{X})$ such that the curvature of hermitian line bundle $\overline{\mathcal{L}}$ is

$$c_1(\overline{\mathcal{L}}) = \frac{\partial\bar{\partial}}{\pi i} \log \|\ell\|. \tag{6.1.3}$$

Let $\widehat{D}_i = (D_i, g_i)$ (i=1, 2) be two arithmetic divisors on $\mathcal{X}$ with disjoint support in the generic fiber:

$$|D_{1F}| \cap |D_{2F}| = \emptyset.$$

Then one can define an *arithmetic intersection pairing*

$$\widehat{D}_1 \cdot \widehat{D}_2 = \sum_v (\widehat{D}_1 \cdot \widehat{D}_2)_v$$

where $v$ runs through the set of places of $F$. To define the intersection we may assume that $D_i$ are irreducible. Then the local intersection is defined as follows:

- if $D_1$ is vertical, and $v$ is finite place
$$(\widehat{D}_1 \cdot \widehat{D}_2)_v = \deg_{D_1}(\mathcal{O}(D_2)) \log q_v,$$
where $\deg_{D_1}(\mathcal{O}(D_2))$ is the *geometric degree.*
- if $D_2$ is horizontal and $v$ is finite, then
$$(\widehat{D}_1 \cdot \widehat{D}_2)_v = \sum_{x \in |X_v|} \log \#\mathcal{O}_{X,x}/(f_1, f_2),$$
where $x$ runs through the set of closed point of $X$ over $v$, and $f_i$ are defining equation of $D_i$ near $x$;
- if $v$ is infinite, then
$$(\widehat{D}_1 \cdot \widehat{D}_2)_v = g_1(D_{2v})\epsilon_v + \int_{X_v(\mathbb{C})} g_2 c_1(\widehat{D}_1)\epsilon_v,$$
where $\epsilon_v = 1$ if $v$ is real and $\epsilon_v = 2$ if $v$ is complex.

One may show that the principal arithmetic divisor has 0-intersection with any other divisors. Thus the intersection pairing only depends on the divisor class. On the other hand, for any two arithmetic divisor classes, we can always find representatives with disjoint support at the generic fiber. It follows that we have a well defined pairing on $\widehat{\mathrm{Pic}}(\mathcal{X})$:

$$(\overline{\mathcal{L}}, \overline{\mathcal{M}}) \longrightarrow \hat{c}_1(\overline{\mathcal{L}}) \cdot \hat{c}_1(\overline{\mathcal{M}}) \in \mathbb{R}.$$

Let $V(\mathcal{X})$ be the group of *vertical metrized line bundles*: namely $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathcal{X})$ with $\mathcal{L} \simeq \mathcal{O}_X$. Then we have an exact sequence

$$0 \longrightarrow V(\mathcal{X}) \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{X}) \longrightarrow \mathrm{Pic}(\mathcal{X}_F) \longrightarrow 0.$$

Define the group of *flat* bundles $\widehat{\mathrm{Pic}}^0(\mathcal{X})$ as the orthogonal complement of $V(\mathcal{X})$. Then we have an exact sequence

$$0 \longrightarrow \widehat{\mathrm{Pic}}^0(\mathcal{O}_F) \longrightarrow \widehat{\mathrm{Pic}}^0(\mathcal{X}) \longrightarrow \mathrm{Pic}^0(X_F) \longrightarrow 0.$$

Recall that the Jacobian $\mathrm{Jac}(X)$ has a Neron-Tate hight papring on its algebraic points defined by theta functions [12]. The following theorem gives a relation between intersection pairing and height pairing:

THEOREM 6.1.1 (Hodge index theorem [12]). *For $\overline{\mathcal{L}}, \overline{\mathcal{M}} \in \widehat{\mathrm{Pic}}^0(\mathcal{X})$,*

$$\langle \mathcal{L}_F, \mathcal{M}_F \rangle = -\hat{c}_1(\overline{\mathcal{L}}) \cdot \hat{c}_1(\overline{\mathcal{M}})$$

*where the left hand side denotes the Neron-Tate height pairing on* $\mathrm{Pic}^0(X) = \mathrm{Jac}(X)(F)$.

In the following we want to introduce a projection formula for the intersection pairing or the height pairing. Let $L$ be a finite extension of $F$ and $\mathcal{Y} \longrightarrow \mathcal{O}_L$ be an arithmetic surface over $\mathcal{O}_L$. Let $f : \mathcal{Y} \longrightarrow \mathcal{X}$ be a morphism over $\mathcal{O}_F$ which is finite at the generic fiber. Then we can define the pull-back map

$$f^* : \quad \widehat{\mathrm{Div}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{Div}}(\mathcal{Y}).$$

The intersection pairing satisfies the following projection formula: for $\widehat{D}_i \in \mathrm{Div}(\mathcal{X})$ $(i = 1, 2)$

$$f^*\widehat{D}_1 \cdot f^*\widehat{D}_2 = \deg f \cdot (\widehat{D}_1 \cdot \widehat{D}_2). \tag{6.1.4}$$

Moreover, if $\widehat{D}_i$ are disjoint at the generic fiber, then projection formula is true for local intersection:

$$\sum_{w|v} (f^*\widehat{D}_1 \cdot f^*\widehat{D}_2)_w = \deg f \cdot (\widehat{D}_1 \cdot \widehat{D}_2)_v. \tag{6.1.5}$$

For $X$ a curve over $F$, let $\widehat{\mathrm{Pic}}(X)$ denote the projective limit of $\widehat{\mathrm{Pic}}(\mathcal{X})$ over all models over $X$. Then the intersection pairing can be extended to $\widehat{\mathrm{Pic}}(X)$. Let $\bar{F}$ be an algebraic closure of $F$ and let $\widehat{\mathrm{Pic}}(X_{\bar{F}})$ be the direct limit of $\mathrm{Pic}(X_L)$ for all finite extensions $L$ of $F$, then the intersection pairing on $\widehat{\mathrm{Pic}}(X_L)$ times $[L : F]^{-1}$ can be extended to an intersection pairing on $\widehat{\mathrm{Pic}}(X_{\bar{F}})$. One still has the Hodge index theorem to relate the normalized heights pairing on $\mathrm{Jac}(X)(\bar{F})$ and the intersection pairing on the *flat* bundles of $\widehat{\mathrm{Pic}}^0(X_{\bar{F}})$.

**Adelic Green's functions.** Let $\mathcal{X}$ be an arithmetic surface as before and let $X$ be the generic fiber of $\mathcal{X}$. Let $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbb{Q}}$ be a fixed class with degree 1 at the generic fiber. Let $x \in X(F)$ be a rational point and let $\bar{x}$ be the corresponding section $\mathcal{X}(\mathcal{O}_F)$. Then $\bar{x}$ can be extended to a unique element $\widehat{x} = (x + D, g)$ in $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ such that

- the bundle $\mathcal{O}(\widehat{x}) \otimes \overline{\mathcal{L}}^{-1}$ is flat;
- for any finite place $v$ of $F$, the component $D_v$ of $D$ on the special fiber of $\mathcal{X}$ over $v$ satisfies
$$D_v \cdot c_1(\mathcal{L}) = 0;$$
- for any infinite place $v$,
$$\int_{X_v(\mathbb{C})} g c_1(\overline{\mathcal{L}}) = 0.$$

We define now the Green's function $g_v(x, y)$ on

$$X(F) \times X(F) \setminus \text{diagonal}$$

by

$$g_v(x, y) = (\widehat{x} \cdot \widehat{y})_v / \log q_v, \tag{6.1.6}$$

where $\log q_v = 1$ or $2$ if $v$ is real or complex. It is easy to see that $g_v(x,y)$ is symmetric, and does not depend on the model $\mathcal{X}$ of $X$ in the following sense: if $\mathcal{X}'$ is different model of $X$ and $\overline{\mathcal{L}}'$ is a hermitian line bundle on $\mathcal{X}'$, such that over some model $\mathcal{X}''$ which dominates both $\mathcal{X}$ and $\mathcal{X}'$, $\overline{\mathcal{L}}'$ and $\overline{\mathcal{L}}$ have the same pull-back, then the Green's functions defined by $(\mathcal{X}, \overline{\mathcal{L}})$ and $(\mathcal{X}', \overline{\mathcal{L}}')$ are same. Also, the Green's function $g_v(x,y)$ is stable under base change. Thus we have a well-defined Green's function on $X(\bar{F})$ for each place $v$ of $F$.

In fact one can define a Green's function $g_v(x,y)$ on $X(\mathbb{C}_v)$ where $\mathbb{C}_v$ is the completion of $\bar{F}$ at a place over $v$. We don't need this fact in this paper.

Practically, one may construct $g_v(x,y)$ in the following manner. If $v$ is a complex place then $g_v(x,y)$ is a solution to the equation

$$\frac{\partial_y \overline{\partial}_y}{\pi i} g_v(x,y) = \delta_x(y) - c_1(\overline{\mathcal{L}})(y). \tag{6.1.7}$$

Let $v$ be a finite place. Then it is easy to see that

$$(\widehat{x} \cdot \widehat{y})_v = (\bar{x} \cdot \bar{y})_v + (D_v \cdot \bar{y}).$$

Thus we have decomposition

$$g_v(x,y) = i_v(\bar{x}, \bar{y}) + j_v(\bar{x}, \bar{y}) \tag{6.1.8}$$

where

$$i_v(\bar{x}, \bar{y}) = (\bar{x} \cdot \bar{y})_v / \log q_v \qquad j_v(\bar{x}, \bar{y}) = (D_v \cdot \bar{y}) / \log q_v. \tag{6.1.9}$$

Notice that $i_v(x,y)$ is the usual *geometric intersection index* in the sense of algebraic geometry over algebraically closed fields, and $j_v(x,y)$ actually depends only on the reductions of $x$ and $y$ in the set of irreducible components of the special fiber of $\mathcal{X}$ over $v$.

The decomposition $g_v = i_v + j_v$ depends on the model $\mathcal{X}$. But if we only work on semistable model, we can actually get a well-defined function $i_v$ and $j_v$ over $X(\bar{F})$. We will not need this fact in this paper.

### 6.2. Global heights of CM-points

**Heights and intersection on tower of Shimura curves.** We now want to apply the general theory of the previous section to intersections of CM-points to Shimura curves $X_U$ over a totally real field $F$ as defined in §1.3. Recall that $X_U$ has the form

$$X_U = G(F)\backslash \mathcal{H}^{\pm} \times G(\mathbb{A}_f)/U \cup \{\text{cusps}\} \tag{6.2.1}$$

which is a smooth and projective curve over $F$ but may not be connected. Let's first try to extend the theory in the last section to the projective limit $X$ of $X_U$. Let $\widehat{\mathrm{Pic}}(X)$ denote the direct limit of $\widehat{\mathrm{Pic}}(X_U)$ with respect to the pull-back maps. We fix one measure on $G(\mathbb{A}_f)$. Then the intersection pairing can be extended to $\widehat{\mathrm{Pic}}(X)$ if we modify the pairings on $\mathrm{Pic}(X_U)$ by the scale $\mathrm{vol}(U)$. Similarly, we can modify local intersection pairing and extend the height pairing to $\mathrm{Jac}(X) = \mathrm{Pic}^0(X)$, which is the direct limit of $\mathrm{Pic}^0(X_U)$ where $\mathrm{Pic}^0(X_U)$ is the subgroup of $\mathrm{Pic}(X_U)$ with class whose degree is 0 on each connected component.

**Hodge classes and Eisenstein classes.** To define Green's function we need to define a canonical class in $\mathrm{Pic}(X)_{\mathbb{Q}}$. On each $X_U$, there is a unique adelic metrized line bundle $\xi_U \in \mathrm{Pic}(X_U)_{\mathbb{Q}}$ of degree 1 on each connected component such that

$$\mathrm{T}_a \xi_U = \sigma_1(a)\cdot \xi, \qquad \sigma_1(a) := \deg \mathrm{T}_a = \sum_{b|a} \mathrm{N}(b), \tag{6.2.2}$$

for any integral idele $a$ prime to the level of $X_U$. The uniqueness is clear as the difference of two such class will be a class in $\mathrm{Pic}^0(X_U)_{\mathbb{Q}} = \mathrm{Jac}(X_U)(F)_{\mathbb{Q}}$ which is cuspidal under the action of the Hecke algebra. For existence, we let $U'$ be a sufficiently small normal subgroup such that every geometric connected component of $X_{U'}$ does not have any elliptic fixed point. Then $[\Omega_{X_{U'}}]$ will have the same degree on each component and satisfies the above equation. Certainly some power of this class will descend to a class $\xi'$ in $\mathrm{Pic}(X_U)$ with the same positive degree on each geometric connected component. We may now define $\xi_U$ to be a constant multiple of $\xi'$ in $\mathrm{Pic}^1(X_U)_{\mathbb{Q}}$. We call $\xi_U$ the *Hodge class* on $X_U$.

It is an interesting question to construct an adelic metric on $\xi_U$ such that the above equation holds for $\bar{\xi}_U$. But in [32], Corollary 4.3.3, we have constructed a metric on $\xi_U$ such that

$$\mathrm{T}_a \widehat{\xi}_U = \sigma_1(a)\widehat{\xi}_U + \phi(a) \tag{6.2.3}$$

where $\phi(a) \in \widehat{\mathrm{Pic}}(F)$ is a $\sigma$-derivation, i.e., for any coprime $a', a''$

$$\phi(a'a'') = \sigma(a')\phi(a'') + \sigma(a'')\phi(a').$$

Let $\mathrm{Pic}(X_U)_{\mathbb{Q}}^{\mathrm{Eis}}$ be the subgroup of elements whose restriction on each connected component is a multiple of the restriction of $\xi$. It is easy to show that

$$\mathrm{Pic}(X_U)_{\mathbb{Q}} = \mathrm{Pic}(X_U)_{\mathbb{Q}}^{\mathrm{Eis}} \oplus \mathrm{Pic}^0(X_U)_{\mathbb{Q}}. \tag{6.2.4}$$

We define $\widehat{\mathrm{Pic}}(X_U)_{\mathbb{Q}}^{\mathrm{Eis}}$ to be the class whose restriction on each irreducible component is a sum of a constant class and a multiple of the restriction of that of $\widehat{\xi}$. Let $\mathrm{Pic}(X)_{\mathbb{Q}}^{\mathrm{Eis}}$ (resp. $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}^{\mathrm{Eis}}$) denote the limit of $\mathrm{Pic}(X_U)_{\mathbb{Q}}^{\mathrm{Eis}}$.

The action of the $G(\mathbb{A}_f)$ on $\mathrm{Pic}(X)_{\mathbb{Q}}^{\mathrm{Eis}}$ is Eisenstein. Indeed, let's define

$$d_U: \quad \mathrm{Pic}(X_U)_{\mathbb{Q}} \longrightarrow \mathcal{S}(Z_U)$$

to be the degree map times $\mathrm{vol}(U)$ where $Z_U = F_+^\times \backslash \mathbb{A}_f^\times / \det(U)$ is the set of connected components of $X_U$. It is easy to extend $d_U$ to a map

$$d: \quad \mathrm{Pic}(X)_{\mathbb{Q}} \longrightarrow \mathcal{S}(F_+^\times \backslash \mathbb{A}_f^\times). \tag{6.2.5}$$

It is easy to see that this map is $G(\mathbb{A}_f)$-equivariant and its restriction on $\mathrm{Pic}(X)_{\mathbb{Q}}^{\mathrm{Eis}}$ is injective. Thus the action of $G(\mathbb{A}_f)$ on $\mathrm{Pic}(X)_{\mathbb{Q}}^{\mathrm{Eis}}$ is Eisenstein. Similarly, one may show that the action of $G(\mathbb{A}_f)$ on $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}^{\mathrm{Eis}}$ is *quasi-Eisenstein.*

We can now define Green's functions $g_v$ on divisors on $X(\bar{F})$ which are disjoint at the generic fiber for each place $v$ of $F$ by multiplying the Green's functions on $X_U$ by $\mathrm{vol}(U)$.

**Height pairing of CM-points.** Let $\eta = \eta_\chi$ be a divisor on $X_U$ defined by an anticyclotomic idele class character $\chi$ of $K$ of degree $\kappa$, where $U = \ker \chi_\Delta$. Notice that $\kappa$ is nonzero only if $\chi$ is trivial. Let $z = [\eta - \kappa \cdot \xi]$ denote the class of $\eta - \kappa \cdot \xi$ in $\mathrm{Jac}(X_U)$. Notice that this class actually lives in $\mathrm{Jac}(X)(L) \otimes \mathbb{C}$ where $L$ is a finite abelian extension fixed by the kernel of $\chi$. The linear functional

$$a \longrightarrow |a|\langle z, \mathrm{T}_a z\rangle$$

is now the Fourier coefficient of a cuspform $\Psi$ of weight 2:

$$\widehat{\Psi}(a) = |a|\langle z, \mathrm{T}_a z\rangle. \tag{6.2.6}$$

In the following we want to express this height in terms of intersections modulo some Eisenstein series and theta series.

Let $\bar{\eta}$ be the arithmetic closure of $\eta$ with respect to $\bar{\xi}$. Then the Hodge index theorem gives

$$\begin{aligned} |a|\langle z, \mathrm{T}_a z\rangle &= -\,|a|\left(\bar{\eta} - \bar{\xi},\, \mathrm{T}_a\bar{\eta} - \deg \mathrm{T}_a\bar{\xi}\right) \\ &= -\,|a|(\bar{\eta}, \mathrm{T}_a\bar{\eta}) + \widehat{E}(a), \end{aligned}$$

where $\widehat{E}(a)$ is the Fourier coefficient of certain derivations of Eisenstein series.

The divisor $\eta$ and $\mathrm{T}_a\eta$ has some common component. We want to compute its contribution in the intersections. Let $r_\chi(a)$ denote the Fourier coefficients of the theta series associated to $\chi$:

$$r_\chi(a) = \sum_{b|a} \chi(b).$$

The we have the following:

LEMMA 6.2.1. *The divisor*

$$\mathrm{T}_a^0\eta := \mathrm{T}_a\eta - r_\chi(a)\eta$$

*is disjoint with $\eta$.*

*Proof.* The multiplicity of $\eta$ in $\mathrm{T}_m\eta$ is given by the following integral

$$\int_{T(\mathbb{A})\backslash G(\mathbb{A}_f)} \mathrm{T}_a\phi(x)\bar{\phi}(x)dx = \mathrm{T}_a\phi(1)$$

where $\phi(x)$ is supported on $T(\mathbb{A}_f)U$ with character $\chi$. In our terminology in §4.2, this is $\ell(m, 0)$ and is computed previously in Lemma 4.2.1. □

In summary, we have shown that the functional

$$a \longrightarrow |a|\langle z, \mathrm{T}_a z\rangle$$

is essentially given by the sum of local intersections

$$-\frac{1}{[L:F]}\sum_v \sum_{\iota \in \mathrm{Gal}(L/F)} g_v(\mathrm{T}_a^0\eta^\iota, \eta^\iota)|a|\log q_v$$

modulo some derivations of Eisenstein series, and $\Pi(\chi)\otimes\alpha^{1/2}$, (where $L$ is the subfield of $\bar F$ fixed by the kernel of $\chi$). The Galois action of $\mathrm{Gal}(K^{\mathrm{ab}}/F)$ is given by class field theory

$$\nu: \quad \mathrm{Gal}(K^{\mathrm{ab}}/F) \longrightarrow N_T(F)\backslash N_T(\mathbb{A}_f),$$

and the left multiplication of the group $N_T(\mathbb{A}_f)$. It follows that if $\eta$ is defined by a function $\phi(g)$ on $T(F)\backslash G(\mathbb{A}_f)$, and $\eta^\iota$ is defined by $\phi(\nu(\iota)^{-1}g)$.

If $\nu(\iota)\in T(F)\backslash T(\mathbb{A}_f)$, then $\phi(\nu(\iota)^{-1}g)=\chi^{-1}(\iota)\phi(g)$. Otherwise,

$$\nu(\iota)=\nu(\iota')\cdot\epsilon\in T(F)\backslash T(\mathbb{A}_f)\epsilon,$$

then

$$\phi(\nu(\iota)g)=\chi(\iota')\phi(\epsilon g)=\mathrm{T}_\epsilon\phi(\epsilon g\epsilon).$$

Notice that $\phi(\epsilon g\epsilon)$ define the divisor $\bar\eta$ corresponding to the character $\bar\chi$. Since $\mathrm{T}_a$ is self-adjoint and commutes with complex conjugation,

$$g_v(\mathrm{T}_a^0\bar\eta,\bar\eta)=g_v(\bar\eta,\mathrm{T}_a^0\bar\eta)=g_v(\overline{\mathrm{T}_a^0\bar\eta},\overline{\bar\eta})=g_v(\mathrm{T}_a^0\eta,\eta).$$

Thus, we have proved the following.

Lemma 6.2.2. *Modulo the derivations of $\sigma_1$ and $r_\chi$, the functional of height pairing*

$$a\longrightarrow |a|\langle z,\mathrm{T}_a z\rangle$$

*is the sum*

$$-|a|\sum_v g_v(\eta,\mathrm{T}_a^0\eta)\log q_v.$$

Notice that for two CM-divisors $A$ and $B$ on $X_U$ with disjoint support represented by two functions $\phi$ and $\psi$ on $T(F)\backslash G(\mathbb{A}_f)$, the Green's function at a place $v$ depends only on $\phi$ and $\psi$. Thus we may simply denote it as

$$g_v(A,B)=g_v(\phi,\psi).$$

### 6.3. Green's functions

In this section we are going to compute the Green's function of CM-points using formulas obtained in Chapter 4.

**Archimedean case.** For each archimedean place $\tau_i$ of $F$, the Riemann surface $X\otimes_{\tau_i}\mathbb{C}$ is actually defined by the same way as $X\otimes_\tau\mathbb{C}$ with $\tau_1$ replaced by $\tau_i$. Thus it suffice to compute the Green's function over the original place $\tau$.

The complex points of $X=X_U$ are identified with

$$X(\mathbb{C})=G(F)\backslash\mathcal{H}^\pm\times G(\mathbb{A}_f)/U \tag{6.3.1}$$

which is really a disjoint union of curves of the type

$$\Gamma\backslash\mathcal{H}.$$

In this case, the $\bar{\eta}$ has curvature proportional to the hyperbolic metric $dxdy/y^2$ for $z = x + yi \in \mathcal{H}$. The set of CM-points is identified with $T(F)\backslash G(\mathbb{A}_f)/U$.

The Green's $g_\tau(x, y)$ on $X$ is nonzero only if both $x$ and $y$ are in the same connected component. In this case, it is given by the constant term as $s \to 0$ of the following convergent series for $\mathrm{Re}(s) > 0$:

$$\sum_{\gamma\in\Gamma} Q_s\left(1 + \frac{|x - \gamma y|^2}{2\mathrm{Im}x\mathrm{Im}\gamma y}\right) \tag{6.3.2}$$

where

$$Q_s(t) = \int_0^\infty \left(x + \sqrt{t^2 - 1}\cosh x\right)^{-1-s} dx. \tag{6.3.3}$$

We refer to Gross [18] and Gross-Zagier [20] for more details.

Notice that if $x = gi, y = hi$ then

$$\frac{|x - y|^2}{2\mathrm{Im}x\mathrm{Im}y} = -2\xi(g^{-1}h),$$

where $\xi$ is a function on $T(\mathbb{R})\backslash \mathrm{GL}_2(\mathbb{R})/T(\mathbb{R})$ defined as before.

Lets define a function $m_s$ on $T(F)\backslash G(F)/T(F)$ as follows:

$$m_s(g) = \begin{cases} Q_s(1 - 2\xi(g_\tau)) & \text{if } \xi(g_\tau) < 0, \\ 0. & \text{otherwise.} \end{cases} \tag{6.3.4}$$

Then

Lemma 6.3.1. *For two CM-points $x, y \in X(\mathbb{C})$, the Green's function at $\tau$ is given by the constant term of a geometric pairing as defined in §4.1 with multiplicity function $m_s$.*

*Proof.* Extend $m$ to a function on $T(F)\backslash G(F)/T(F) \times G(\mathbb{A}_f)$ with support on $T(F)\backslash G(F)/T(F) \times \{e\}$. Then we need to show that

$$g_s(x, y) = \sum_{\gamma\in G(F)} m_s(g^{-1}\gamma h),$$

where $g, h$ are two elements in $G(\mathbb{A}_f)$ representing $x$ and $y$. Indeed, if the right hand side is nonzero then there is a $\gamma_0 \in G(F)$ such that $g^{-1}\gamma h$ has finite component in $U$ and such that $\gamma$ has positive determinant. It follows that $x$ and $y$ are in the same connected component. It is easy to show that $g_s(x, y)$ has the same expression as before. □

**Ordinary case.** We now want to consider the Green's function at a prime $\wp$ of $F$ which is split in $K$. For $U = U_\wp U^\wp$ we have shown the following for the model $\overline{\mathcal{M}}_U$:

1. the set of ordinary points is given by

$$K^\times\backslash(N(F_\wp)\backslash \mathrm{GL}_2(F_\wp)) \times G(\mathbb{A}_f^\wp)/U;$$

2. the special fiber $\bar{\mathcal{M}}_{U,\mathbb{F}}$ over $\wp$ has connected components indexed by

$$G(F)_+\backslash G(\mathbb{A}_f)/U;$$

3. each component is a union of irreducible components indexed by
$$\mathbb{P}^1(F_\wp)/U_\wp;$$
4. every two irreducible components intersects at the set of supersingular points indexed by
$$G(F)\backslash G(\mathbb{A}_f)/U',$$
where $U' = U'_\wp \cdot U^\wp$ with $U'_\wp$ is the maximal compact subgroup of $G(F_\wp)$.

In the following we want to compute the Green's function $g_\wp$ for CM-divisors $A, B$ represented by functions $\phi$ and $\psi$ on $T(F)\backslash G(\mathbb{A}_f)/U$. Let $L$ be a finite extension of $F$ where every point in $A, B$ is rational and let $\mathcal{X}_{U,L}$ be the minimal resolution of singularity of $\overline{\mathcal{M}} \otimes \mathcal{O}_L$. Then we have the decomposition
$$g_\wp(A,B) = i_\wp(A,B) + j_\wp(A,B).$$
Notice that in general $i, j$ depends on $U$ but when $U^\wp$ is sufficiently small, then $i(A,B), j(A,B)$ will not depend on $U$ for fixed $\phi$ and $\psi$. This is because the morphism
$$\overline{\mathcal{M}}_{U_1,\wp} \longrightarrow \overline{\mathcal{M}}_{U_2,\wp}$$
is smooth at ordinary points when $U_i^\wp$ are sufficiently small. So we have a well defined decomposition
$$g_\wp(\phi,\psi) = i_\wp(\phi,\psi) + j_\wp(\phi,\psi).$$

First let's start to compute the geometric intersection index $i_\wp(\bar\eta, \mathrm{T}_a^0\bar\eta)$ using Lemma 5.5.1. Let $d\mu(g)$ be a distribution on $G(\mathbb{A}_f)$ supported on $N(F_\wp)$ over which it is induced by the *multiplicative* measure on $F_\wp^\times$:
$$d\mu\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} = d^\times x = \frac{1}{1-q^{-1}}\frac{dx}{|x|}.$$
Define a distribution $d\mu(x,y)$ on $[T(F)\backslash G(\mathbb{A}_f)]^2$ such that for any $\phi(x,y) \in \mathcal{S}(T(F)\backslash G(\mathbb{A}_f))$
$$\int \phi(x,y)d\mu(x,y) = \int_{T(F)\backslash G(\mathbb{A}_f)} dx \int_{G(\mathbb{A}_f)} \phi(x,gx)d\mu(g), \tag{6.3.5}$$
where $dx$ is a measure on $G(\mathbb{A}_f)$.

Lemma 6.3.2. *The geometric intersection index of CM-divisors is given by the following distribution. Let $A$ and $B$ be two CM-divisors on $X_U$ represented by two functions $\phi$ and $\psi$ on $T(F)\backslash G(\mathbb{A}_f)/U$. Then*
$$i_\wp(A,B) = \int \phi(x)\bar\psi(y)d\mu(x,y).$$

*Proof.* It is easy to see that both sides are additive in both $B$ and $A$ and are invariant under the action by $G(\mathbb{A}_f)$. Thus it is sufficient to prove the lemma in the case where $A = U$, $B = \begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}U$,
$$U_\wp = (1 + \wp^n M_2(\mathcal{O}_\wp)),$$

and $n \geq \mathrm{ord}(a)+1$. Then Lemma 5.5.1 gives

$$i_\wp(A,B) = \mathrm{vol}(U)\frac{1}{q^{n-\mathrm{ord}(x)-1}(q-1)},$$

where $q = \mathrm{N}(\wp)$. On the other hand, its is easy to obtain that

$$\int \phi(x)\bar{\psi}(y)d\mu(x,y) = \mathrm{vol}(U)\int_{|x-a|\leq|\pi|^n} d^\times x = \mathrm{vol}(U)\frac{|\pi|^n}{(1-q^{-1})|a|}.$$

The lemma now follows. □

Lemma 6.3.3. *The local intersection index is given by*

$$i_\wp(\eta, \mathrm{T}_a^0\eta) = r_\chi(a')\sum_{i+j=n_v}\mu(\pi)^{i-j} j \log \mathrm{N}(\wp) =: r_\chi^\wp(a)$$

*where $a = a'\wp^{n_\wp}$ is the primary decomposition.*

*Proof.* The intersection we want is

$$\begin{aligned}
(\eta, \mathrm{T}_a^0\eta)_\wp =& \int_{[T(F)\backslash G(\mathbb{A}_f)]^2} \mathrm{T}_a^0\phi(x)\bar{\phi}(y)d\mu(x,y)\\
=& \int_{T(F)\backslash T(\mathbb{A}_f)} \bar{\chi}(y)dy \int_{G(\mathbb{A}_f)} \mathrm{T}_a^0\phi(gy)d\mu(g)\\
=& \mathrm{vol}(T(F)\backslash T(\mathbb{A}_f)) \int_{G(\mathbb{A}_f)} \mathrm{T}_a^0\phi(g)d\mu(g).
\end{aligned}$$

Write $a = a'\wp^n$ with $a'$ prime to $\wp$. Recall that $\mathrm{T}_m^0\phi$ is simply the part of $\mathrm{T}_a\phi$ restricted to the complement of $T(\mathbb{A}_f)U$. Thus, on the support of the distribution of $m$, $\mathrm{T}_a^0\phi$ is simply

$$\mathrm{T}_{\wp^n}^0\phi_\wp \cdot \mathrm{T}_{a'}\phi^\wp.$$

Thus the last integral here is a product of two integrals

$$\int_{G(\mathbb{A}_f^\wp)} \mathrm{T}_{a'}\phi^\wp(x)d\mu(x) = \mathrm{T}_{a'}\phi^\wp(1) = r_\chi(a'),$$

and

$$\int_{F_v} \mathrm{T}_{\wp^n}^0\phi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} dx^\times.$$

This integral is zero if $n = 0$.

If $n > 0$, then $U_\wp$ is the maximal $\mathrm{GL}_2(\mathcal{O}_\wp)$, and this last integral is

$$\begin{aligned}
&\int_{|x|>1} \mathrm{T}_{\wp^n}\phi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} dx^\times \\
=&\int_{|x|>1} \sum_{\substack{i+j=n_v\\ y\in\mathcal{O}_v/\pi^i}} \phi_\wp\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\pi^i & y\\ 0 & \pi^j\end{pmatrix}\right) dx^\times \\
=&\int_{|x|>1} \sum_{\substack{i+j=n_v\\ y\in\mathcal{O}_v/\pi^i}} \phi_\wp\left(\begin{pmatrix}\pi^i & 0\\ 0 & \pi^j\end{pmatrix}\begin{pmatrix}1 & y\pi^{-i}+x\pi^{j-i}\\ 0 & 1\end{pmatrix}\right) dx^\times \\
=&\sum_{i+j=n_v} \mu(\pi)^{i-j} \sum_{x\in\pi^{-j}\mathcal{O}_v/\mathcal{O}_v-\{0\}} q^{\mathrm{ord}(x)}(q-1)^{-1} \\
=&\sum_{i+j=n_v} \mu(\pi)^{i-j} j.
\end{aligned}$$

□

Write $r'_\chi(a)$ for the sum of all $r^\wp_\chi(a)$, which is a finite sum over $\wp \mid a$. Then it is easy to see that $r'_\chi(a)$ is one derivative for $r_\chi(a)$, i.e., for any coprime $a$, $n$,

$$r'_\chi(ab) = r_\chi(a)r'_\chi(b) + r'_\chi(a)r_\chi(b).$$

We now compute the Green's function for CM-points. Since the Hecke operator $\mathrm{T}_\ell$ for $\ell \neq \wp$ acts trivially on $\mathbb{P}^1(F_\wp)/U_\wp$, the set of ordinary components. Thus, we have the following identity of the pairings

$$j(\eta, \mathrm{T}^0_a\eta) = \deg(\mathrm{T}^0_a)j(\eta,\eta).$$

In summary, we have shown the following

LEMMA 6.3.4. *For an ordinary place $\wp$, the function*

$$a \longrightarrow |a|g_\wp(\eta, \mathrm{T}^0_a\eta)$$

*is a sum of an Eisenstein series and a derivation of the theta series*

$$\Pi(\chi)\otimes\alpha^{1/2}.$$

**Supersingular case.** We now want to handle the case where $\wp$ is a finite prime of $F$ which is split in $B$ but not split in $K$. Then all CM-points will have supersingular reduction. The reduction takes the following form:

$$T(F)\backslash G(\mathbb{A}_f)/U \longrightarrow G'(F)\backslash F^\times_\wp \times G(\mathbb{A}^\wp_f)/U, \tag{6.3.6}$$

where $G' = (B')^\times$ with $B'$ a definite quaternion algebra obtained from $B$ by changing invariants at $\wp$ and $\tau$. Notice that this reduction is taken on some base changes of the original models. So the reductions may not be regular points. To do intersection theory one must use the minimal regular models. The reduction should then take a different form.

First, let's treat the simplest case where $K/F$ is unramified at $\wp$ and where $U_\wp$ is the maximal compact subgroup $\mathrm{GL}_2(\mathcal{O}_\wp)$. In this case the reduction is given by

$$T(F)\backslash G(\mathbb{A}_f)/U \longrightarrow G'(F)\backslash G'(\mathbb{A}_f)/U', \tag{6.3.7}$$

where $U' = U'_\wp \cdot U^\wp$ with $U'_\wp = \mathcal{O}^\times_{B',\wp}$. Here we have used the identification

$$G(F_\wp)/U_\wp \simeq G'(F_\wp)/U'_\wp \simeq \mathbb{Z}/2\mathbb{Z}.$$

Notice that $\overline{\mathcal{M}}_{U,\wp}$ has smooth spacial fiber if $U^\wp$ is sufficiently small. The intersection is given by a distribution on

$$\mathcal{X}^0_\wp := \left\{(x,y) \in G'(F_\wp) \times_{T(F_\wp)} G(F_\wp) : |\det(x)\det(y)|_\wp = 1\right\}$$

given in Lemma 5.5.2. More precisely, we have:

LEMMA 6.3.5. *For any $g_2 \in G(F_\wp)$, Let $m(g_1, g_2)$ be a function on*

$$G'(F) \times G'(\mathbb{A}_f)$$

*with support on $G'(F)$ given by Lemma 5.5.2. For two disjoint CM divisors represented by two functions $\phi$ and $\psi$ on $T(F)\backslash \widetilde{G}(\mathbb{A}_f^\wp)/U$ supported on*

$$T(F)tU_\wp \times G(\mathbb{A}_f^\wp), \qquad T(F)\alpha U_\wp \times G(\mathbb{A}_f^\wp)$$

*respectively, with $t \in T(F_\wp)$ and $\alpha \in G(F_\wp)$. Then, the Green's function is given by*

$$g_\wp(\phi,\psi) = \int_{[T(F)\backslash G'(\mathbb{A}_f)]^2} \phi'(x) \sum_{\gamma \in G'(F)} m(x^{-1}\gamma\alpha^{-1}y)\bar{\psi}'(y)dxdy$$

*where $\phi'$ and $\psi'$ are functions on $T(F)\backslash G'(\mathbb{A}_f)/U'$ supported on*

$$T(F)U'_\wp \times G'(\mathbb{A}_f^\wp),$$

*such that*

$$\phi'(1, g^\wp) = \phi(t, g^\wp), \qquad \psi'(1, g^\wp) = \psi(\alpha, g^\wp).$$

*Proof.* It is easy to see that both sides are additive in $\phi$ and $\psi$ and invariant under the action of $G(\mathbb{A}_f^\wp)$. Thus, we may assume that $\phi$ is the characteristic function of $T(F)tU$, and that $\psi$ is the characteristic function of some $T(F)yU$ for some $y \in G(\mathbb{A}_f)$ with $y_\wp = \alpha$. Now $g_\wp(\phi,\psi) \neq 0$ only if they have the same reduction or equivalently, for some $\gamma_0 \in G'(F)$, $y \in \gamma_0 U'$. In this case, the intersection is given by

$$\mathrm{vol}(U)m(\gamma_0^{-1}x, y_\wp) = \mathrm{vol}(U)m(\gamma_0^{-1}, y_\wp).$$

On the other hand the integral is given by

$$\mathrm{vol}(U)\sum_{\gamma \in G'(F)} m(\gamma y) = \mathrm{vol}(U)m(\gamma_0^{-1}, y_\wp). \qquad \square$$

LEMMA 6.3.6. *Assume that $K_\wp$ is unramified over $F_\wp$. For $n$ a non-negative integer, define a function on $G'(F)$ by*

$$m_n(\gamma) = \begin{cases} \frac{1}{2}\mathrm{ord}_\wp(\xi(\gamma)\pi^{1+n}) & \text{if } \xi(\gamma) \neq 0,\ \mathrm{ord}_\wp(\xi(\gamma)\pi^n) \text{ is odd,} \\ n/2 & \text{if } \xi(\gamma) = 0,\ n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

*For two disjoint CM divisors represented by two functions $\phi$ and $\psi$ on*

$$T(F)\backslash T(F_\wp)U_\wp \times G(\mathbb{A}_f^\wp)/U \simeq T(F)\backslash T(F_\wp)U'_\wp \times G(\mathbb{A}_f^\wp)/U'$$

*which are invariant under the action from left hand side by $T(F_\wp)$, the local intersection index is given by the geometric pairing for the multiplicity function $m_n$:*

$$g_\wp(\phi, \mathrm{T}_{\wp^n}\psi) = \int_{[T(F)\backslash G'(\mathbb{A}_f)]^2} \phi(x) \sum_{\gamma\in G'(F)} m_n(x^{-1}\gamma y)\bar{\psi}(y)dxdy.$$

*Proof.* Consider the decomposition

$$G(F_\wp) = \coprod_{c=0}^{\infty} T(F_\wp) \begin{pmatrix} \pi^c & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_\wp),$$

and define constants,

$$\gamma_c = \mathrm{T}_{\wp^n}\psi_\wp \begin{pmatrix} \pi^c & 0 \\ 0 & 1 \end{pmatrix} \cdot \mathrm{vol}\left(T(F_\wp)\begin{pmatrix} \pi^c & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_\wp)\right).$$

Then by Lemma 6.3.5,

$$g_\wp(\phi, \psi) = \int_{[T(F)\backslash G(\mathbb{A}_f)]^2} \phi(x) \sum_{\gamma\in G'(F)} m'\left(x^{-1}\gamma y\right)\bar{\psi}(y)dxdy,$$

where $m'(g)$ is a distribution on $G'(\mathbb{A}_f)$ supported on $G'(F_\wp)$ such that

$$m'(\gamma) = \sum_{c\geq 0} m\left(\gamma\begin{pmatrix} \pi^{-c} & 0 \\ 0 & 1 \end{pmatrix}\right)\gamma_c.$$

We now want to compute $\gamma_c$. Notice that in our case, $\phi_\wp$ is actually the characteristic function of $\mathrm{PGL}_2(\mathcal{O}_\wp)$. It follows that

$$\begin{aligned}
\mathrm{T}_{\wp^n}\phi_\wp \begin{pmatrix} \pi^c & 0 \\ 0 & 1 \end{pmatrix} &= \sum_{\substack{i+j=n \\ x \mod \pi^i}} \phi_\wp\left(\begin{pmatrix} \pi^c & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi^i & x \\ 0 & \pi^j \end{pmatrix}\right) \\
&= \sum_{\substack{i+j=n \\ x \mod \pi^i}} \phi_\wp \begin{pmatrix} \pi^{i+c} & x\pi^c \\ 0 & \pi^j \end{pmatrix} \\
&= \begin{cases} 1 & \text{if } n-c \text{ is even and } \geq 0, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathrm{vol}\left(T(F_\wp)\begin{pmatrix} \pi^c & 0 \\ 0 & 1 \end{pmatrix}\mathrm{GL}_2(\mathcal{O}_\wp)\right) &= \#(\mathcal{O}_{K,\wp}/\pi^c)^\times/(\mathcal{O}_\wp/\pi^c)^\times \\
&= \begin{cases} 1 & \text{if } c = 0, \\ q^{c-1}(q+1) & \text{if } c > 0. \end{cases}
\end{aligned}$$

It follows that

$$\gamma_c = \begin{cases} 1 & \text{if } c = 0,\ n \text{ is even,} \\ q^{c-1}(q+1) & \text{if } n - c \text{ is even, } n \geq c > 0, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 5.5.2, we have

$$m'(\gamma) = \begin{cases} \frac{1}{2}\mathrm{ord}_\wp(\xi(\gamma)\pi^{n+1}) & \text{if both } n,\ \mathrm{ord}_\wp(\det\gamma) \text{ are even, and } \xi(\gamma) \neq 0, \\ n/2 & \text{if both } n,\ \mathrm{ord}_\wp(\det\gamma) \text{ are even, } \xi = 0, \\ (n+1)/2 & \text{if both } n,\ \mathrm{ord}(\det\gamma) \text{ are odd} \\ 0 & \text{otherwise.} \end{cases}$$

We want to show that $m' = m_n$. Write $\gamma = a + b\epsilon$ with $a, b \in \mathcal{O}_{K,\wp}$, $\epsilon x = \bar{x}\epsilon$ $\epsilon^2 \in \pi$, $(a, b) = 1$. Then

$$\xi(\gamma) = -\mathrm{N}(b)\pi/\det(\gamma), \qquad \det\gamma = \mathrm{N}(a) - \mathrm{N}(b)\pi.$$

If $a$ is invertible then $\mathrm{ord}_\wp(\det\gamma) = 0$ and $\xi(\gamma) = 0$ or $\mathrm{ord}_\wp(\xi(\gamma))$ is odd. If $a$ is not invertible, then $\mathrm{ord}_\wp(\det(\gamma)) = 1$, and $\mathrm{ord}_\wp(\xi(\gamma)) = 0$. □

We want now to treat the case where $U_\wp$ is not maximal. We will only consider so called special CM-points. By blowing up the models we may assume that the reduction factors the following map

$$T(F)\backslash T(F_\wp)U_\wp \times G(\mathbb{A}_f^\wp)/U \longrightarrow G'(F)\backslash G'(\mathbb{A}_f)/U', \tag{6.3.8}$$

where $U' = U'_\wp \cdot U^\wp$ with

$$U'_\wp = (\mathcal{O}_{K,\wp} + c(\chi)\mathcal{O}_{K,\wp}\epsilon)^\times, \tag{6.3.9}$$

where $\epsilon$ is as before: $\epsilon x = \bar{x}\epsilon$ for any $x \in K$, and $\epsilon^2 \in F_\wp$ with $\mathrm{ord}(\epsilon^2) = 0, 1$.

Lemma 6.3.7. *The local intersection is given by a certain distribution $m$ on $G'(\mathbb{A}_f)$. For any two CM-divisor represented by functions $\phi$ and $\psi$ on $T(F)\backslash G(\mathbb{A}_f)/U$ whose components at $\wp$ are supported on $T(F_\wp)U_\wp$ with character $\chi$, we have*

$$g_\wp(\phi, \psi) = \langle\phi, \psi\rangle^0 + \int_{[T(F)\backslash G'(\mathbb{A}_f)]^2} \phi(x)k(x, y)\bar{\psi}(y)dxdy.$$

*Here $\langle\cdot,\cdot\rangle^0$ is the geometric pairing defined by the multiplicity function*

$$m(\gamma) = \begin{cases} \frac{1}{2}\mathrm{ord}_\wp(\xi(\gamma)) & \text{if } 0 < |\xi(\gamma)| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

*and $k(x, y)$ is a locally constant function on $[G'(F)\backslash G'(\mathbb{A}_f)]^2$.*

*Proof.* We will use the minimal resolution of the singularly $\mathcal{X}_{U,L}$ of $X_U \otimes L$ to compute Green's function. Thus we have a decomposition

$$g_\wp(\phi, \psi) = i_\wp(\phi, \psi) + j_\wp(\phi, \psi).$$

By Lemma 5.5.2, 5.5.4, 6.3.6, the intersection index $i_\wp(\phi, \psi)$ can be given by a formula with the same property described in the lemma. The function $j_\wp(\phi, \psi)$ is locally constant so must be given by a locally constant kernel. □

**Superspecial case.** We now assume that $\wp$ is not split in $B$. The reduction of CM-points which is secial at $\wp$ factors the following map

$$T(F_\wp)\backslash T(F_\wp)U_\wp G(\mathbb{A}_f^\wp)/U \longrightarrow G'(F_\wp)\backslash G(\mathbb{A}_f^\wp)/U' \tag{6.3.10}$$

where $U' = U'_\wp \cdot U^\wp$ with

$$U'_\wp = (\mathcal{O}_{K,\wp} + c(\chi)\mathcal{O}_{K,\wp}\epsilon)^\times. \tag{6.3.11}$$

LEMMA 6.3.8. *The local intersection is given by a certain distribution $m$ on $G'(\mathbb{A}_f)$. For any two CM-divisor represented by functions $\phi$ and $\psi$ on $T(F)\backslash G(\mathbb{A}_f)/U$ whose components at $\wp$ are supported on $T(F_\wp)U_\wp$ with character $\chi$, we have*

$$g_\wp(\phi,\psi) = \langle\phi,\psi\rangle^0 + \int_{[T(F)\backslash G'(\mathbb{A}_f)]^2} \phi(x)k(x,y)\bar{\psi}(y)dxdy.$$

*Here $\langle\cdot,\cdot\rangle^0$ is the geometric local pairing defined by the multiplicity function*

$$m(\gamma) = \begin{cases} \frac{1}{2}\mathrm{ord}_\wp(\xi(\gamma)) & \text{if } 0 < |\xi(\gamma)| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

*and $k(x,y)$ is a locally constant function on $[G'(F)\backslash G'(\mathbb{A}_f)]^2$.*

*Proof.* Use Lemma 5.5.5, 5.5.6 and the same argument as in the proof of Lemma 6.3.7. □

## 6.4. Gross-Zagier formula for central derivatives

In this section we will complete the proof of Gross-Zagier formula (Theorem 1.3.2) for the derivatives of Rankin's L-series, by comparing heights of CM-points and Fourier coefficients of the kernel function of the Rankin-Selberg convolution. The principle is as same as that in Gross-Zagier's original paper [20]. Up to a constant and modulo some *negligible forms*, the new form $\Psi$ with Fourier coefficient

$$\widehat{\Psi}(a) := |a|\langle\eta, \mathrm{T}_a\eta\rangle \tag{6.4.1}$$

is equal to the holomorphic cusp form $\Phi$ defined in §3.5 which represents the derivative of Rankin L-function $L'(1/2, \Pi\otimes\chi)$. Thus we need to show that the functional $a \longrightarrow \widehat{\Psi}(a)$ is equal to the Fourier coefficient $a \longrightarrow \widehat{\Phi}(a)$ for for $a \in \mathbb{N}_F(ND)$, the semigroup of integral ideals of $\mathcal{O}_F$ prime to $ND$.

In §3.5 and §6.2, up to derivations of Eisenstein series and theta series $\Pi(\chi)\otimes\alpha^{1/2}$, we have decomposed both $\widehat{\Phi}(a)$ and $\widehat{\Psi}(a)$ into a sum of local terms $\widehat{\Phi}_v(a)$ and $\widehat{\Psi}_v$, where

$$\widehat{\Psi}_v(a) := -|a|g_v(\eta_v, \mathrm{T}^0_a\eta)_v \log q_v. \tag{6.4.2}$$

Thus, it suffices to compare these local terms for each place $v$ of $F$ and each idele class $a \in \mathbb{N}_F(ND)$. We need only consider $v$ which is not split in $K$, since $\widehat{\Phi}_v = 0$ and $\widehat{\Psi}_v$ is *quasi-Eisenstein.*

Our main tool is the pre-Gross-Zagier formula, Corollary 4.3.3, for quaternion algebra ${}_vB$ with ramification set

$${}_v\Sigma = \begin{cases} \Sigma\cup\{v\} & \text{if } v\notin\Sigma, \\ \Sigma\setminus\{v\} & \text{if } v\in\Sigma. \end{cases} \tag{6.4.3}$$

Let ${}_vG$ denote the algebraic group ${}_vB^\times/F^\times$.

**Archimedean case.** LEMMA 6.4.1. *For $v$ an infinite place,*

$$\widehat{\Phi}_v(a) = 2^{g+1}|c(\omega)|^{1/2}\widehat{\Psi}_v(a).$$

*Proof.* By Proposition 3.5.5, $\widehat{\Phi}_v(a)$ is the constant term at $s=0$ of a sum over $\xi \in F$ such that $0 < \xi_w < 1$ for all infinite place $w \neq v$ and $\xi_v < 0$ of the following terms:

$$(2i)^g \omega_\infty(\delta_\infty)|\xi\eta|_\infty^{1/2} \cdot \bar{W}_f\left(1/2, \xi, \eta, \begin{pmatrix} a\delta_f^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \cdot \int_1^\infty \frac{-dx}{x(1+|\xi|_v x)^{1+s}}.$$

By the pre-Gross-Zagier formula, Corollary 4.3.2, $\widehat{\Phi}_v(a)$ is thus equal to the constant term at $s=0$ of

$$-2^g|c(\omega)|^{1/2}|a|\langle \mathrm{T}_a\phi, \phi\rangle_s,$$

for a geometric pairing of CM-points $T(F)\backslash_v G(\mathbb{A}_f)$ with multiplicity function $m_s^v$ on $T(F)\backslash_v G(F)/T(F)$. Further, $m_s^v(g) \neq 0$ only if $\xi(g)_v < 0$; in this case it is given by

$$m_s^v(g) = \int_1^\infty \frac{dx}{x(1+|\xi|_v x)^{1+s}}.$$

Now, by Lemma 6.3.1, $g_v(\eta, \mathrm{T}_a^0\eta)$ is the constant term of a geometric pairing of $\phi$ and $\mathrm{T}_a\phi$ with multiplicity function $m_s = Q_s(1-2\xi)$ supported on $\xi < 0$. Notice that as a function of $\xi$, one has

$$2Q_s(1+2|\xi|) = \int_\infty^\infty \frac{dt}{(z+\sqrt{z^2-1}\cosh t)^{1+s}} = \int_1^\infty \frac{(x-1)^s dx}{x^{1+s}(1+|\xi|x)^{1+s}}.$$

It follows that

$$\widehat{\Psi}(a) - 2^{g+1}|c(\omega_v)|^{1/2}\widehat{\Phi}(a)$$

is the constant term of a geometric pairing of $\phi$ and $\mathrm{T}_a\phi$ with multiplicity function

$$m_s - 2Q_s.$$

It is not difficult to show that

$$m_s - 2Q_s = O(|\xi|^{-s-2})$$

as $|\xi| \longrightarrow \infty$, and vanishes at $s=0$. Thus if we use the difference to defined the intersection pairing, then it vanishes at $s=0$. ◻

**Unramified case.** If $v$ is a finite place, by Proposition 3.5.5 $\widehat{\Phi}_v(a)$ is a sum over $\xi \in F$ with $0 < \xi < 1$ of the following terms:

$$(2i)^g|\eta\xi|_\infty^{1/2} \cdot \bar{W}_f^v\left(1/2, \xi, \eta, \begin{pmatrix} a\delta_f^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \cdot \bar{W}_v'\left(1/2, \xi, \eta, \begin{pmatrix} a\delta_f^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

We want to write this as the geometric pairing on $T(F)\backslash_v G(\mathbb{A}_f)$ of $\phi$ and $\mathrm{T}_{a'}\phi$ where $a = \pi^n a'$ ($\wp \nmid a'$), and $\phi$ is the standard function on $T(F)\backslash G'(\mathbb{A}_f)/U'$ with character $\chi$, and $U' = U'_\wp U^\wp$ with

$$U'_\wp = (\mathcal{O}_{K,\wp} + c(\chi)\mathcal{O}_{K,\wp}\epsilon)^\times.$$

First we consider the unramified case

LEMMA 6.4.2. *Let $v$ be a place of $F$ where $\omega$ and $\chi$ are both unramified and* $\mathrm{ord}_v(N) = 0$. *Then there is a constant $c$ such that*

$$\widehat{\Phi}_v(a) - 2^{g+1}|c(\omega)|^{1/2}\widehat{\Psi}_v(a) = c\log|a|_v \cdot |a|^{1/2}\widehat{\Pi}(\chi)(a).$$

*Proof.* By Lemma 3.4.5,

$$W_v'\left(1/2, \xi, \eta, \begin{pmatrix} a\delta_f^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \neq 0,$$

only if $\mathrm{ord}_v(\eta a)$ is even and nonnegative, and $\mathrm{ord}_v(\xi a)$ is odd and positive; in this case it is given by

$$\epsilon(\omega_v, \psi_v)|\eta\xi|_v^{1/2}|a|_v \log|\xi a|_v.$$

Thus, we see that up to a multiple of

$$|a|^{1/2}\widehat{\Pi}(\chi)(a)\log|a|_v,$$

the functional $\widehat{\Phi}(a)$ is equal to

$$2^g|c(\omega)|^{1/2}\langle\phi, \mathrm{T}_a\phi\rangle$$

for a geometric local pairing on $T(F)\backslash_v G^s(\mathbb{A}_f)$ with multiplicity function $m^a(\xi)$ which is nonzero only if $\mathrm{ord}_v(\xi a)$ is odd and positive, and $\mathrm{ord}_v(\eta a)$ is even and nonnegative. In this case

$$m^a(\xi) = \log|\xi a\pi|_v.$$

Here $\phi$ is the standard function on $G(\mathbb{A}_f)$ with maximal support at $T(\mathbb{A}_f)\widehat{R}^\times$ with character $\chi$, where $R_w$ is as before for $w \neq v$, and $R_v$ is the maximal order of the definite quaternion algebra ${}_vB_v$.

As a function of $\xi = \xi(\gamma)$, we claim that

$$m^a(\gamma) = -2m_n(\gamma)\log q_v,$$

if $\xi \neq 0, 1$, where $m_n$ is given by Lemma 6.3.6. In other words, we want to show that $m_n(\gamma) \neq 0$ only if $\mathrm{ord}((1-\xi)\pi^n)$ is even and nonnegative, and $\mathrm{ord}(\xi\pi^n)$ is positive and odd, and in this case In this case,

$$m_n(\gamma) = \frac{1}{2}\mathrm{ord}(\xi\pi^{n+1}).$$

We need only check the positivity. Write

$$\gamma = a + b\epsilon$$

whose norm at $\wp$ is either 0 or 1. In the first case, $\mathrm{ord}(\xi)$ is odd and positive, and in the second case, $\mathrm{ord}(1-\xi)$ is odd and positive. $\square$

**Ramified case.** We now want to treat the case where $v$ is a ramified place for $f$, $\chi$, or $\omega$. In this case we will not be able to prove the identity as in the archimedean case, or the unramified case. But we can prove the following:

LEMMA 6.4.3. *For $v$ a finite place, the difference*

$$\widehat{\Phi}(a)-2^{g+1}|c(\omega)|^{1/2}\widehat{\Psi}(a)=c|a|^{1/2}\widehat{\Pi}(\chi)(a)+_v\widehat{f}$$

*where $c$ is a constant, and ${}_v\widehat{f}$ is a form on ${}_vG(F)\backslash_vG(\mathbb{A}_f)$. Moreover, the function ${}_vf$ has character $\chi$ under the right translation by $K_v^\times$.*

*Proof.* We will only consider so called special CM-points. As in the unramified case, using Lemma 3.4.6 and 3.4.7, one can show that $\widehat{\Phi}$ is equal the geometric local pairing of

$$2^g|c(\omega))|^{1/2}|a|\langle\phi,\mathrm{T}_a\phi\rangle$$

for a multiplicity function $m(g)$ on ${}_vG(F)$ with singularity

$$\log|\xi|_v.$$

On other hand, by Lemma 6.3.7, and 6.3.8, we know that

$$\widehat{\Psi}(a)=-g_v(\phi,\mathrm{T}_a^0\phi)\log q_v$$

is also a geometric pairing with singularity

$$\frac{1}{2}\log|\xi|_v.$$

Thus,

$$\widehat{\Phi}(a)-2^{g+1}|c(\omega)|^{1/2}\widehat{\Psi}(a),$$

is a geometric pairing without *singularity.* In other words, it is given by

$$\int_{[T(F)\backslash_vG(\mathbb{A}_f)]^2}\phi(x)k(x,y)\mathrm{T}_a\phi(y)dxdy,$$

for $k(x,y)$ a locally constant function of $({}_vG(F)\backslash_vG(\mathbb{A}_f))^2$. The lemma now follows, since we decompose

$$k(x,y)=\sum_i c_i(x)f_i(y)$$

into eigenfunctions $f_j$ for Hecke operators on ${}_vG(F)\backslash_vG(\mathbb{A}_f)$ to obtain

$$\sum_i\lambda_i(a)\int_{T(F)\backslash_vG(\mathbb{A}_f)}\phi(x)c_i(x)dx\cdot\int_{T(F)\backslash_vG(\mathbb{A}_f)}f_i(y)\bar{\phi}(y)dy,$$

where $\lambda(a)$ is the eigenvalue of $\mathrm{T}_a$ for $f_i$. Thus we may take

$${}_vf=\sum_i\int_{T(F)\backslash_vG(\mathbb{A}_f)}\phi(x)c_i(x)dx\cdot\int_{T(F)\backslash_vG(\mathbb{A}_f)}f_i(y)\bar{\phi}(y)dy.$$

□

**Conclusion of Proof of Theorem 1.3.2.** In summary, at this stage we have shown that the quasi-newform

$$\Phi - 2^{g+1}|c(\omega)|^{1/2}\Psi$$

has Fourier coefficients which are a sum of the following terms:

- derivations $A$ of Eisenstein series,
- derivations $B$ of theta series $\Pi(\chi) \otimes \alpha^{1/2}$,
- functions ${}_vf$ appearing in ${}_vG(F)\backslash {}_vG(\mathbb{A}_f)$ with character $\chi$ under the right translation of $K_v^\times$, where $v$ are places dividing $DN$.

By linear independence of Fourier coefficients of derivations of forms [31] Proposition 4.5.1, we may conclude that $A = B = 0$.

Let $\Pi$ now be the representation defined by the form $f$ in the introduction and let ${}_vf_\Pi$ be its projection in $\Pi$. If ${}_vf_\Pi \neq 0$ then both $\Pi_v^\chi$ and $(\Pi_v')^\chi$ are nonzero.

If $\chi$ is trivial, then this is a contradiction by Theorem 2.3.2.

If $\chi$ is nontrivial then $\Pi_v$ must be special with unramified twist. Thus, $(\Pi_v')$ is given by an unramified character. Thus $\chi$ is unramified and $K/F$ is ramified at $v$. This contradicts Lemma 2.3.4.

In summary we have shown that $\Phi - 2^{g+1}|c(\omega)|^{1/2}\Psi$ has trivial quasi-newform projection. By Proposition 3.1.3, we thus obtain

$$L'(1/2, \Pi \otimes \chi) = 2^{g+1}|c(\omega)|^{1/2} \cdot (\phi^\sharp, \phi^\sharp) \cdot \langle y_\chi, y_\chi \rangle.$$

## REFERENCES


[1] J.-F. Boutot and H. Carayol, *Uniformisation p-adique des courbes de Shimura: les thérèmes de Cerednik et de Drinfeld*, Astérisque, 196-197 (1991), pp. 45–158.

[2] H. Carayol, *Sur la mauvaise réduction des courbes de Shimura*, Comp. Math., 59 (1986), pp. 151–230.

[3] W. Casselman, *On some results of Atkin and Lehner*, Math. Ann., 201 (1973), pp. 301–314.

[4] Cogdell, Piateski-Shapiro, and Sarnak, *Esitmates for Hilbert modular L-functions and applications*, in preparation.

[5] P. Cohen, *Hyperbolic distribution problems on Siegel 3-folds and Hilbert modular varieties*, in preparation.

[6] Bertolini and H. Darmon, *Kolyvagin's descent and Mordell-Weil groups over ring class fields*, Journal für die rein und angewande Mathematik, 412 (1990), pp. 63–74.

[7] Bertolini and H. Darmon, *Iwasawa's main conjecture for elliptic curves over anticyclotomic $\mathbb{Z}_p$-extensions*, manuscript, 2001.

[8] Bertolini and H. Darmon, *A rigid analytic Gross-Zagier formula and arithmetic application (with an appendix by Bas Edixhoven)*, Ann. Math. (2), 146:1 (1997), pp. 111–147.

[9] P. Deligne, *Travaux de Shimura*, Séminaire Bourbaki, ed., in Lect. Notes Math. 244, Springer-Verlag, 1971, pp. 123–165.

[10] V. G. Drinfeld, *Coverings of p-adic symmetric regions*, Funct. Anal. Appl., 10 (1976), pp. 29–40.

[11] W. Duke, *Hyperbolic distribution problems and half-integral weight Maass forms,* Invent. Math., 92:1 (1988), pp. 73–90.

[12] G. Faltings, *Calculus on arithmetic surfaces*, Ann. Math., 119 (1984), pp. 387–424.

[13] H. Gillet and C. Soulé, *Arithmetic intersection theory*, I.H.E.S. Publ. Math., 72 (1990), pp. 94–174.

[14] D. Goldfeld, *The class numbers of quadratic fields and the conjectures of Birch and Swinnerton-Dyer*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 4 (1976), pp. 624–663.

[15] B. H. Gross, *On canonical and quasi-canonical liftings*, Invent. Math., 84 (1986), pp. 321–326.

[16] B. H. Gross, *Kolyvagin's work on modular elliptic curves*, in L-function and Arithmetic, J. Coates and M. J. Taylor, ed., Cambridge University Press, 1991, pp. 253–356.

[17] B. H. Gross, *Heegner points on $X_0(N)$*, in Modular Forms, R. A. Rankin, ed., Ellis Horwood, 1984, pp. 87–105.

[18] B. H. Gross, *Local Heights on curves*, in Arithmetic Geometry, Cornell and Silverman, ed., Springer-Verlag, New York, 1986, pp. 327–339.
[19] B. H. Gross and D. Prasad, *Test vectors for linear forms*, Math. Ann., 291:2 (1991), pp. 343–355.
[20] B. H. Gross and D. B. Zagier, *Heegner points and derivatives of L-series*, Invent. Math., 84 (1986), pp. 225–320.
[21] H. Iwaniec, *Introduction to the Spectral Theory of Automorphic Forms,* Bibl. Rev. Mat. Iber., Madrid, 1995.
[22] H. Jacquet, *Automorphic Forms on* $\mathrm{GL}_2$ *II*, Lect. Notes. Math. 289, Springer-Verlag, 1972.
[23] H. Jacquet and R. Langlands, *Automorphic Forms on* $\mathrm{GL}_2$, Lect. Notes. Math. 114, Springer-Verlag, 1971.
[24] N. Katz and B. Mazur, *Arithmetic Moduli of Elliptic Cuvres,* Ann. Math. Studies 108, 1985.
[25] V. A. Kolyvagin, *Euler Systems*, The Grothendieck Festschrift. Prog. in Math., Boston, Birkhauser, 1990.
[26] V. A. Kolyvagin and D. Yu. Logachev, *Finiteness of* Ш *over totally real fields*, Math. USSR Izvestiya, 39:1 (1992), pp. 829–853.
[27] E. Kowalski, P. Michel, and J. Vanderkam, *Rankin-Selberg L-functions in the level aspects*, preprint, 2000.
[28] L. Szpiro, (ed.), *Séminaire sur les pinceaux arithmétiques: la conjecture de Mordell*, Asterique 127, 1985.
[29] J. Tate, *Fourier analysis in number fields and Hecke's zeta-functions*, in Algebraic number theory, J. W. S. Cassels and A. Fröhlich, ed., Academic Press, 1967, pp. 305–347.
[30] J. -L. Waldspurger, *Correspondences de Shimura et quaternions*, Forum Math., 3 (1991), pp. 219–307.
[31] S. Zhang, *Heights of Heegner points on Shimura curves*, Annals of Mathematics (2), 153:1 (2001), pp. 27–147.
[32] S. Zhang, *Admissible pairings on curves*, Invent. Math., 112 (1993), pp. 171–193.