The L-series of certain rigid Calabi-Yau threefolds

H. A. Verrill

Abstract. The L-series of two Calabi-Yau three-folds are found. In each case it is shown that the middle cohomology has dimension 2, and that (up to Euler factors at primes of bad reduction) the Mellin transform of the L-series of the middle cohomology is given by a modular form of weight four.

1. Introduction

A smooth projective variety $\mathcal{Z}$ of dimension n is called a *Calabi–Yau* variety if $H^i(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) = 0$ for $0 < i < n$, and $K_{\mathcal{Z}} \equiv 0$, i.e., the canonical bundle is trivial. A Calabi-Yau variety of dimension 1 is an elliptic curve. A Calabi-Yau variety of dimension 2 is a K3 surface. Calabi-Yau three-folds have been important in physics, and a source of basic results and examples can be found in [H]. A rigid Calabi-Yau three-fold is one with no complex deformation, i.e., $h^{2,1} = 0$, which means that $H^3(\mathcal{Z})$ is two dimensional. In this case, it is predicted that the Mellin transform of the L-series of $H^3(\mathcal{Z}, \mathbb{Q}_\ell)$ is a weight four modular form.

In this paper I determine (up to Euler factors at the primes of bad reduction) the L-series of two Calabi-Yau three-folds, $\mathcal{Z}_{A_1^3}$ and $\mathcal{Z}_{A_3}$. The first of these varieties was also studied in [PS], where the Picard-Fuchs equation was found.

The Calabi-Yau varieties described in this paper have K3 fibrations, and are double covers of the varieties described in [V1]. The construction in [V1] starts with a root lattice R of dimension n, and uses toric geometry to produce a pencil of $n-1$ dimensional Calabi-Yau varieties, $\mathcal{X}_R$. Below we describe how to take a double cover to produce a Calabi-Yau n-fold, $\mathcal{Z}_R$. For example, from the lattice $A_1 \times A_1 \cong \mathbb{Z} \times \mathbb{Z}$ we obtain a pencil of elliptic curves. Several of these families turn out to coincide with families studied by Beukers, Peters and Stienstra ([SB], [PS] and [Be]); this is indicated in Table 1. This table shows information for four fibred varieties, giving the solution of the Picard-Fuchs equation, and the L-series, to illustrate the relationship between them.

1.1. Notation. We now explain the notation in Table 1.

• In each case, the Mellin transform of the L-series of the middle cohomology is given in terms of the Dedekind eta function, $\eta(\tau) = q^{\frac{1}{24}} \prod(1-q^n)$, where $q = e^{2i\pi\tau}$. Note that we are interested in calculating the L-series of the representation given by the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the ℓ-adic cohomology of the variety $\mathcal{Z} = \mathcal{Z}_{A_3}$ and $\mathcal{Z}_{A_1^3}$. But $\mathcal{Z}$ will be a variety defined over $\mathbb{Z}$, and we can consider it as a complex

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Table 1. Examples

	Variety			
	$\mathcal{Z}_{A_1\times A_1}$	$\mathcal{Z}_{A_2}$	$\mathcal{Z}_{A_1\times A_1\times A_1}$	$\mathcal{Z}_{A_3}$
Variety	K3 surface	K3 surface	Calabi-Yau three-fold	Calabi-Yau three-fold
fibre, Z_t	elliptic curve	elliptic curve	K3 surface	K3 surface
Local equation	$x+\frac{1}{x}+y+\frac{1}{y}=t^2$	$x+\frac{1}{x}+y+\frac{1}{y}+xy+\frac{1}{xy}=t^2$	$x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}=t+\frac{1}{t}$	$x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}+xy+\frac{1}{xy}+zy+\frac{1}{zy}+xyz+\frac{1}{xyz}=t+\frac{1}{t}-2$
Reference	$\mathcal{A}'$ in [SB]	$\mathcal{C}$ in [SB]	$\mathcal{Z}$ in [PS]	§ 3 below
monodromy group	$\overline{\Gamma_0(16)}$	$\overline{\Gamma_0(12)}$	$\overline{\Gamma_0(12\|2)+6}$	$\overline{\Gamma_0(12)+12}$
parameter, t	$\frac{\eta(16\tau)^4\eta(4\tau)^2}{\eta(8\tau)^6}$	$\frac{\eta(12\tau)^4\eta(2\tau)^2}{\eta(6\tau)^2\eta(4\tau)^4}$	$\frac{\eta(2\tau)^6\eta(12\tau)^6}{\eta(4\tau)^6\eta(6\tau)^6}$	$\frac{\eta(3\tau)^4\eta(12\tau)^8\eta(2\tau)^{12}}{\eta(\tau)^4\eta(4\tau)^8\eta(6\tau)^{12}}$
periods, f	$\frac{\eta(8\tau)^{10}}{\eta(16\tau)^4\eta(4\tau)^4}$	$\frac{\eta(4\tau)^6\eta(6\tau)}{\eta(2\tau)^3\eta(12\tau)^2}$	$\eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau)$	$\frac{\eta(2\tau)^4\eta(6\tau)^4}{\eta(\tau)^2\eta(3\tau)^2}$
v_n, where $f=\sum v_n t^n$	$\binom{2m}{m}^2$ $n=4m$ 0 otherwise	$\sum_{p+q+r=n}\left(\frac{n!}{p!q!r!}\right)^2$ $n=2m$ 0 otherwise	$\sum_{k=0}^{m}\binom{m}{k}^2\binom{m+k}{k}^2$ $n=2m+1$ 0 otherwise	$\sum_{m=0}^{n-1}\sum_{\substack{p+q+\\r+s=m}}(-1)^m\binom{n+m}{2m+1}\left(\frac{m!}{p!q!r!s!}\right)^2$
Reference	[SB]§ 14	[SB]§ 14	[PS] Theorem 6	[$V1$], [$V2$], [$V3$]
Mellin transform of L-series	$\eta(4\tau)^6$	$\eta(2\tau)^3\eta(6\tau)^3$	$\eta(2\tau)^4\eta(4\tau)^4$	$\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$
Reference	[SB] Theorem 14.2	[SB] Theorem 14.2	§ 4 below, and [vGN], [AO]	§ 3 below

variety. To calculate the dimensions of the cohomology groups, it will be easier to work with singular cohomology. Then we use the fact that $H^i(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$ and $H^i(\mathcal{Z}_{A_3}, \mathbb{R})$ have the same dimension. We will use the same notation $\mathcal{Z}_R$ to refer to the variety, whether we are looking at it over $\mathbb{C}$ or some other field, and what is meant should be clear from the context. Note also we will not calculate the Euler factors at the primes of bad reduction, so the L-series listed in the table are the L-series of the varieties up to possible Euler factors at 2 and 3.

• The v_n in the table is the coefficient of the t expansion of the period f. The v_n are related to the L-series as follows: If g is the Mellin transform of the L-series, and $g = \sum \gamma_n q^n$, and $f = \sum v_n t^n$ then for $\mathcal{Z}_{A_1^2}$ and $\mathcal{Z}_{A_2}$, $\gamma_p \equiv v_{p-1} \bmod p$, and for $\mathcal{Z}_{A_1^3}$ and $\mathcal{Z}_{A_3}$, $\gamma_p \equiv v_p \bmod p$; this holds for all primes p, except possibly those of bad reduction. This can be verified for the examples in the table from the given data, and proofs of the congruences can be found in [V3]. This relationship between the L-series and the solution of the Picard-Fuchs equation is expected because of the result in [S2]. More examples of this kind of congruence can be found in [V3].

• The monodromy group Γ is given using the notation in [CN], where $\Gamma_0(a)+b$ is written to mean the group generated by $\Gamma_0(a)$ together with certain Fricke involutions, and $\overline{\Gamma_0(a)+b}$ means we have quotiented by $\pm I$. For example,

$$\Gamma_0(12)+12 := \left\{ \begin{pmatrix} a & b \\ 12c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | a,b,c,d \in \mathbb{Z} \right\}$$

$$\cup \left\{ \frac{1}{\sqrt{12}} \begin{pmatrix} 12a & b \\ 12c & 12d \end{pmatrix} \in SL_2(\mathbb{R}) | a,b,c,d \in \mathbb{Z} \right\}.$$

Note that monodromy groups are only defined up to conjugacy, and the groups listed are sometimes slightly different from the ones in the papers referred to, e.g., it is shown in [SB] § 14, that the K3 surface $\mathcal{A}'$ is an elliptic modular surface associated with the group $\Gamma_0(16) \cap \Gamma_1(4)$. But when we quotient this group by $\pm I$, (since we are considering the action on the upper half plane $\mathcal{H}$), this is the same as $\overline{\Gamma_0(16)}$. Note that the parameter t given in the case of $\mathcal{Z}_{A_3}$ is a uniformizing parameter for $\Gamma_0(12)+12$ conjugated by $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$, rather than for $\Gamma_0(12)+12$, since the conjugated group is a subgroup of $\Gamma_0(6)+6$, the group found in [V1] to be the monodromy group for $\mathcal{X}_{A_3}$. Since $\mathcal{Z}_{A_3}$ is constructed from $\mathcal{X}_{A_3}$ it is most natural to use the parameter given in the table. In each case, t is a uniformizing parameter for the given monodromy group, and is a (non holomorphic) modular function for Γ, and generates the space of modular functions for Γ. The function f is a modular form of weight 2 for Γ.

• The row for the periods gives a modular form $f(\tau)$ such that when the family is given as a fibration over $\mathcal{H}/\Gamma$, so the parameter t is given as a uniformizing parameter $t(\tau)$, then the period lattice of the fibre over $\tau \in \mathcal{H}/\Gamma$ is given by

$$f(\tau)(\mathbb{Z} \times \tau\mathbb{Z} \times \tau^2\mathbb{Z}),$$

for $\mathcal{Z}_{A_1 \times A_1 \times A_1}$ and $\mathcal{Z}_{A_3}$, and $f(\tau)(\mathbb{Z} \times \tau\mathbb{Z})$ for $\mathcal{Z}_{A_1 \times A_1}$ and $\mathcal{Z}_{A_2}$.

2. Pencils of varieties defined by root lattices

Given a root lattice R, the Weyl chambers form cones of a fan Σ_R, which defines a toric variety, $X(\Sigma_R)$. The character of the adjoint representation defines a map $\phi : X(\Sigma_R) \to \mathbb{P}^1$. Blowing up the base locus of this map gives a variety $\mathcal{X}_R$ with a fibration $\phi : \mathcal{X}_R \to \mathbb{P}^1$. This construction is described in [V1], where it is shown that if R is of type A_n, or products of A_n root lattices, then the general fibre is a Calabi-Yau variety. The variety $\mathcal{X}_R$ does not have trivial canonical class, and so cannot be a Calabi-Yau variety, but for $R = A_3$, $A_1^3 := A_1 \times A_1 \times A_1$, or $A_1 \times A_2$, we can easily obtain a Calabi-Yau variety $\mathcal{Z}_R$:

THEOREM 2.1. *For the root lattices $R = A_3$, A_1^3, or $A_1 \times A_2$, there is a Calabi-Yau variety $\mathcal{Z}_R$, which is a double cover of the variety $\mathcal{X}_R$, (described above) and is given by the pull back in the following diagram, where $F(t)$ is a certain rational function of degree 2:*

$$\begin{tikzcd} \mathcal{Z}_R \arrow[r, "\pi"] \arrow[d] & \mathcal{X}_R \arrow[d, "\phi"] \\ \mathbb{P}^1 \arrow[r, "t \mapsto \lambda = F(t)"] & \mathbb{P}^1 \end{tikzcd}$$

PROOF. The canonical divisor of the toric variety $X(\Sigma_R)$ is linearly equivalent to $-X_\infty$. In blowing up the base locus of ϕ, we still have the canonical divisor of $\mathcal{X}_R$ given by $K_{\mathcal{X}_R} \sim -X_\infty$. By the adjunction formula, we have

$$K_{\mathcal{X}_R} = \pi^* K_{\mathcal{X}_R} + \tilde{B},$$

where $\tilde{B} = \pi^{-1} B \subset \mathcal{X}$ is the ramification locus, and B is the branch locus, so B is the union of the fibres over the branch points of the double cover. We have $\lambda = F(t)$, a rational function of degree 2, so there are two branch points of the cover, α, β, and $B = X_\alpha + X_\beta$. Hence $\tilde{B} = \tilde{X}_\alpha + \tilde{X}_\beta$, where $\tilde{C}$ just means lifting the divisor C in $\mathcal{X}_R$ to something in $\mathcal{Z}_R$. Since X_α and X_β are both fibres of $\phi : \mathcal{X} \to \mathbb{P}^1$, we have $X_\alpha \sim X_\beta$, and $\tilde{X}_\alpha \sim \tilde{X}_\beta$. Hence $\tilde{B} \sim 2\tilde{X}_\alpha \sim \pi^* X_\beta$, so $K_{\mathcal{Z}_R} = \pi^*(-X_\infty) + \pi^* X_\beta = 0$.

Since $\mathcal{Z}_R$ is connected, $h^0 = 1$. The first Betti number $h^1 = 0$ since the base $\mathbb{P}^1$ is simply connected, and so are the fibres, which are Calabi-Yau varieties, or degenerations of Calabi-Yau varieties. In the case that $\mathcal{Z}_R$ is three dimensional, this is enough to conclude that it is a Calabi-Yau three-fold, and has Hodge diamond of the form:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2} & & 0 & \\ 1 & & h^{1,2} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Here we are using Hodge star duality, $h^{p,q} = h^{n-q,n-p}$, complex conjugation, $h^{p,q} = h^{q,p}$, and holomorphic duality, $h^{0,q} = h^{0,3-q}$, see [H] §1.2. □

REMARK 2.2. If $R = A_{m_1} \times A_{m_2} \dots \times A_{m_k}$, and $\mathcal{Z}_R$ has higher dimension than 3, then although the canonical class is trivial, this is not enough to show $H^i(\mathcal{Z}, \mathcal{O}_\mathcal{Z}) = 0$ for $0 < i < n$. However, if we take $F(t) = at + b + \frac{c}{t}$, then $\mathcal{Z}_R$ is birational to a member of the family of toric Calabi-Yau hypersurfaces in the toric variety $X(\Sigma_{R \times A_1})$. This is the variety described in [V1], but instead of taking a family defined by the character of the adjoint representation of R, where all coefficients of the monomials in the defining equation of the fibres are 1, we allow other coefficients, as in the more general case described in [Ba1]. The polyhedron defining the family of Calabi-Yaus is reflexive (similar to Lemma 1, [V1]), and so by Theorem 4.1.9 in [Ba1], the fibres are Calabi-Yau three-folds. So, $\mathcal{Z}_R$ is birational to a Calabi-Yau n-fold, $\mathcal{Y}$, and has trivial canonical class. From [Ba2] Theorem 1.1, this means that the Betti numbers of $\mathcal{Z}_R$ and $\mathcal{Y}$ are equal, and in remark 4.5 of [Ba2] is is stated that further, the Hodge numbers of $\mathcal{Z}_R$ and $\mathcal{Y}$ are also equal, so in fact $\mathcal{Z}_R$ should be a Calabi-Yau variety.

Note that different choices of double cover will give different Calabi-Yau varieties. The choices of double cover for $R = A_3$ and A_1^3 will be given below. In [V1] a description of $\mathcal{X}_{A_1^3}$ and $\mathcal{X}_{A_3}$ are given in more detail. In this paper we just give the equations in local coordinates.

PROPOSITION 2.3. *For $R = A_3, A_1^3$ or $A_1 \times A_2$, $H^2(\mathcal{Z}_R, \mathbb{R})$ spanned by algebraic cycles, and $h^2 = \rho(\mathcal{Z}_R)$, the Picard number of $\mathcal{Z}_R$.*

PROOF. Since $\mathcal{Z}_R$ is a Calabi-Yau variety, $h^{2,0} = h^{0,2} = 0$, so since $\mathcal{Z}_R$ is simply connected, $H^2(\mathcal{Z}_R, \mathbb{Z})$ has no torsion, and $H^2(\mathcal{Z}_R, \mathbb{Z}) \subset H^{1,1}(\mathcal{Z}_R, \mathbb{R})$, so $H^{1,1}(\mathcal{Z}_R, \mathbb{R})$ is spanned by algebraic cycles. □

For $R = A_3$ and A_1^3, we show below that we obtain smooth Calabi-Yau three-folds. By Proposition 2.3, the cycles in $H^2(\mathcal{Z}_R, \mathbb{Q}_\ell)$ correspond to classes in $\mathrm{Pic}(\mathcal{Z}_R)$. We will see Lemma 3.8 and 4.1 that the way the double cover is chosen means that this space is spanned by cycles represented by subvarieties defined by equations with coefficients in $\mathbb{Z}$, and so is fixed by the Frobenius action. This means that in all even dimensions, the action of the Frobenius on $H^{2i}(\mathcal{Z}_R, \mathbb{Q}_\ell)$ is just multiplication by p^i. The interesting part of the Galois action happens in odd dimensions.

3. Calculation of the L-series of $H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$

Now we consider $\mathcal{Z}_{A_3}$, a Calabi-Yau threefold, defined over $\mathbb{Q}$. We will show that the Mellin transform of $H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$ is a weight 4, level 6, modular form, $(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$. Locally $\mathcal{Z}_{A_3}$ is given by

$$\mathcal{Z}_{A_3} : (1 + x + xy + xyz)(1 + z + zy + xyz)t = (t + 1)^2 xyz,$$

plus some blowing up. In [V1] a detailed description of the variety $\mathcal{X}_{A_3}$ was given. The Calabi-Yau variety $\mathcal{Z}_{A_3}$ is obtained from $\mathcal{X}_{A_3}$ by taking a pull back over a double cover of the base of the fibration, as described in Theorem 2.1. In § 3.3 we describe the resolution of the singularities of this variety.

3.1. Why we expect the Galois representation attached to $H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$ to be modular. There are two reasons for this, which also apply in the case of $\mathcal{Z}_{A_1^3}$. First, the fact that $\mathcal{Z}_{A_3}$ is rigid (Lemma 3.7) means that $H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$ admits a two dimensional Galois representation; Fontaine and Mazur conjecture (roughly speaking) that all irreducible odd 2-dimensional Galois representations “coming from geometry” should be modular, up to a Tate twist—see [FM] conjecture 3 for the precise statement relevant here.

Further, there is a relationship between $\mathcal{Z}_{A_3}$ and a pencil of elliptic curves, which also leads us to expect $\mathcal{Z}_{A_3}$ to be modular. The two dimensional Galois representation associated to f by Deligne [D] to a new form f of weight k, is a piece of the ℓ-adic cohomology $H^{k+1}(X^{(k)}, \mathbb{Q}_\ell)$, where $X^{(k)}$ is the k-fold fibre product over C, of the universal family X of elliptic curves over a modular curve C. In the situation here, one might expect $\mathcal{Z}_{A_3}$ to be related to the fibred product of the universal family of elliptic curves for $\Gamma_0(6)$, because of the relationship between the transcendental lattices of the fibres, and the relationship between the Picard-Fuchs equations, which is mentioned in [V1] and [V2]. So far I have not been able to construct a suitable map using this approach. However, an explicit correspondence given by a birational map between $\mathcal{Z}_{A_3}$ and the fibred product of the universal family of elliptic curves for $\Gamma_0(6)$ has been constructed by M. H. Saito and N. Yui [SY], though this map does not seem to relate to the monodromy.

3.2. The L-series predicted by the periods and monodromy of $\mathcal{Z}_{A_3}$. The Picard-Fuchs equation for a pencil of K3 surfaces, with fibre X_t is a differential equation satisfied by the periods, which are defined as follows: Let ω_t be the unique differential 2,0 form on X_t. A period is a function of $t \in B := \mathbb{P}^1 \setminus \{s \text{ with } X_s \text{ singular}\}$, given by

$$\int_\gamma \omega_t,$$

where $\gamma \in H_2(X_t)$.

A relationship between the Picard-Fuchs equation and the L-series is given in [S2]. That result does not quite apply to the variety $\mathcal{Z}_{A_3}$, but the result of [S2] should be valid for a larger class of varieties. Although this result is not used in the proof of Theorem 3.14, it is important in that it predicted the result before any work was done towards resolving singularities of $\mathcal{Z}_{A_3}$, and counting points, done in § 3.3. Rather than state the result of [S2], we state a related conjecture:

CONJECTURE 3.1. Let $f(x,y,z) \in \mathbb{Z}[x,y,z]$. Define a variety $\mathcal{X}$ in $\mathbb{A}^4$ by $f(x,y,z)t = xyz(t+1)^2$. Let a_n denote the coefficient of $(xyz)^n$ in the expansion of $f(x,y,z)^n$. Let $b_n = \sum_{m=0}^{n}(-1)^m\binom{n+m}{2m+1}a_m$. Let N_p denote the number of points on $\mathcal{X}$ over $\mathbb{F}_p$. Then (under certain additional assumptions on f) there is a constant $c \in \mathbb{Z}$ so that

$$b_p + N_p \equiv c \bmod p$$

This conjecture relates the coefficients of the Picard-Fuchs differential equation to the coefficients of the L-series, since b_n is the coefficient of t^n in the expansion in t of a certain period, and in certain situations, including the one here, $-N_p$ will be equal to the trace of the Frobenius, up to some polynomial in p. Unfortunately it is not clear for what class of polynomials $f(x,y,z)$ this conjecture may hold; it seems to be true for many, but not all, simple polynomials with low degree and few terms. To save space, it is stated in the above elementary form, rather than in terms of L-series and Picard-Fuchs equations.

The following results establish this kind of congruence between the coefficients of the expansion of a period, and the coefficients of a certain weight 4 modular form.

LEMMA 3.2. *There is a period for the pencil of K3 surfaces $\mathcal{Z}_{A_3}$ which has an expansion in terms of the local co-ordinate t:*

$$\sum_{n\geq 1} b_n t^n,$$

where

$$b_n = (-1)^{n+1} \sum_{m=0}^{n-1} \sum_{p+q+r+s=m} (-1)^m \binom{n+m}{2m+1} \left(\frac{m!}{p!q!r!s!} \right)^2 .$$

The b_n *are also given thus:* t *is given by* $(\Gamma_0(12)+12)^{\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}}$

$$-\frac{\eta(3\tau)^4 \eta(12\tau)^8 \eta(2\tau)^{12}}{\eta(\tau)^4 \eta(4\tau)^8 \eta(6\tau)^{12}}.$$

A period is given by a modular form of weight 2

$$-\frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2}$$

and so for q *in a small enough neighbourhood of* 0*, if we express the following in terms of* q*-expansions,*

$$-\frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2} = \sum b_n \left(-\frac{\eta(3\tau)^4 \eta(12\tau)^8 \eta(2\tau)^{12}}{\eta(\tau)^4 \eta(4\tau)^8 \eta(6\tau)^{12}} \right)^n .$$

Proof. In [V1] the solution to the Picard-Fuchs equation for $\mathcal{X}_{A_3}$ was found, and an expression for the period was given in [V1] Proposition 6. There, the period was given by $\sum \nu^{n+1} a_n$, where $a_n = \sum_{p+q+r+s=n} \left(\frac{n!}{p!q!r!s!} \right)^2$. The variable ν is given by $\nu = \frac{1}{\lambda+4}$, where λ is the parameter for $\mathcal{X}_{A_3}$. Since $\mathcal{Z}_{A_3}$ is obtained from $\mathcal{X}_{A_3}$ by taking a double cover, the Picard-Fuchs equation is obtained by making a change of variables, and the solution will be given by the same modular form of weight 2, but in terms of a different variable t, where $\lambda = \frac{(t-1)^2}{t}$. By making a change of variables, the expression for the coefficients b_n is obtained from the a_n. To find an appropriate expression for t as a modular function, note that since λ is a degree 2 function of t, then we are looking for a subgroup of index 2 in the monodromy group $\Gamma_0(6)+6$ of $\mathcal{X}_{A_3}$. The group $\Gamma_0(12)+12$ conjugated by $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$ is an index two subgroup of $\Gamma_0(6)+6$, with uniformizing parameter $t(\tau)$ satisfying $\lambda = \frac{(1-t)^2}{t}$, given as above. □

Note, changing the sign of t and f is convenient, still gives a parameter and period for the family, and does not have an effect on the following result.

Theorem 3.3. *If* b_n *is defined as in Lemma 3.2, and*

$$(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2 = \sum \gamma_n q^n,$$

then for p *prime* $\neq 2, 3$*,*

$$b_{mp^r} - \gamma_p b_{mp^{r-1}} + p^3 b_{mp^{r-2}} \equiv 0 \bmod p^r.$$

In particular, for all primes p*,*

$$b_p \equiv \gamma_p \ mod\ p.$$

Proof. See [V3]. □

Since $\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$ has the expected weight, 4, and the right level 6, (only divisible by the primes of bad reduction), and satisfies the congruence suggested by [S2], we expect that this will be the Mellin transform of the L-series, and this is what will be proved in the next sections.

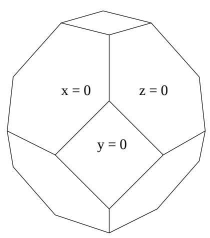
3.3. Resolution of singularities and counting of points. Here we give a qualitative description of the process of constructing the variety and resolving singularities. The variety is given by equations with coefficients in $\mathbb{Z}$, and so we can consider it over any field. For counting points, we work over $\mathbb{F}_p$. For calculating cohomology, it will be easier to work over $\mathbb{C}$.

- $X(\Sigma_{A_3})$ is a smooth toric variety, as described in [V1]. There is a map $\phi : X(\Sigma_{A_3}) \to \mathbb{P}^1$.
- $\mathcal{X}_{A_3}$, also described in [V1], is obtained from $X(\Sigma_{A_3})$ by blowing up the base locus of ϕ, and resolving singularities.
- $\mathcal{Z}_{A_3}$ is obtained by taking a pull back over a double cover of $\mathbb{P}^1$, and resolving the singularities of the resulting variety.

Note: In characteristic 2 and 3 the singular fibres X_0, X_{-4}, X_{12} described below, are not all distinct, and the following does not resolve the singularities. However, in all other characteristics, a smooth Calabi-Yau three-fold is obtained.

A: The variety $X(\Sigma_{A_3})$ The toric variety $X(\Sigma_{A_3})$ is the disjoint union of the fibre at infinity, pictured in Figure 1, together with the open torus $\mathbb{F}_p^n$. In terms of local coordinates x, y, z, the map ϕ is given by

$$\phi(x,y,z) = \frac{(1+x+xy+xyz)(1+z+zy+zyx) - 4xyz}{xyz}$$

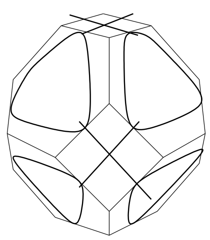


The fibre at infinity of the toric variety $X(\Sigma_{A_3})$ consists of a union of 14 rational surfaces, represented by the faces of a truncated octahedron. Square faces represent $\mathbb{P}^1 \times \mathbb{P}^1$; hexagonal faces represent $\mathbb{P}^2$ blown up in three points. Locally, these faces are $x = 0, y = 0, z = 0$, as shown. Over $\mathbb{F}_p$, this gives $8(p^2+4p+1) + 6(p+1)^2 - 36(p+1) + 24 = 14p^2 + 8p + 2$ points.

FIGURE 1. The fibre at infinity

The K3 fibres are given by $X_\lambda = \overline{\phi^{-1}(\lambda)}$. These are smooth K3 surfaces, expect for X_λ with $\lambda = 0, -4, 12, \infty$. A description of the singular fibres is given in Table 3.3, and they are also pictured in Figure 3.

B: The variety $\mathcal{X}_{A_3}$. In passing to $\mathcal{X}_{A_3}$, we blow up the base locus B, which is given by the intersection of X_λ with X_∞, pictured in Figure 2.



The base locus, $X_\lambda \cap X_\infty$ consists of 20 lines, 30 points being on two lines.
On each $\mathbb{P}^1 \times \mathbb{P}^1$, two lines; locally, on $y = 0$, $x = -1$ and $z = -1$.
On each blown up $\mathbb{P}^3$, one line; locally, on $x = 0$, $1 + z + zy = 0$.
In total, $20(p+1) - 30 = 20p - 10$ points.

FIGURE 2. The base locus of $\phi : X(\Sigma_{A_3}) \to \mathbb{P}^1$

λ	description of fibre	singularities
0	K3 surface with 4 isolated singularities. In passing to $\mathcal{Z}_{A_3}$ these singularities are all resolved.	$(1, \pm 1, -1), (-1, \pm 1, 1)$
12	K3 surface with 1 isolated singularity	$(1, 1, 1)$
-4	Two surfaces intersecting in three lines. After resolution, these three lines are blown up, so this fibre has 5 components.	lines $(-1, r, -1)$, $(r, -r^{-1}, r), (r, -1, r^{-1})$
∞	A union of 8 copies of $\mathbb{P}^2$ blown up in three points, and 6 copies of $\mathbb{P}^1 \times \mathbb{P}^1$. An additional 30 copies of $\mathbb{P}^1$ are added in resolving singularities	locally, the x, y and z axes. In total 36 lines.

TABLE 2. Description of singular fibres

Locally, $\mathcal{X}_{A_3}$, which is described in [V1], is given by

$$\left\{(x, y, z, \lambda) \in \mathbb{A}^4_{\mathbb{F}} | (1 + x + xy + xyz)(1 + z + zy + zyx) = (\lambda + 4)xyz\right\}.$$

Over $\mathbb{F}_p$, the variety is now a disjoint union of the open torus, the fibre at infinity, and the blown up base locus. The later has $(|\mathbb{P}^1(\mathbb{F}_p)| - 1).|B(\mathbb{F}_p)|$ points. The calculation $|B(\mathbb{F}_p)| = 20p - 10$ is shown in Figure 2. In blowing up the base locus B, singularities are introduced at the 30 singularities of B. Of these 30 singularities, 24 are on the fibre X_∞, and 6 are on X_{-4}. These are resolved by blowing up, which adds 30 copies of $\mathbb{P}^1$. Figure 3 shows the singular fibres of $\phi : \mathcal{X}_{A_3} \to \mathbb{P}^1$.

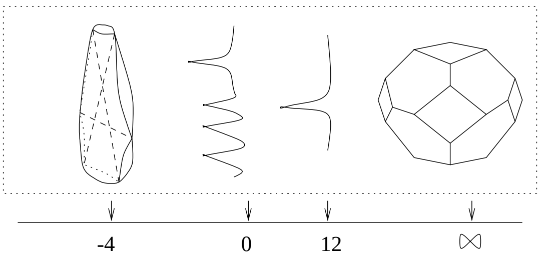


FIGURE 3. Singular fibres of the fibration of $\mathcal{X}_{A_3}$

C: The variety $\mathcal{Z}_{A_3}$. Away from singularities, $\mathcal{Z}_{A_3}$ is locally given by

$$(1 + x + xy + xyz)(1 + z + zy + zyx)t = (1 + t)^2 xyz.$$

Figure 4 shows the singular fibres of $\mathcal{Z}_{A_3}$, and the double cover. When the double cover $\lambda = \frac{(t-1)^2}{t}$ is taken, singularities can only occur at singularities of the fibres over the branch points, i.e., at $\lambda = 0$ or -4. For X_0, the four singularities are resolved by blowing up a surface passing through them in the ambient space, which adds a copy of $\mathbb{P}^1$ at each singularity. The singularities of $\mathcal{Z}_{A_3}$ on the fibre X_{-4} consist of the three copies of $\mathbb{P}^1$, which are the intersection of the two components of X_{-4}. To resolve the singularities, first blow up these three lines, adding three copies of $\mathbb{P}^1 \times \mathbb{P}^1$. Then, in the blow up, over each of the three points of intersection of these three lines, there are two singular points, which are resolved by blowing up, which adds 6 copies of $\mathbb{P}^1$.

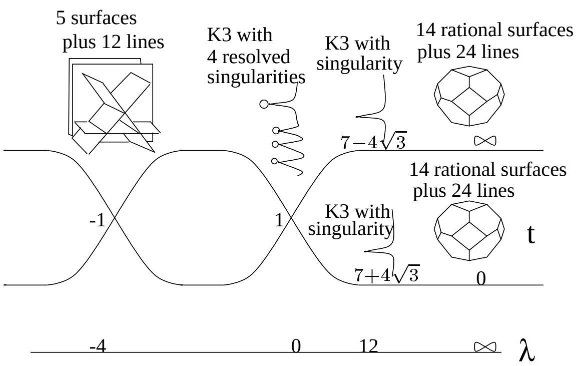


FIGURE 4. The singular fibres of $\mathcal{Z}_{A_3}$, which is obtained from $\mathcal{Z}_{A_3}$ by taking the pull back over the double cover of $\mathbb{P}^1$, $\lambda = \frac{(t-1)^2}{t}$.

To count points, we may consider $\mathcal{Z}_{A_3}$ to be a disjoint union of:
(i) The part from the open torus:

$$\left\{(x,y,z,t) \in (\mathbb{F}_p^{\times})^4 | (1+x+xy+xyz)(1+z+zy+zyx)t = (1+t)^2 xyz\right\}.$$

(ii) The fibres at Z_∞, Z_0: $2(14p^2+8p+2+24p)$ points.
(iii) The base locus, in each fibre: $(p-1)(20p-10)$ points.
(iv) Four copies of $\mathbb{P}^1$, added when singularities of X_0 are resolved: $4p$ points
(v) Three copies of $\mathbb{P}^1 \times \mathbb{P}^1$, and 12 copies of $\mathbb{P}^1$, added in resolving the singularities on X_{-4}: $3p^2+12p$ points.

From the above discussion, we have the following:

PROPOSITION 3.4. *For $p \geq 5$ a prime, the number of points on $\mathcal{Z}_{A_3}(\mathbb{F}_p)$ is given by*

$$N_p + 2(14p^2+8p+2+24p) + (p-1)(20p-10) + 4p + 3p^2 + 12p$$
$$= N_p + 51p^2 + 50p + 14$$

where

$$N_p = \left|\left\{(x,y,z,t) \in (\mathbb{F}_p^{\times})^4 | (1+x+xy+xyz)(1+z+zy+zyx)t = (1+t)^2 xyz\right\}\right|.$$

3.4. The action of the Frobenius automorphism on the Cohomology. First we determine what the cohomology is, then we look at how the Frobenius acts.

A: Hodge structure

LEMMA 3.5.

$$h^{1,1}(\mathcal{Z}_{A_3}) = 50$$

PROOF. The Picard number $\rho(\mathcal{Z}_{A_1^3})$ is given by

$$\rho(\mathcal{Z}_{A_1^3}) = \rho(\text{general fibre}) + 1 + \sum_{Z_s}(n(Z_s)-1),$$

where $n(Z_s)$ is the number of components of Z_s. The fibres Z_∞, and Z_0 have 14 components, and Z_{-1} has 5 components. By Proposition 2.3, we have $h^{1,1} = \rho(\mathcal{Z}_{A_3})$. The general fibre has $\rho = 19$ (by [V1] Proposition 2), so

$$h^{1,1} = \rho(\mathcal{Z}_{A_3}) = 19 + 1 + 14 - 1 + 14 - 1 + 5 - 1 = 50.$$

$\square$

LEMMA 3.6. *The Euler characteristic of $\mathcal{Z}_{A_3}$ is 100.*

PROOF. We have

$$\chi(\mathcal{Z}_{A_3}) = \chi(\mathbb{P}^1) \times \chi(\text{K3 surface}) + \sum_{Z_t \text{ a singular fibre}} (\chi(Z_t) - \chi(\text{general fibre}))$$

Since the singular fibres over $\lambda = 12$ have one singularity each, which can be resolved by blowing up, adding a $\mathbb{P}^1$, they have Euler characteristic 23. For the singular fibres X_∞ and X_{-4}, each triple point reduces the Euler characteristic by 1. Hence $\chi(Z_\infty) = \chi(Z_0) = 24 - 24$ and $\chi(Z_{-1}) = 24 - 6$.

Hence we have

$$\chi(\mathcal{Z}_{A_3}) = 2 \times 24 + 2 \times (23 - 24) + 2 \times 24 + 6 = 100$$

$\square$

PROPOSITION 3.7. *$\mathcal{Z}_{A_3}$ is a rigid Calabi–Yau threefold with Hodge diamond:*

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 50 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 50 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

PROOF. By Theorem 2.1 we already know most of the values of the Hodge numbers. We have $h^{1,1} = 50$ and $\chi(\mathcal{Z}_{A_3}) = 100$, by Lemmas 3.5,3.6. Then since $\chi(\mathcal{Z}_{A_3}) = \sum(-1)^i h^i$, where $h^i = \dim H^i(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$, we have $h^3 = 2(50 - 0 + 1) - 100 = 2$. The rest will follow by the symmetry of the Hodge diamond. Since the third Betti number is two $\mathcal{X}_{A_3}$ is rigid by definition. $\square$

B: The L-series

The zeta function for $\mathcal{Z}_{A_3}$ is given by:

$$Z_p(\mathcal{Z}_{A_3}, s) = \frac{P_1(t)P_3(t)P_5(t)}{P_0(t)P_2(t)P_4(t)P_6(t)},$$

where $P_i(t) = det(1 - t\,\mathrm{Frob}_p^*|_{H^i})$, so $degP_i = h^i = \dim H^i(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$. This holds for $p \neq 2, 3$. For $i \neq 3$, the polynomials $P_i(t)$ are determined by finding the action of the Frobenius on $H^i(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$

LEMMA 3.8. *Let p be a prime, $p \neq 2, 3$. For $k = 1, 5$, $H^k(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$ is trivial, and $P_k(t) = 1$. For $k = 0, 2, 4, 6$, The action of* Frob_p *on $H^k(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$ is multiplication by $p^{\frac{k}{2}}$,*

$$P_k(t) = (1 - p^{\frac{k}{2}} t)^{h^i}.$$

Proof. Theorem 2.1 gave $h^1 = 0$, and $h^0 = 1$, which determines the action there. By Lemma 3.7, $h^2 = 50$. The Picard lattice is spanned by the components of the singular fibres of $\mathcal{Z}_{A_3}$, and the general fibre, and the sections coming from the Picard lattice of the general fibre. Curves spanning the Picard lattice of the fibre X_λ of $\mathcal{X}_{A_3}$ are described explicitly in [V1] §3. These all are given by equations with coefficients in $\mathbb{Z}$, except for some involving $\sqrt{\lambda(\lambda+4)}$, e.g., on X_λ we have the curve $x=-1$, $2z = \lambda+2+\sqrt{\lambda(\lambda+4)}$. But in $\mathcal{Z}_{A_3}$, we have $\sqrt{\lambda(\lambda+4)} = \frac{t^2-1}{t}$. Hence we have curves spanning $\mathrm{Pic}(\mathcal{Z}_{A_3})$ which are all defined over $\mathbb{Z}$, which means that the Frobenius action on H^2 is just multiplication by p, and $P_2(t) = (1-pt)^{h^2} = (1-pt)^{50}$. By duality this determines the action on the remaining H^k. □

So, we have:

$$Z_p(\mathcal{Z}_{A_3}, s) = \frac{P_3(t)}{(1-t)(1-pt)^{50}(1-p^2t)^{50}(1-p^3t)}.$$

Here, for $p \neq 2, 3, \ell$,

$$P_3(t) = det(1 - t\,\mathrm{Frob}_p^*\,|_{H^3}) = 1 + a_pt + p^3t^2,$$

where

$$a_p = Tr\,\mathrm{Frob}_p\,|_{H^3}.$$

At the primes of bad reduction we have to define $P_{p,3}$ in a different way. For $p = \ell$, we can define a_ℓ by taking the trace of the Frobenius action on ℓ'-adic cohomology. It is more complicated to find Euler factors for $p = 2, 3$. We would like the L-series of $L(H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell), s)$ to be given by

$$\prod_{p\neq 2,3} \frac{1}{P_{p,3}(p^{-s})}\frac{1}{(1-2.2^{-s})}\frac{1}{(1-3.3^{-s})},$$

then we could show this is the L-series for the modular form $\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$. However, we do not give any justification for why the Euler-factor at 2 and 3 should be given in this way, and so we are not proving in this paper that the global L-series is equal to the stated modular form, rather that it is equal up to Euler factors at the primes of bad reduction, and that the Galois representation on $H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$ has semisimplification isomorphic to the semisimplification of the representation associated by Deligne to the modular form $\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$.

Proposition 3.9. *For p a prime $p \neq 2, 3$, the L-series of $H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$ has coefficients a_p given by*

$$a_p = p^3 - p^2 - 13 - N_p$$

where

$$N_p = |\left\{(x,y,z,t) \in (\mathbb{F}_p^\times)^4 | (1+x+xy+xyz)(1+z+zy+zyx)t = (1+t)^2xyz\right\}|.$$

Proof. The Lefschetz fixed point theorem tells us that

$$|\mathcal{Z}_{A_3}(\mathbb{F}_p)| = \sum(-1)^i Tr\,\mathrm{Frob}_p\,|_{H^i}.$$

The Frobenius action on $H^i(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$, for $i \neq 3$ was described in Lemma 3.8, and h^1 are given in Proposition 3.7, so

$$a_p = 1 - 0 + 50p + 50p^2 - 0 + p^3 - |\mathcal{Z}_{A_3}(\mathbb{F}_p)|.$$

From Proposition 3.4 we have

$$|\mathcal{Z}_{A_3}(\mathbb{F}_p)| = N_p + 51p^2 + 50p + 14.$$

Putting these together we have given value of a_p. $\square$

A computer calculation finds the first few coefficients a_p of the L-series of $H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$:

p	N_p	a_p
5	81	6
7	297	-16
11	1185	12
13	1977	38
17	4737	-126
19	6465	20
23	11457	168
73	383457	218

These coefficients agree, as far as calculated, with the coefficients of the modular form $\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2 = q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + 6q^6 - 16q^7 - 8q^8 + 9q^9 - 12q^{10} + 12q^{11} - 12q^{12} + 38q^{13} + 32q^{14} - 18q^{15} + 16q^{16} - 126q^{17} + O(q^{18})$, which was also predicted to be the L-series in § 3.1. In the next section we show how to prove we actually have equality.

3.5. Proof of equality of two L-series. Throughout this section, we use a_n to denote the coefficients of the L-series of $H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_2)$, and b_n to denote the coefficients of $(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$. So, the L-series is $\sum \frac{a_n}{n^s}$, and the eta product is $\sum b_n q^n$. In this section we prove that $a_n = b_n$ for all n, thus identifying the Mellin transform of the L-series of $H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$, and showing its modularity.

We use Livné's method to prove the equality of the L-series. For convenience we restate the theorem of Serre, based on Faltings results, and taken from Livné's paper [L]:

Theorem (Specialization of [L], Theorem 4.3) *Let S a finite set of rational primes, and $\rho_1, \rho_2 : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Q}_2)$ representations unramified outside S. Denote the composite of all quadratic extensions of $\mathbb{Q}$ unramified outside S by $\mathbb{Q}_S$. Suppose that*

1. $\mathrm{Tr}\,\rho_1 \equiv \mathrm{Tr}\,\rho_2 \equiv 0 (\mathrm{mod}\, 2)$ *and* $\det \rho_1 \equiv \det \rho_2 (\mathrm{mod}\, 2)$.

2. There is a finite set T of primes, disjoint from S, for which

i. *The image of the set* $\{\mathrm{Frob}_t\}_{t\in T}$ *in* $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ *is surjective.*

ii. $\mathrm{Tr}\,\rho_1(\mathrm{Frob}_t) = \mathrm{Tr}\,\rho_2(\mathrm{Frob}_t)$ *and* $\det \rho_1(\mathrm{Frob}_t) = \det \rho(\mathrm{Frob}_t)$ *for all $t \in T$.*

Then ρ_1 and ρ_2 have isomorphic semisimplifications.

We will apply this theorem to show that the Mellin transform of the L-series of $H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$ is equal to the eta product $(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$. To do this we need to produce representations, $\rho_1, \rho_2 : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_2)$, with the required L-series.

The representation ρ_1 is given by the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the 2-adic cohomology, $H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_2)$, which is a two dimensional vector space by Proposition 3.7.

The representation ρ_2 is the 2-adic representation associated by Deligne's theorem to the eigenform $f = (\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$. So the Mellin transform of the L-series of ρ_2 is $(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$. To apply Livné's result, we now prove some parity results.

LEMMA 3.10. *The number of points*

$$N_p = |\{(x,y,z,t) \in (\mathbb{F}_p^\times)^4 | (1+x+xy+xyz)(1+z+zy+zyx)t = (1+t)^2 xyz\}|$$

is odd for all primes $p \geq 5$.

PROOF. For fixed t, we have an involution, $(x,y,z) \mapsto (x^{-1}, y^{-1}, z^{-1})$. The 8 fixed points are $(\pm 1, \pm 1, \pm 1)$. Of these, $(1,1,1)$ is on the fibres over $\lambda = 12$, so gives 2 points on $\mathcal{Z}_{A_3}$. The remaining 7 fixed points are on the fibres over the branch points of $\lambda = (t-1)^2/t$. So in total this is an odd number of points. ☐

LEMMA 3.11. *If* $\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Q}_\ell)$ *is unramified outside* 2 *and* 3*, and* $Tr\rho(\mathrm{Frob}_5) \equiv Tr\rho(\mathrm{Frob}_7) \equiv Tr\rho(\mathrm{Frob}_{11}) \equiv Tr\rho(\mathrm{Frob}_{13}) \equiv 0 \bmod 2$*, then we have* $Tr\rho(\mathrm{Frob}_p) \equiv 0 \bmod 2$ *for all primes* $p \geq 5$.

PROOF. This works in the same way as Livne's Proposition 4.10. Let $\overline{\rho} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Z}/2\mathbb{Z})$ be the reduction mod 2 of ρ. Let L be the extension of $\mathbb{Q}$ which is fixed by the kernal of $\overline{\rho}$. Then, assuming the trace is not always 0 mod 2, the image of $\overline{\rho}$ must contain an element of order 3, since these are the only ones with non zero trace. So, the image is isomorphic to C_3 or S_3. There are only 9 extensions of $\mathbb{Q}$ unramified outside $2,3$, with Galois group C_3 or S_3. The cubics which these are the splitting fields of can be found, and shown to be irreducible mod p for some $p \in \{5,7,11,13\}$, which means that for such a p, the order of Frob_p in $\mathrm{Gal}(L/\mathbb{Q})$ is 3, and the trace will be odd. But we have assumed all these traces are even, so this is a contradiction. ☐

LEMMA 3.12.

$$\mathrm{Tr}\,\rho_1 \equiv \mathrm{Tr}\,\rho_2 \equiv 0 (\mathrm{mod}\, 2)$$

PROOF. For $p \geq 5$, Lemma 3.9 showed that the coefficients of the L-series of $H^3(\mathbb{Z}_{A_3}, \mathbb{Q}_\ell)$ are given by $a_p = p^3 - p^2 - 13 - N_p$. These are all even, because $p^3 - p^2 - 13$ is always odd, and N_p is odd, by Lemma 3.10. The coefficients of the modular form are all even, since the coefficients of q^5, q^7, q^{11}, q^{13} are all even, so we can apply Lemma 3.11. So we deduce that $\mathrm{Tr}\,\rho_1(\mathrm{Frob}_p) \equiv \mathrm{Tr}\,\rho_2(\mathrm{Frob}_p) \equiv 0 (\mathrm{mod}\, 2)$ for all primes $p \geq 5$. But by the Tchebotarev density theorem, we can then deduce that $\mathrm{Tr}\,\rho_1 \equiv \mathrm{Tr}\,\rho_2 \equiv 0 (\mathrm{mod}\, 2)$.

☐

LEMMA 3.13.

$$\det \rho_1 \equiv \det \rho_2 (\mathrm{mod}\, 2).$$

PROOF. From Deligne's construction ([D]), $\det \rho_2 = \chi^3$, where χ is the cyclotomic character. By Poincaré duality, there is a perfect pairing

$$H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_2) \times H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_2) \to H^6(\mathcal{Z}_{A_3}, \mathbb{Q}_2) \cong \mathbb{Q}_2,$$

which respects the Galois action. On $H^6(\mathcal{Z}_{A_3}, \mathbb{Q}_2)$, the action of Frob_p is given by multiplication by p^3. (See for example [Mi] §11 and §12 Theorem 12.6.) So if the eigenvalues of Frob_p on $H^3(\mathcal{Z}_{A_3}, \mathbb{Q}_2)$ are α_1, α_2, then by duality, they are also $p^3/\alpha_1, p^3/\alpha_2$, and so $\det(\rho_1(\mathrm{Frob}_p)) = \alpha_1\alpha_2 = p^6/\alpha_1\alpha_2$, which implies that $\det(\rho_1(\mathrm{Frob}_p)) = p^3$, for all primes p. So for all primes $p > 3$, $\det(\rho_1(\mathrm{Frob}_p)) = \det(\rho_2(\mathrm{Frob}_p))$, and so by the Tchebotarev density theorem for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have $\det(\rho_1(\sigma)) = \det(\rho_2(\sigma)) = \chi^3(\sigma)$. Hence also $\det \rho_1 = \chi^3$, and the determinants of the two representations are equal. ☐

THEOREM 3.14. *Up to Euler factors at $p = 2, 3$, the Mellin transform of the L-series of the middle cohomology of the variety $\mathbb{Z}_{A_3}$ is the weight 4 level 6 modular form $\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$.*

PROOF. The primes of bad reduction of $\mathcal{Z}_{A_3}$ are 2 and 3, and the level of $\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$ is 6 The field $\mathbb{Q}_{\{2,3\}}$ needed to apply Livné's result is $\mathbb{Q}[\xi_{24}]$, and so we can take $T = \{5, 7, 11, 13, 17, 19, 23, 73\}$, since this is a set of representatives for $(\mathbb{Z}/24\mathbb{Z})^\times$. From Table 3.4 for the values of a_p, and the q-expansion of $\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$, we have the required equality of traces at Frob_p for $p \in T$. By Lemma 3.12 we have the required condition on traces of the representations, and by Lemma 3.13 we have the required condition on determinants. Thus we are able to obtain the equality of the L-series by applying Livné's result. □

4. Calculation of the L-series of $H^3(\mathcal{Z}_{A_1^3}, \mathbb{Q}_\ell)$

The variety $\mathcal{Z}_{A_1\times A_1\times A_1}$ is isomorphic the the pencil $\mathcal{Z}$ described in [PS], which is also studied in [SB], [Be], and also [vGN]. Although this has been studied in several other places, the approaches are slightly different, so this calculation is included here. The variety $\mathcal{Z}$ in [PS] is birational to $\mathcal{Z}_{A_1^3}$, but the L-series is not calculated, though in [Be], the congruence relationship between the modular form $\eta(2\tau)^4\eta(4\tau)^4$ and the solution of the Picard-Fuchs equation of the fibration of the family is given, which as in § 3.2, following [S2], should suggest this would be the L-series of $\mathcal{Z}_{A_1^3}$. In [vGN], the variety X' has L-series $\eta(q^2)^4\eta(q^4)^4$, but X' is constructed in very a different way to $\mathcal{Z}_{A_3}$, and has a different Hodge numbers, e.g., $h^2 = 128$, and the Euler characteristic is 254 ([Proposition 2.2 [vGN]). However, in the remark after Theorem 2.4 in [vGN], it is shown that there is a birational map between X' and $\mathcal{X}$ of [PS]. In [AO] there is a completely different proof of the modularity of the L-series of $\mathbb{Z}_{A_1^3}$, (given the description of the values of the L-series in Proposition 4.2), using class field theory.

4.1. Structure of $\mathcal{Z}_{A_1^3}$. We start with a toric variety $X(\Sigma_{A_1^3}) = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, with a map $\phi : X(\Sigma_{A_1^3}) \to \mathbb{P}^1$ given by

$$\phi(x, y, z) = x + x^{-1} + y + y^{-1} + z + z^{-1}.$$

The base locus of ϕ is blown up to obtain $\mathcal{X}_{A_1^3}$, which away from $xyz = 0$, is given by

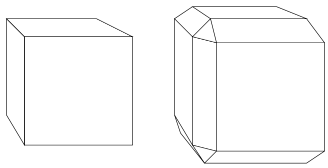
$$x + x^{-1} + y + y^{-1} + z + z^{-1} = \lambda.$$

The fibre $X_\lambda = \overline{\phi^{-1}(\lambda)}$ is a smooth K3 surface for general λ. Taking a pull back of the double cover $\lambda = -t - t^{-1}$, we obtain $\mathcal{Z}_{A_1^3}$, given (when $xyz \neq 0$) by

$$x + x^{-1} + y + y^{-1} + z + z^{-1} + t + t^{-1} = 0.$$

The fibres are denoted by Z_t.

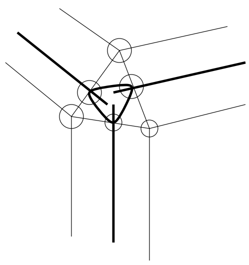
Now we go through this construction in more detail. Since $X(\Sigma_{A_1^3})$ is a toric variety, we consider it composed of the open torus, plus the complement of the open torus, which is fibre at infinity X_∞, and which consists of 6 copies of $\mathbb{P}^1 \times \mathbb{P}^1$. To obtain the same fibre at infinity as given in the compactification described in [PS], before blowing up the base locus of ϕ, we blow up the lines of singularities in the fibre at infinity, where its components intersect. This adds 12 copies of $\mathbb{P}^1 \times \mathbb{P}^1$, and 8 copies of $\mathbb{P}^3$, and gives a variety with 24 singularities, all on X_∞.



In $X(\Sigma_{A_1^3})$, X_∞ consists of 6 copies of $\mathbb{P}^1 \times \mathbb{P}^1$, represented by the faces of the cube in the first picture. After the blow up of the lines where these faces intersect, X_∞ consists of 12 copies of $\mathbb{P}^1 \times \mathbb{P}^1$, and 8 copies of $\mathbb{P}^2$, represented respectively by rectangles and triangles in the second picture.

FIGURE 5. The Fibre at infinity.

Apart from the fibre at infinity, the other singular fibres are at $\pm 2, \pm 6$, where the only singularities are rational double points. When the base locus is blown up, we introduce 24 more singularities, also all on X_∞.



Part of the fibre X_∞ is shown. The thicker lines are where the general fibre intersects X_∞ before the base locus is blown up, and the 6 singularities on one of the eight $\mathbb{P}^2$s in X_∞, are circled.

FIGURE 6. Singularities of $\mathcal{X}_{A_1^3}$, on X_∞.

Of the 48 singularities, 6 are circled in Figure 6. There are 6 such for each of the 8 corners, in the same configuration. These singularities are resolved by blowing up all eight $\mathbb{P}^2$s.

Now we take the double cover $\lambda = -t - t^{-1}$. There will be singularities over the branch points, $s = \pm 2$ corresponding to the singularities of the fibre. These singularities are at

$$\begin{gathered}(t, x, y, z) = (1, 1, -1, -1), (1, -1, 1, -1), (1, -1, -1, 1),\\ (-1, -1, 1, 1), (-1, 1, -1, 1), (-1, 1, 1, -1)\end{gathered}$$

and are resolved by blowing up the surface $t + x = 0, y + z = 0$, and then the strict transform of the similar surfaces, obtained by permuting the indices. This resolves the singularities in all odd characteristics.

LEMMA 4.1. *The variety $\mathcal{Z}_{A_1^3}$ is a rigid Calabi-Yau three-fold, with*

$$\chi(\mathcal{Z}_{A_1^3}) = 140, \;\; \rho(\mathcal{Z}_{A_1^3}) = 70,$$

and Hodge diamond:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 70 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 70 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

PROOF. Theorem 2.1 tells us most of the hodge numbers. We have

$$\rho(\mathcal{Z}_{A_1^3}) = \rho(\text{general fibre}) + 1 + \sum_{Z_s} (n(Z_s) - 1),$$

where $n(Z_s)$ is the number of components of Z_t. Here $n(Z_s) = 1$, except for Z_∞, and Z_0, which have 26 components. The general fibre has $\rho = 19$ (by [PS] Theorem 1), so, with Proposition 2.3 we have

$$h^{1,1} = \rho(\mathcal{Z}_{A_1^3}) = 19 + 1 + 26 - 1 + 26 - 1 = 70.$$

Next we calculate the Euler characteristic. For the fibre at infinity, we add up the Euler characteristic of the components, to get $\chi(Z_\infty) = \chi(Z_0) = 72$. The singular fibres over $\lambda = \pm 2$ have been resolved. Over each of $\lambda = 6, -6$, there are two fibres with one singularity, and Euler characteristic 23. So we have

$$\begin{aligned}\chi(\mathcal{Z}_{A_1^3}) &= \chi(\mathbb{P}^1) \times \chi(\text{K3 surface}) + \sum_{Z_t \text{ a singular fibre}} (\chi(Z_t) - \chi(\text{general fibre})) \\ &= 2 \times 24 + 4 \times (23 - 24) + 2 \times (72 - 24) = 140.\end{aligned}$$

Since $\chi(\mathcal{Z}_{A_3}) = \sum (-1)^i h^i$, $h^i = \dim H^i(\mathcal{Z}_{A_1^3}, \mathbb{Q}_\ell)$, we have $h^3 = 2(70-0+1) - 140 = 2$, so $\mathcal{Z}_{A_3}$ is a rigid Calabi-Yau three-fold. $\square$

We denote by $|\mathcal{Z}_{A_1^3}(\mathbb{F}_p)|$ the total number of points on $\mathcal{Z}_{A_1^3}(\mathbb{F}_p)$, and define N_p to be the number of solutions of

$$t + t^{-1} + x + x^{-1} + y + y^{-1} + z + z^{-1} = 0$$

in $\mathbb{F}_p$ with $txyz \neq 0$.

LEMMA 4.2. *The L-series of $H^3(\mathcal{Z}_{A_1^3}, \mathbb{Q}_\ell)$ is given by $\sum_{n \geq 1} \frac{a_n}{n^s}$, where for p prime, $p \neq 2$, a_p is even, and given by $a_p = p^3 - 2p^2 - 7 - N_p$.*

PROOF. There are $26p^2 + 44p + 24$ points on X_∞, which is pictured in Figure 5. The base locus consists of 20 copies of $\mathbb{P}^1$, however, 24 points are counted twice, so the number of points on the base is $20p - 4$. After blowing up, each X_t, for $t \neq 0, \infty$, consists of a copy of the base locus, and on open part, where $xyz \neq 0$. So, in counting points, we add up the number of points on X_0 and X_∞, the number of points on the base locus for all fibres; the number of points in the open part with $xyzt \neq 0$, and we must also add $6p$ points for the 6 copies of $\mathbb{P}^1$ added in the resolution of the surfaces over ± 2. So we have

$$\begin{aligned}|\mathcal{Z}_{A_1^3}| &= 2(26p^2 + 24p + 2) + (p-1)(20p - 4) + 6p + N_p \\ &= 72p^2 + 70p + 8 + N_p\end{aligned}$$

The Lefschetz fixed point theorem, applied to the Frobenius action on the ℓ-adic cohomology, gives

$$|\mathcal{Z}_{A_1^3}| = 1 + 70p - a_p + 70p^2 + p^3$$

Here, $a_p = Tr\,\mathrm{Frob}_p|_{H^3}$. The action on H^2 and H^4 is multiplication by p and p^2. To show this, we can find explicit divisors spanning the Picard lattice of the fibres, given in [PS] §2, which are all rational curves, and we also know the components of the degenerate fibres, Z_0 and Z_∞, so we can give a spanning set for $\mathrm{Pic}(\mathcal{Z}_{A_1^3})$. Since these are all represented by subvarieties defined by equations with coefficients in $\mathbb{Z}$, they are fixed by the action of the Frobenius automorphism. By Proposition 2.3, images of elements of $\mathrm{Pic}(\mathcal{Z}_{A_1^3})$ span $H^2(\mathcal{Z}_{A_3}, \mathbb{Q}_\ell)$, so the action of the Frobenius on H^2 is just multiplication by p. By duality, the action on H^4 is multiplication by p^2. Hence,

$$72p^2 + 70p + 8 + N_p = 1 + 70p - a_p + 70p^2 + p^3.$$

So

$$a_p = p^3 - 2p^2 - 7 - N_p,$$

where N_p is the number of solutions of

$$t + t^{-1} + x + x^{-1} + y + y^{-1} + z + z^{-1} = 0$$

in $\mathbb{F}_p$ with $txyz \neq 0$.

The a_p are all even, since $p^3 - 2p^2 - 7$ is even, and N_p is even, because any solution (t, x, y, z) can be paired up with the solution $(-t, -x, -y, -z)$.

$\square$

4.2. Proof of equality of two L-series. In this section, we use a_n to denote the coefficients of the L-series of $H^3(\mathcal{Z}_{A_1^3}, \mathbb{Q}_2)$, and b_n to denote the coefficients of $(\eta(2\tau)\eta(4\tau))^4 = q\prod(1-q^{2n})^4(1-q^{4n})^4 = q - 4q^3 - 2q^5 + O(q^6)$

We will once again apply the Faltings/Serre/Livné result. For application of this theorem, the representation ρ_1 is defined to be that given by the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the 2-adic cohomology, $H^3(\mathcal{Z}_{A_1^3}, \mathbb{Q}_2)$, which is a two dimensional vector space by Lemma 4.1 The representation ρ_2 is the 2-adic representation associated by Deligne's theorem, to the eigenform $(\eta(2\tau)\eta(4\tau))^4$.

Lemma 4.3.

$$\operatorname{Tr}\rho_1 \equiv \operatorname{Tr}\rho_2 \equiv 0 (\operatorname{mod} 2)$$

Proof. The fact that all the coefficients a_p of the L-series of the representation on $H^3(\mathcal{Z}_{A_1^3}, \mathbb{Q}_2)$ are even was given in Lemma 4.2.

Next we want to prove that if $(\eta(2\tau)\eta(4\tau))^4 = \sum b_n q^n$, then for primes $p > 2$, b_p is even. We can deduce this from Proposition 4.10 of [L], which tells us that if the representation ρ_2 is unramified outside $\{2, 5\}$, and $\operatorname{Tr}\rho(\mathrm{Frob}_3) \equiv 0 (\operatorname{mod} 2)$, then $\operatorname{Tr}\rho_2 \equiv 0$ identically. The level of $(\eta(2\tau)\eta(4\tau))^4$ is 8, so the corresponding representation is unramified outside 2, and the coefficient of q^3 is $-4 \equiv 0 \bmod 2$, and so we deduce that for all primes $p > 2$, b_p is even. An alternative way to see that all b_p are even is to note that we have $\eta(2\tau)^4\eta(4\tau)^4 \equiv \eta(8\tau)^3 \bmod 2$. Then at least $3/4$ of the b_p must be even, but then by Tchebotarev density theorem, this means they all are. Alternatively, we can apply Jaccobi's formula for $\eta(\tau)^3$, which also implies that the b_p are even. $\square$

Theorem 4.4. *Up to an Euler factor at $p = 2$, the Mellin transform of the L-series of $H^3(\mathcal{Z}_{A_1^3}, \mathbb{Q}_\ell)$ is equal to*

$$(\eta(2\tau)\eta(4\tau))^4.$$

Proof. We apply theorem 4.3 of [L], which was restated in § 3.5. We take $S = \{2\}$. The representations ρ_1 and ρ_2 are as described above. The determinants of both representations are $\det(\rho_i(Frob_p)) = p^3$, in exactly the same way as for $\mathcal{Z}_{A_3}$ in Lemma 3.13. By Lemmas 4.3 the trace in both cases is even. The field $\mathbb{Q}_{\{2\}} = \mathbb{Q}[i, \sqrt{2}] = \mathbb{Q}[\xi_8]$, where ξ_8 is a primitive eighth root of unity. So we have $\mathrm{Gal}\,\mathbb{Q}[\xi_8] \cong (\mathbb{Z}/8\mathbb{Z})^\times$, and since $(\mathbb{Z}/8\mathbb{Z})^\times = \{[3], [5], [7], [17]\}$, $T = \{3, 5, 7, 17\}$

satisfies the required condition of [L] 4.3. The first few terms of the q expansion of $(\eta(2\tau)\eta(4\tau))^4 = \sum_{n\geq 1} b_n q^n$ are given by

$$q - 4q^3 - 2q^5 + 24q^7 - 11q^9 - 44q^{11} + 22q^{13} + 8q^{15} + 50q^{17} + O(q^{19}).$$

From Lemma 4.2 we can calculate the a_p, and show that $a_p = b_p$ for $p = 3, 5, 7, 17$. Hence Theorem 4.3 applies, and we deduce that ρ_1 and ρ_2 have isomorphic semisimplifications, and hence modulo possible factors at $p = 2$, their L-series are equal. □

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