Root Lattices

and

Pencils of Varieties.

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Declaration

Except where otherwise stated in the text, this dissertation is the result of my own work and is not the outcome of work done in collaboration.

I hereby declare that this dissertation is not substantially the same as any I have submitted for a degree or diploma or any other qualification at any other university.

I further state that no part of my dissertation has already been, or is being currently submitted for any such degree, diploma or other qualification.

Helena Anne Verrill

July 20, 1994.

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Introduction

In this dissertation I study how a root lattice can give rise to a pencil of varieties, and I look at a few examples in detail.

This work is motivated by the example of C. Peters and J. Stienstra, in [P,S]. Here an interesting family of K3 surfaces is studied. which can be thought of as coming from a lattice $A_1 \times A_1 \times A_1$.

A natural question is what will happen for other root lattices?

In chapter 1, I start by giving basic definitions and notation. The construction of a pencil of varieties from a root lattice is described, and there are a few examples.

In chapters 2 and 3, the A_3 case is looked at in detail.

In chapter 2, the general fibre is described and shown to be a K3 surface, with Picard number 19.

In chapter 3, the structure of the family is examined. The Picard-Fuchs equation and the monodromy group are found (Corollary 1, page 48 and Theorem 2, page 58. Lemma 14), page 67 gives the solution space of the Picard-Fuchs equation..

For some of the calculations were made with the aid of a computer. I used the computer algebra packages Maple and Pari, in an interactive fashion.

Chapter 1

Preliminaries

1.1 Very basic notation

- $Bl_n(V)$ means the variety V blown up in n points; which n points will be apparent when the term is used.

- $\mathcal{H}$ is the upper half-plane.

- η is the Dedekind eta function, given by

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1 - e^{2\pi i\tau n}), \quad \tau \in \mathcal{H}.$$

  η satisfies the following relations

$$\eta(-1/\tau) = \sqrt{(-i\tau)}\eta(\tau)$$

  and

$$\eta(\tau + 1) = e^{\pi i/12}\eta(\tau)$$

  (see [R] for more details). We write $q = e^{\pi i\tau/12}$.

- If R is a ring, then $SL_2(R)$ is the group of 2×2 matrices with entries in R and determinant 1.

- Subgroups of $SL_2(\mathbb{R})$.

  - $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| c \equiv 0(\mathrm{mod} N) \right\}$
  - $\Gamma_1(N) = \left\{ \begin{pmatrix} a+1 & b \\ c & d+1 \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| a \equiv c \equiv d \equiv 0(\mathrm{mod} N) \right\}$
  - For $N, e \in \mathbb{Z}, e \| N$, define
    $w_e = \left\{ \left\{ \begin{matrix} a & b \\ c & d \end{matrix} \right\} := \frac{1}{\sqrt{e}} \begin{pmatrix} ae & b \\ cN & de \end{pmatrix} \in SL_2(\mathbb{R}) \middle| a, d, b, c \in \mathbb{Z} \right\}$
    w_e is a coset of $\Gamma_0(N)$, and is called an Atkin-Lehner involution of $\Gamma_0(N)$.

  - For $N, e, f, g, \dots \in \mathbb{Z}, e, f, g, \ldots \| N$, define

    $$\Gamma_0(N) + e, f, g, \ldots$$

    to be the group obtained by adjoining $w_e, w_f, w_g, \ldots$ to $\Gamma_0(N)$. $\Gamma_0(N)+$ means that all w_e for $e \| N$ have been added. This notation is taken from [C,N].

  -

    $$\Gamma^0(N) + e, f, g, \dots = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma_0(N) + e, f, g, \ldots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  - For G a subgroup of $SL_2(\mathbb{C})$, the notation $\overline{G}$ is used to denote $G/(G \cap \{\pm I\})$. To denote that a matrix is being considered up to ± 1, square brackets are used.

- $SL_2(\mathbb{R})$ acts on $\mathcal{H}$ in the usual way, via fractional linear transformations.

- $SL_2(\mathbb{R})$ acts on meromorphic functions on $\mathcal{H}$ in the usual way, and modular forms are defined as usual. When I talk about modular functions for a group, I mean modular forms of weight 0. See [Shim] for details of definitions.

- If $\Gamma \subseteq SL_2(\mathbb{R})$, then we can define a surface $\mathcal{H}/\Gamma$, but quotienting out by the action of Γ. I use the notation $\mathcal{H}^*/\Gamma$ to denote the compactification, obtained by adding in the cusps.

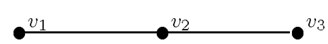
- If $\Gamma \subseteq SL_2(\mathbb{R})$ is such that the genus of $\mathcal{H}^*/\Gamma$ is zero, then a **Hauptmodul** for Γ is a function f such that the space of modular functions on $\mathcal{H}^*/\Gamma$ is equal to $\mathbb{C}(f)$.

- If $\Gamma \subseteq SL_2(\mathbb{R})$ then $M_k(\Gamma)$ is the space of modular forms of weight k for Γ.

* $Pic(X) \subseteq H_2(X, \mathbb{Z})$ is the Picard group of the variety X, the group of algebraic cycles. $T = T(X)$, the transcendental lattice, is the orthogonal complement of $Pic(X)$ in $H_2(X, \mathbb{Z})$, with respect to the intersection product.

1.2 Lattice theory

For the most part, this thesis is concerned with a variety arising from a construction involving the root lattice A_3. This is a lattice with basis v_1, v_2, v_3, (this is a set of **simple roots**) having a bilinear form with values as determined by the corresponding **Dynkin diagram**, pictured thus:



$$\begin{array}{lll} v_i.v_j & = 2 & \text{if } i = j \\ & = -1 & \text{if } i \neq j \text{ and there is an edge joining } v_i \text{ and } v_j \\ & = 0 & \text{otherwise} \end{array}$$

This lattice may be taken as a 3-dimensional sublattice of $\mathbb{R}^4$, with $v_1 = e_1 - e_2, v_2 = e_2 - e_3$ and $v_3 = e_3 - e_4$. The **root system** is pictured in Figure 2.1, page 20. The **root lattice** is the lattice generated by the root system. One of the **Weyl chambers** (truncated to within a cube) is pictured as σ (Figure 2.2). Figure 2.3 gives an idea of how the 24 Weyl chambers for A_3 fit together. The **Weyl group** acts transitively on the Weyl chambers, as a group of transformations of Euclidean space, preserving the dual lattice (Figure 2.2). It is a subgroup of $GL(\mathbb{R}^3)$ generated by symmetries s_r for r in the root system, where s_r is the reflection in the subspace of $\mathbb{R}^3$ perpendicular to r.

For any Dynkin diagram D, there is a corresponding semi-simple **Lie algebra** $\mathfrak{g}_D$. A_3 corresponds to the Lie algebra $\mathfrak{g}_{A_3} = \mathfrak{sl}_4(\mathbb{C})$ of traceless

4×4 matrices over $\mathbb{C}$. In this case the **Cartan subalgebra** $\mathfrak{h}$ is the space of 4×4 traceless diagonal matrices, so it has a basis $e_1 - e_2, e_2 - e_3, e_3 - e_4$, where e_i is a matrix with 1 in the (i,i) place, and zeros elsewhere. The root lattice is a sublattice of $\mathfrak{h}^*$. The Weyl chambers are found in $\mathfrak{h}^*$.

The **weights of the adjoint representation** are, in the A_3 case, given by the $e_i - e_j$. These correspond to homogeneous polynomials $X_i X_j^{-1}$, called the **weights** of the representation, and their sum gives the **character** χ of the adjoint representation of the associated Lie algebra,

$$\sum_{\substack{i \neq j \\ 0 \leq i,j \leq 4}} X_i X_j^{-1}.$$

This is the simplest non-trivial polynomial invariant under the action of the Weyl group. For more general definitions of the terms in bold face, and results about lattices, see [Se] or [Hu].

1.3 Toric Varieties

A **toric variety** X is a variety that contains a torus $\mathbb{T} = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ as a dense open subset, such that the natural action of $\mathbb{T}$ on itself extends to an action on X. A toric variety X_Σ can be defined from a **fan** Σ in a lattice N. A cone $\sigma \in \Sigma$ corresponds to an affine subvariety $X_\sigma \subset X_\Sigma$. Fixing notation,

V is an n-dimensional real vector space.

$N \cong \mathbb{Z}^n$ is a sublattice of V

$N^\vee := \{v \in V^\vee | (v.\mathrm{N}) \in \mathbb{Z} \forall\, \mathrm{N} \in N\}$, with basis $m_1, \ldots, m_n$.

$$\mathbb{C}[\sigma^\vee \cap N^\vee] := \mathbb{C}[\prod_{i=1}^{r} x_i^{\lambda_i} | \sum \lambda_i e_i \in \sigma^\vee \cap N^\vee]$$

$$X_\sigma := \text{Spec } \mathbb{C}[\sigma^\vee \cap N^\vee].$$

X is a union of orbits O_τ, $\tau \in \Sigma$. The closure of O_τ is a toric variety $V(\tau)$, corresponding to the fan $St(\tau)$, the star of τ (see [F] §3.1, page 52). We have $\dim(V(\tau)) = n - \dim(\tau)$, and

$$X = \mathbb{T} \sqcup \bigcup_{e \text{ an edge of } \Sigma} V(e).$$

Let D_e be the divisor class of $V(e)$, and let

$$\Delta = \sum_{e \text{ an edge of } \Sigma} D_e;$$

then the canonical divisor can be given by $K \sim -\Delta$ (see [F] page 85 §4.3).

See [F] or [D] for details.

1.4 K3 surfaces

A K3 surface X is a compact, connected 2-dimensional complex manifold with $\mathcal{K}_X$ (the canonical line bundle) trivial, and the first Betti number, $b_1(X)$, equal to 0.

Proposition 1 *If X is a K3 surface, then the Hodge numbers are as in the following array:*

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 20 & 0 \\ 2 & 1 & 0 & 1 \end{array} \quad ((i,j)^{th} entry = dim_{\mathbb{C}} H^{i,j}(X))$$

Proof. See [BPV] Proposition (3.3), page 241.

■

So there exists a unique (2,0)-form (up to scalars), $\omega \in H^{2,0}$, and

$$H^{2,0} = \langle \omega \rangle,$$

$$\omega \wedge \omega = 0 \, (\in H^{4,0} = 0),$$

and

$$\langle \omega \wedge \overline{\omega} \rangle = H^{2,2}.$$

$Pic(X) \subseteq H^{1,1}$, so $\rho(X) = rank(Pic(X)) \leq 20$. For more details about K3 surfaces see [BPV].

1.5 From a root lattice to a pencil of varieties

Given a root lattice L, we have a fan Σ_L, defined by the Weyl chambers. This fan defines a toric variety $X = X_L = X_{\Sigma_L}$. The character χ of the lattice defines a function on the toric variety, since affine pieces of X are given by

$$\text{Spec } \mathbb{C}[\sigma^{\vee} \cap \mathfrak{h}^{*}],$$

(where σ is a Weyl chamber), and the character is an element of $\mathbb{Z}[\Lambda]$ (see [F,H] page 375), where $\Lambda \subseteq \mathfrak{h}^*$ is the weight lattice ([F,H] page 172). In [F,H] a Weyl chamber is defined as a subset of $\mathfrak{h}^*$, but it is dual to a cone spanned by a set of simple roots for the root lattice, which is in the weight space, so one could view as it being in $\mathfrak{h}^{**}$, and then $\mathfrak{h}^{***}$ is canonically isomorphic to $\mathfrak{h}^*$, and writing $[\sigma^\vee \cap \mathfrak{h}^*]$ above makes sense. X is fibred by

$$X_\lambda = (\chi = \lambda).$$

There is a base locus; all the X_λ have the same intersection with $\Delta(= X - S)$. But this can just be blown up, to give $\tilde{X}$, i.e., take

$$\tilde{X} \subseteq \mathbb{P}^1 \times X$$

$$\tilde{X} := \left\{(\lambda, \boldsymbol{x}) \in \mathbb{P}^1 \times X | \phi(\boldsymbol{x}) = \lambda\right\}.$$

The fibres of $\tilde{X}$, given by the projection to $\mathbb{P}^1$, are the same as the fibres for X, but are now separated, to give a pencil of varieties.

1.5.1 The one-dimensional case: A_1

This is the trivial case. The Dynkin diagram for A_1 is a single point:

$$\bullet$$

The root lattice is 1-dimensional, and the toric variety will be $\mathbb{P}^1$, fibred by $X_\lambda = \{(x : y) \in \mathbb{P}^1 | x^2 + y^2 = \lambda xy\}$.

1.5.2 Two-dimensional cases

See Table 1.1.

1.5.3 Three-dimensional cases

Chapters 2 and 3 deal with A_3, and the A_1^3 case is the family in [P,S]. These and the $A_1 \times A_2$ case give a families of K_3 surfaces.

1.5.4 The A_n case.

Lemma 1 *For A_n, $X_\infty = \bigcup V(e)$.*

Proof. In this case, $\mathfrak{h}^*$ can be given by

$$\mathfrak{h}^* = \mathbb{C}\{L_1, L_2, \ldots, L_{n+1}\}/(L_1 + L_2 + \cdots + L_{n+1} = 0).$$

([F,H], page 212), and the roots are the differences $L_i - L_j$ ([F,H], page 213). In the toric construction, L_i corresponds to X_i, with $\prod X_i = 1$, and the roots correspond to $X_i X_j^{-1}$, and the sum of these gives the character

$$\sum_{\substack{i \neq j \\ 1 \leq i,j \leq n+1}} X_i X_j^{-1}.$$

A typical Weyl chamber σ (cone from the toric point of view) is spanned by

$$L_1, L_1 + L_2, L_1 + L_2 + L_3, \ldots, L_1 + \cdots + L_n$$

The dual, $\sigma^\vee$, is spanned by

$$L_1 - L_2, L_2 - L_3, \ldots, L_n - L_{n+1}$$

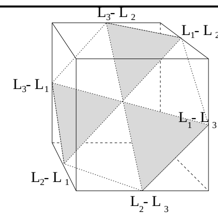
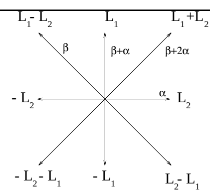
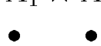
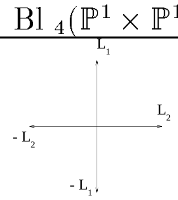
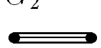
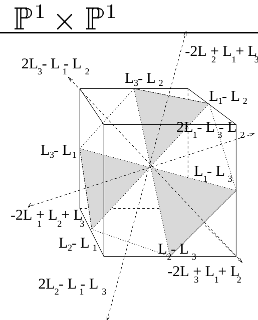
Name and Dynkin diagram	Σ_L and X_{Σ_L}	the fibre X_λ, and reference
A_2	 $Bl_3(\mathbb{P}^2)$	$X_\lambda : XY^{-1} + YZ^{-1} + ZX^{-1}$ $+ XZ^{-1} + ZY^{-1} + YX^{-1} = \lambda$ i.e., $(X+Y+Z)(XY+YZ+ZX) = (\lambda+3)XYZ$, which appears in [S-B] as $\mathcal{C}$
B_2	 Bl $_4(\mathbb{P}^1 \times \mathbb{P}^1)$	$X_\lambda : (1+x+x^2)(1+z+z^2) = zx(1+\lambda)$ where x comes from L_1, and z from L_2
$A_1 \times A_1$ 	 $\mathbb{P}^1 \times \mathbb{P}^1$	$X_\lambda : x + x^{-1} + y + y^{-1} = \lambda$ or $(x+y)(xy+1) = \lambda xy$, This is example $\mathcal{A}'$ in [S-B].
G_2 	 Bl $_3$(Bl $_6(\mathbb{P}^2))$	$X_\lambda : (\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + 1)(\frac{y}{x} + \frac{z}{y} + \frac{x}{z} + 1) = \lambda + 1$

Table 1.1: The 2-dimensional lattices

([F,H] page 216). If $L_i - L_{i+1}$ corresponds to $x_i = X_i/X_{i+1}$, then $X_\sigma =$ Spec $\mathbb{C}[x_1, \dots x_n]$, and the character is given by

$$\begin{aligned}
&\sum_{\substack{i \neq j \\ 1 \leq i,j \leq n-1}} X_i X_j^{-1} \\
&= (X_1 + \cdots + X_{n+1})(X_1^{-1} + \cdots + X_{n+1}^{-1}) - n - 1 \\
&= (1 + X_2/X_1 + \cdots + X_{n+1}/X_1)(1 + X_1/X_2 + \cdots + X_1/X_{n+1}) - n - 1 \\
&= (1 + x_1^{-1} + \cdots + (x_1 x_2 \dots x_n)^{-1})(1 + x_1 + x_1 x_2 + \cdots + x_1 x_2 \dots x_n) - n - 1 \\
&= \tfrac{(1 + x_n + x_{n-1} x_n + \cdots + x_1 x_2 \dots x_n)(1 + x_1 + x_1 x_2 + \cdots + x_1 x_2 \dots x_n)}{(x_1 x_2 \dots x_n)} - n - 1
\end{aligned}$$

Hence the fibre X_λ is given locally by

$$(\lambda + n + 1) x_1 x_2 \dots x_n =$$

$$(1 + x_n + x_{n-1} x_n + \cdots + x_1 x_2 \dots x_n)(1 + x_1 + x_1 x_2 + \cdots + x_1 x_2 \dots x_n).$$

This means that X_∞ is given by $x_1 x_2 \dots x_n$ in these local coordinates. So if this affine piece is A, then $X_\infty \cap A = \bigcup V(e) \cap A$ (each piece taken with multiplicity 1). The same happens for each affine piece associated with a cone of the fan, and the union of these is X. Hence X_∞ is exactly the same as $\bigcup V(e)$.

$\blacksquare$

Proposition 2 *For the lattices A_n, the corresponding families are of varieties with trivial canonical divisor.*

Proof. We have $K_X \sim -\Delta$ (fact about toric varieties), $X_\infty \sim \Delta$ (by Lemma 1), and $X_\lambda \sim X_\infty$ (by definition of linear equivalence). So, by adjunction ([GH] page 147) we have

$$\begin{aligned} K_{X_\lambda} &\sim (K_X + X_\lambda)|_{X_\lambda} \\ &\sim (-\Delta + \Delta)|_{X_\lambda} \\ &\sim 0 \end{aligned}$$

■

Each Weyl chamber determines a copy of $\mathbb{C}^n$, and a set of local coordinates. Because of the symmetry of the fan under the action of the Weyl group, X_λ is given by the same equations in each local coordinate system. By considering one affine piece of X_λ, the global structure can be determined by symmetry.

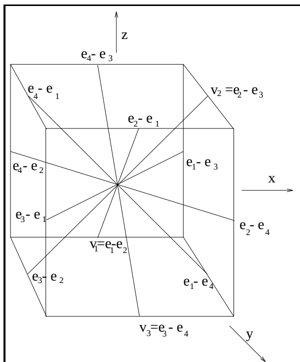
Chapter 2

The A_3 case: the general fibre.

We now turn out attention to a more detailed study of the A_3 case. This chapter is concerned with the general fibre, and the following chapter deals with the structure of the pencil.

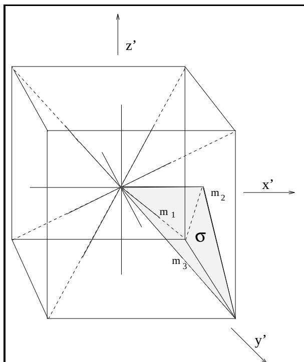
2.1 The total space, X_{A_3}

In this section I describe the 3-fold $X = X_{A_3}$. As described in Section 1.2, we have a lattice $N \hookrightarrow \mathbb{R}^4$ spanned by the v_i. (Fig 2.1). $X_1 \dots X_4$ (respectively x, y, z) are the variables associated to the vectors $e_1 \dots e_4$ (respectively v_1, v_2, v_3), which appear in Figure 2.1. The relationship between the e_i and the v_i gives the relations $x = X_1/X_2, y = X_2/X_3, z = X_3/X_4$. X_{A_3} is the toric variety defined by the fan in Figure 2.3.



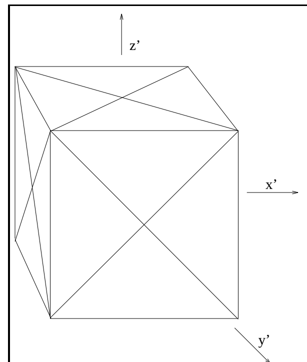
Here N is identified with a lattice in $\mathbb{R}^3$.
$v_1 = e_1 - e_2 = (0, -1, -1)$
$v_2 = e_2 - e_3 = (1, 0, 1)$
$v_3 = e_3 - e_4 = (0, 1, -1)$
The root system consists of the 12 vectors to midpoints of edges of the the cube.

Figure 2.1:
The root system A_3



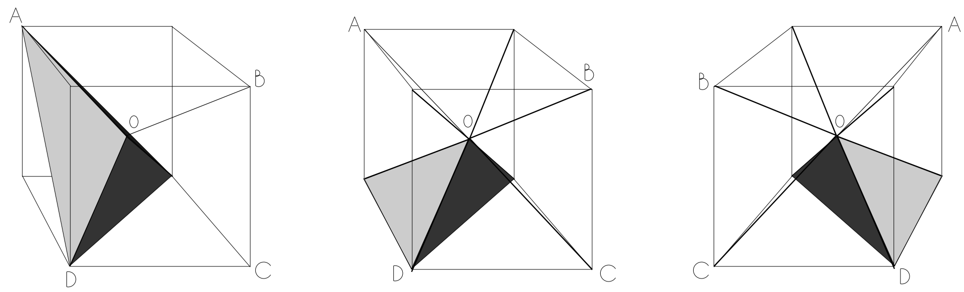
$m_1 = \frac{1}{2}(1, -1, -1)$
$m_2 = (1, 0, 0)$
$m_3 = \frac{1}{2}(1, 1, -1)$
The m_i span σ, a typical cone in the fan defining X. σ is dual to the cone spanned by v_1, v_2, v_3, and can also be thought of as a Weyl chamber for A_3.

Figure 2.2:
The dual lattice.



This figure shows the fan of the Weyl chambers restricted to the surface of a cube, centred at the origin

Figure 2.3:
The fan defining X.



The fan given by taking cones on triples of the vectors A, B, C, D, is $\mathbb{P}^1$. ([F] page 22.)

Each of these four cones is subdivided, resulting in a fan with typical cone as above. This corresponds to the operation of blowing up four points. (see [F] page 40).

The mirror image also gives $Bl_4(\mathbb{P}^3)$. These cones both subdivide to give the cone for X. By considering the morphisms between these varieties, a description of X is obtained.

Figure 2.4: The construction of X_{A_3}

Proposition 3 *X can be given as a subvariety of $(\mathbb{P}^3 \times (\mathbb{P}^2)^4)^2$; if the variables are*

$$(X_1^k : X_2^k : X_3^k : X_4^k), (R_{i,1}^k : R_{i,2}^k : R_{i,3}^k),\ i = 1, 2, 3, 4, k = 1, 2,$$

then X is given by

$$X_i^k R_{j,l}^k = X_l^k R_{j,i}^k,\ X_i^1 X_j^2 = X_j^1 X_i^2, \qquad (i, j, l \text{ distinct.})$$

Proof. Figure 2.4 justifies this. In the figure just one of the cones of each fan is shaded. Note that we have here an example of a flip; blowing down the $\mathbb{P}^1$s of X in one way gives the variety defined by the second cone in Figure 2.4, and the other way we get the variety defined by the third cone of this figure.

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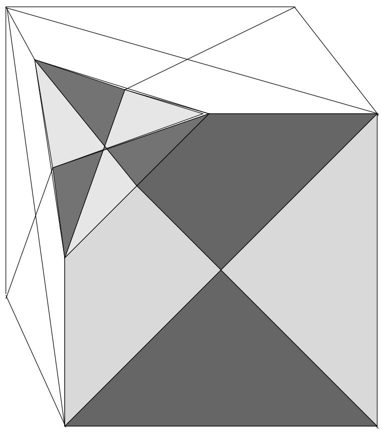


Figure 2.5: The stars of $-m_1$ and $m_3 - m_1$

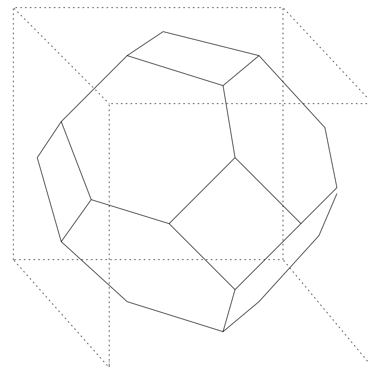


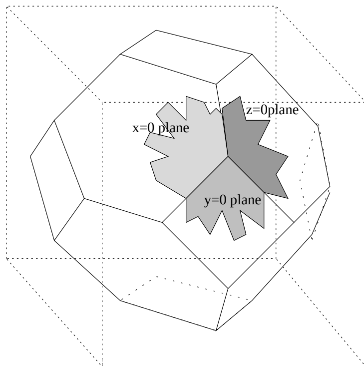
Figure 2.6: $X_{A_3} \backslash \mathbb{T}$

2.2 $X_{A_3} \backslash \mathbb{T}$

Proposition 4 *$X_{A_3} \backslash \mathbb{T}$ consists of eight copies of $Bl_3(\mathbb{P}^2)$ and six copies of $\mathbb{P}^1 \times \mathbb{P}^1$, configured as in Figure 2.6.*

Proof. By looking at the stars, as in Figure 2.5, one sees that the closure of the subvariety locally defined by $y = 0$, is $\mathbb{P}^1 \times \mathbb{P}^1$, and for $x = 0$ and $z = 0$ one has $\mathbb{P}^2$ blown up in 3 points. There is a $Bl_3(\mathbb{P}^2)$ for each corner of the cube, and a $\mathbb{P}^1 \times \mathbb{P}^1$ for each face. By considering how these pieces fit together locally (see Figure 2.7), the global situation is deduced by symmetry.

■



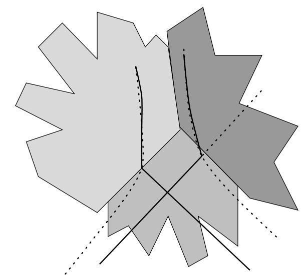


Figure 2.7: $X_{A_3}\backslash\mathbb{T}$, globally, and restricted to part of one of the 24 affine pieces defined by cones of the fan.

2.3 The general fibre is a K3 surfaces

2.3.1 Definition of the pencil of K3 surfaces

As in Subsection 1.5, the fibres are defined by

$$X_\lambda = ((\phi - \lambda) = 0),$$

where ϕ comes from the character of the adjoint representation.

ϕ can be defined on the $\mathbb{C}^{*3}$ piece of the toric variety, which corresponds to the origin of the fan, or on the local affine pieces. When considering ϕ restricted to $\mathbb{C}^3$, I consider $\mathbb{C}^3$ to be identified with an open affine subvariety of $\mathbb{P}^3$, $(X_1 : X_2 : X_3 : X_4) \in \mathbb{P}^3 | X_1 \neq 0, i = 1,2,3,4$, and write $\Phi = \phi|_{\mathbb{C}^{*3}}$. We have

$$\Phi(X_1 : X_2 : X_3 : X_4) = (X_1 + X_2 + X_3 + X_4)(X_1^{-1} + X_2^{-1} + X_3^{-1} + X_4^{-1}) - 4$$

For any i, j, k, l with $i, j, k, l = 1, 2, 3, 4$, let $\sigma_{(i,j,k,l)}$ be the cone in N

spanned by $e_i - e_j, e_j - e_k, e_k - e_l$. Let $A_{(i,j,k,l)}$, be the affine piece corresponding to $\sigma_(i,j,k,l)^\vee$. We can take local coordinates $x_{i,j} = X_i/X_j, y_{j,k} = X_j/X_k, z_{k,l} = X_k/X_l$. Foe example, $A_{(1,2,3,4)}$, has coordinates $x = X_1/X_2, y = X_2/X_3, z = X_3/X_4$.

X_λ restricted to $A_{(1,2,3,4)}$ is given by

$$(1 + x + xy + xyz)(1 + z + zy + zyx) = (\lambda + 4)xyz. \tag{2.1}$$

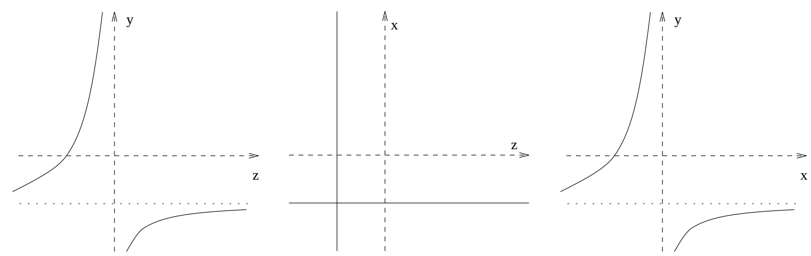
2.3.2 The base locus

I shall now describe $X_\lambda \cap X_\infty$; this will be used to show that the general X_λ is smooth, and gives a large proportion of the Picard group on X_λ.

Proposition 5 *The base locus consists of* 20 *rational curves, in configuration as indicated in Figure 2.9(b), with intersection graph as in Figure 2.10.*

Proof. Locally X_λ is defined by Equation 2.1. The base locus is the part where $xyz = 0$. So locally the base locus consists of the 3 parts drawn in Figure 2.8, which fit together as in Figure 2.9(a). This is drawn on $X_{A_3}\backslash\mathbb{T}$ in Figure 2.7. By symmetry, the same picture can be drawn for each similar part of $X_{A_3}\backslash\mathbb{T}$, resulting in Figure 2.9(b), from which the intersection graph is easily constructed.

■



$1 + z + xy = 0$ $\quad$ $(1 + x)(1 + z) = 0$ $\quad$ $1 + x + xy = 0$

$X_\lambda|_{x=0}$ $\quad$ $X_\lambda|_{y=0}$ $\quad$ $X_\lambda|_{z=0}$

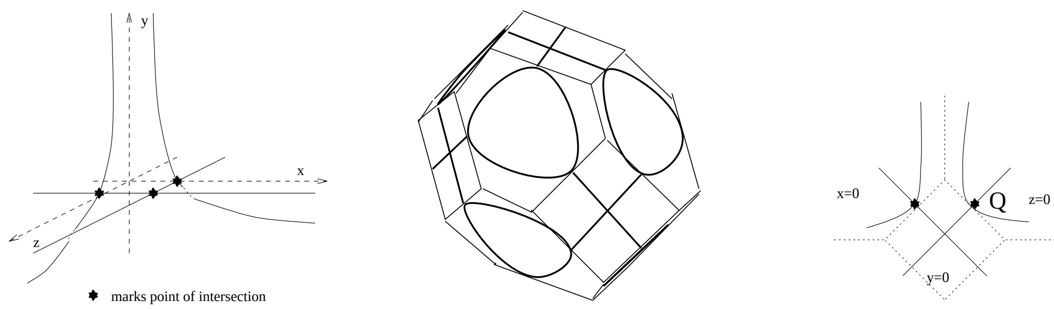
Figure 2.8: Local components of the base locus

Lemma 2 *X_λ is smooth for general λ.*

Proof. Bertini's Theorem says that in a linear system the general divisor is smooth away from the base locus (see [Ha] page 274). The pencil of X_λ is in the linear system $|-K_X|$, and the base locus is as in Figure 4.2. Where X_∞ is smooth, the general member cannot be singular, so the only points where the general member could possibly be singular are like those marked in Figure 4.3.

If X_λ is singular at Q (see Figure 2.9(c)), then X_λ and X_∞ must both have multiplicity ≥ 2 there, Hence at Q the base locus would have multiplicity 4. But at Q, $X_\lambda \cap X_\infty$ is defined by $(zy, 1 + x + xy)$, so locally consists of two reduced components meeting transversally at Q, so the multiplicity here is 2. Hence the general X_λ is smooth.

■



(a)
$X_\lambda \cap X_\infty$
in x, y, z system

(b)
$X_\lambda \cap \Delta$

(c)
part of $X_\lambda \cap \Delta$
with possible
singularities of X_λ
marked

Figure 2.9: The base locus, globally

Remark 1 *Singular Fibres*

It can be shown directly from the local equations for X_λ that the values of λ for which there are singularities are $\lambda = 0, 4, -12$ and ∞. This information also comes out of calculating the Picard-Fuchs equation, and observing that these are its singular values.

Theorem 1 *X_λ is a smooth K3 surface for general λ.*

Proof. By Lemma 2 the general X_λ is smooth. By Proposition 2 the canonical divisor is trivial. A smooth surface with trivial canonical class is either a K3 surface or an Abelian surface (see [H] page 422 Theorem 6.3). But these surfaces contain rational curves, in the base locus, for example, and Abelian surfaces do not (since any non-trivial morphism from $\mathbb{P}^1$ to an Abelian surface must factor through the Albanese variety of $\mathbb{P}^1$, which is the same as its Jacobian, which is trivial, since the genus is zero).

■

2.4 The Picard group of X_λ

In this section I show that the Picard group of general X_λ has rank 19, and in doing so I show that the family is non-constant.

Lemma 3 $\rho(X_\lambda) \leq 19$ *for general* λ.

Proof. X_λ is a K3 surface, so $\rho(X_\lambda) \leq 20$. [Sh-I] show that K3 surfaces with $\rho = 20$ from a discrete countable family, so if this were the case, our family would have to be locally constant, and the monodromy would be finite. After some base change, which we can assume to have been made, the monodromy would be trivial.

To show that the monodromy is infinite, I look at the kind of degeneration that occurs. At ∞, the degeneration is good in the sense of Kulikov

In this case, p_g of the general fibre, a K3 surface, is 1 ([H] page 422, Theorem 6.3), and the components of X_∞ are all rational surfaces, and so have $p_g = 0$ ([H] page 183). Hence the monodromy is not trivial. Hence result.

■

Proposition 6 *$\rho(X_\lambda) = 19$ for general λ, and the discriminant of the Picard lattice is* 6.

Proof. Since $\rho(X_\lambda) \leq 19$ (Lemma 3), we just have to find a set of lines which have an intersection matrix of rank 19. This is because the matrix is of rank 19 if and only if there is a sub-collection of 19 curves, with intersection matrix 19. If these curves $c_1, \dots c_{19}$ were linearly dependent, there would be a relation $\lambda_1 c_1 + \cdots \lambda_{19} c_{19} = 0$ for some $\lambda_i \in \mathbb{Z}$, so $\lambda_1 c_1.c_i + \cdots \lambda_{19} c_{19}.c_i = 0$ for each $i, 1 \leq i \leq 19$. But then columns of the intersection matrix would be linearly dependent.

The base locus gives 20 lines on X_λ. See Figure 2.9, and Figure 2.10 for the intersection graph. The corresponding intersection matrix has rank 16 by a computer calculation.

For some more lines, take the intersection of

(2.2) $X_\lambda : (X_1 + X_2 + X_3 + X_4)(X_1^{-1} + X_2^{-1} + X_3^{-1} + X_4^{-1}) = (\lambda + 4)$

with the hyperplane

$$X_i = -X_j,$$

to give a quadratic curve.

For example, substituting $X_1 = -X_2$ into Equation 2.2 gives

$$(X_3 + X_4)(X_3^{-1} + X_4^{-1}) = (\lambda + 4)$$
$$\Rightarrow X_3/X_4 = \frac{\lambda + 4 \pm \sqrt{(\lambda(\lambda + 4))}}{2}, \text{a constant.}$$

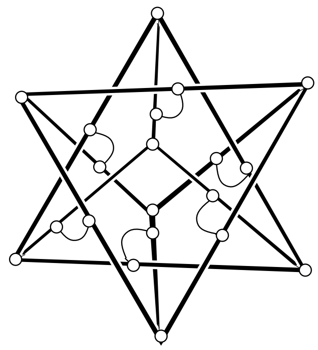


Figure 2.10: Intersection graph for curves in $X_\lambda \cap X_\infty$.

The only permutation fixing one of these curves is (i, j).

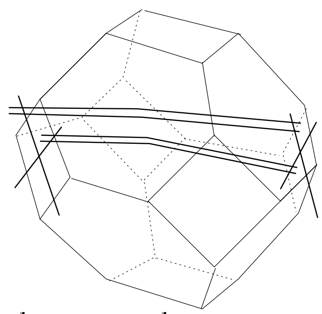
Locally, we have $(X_i/X_j, X_k/X_l)$ replaced by $(x, z), (xy, zy)$, or (y, xyz) as appropriate (I mean $(X_1/X_2, X_3/X_4) = (x, z)$, which gives curves $x = -1, z =$ constant (two choices of constant); $(X_2/X_3, X_1/X_4) = (y, xyz)$, etc.)

There are twelve curves obtained in this way since S_4, the symmetric group on four symbols acts transitively, and has order 24, and each curve has stabiliser of order 2.

This gives 12 more lines. These come in three sets of four, each set 'stretching between opposite faces of the cube'; this is pictured in Figure 2.11.

Together this gives 32 lines. A computer calculation shows that the intersection matrix has rank 19 and discriminant 6.

6 is square free, so this means that these 32 lines generate the Picard group over $\mathbb{Z}$, since if they generated a sublattice of index d in $Pic(X_\lambda)$, the discriminant of the intersection matrix would be divisible by n^2. ∎



drawn on the two opposite faces, we have
$x = -1$ and $z = -1$ on $y = 0$ and $y = \infty$
the horizontal lines represent
$x = -1, \; z = \frac{\lambda+4 \pm \sqrt{(\lambda(\lambda+4))}}{2}$
$z = -1, \; x = \frac{\lambda+4 \pm \sqrt{(\lambda(\lambda+4))}}{2}$

Figure 2.11: Representation of some curves.

Proposition 7 *With respect to some basis, the intersection matrix of the lattice of transcendental cycles, $T = T(X_\lambda) := Pic(X_\lambda)^\perp$ is*

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Proof.

For a K3 surface X, $H^2(X, \mathbb{Z})$ is an even unimodular lattice, with signature (3,19) (see [Kon] page 58).

So now The Picard lattice is even, with rank 19, signature (1,18) (Hodge index theorem, [H] Theorem 1.9, page 364), discriminant 6 (Proposition 6), hence the lattice of transcendental cycles, $T(X_\lambda) = Pic(X_\lambda) \perp$ is even, rank 3, signature (2,1), discriminant ± 6.

In [C,S] page 378 §7, two quadratic forms are defined to have the same genus if they are equivalent over the p-adic integers for all primes p.

In [C,S] table 15.4 page 387, genera of forms with $|det| \leq 11$ are listed. For determinant ± 6, the genera are $I_{r,s}(2 \times 3^{\pm})$ and $II_{r,s}(2 \times 3^{\pm})$. This notation is described in [C,S] page 384 §7.8.

$(r, s) = (2, 1)$, so the determinant is -6 and we are in the $2 \times 3^{-}$ case. The lattice is even, so of type II. Hence the genus is $II_{2,1}(2 \times 3^{-})$.

The above matrix is also of this genus.

By [C,S] Theorem 21, page 395, there is only one genus in this class.

Hence result.

$\blacksquare$

Proposition 8 *$H_2(X_\lambda, \mathbb{C})$ has a basis*

$$\gamma_i = \gamma_i^\lambda \in H_2(X_\lambda, \mathbb{Z}),$$

$i = 1, \ldots, 22$, such that $\gamma_1, \gamma_2, \gamma_3$ is a basis of $T(X_\lambda)$, and with respect to which the intersection matrix is as in Proposition 7, and is independent of λ.

Proof. From Proposition 7, for each λ there is an integral basis as required, since the above results (from [C,S]) are over $\mathbb{Z}$. For any continuous choice of basis the intersection matrix is independent of λ, since the curves vary continuously, so the intersections vary continuously, but the intersections of the curves are in $\mathbb{Z}$. $\blacksquare$

Chapter 3

The A_3 case: monodromy and the Picard-Fuchs Equation

To investigate how X_λ varies as λ varies, one looks at the Hodge structure. In this case I shall be considering the Picard-Fuchs equation and the monodromy group. The details of the definitions of the monodromy and the Picard-Fuchs equation are given below. For more general details on variation of Hodge structure see [Tu].

X_λ is a K3 surface, so up to scalar multiplication, there is a unique 2-form $\omega_\lambda \in H^{2,0}(X_\lambda, \mathbb{C})$.

${\gamma_1}_\lambda, \ldots, {\gamma_{22}}_\lambda$ is a basis of $H_2(X_\lambda, \mathbb{Z})$ as in Proposition 8. ${\gamma_1}_\lambda{}^*, \ldots, {\gamma_{22}}_\lambda{}^*$ is the corresponding dual basis for $H^2(X_\lambda, \mathbb{C})$.

The duality is defined (see [GH] page 56) so that for $\gamma \in H_2(X_\lambda, \mathbb{Z})$, then $\gamma^* \in H^2(X_\lambda, \mathbb{C})$ is such that for all $\alpha \in H_2(X_\lambda, \mathbb{Z})$,

$$\int_\alpha \gamma^* = {}^{\#}(\gamma.\alpha)$$

where ${}^{\#}(-.-)$ is the intersection product.

The intersection of cycles in homology is Poincaré dual to the wedge product in cohomology, i.e., for $\alpha, \gamma \in H_2(X_\lambda, \mathbb{Z})$ we have

$$^{\#}(\gamma.\alpha) = \int_{X_\lambda} \alpha^* \wedge \gamma^*.$$

(See [GH] page 59, and [GH] page 53 for Poincaré duality.)

Hence the intersection matrix for the dual basis is also as in Proposition 7.

The integral of ω_λ around any algebraic cycle is zero, hence when $H_2(X_\lambda, \mathbb{Z}) \otimes \mathbb{C}$ and $H^2(X_\lambda, \mathbb{C})$ are identified, ω_λ is in $T(X_\lambda) \otimes \mathbb{C}$.

As λ varies, γ_{i_λ} smoothly varies, as determined by the local trivialisation and the transition functions, so one might try to define a map

$$\varphi_m : B \to \mathbb{P}^{21} = \mathbb{P}(H_2(X_\lambda, \mathbb{Z}))$$

by

$$\lambda \mapsto (\gamma_{i_\lambda}.\omega_\lambda) = \left(\int_{\gamma_{1_\lambda}} \omega_\lambda : \cdots : \int_{\gamma_{22_\lambda}} \omega_\lambda\right) \tag{3.1}$$

$$(B = \mathbb{C} \backslash \{0, -4, 12, \infty\})$$

But if the cohomology is not trivial, this is not well-defined, since you could start at λ, with $\gamma_{1_\lambda}, \ldots, \gamma_{22_\lambda}$, and moving round a loop α in B get back to a different integral basis, $\gamma'_{1_\lambda}, \ldots, \gamma'_{22_\lambda}$, (see Figure 3.1). To get a well-defined map, we must quotient out by the group determined by the action of these

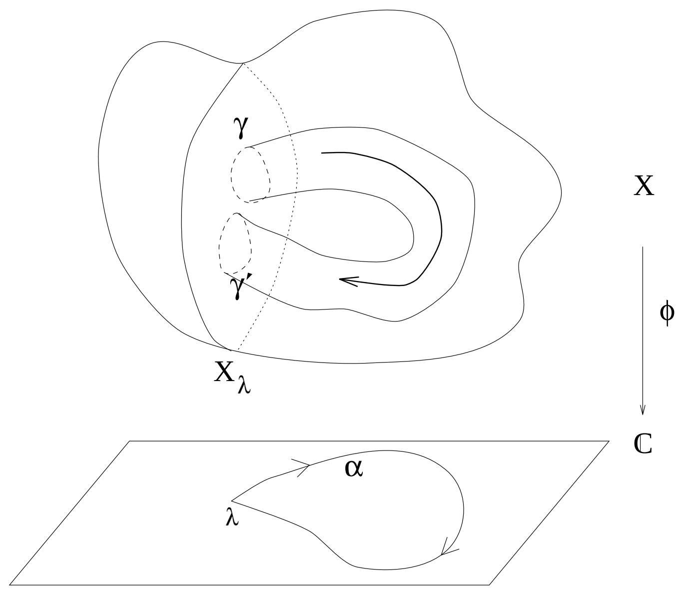


Figure 3.1: Illustration of monodromy.

loops on the image. Thus, a loop α in B gives rise to an automorphism of $H_2(Z_\lambda, \mathbb{Z})$, so we have a map

$$\pi_1(B) \to Aut(\mathbb{P}H_2(X_\lambda, \mathbb{Z}))$$

The image of this map, Γ_{mon}, is the **monodromy group**.

The **Picard-Fuchs equation** is a differential equation, with the same monodromy. (See [P] page 599 §6 for more details of Picard-Fuchs equations.)

The (well-defined) map

$$\varphi_m : B \to \mathbb{P}^{21}/\Gamma_{mon}$$

is the **period map**.

3.1 The monodromy group is isomorphic to a subgroup of $\Gamma_0(3)+$

First it will be shown that Γ_{mon} is isomorphic to a subgroup of $SO(T,\mathbb{Z})$, then an injective map $j : \overline{SL(2,\mathbb{C})} \to SL(3,\mathbb{C})$ is constructed, the image of which is $SO(T \otimes \mathbb{C})$, and $j^{-1}SO(T,\mathbb{Z}) = \overline{\Gamma_0(3)+}$.

Here, T is the transcendental lattice, with basis chosen so that the intersection matrix M, is as in Proposition 7, so that

$$SO(T) = \{A \in SL(3,\mathbb{R}) | A^T M A = M\},$$

and $SO(T,\mathbb{Z}) = SO(T) \cap SL(3,\mathbb{Z})$.

Proposition 9 *The monodromy group Γ_{mon} is isomorphic to a subgroup Γ of $SO(T,\mathbb{Z})$. The period map reduces to*

$$\varphi_\Gamma : B \to \mathbb{P}^2/\Gamma,$$

in the sense that φ_Γ is the restriction of φ_m to the first three components, and there is an isomorphism θ such that the following diagram commutes:

$$\begin{tikzcd} & \mathbb{P}^{21}/\Gamma_{mon} \arrow[dd, "\theta"] \\ B \arrow[ur, "\varphi_m"] \arrow[dr, "\varphi_\Gamma"'] & \\ & \mathbb{P}^2/\Gamma \end{tikzcd}$$

Proof. First the restriction from $\mathbb{P}^{21}$ to $\mathbb{P}^2$ is because $\int_{\gamma_i} \omega = 0$ for $i > 3$, so Γ is isomorphic to a subgroup of $SL_3(\mathbb{R})$.

Now, $\gamma \in \Gamma$ takes an integral basis to an integral basis since moving round a loop on B takes curves to other curves, so $\Gamma \in SL(T, \mathbb{Z})$. The inner product is preserved, since everything varies smoothly, and the intersections are in $\mathbb{Z}$ and hence constant. Hence $\Gamma \subseteq SO(T, \mathbb{Z})$.

$\blacksquare$

Proposition 10 *If $\omega \in H^{2,0}$, with $\omega = a\gamma_1^* + b\gamma_2^* + c\gamma_3^*$, with a, b, c functions of λ, then a, b, c satisfy*

$$ab + 3c^2 = 0$$

and

$$\Im(b(\lambda)/c(\lambda)) \neq 0 \ \forall \lambda \in B.$$

Proof. Suppose $\omega = a\gamma_1^* + b\gamma_2^* + c\gamma_3^*$ then $\omega \wedge \omega = 0$ (as it will be in $H^{4,0}$, which is zero), so

$$(a, b, c) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0.$$

Hence $ab + 3c^2 = 0$.

But $\omega \wedge \overline{\omega} > 0$, so

$$(a, b, c) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \overline{a} \\ \overline{b} \\ \overline{c} \end{pmatrix} > 0$$

so

$$\overline{a}b + a\overline{b} + 6\overline{c}c = -3b\overline{b}(i(\frac{c}{b} - \frac{\overline{c}}{\overline{b}}))^2 > 0.$$

$$-3b\overline{b}(2\Im(\frac{c}{b}))^2 = 12|b|^2 \left(\Im(\frac{c}{b})\right)^2 > 0.$$

Hence result.

$\blacksquare$

Proposition 11 *Γ is isomorphic to a subgroup Δ of $\overline{SL_2(\mathbb{R})}$. There is an isomorphism ϑ such that the following diagram commutes:*

$$\begin{tikzcd} & \mathbb{P}^{21}/\Gamma_{mon} \arrow[dd, "\vartheta"] \\ B \arrow[ur, "\varphi_m"] \arrow[dr, "\varphi_\Delta"'] & \\ & \mathcal{H}/\Delta \end{tikzcd}$$

Proof. By the above result, a is determined by b, c, and the map φ_m induces

$$\tilde{\varphi}' : B \to \mathbb{P}^1/\Gamma|_{\mathbb{P}^1}$$

$$\lambda \mapsto (c : b) = (\int_{\gamma_3} \omega : \int_{\gamma_2} \omega)$$

By Proposition 10, $\Im(\frac{c}{b}) \neq 0$, so by changing γ_3 to $-\gamma_3$ if necessary, this map becomes

$$\varphi_\Delta : B \to \mathcal{H}/\Delta, \tag{3.2}$$

where Δ is some discrete subgroup of $\overline{SL_2(\mathbb{R})}$, and φ_Δ is the quotient of two integrals, $\int_{\gamma_{\lambda_3}} \omega_\lambda$ and $\int_{\gamma_{\lambda_2}} \omega_\lambda$.

■

Γ and Δ are related as follows.

Proposition 12 *There is an isomorphism*

$$j : \overline{SL(2,\mathbb{C})} \to SO(T \otimes \mathbb{C})$$

given by

$$j : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} d^2 & -\frac{1}{3}c^2 & 2dc \\ -3b^2 & a^2 & -6ba \\ db & -\frac{1}{3}ca & ad+cb \end{bmatrix}$$

Proof.

If $\tilde{A}$ is in $SO(T \otimes \mathbb{C})$, then $\tilde{A}^T M \tilde{A} = M$, where M is the intersection matrix of T, as in Proposition 7. In particular, if $\boldsymbol{x}^T M \boldsymbol{x} = 0$, then $\boldsymbol{x}^T \tilde{A}^T M \tilde{A} \boldsymbol{x} = 0$.

There is an isomorphism

$$\mathbb{P}^1 \to \left\{ \boldsymbol{x} = (x:y:z) \in \mathbb{P}^2 \,\middle|\, \boldsymbol{x}^T M \boldsymbol{x} = xy + 3z^2 = 0 \right\},$$

$$(\xi : \zeta) \mapsto (\zeta^2 : -3\xi^2 : \zeta\xi).$$

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \overline{SL_2(\mathbb{Z})},$$

so under A

$$\begin{bmatrix} \xi \\ \zeta \end{bmatrix} \mapsto \begin{bmatrix} a\xi + b\zeta \\ c\xi + d\zeta \end{bmatrix},$$

and

$$\begin{bmatrix} \zeta^2 \\ -3\xi^2 \\ \zeta\xi \end{bmatrix} \mapsto \begin{bmatrix} (c\xi + d\zeta)^2 \\ -3(a\xi + b\zeta)^2 \\ (a\xi + b\zeta)(c\xi + d\zeta) \end{bmatrix}$$

$$= \begin{bmatrix} d^2 & -\frac{1}{3}c^2 & 2dc \\ -3b^2 & a^2 & -6ba \\ db & -\frac{1}{3}ca & ad + cb \end{bmatrix} \begin{bmatrix} \zeta^2 \\ -3\xi^2 \\ \zeta\xi \end{bmatrix}.$$

j is clearly injective, and maps to $SL_3(\mathbb{C})$, since $det(j(A)) = det(A)^3$. If a linear map preserving the intersection form of T is induced by an element of $SL_2(\mathbb{C})$, then it must necessarily be of this form. By computing $j(A)^T M j(A)$, we see that we have sufficiency, so j is an injection from $\overline{SL(2,\mathbb{C})}$ to $SO(T \otimes \mathbb{C})$.

We just need to check that j is surjective. This can be done by taking any $\tilde{A} \in SO(T \otimes \mathbb{C})$, and looking at $\tilde{A}^T M \tilde{A} = M$.

■

Lemma 4

$$j^{-1}\Big(SO(T, \mathbb{Z})\Big) = \overline{\Gamma_0(3)^+},$$

where

$$\Gamma_0(3)^+ = \left\{ \begin{pmatrix} a & b \\ 3c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, a, b, c, d \in \mathbb{Z} \right\}$$

$$\bigcup \left\{ \sqrt{3} \begin{pmatrix} a & b/3 \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \,\middle|\, a, b, c, d \in \mathbb{Z} \right\}$$

Proof. Take $A \in SL_2(\mathbb{C})$, $\tilde{A} = j(A)$. $a^2, ac \in \mathbb{Z} \Rightarrow a/c \in \mathbb{Q}$. Similarly for the other pairs, so the ratios of a, b, c, d are rational, so $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = R \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, $R \in \mathbb{R}$, $a', b', c', d' \in \mathbb{Z}$, with no common factor (this would be swallowed in R). $a^2 = R^2 a'^2 \in \mathbb{Z} \Rightarrow R^2 \in \mathbb{Q}$. Say $R^2 = \frac{p}{q}, p, q \in \mathbb{Z}$, $(p,q) = 1$, so

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sqrt{\frac{p}{q}} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \mapsto$$

$$\begin{pmatrix} \frac{p}{q}d'^2 & -\frac{p}{q}\frac{1}{3}c'^2 & \frac{p}{q}2d'c' \\ -\frac{p}{q}3b'^2 & \frac{p}{q}a'^2 & -\frac{p}{q}6b'a' \\ \frac{p}{q}d'b' & -\frac{p}{q}\frac{1}{3}c'a' & \frac{p}{q}(d'a' + c'b') \end{pmatrix} = \tilde{A} \in SO_3(T, \mathbb{Z}).$$

So, since all the entries of $\tilde{A}$ are in $\mathbb{Z}$ the only possibility for $\frac{p}{q}$ is $\frac{p}{q} = p3^r$ for some $r \in \mathbb{Z}$. But 3 does not divide all of a', b', c', d', so $r \geq -1$.

Case 1: $A = \sqrt{\frac{p}{3}} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, $3 \not| p$

Case 2: $A = \sqrt{p} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$

Now, $(Det(A))^3 = Det(\tilde{A}) = 1$, and $Det(A) \in \mathbb{R}$ so $Det(A) = 1$. Hence in each case $p = 1$.

Case 1: now $a = \frac{1}{3}a'^2, c = \frac{1}{3}c'^2, d = \frac{1}{3}d'^2$, are entries of $\tilde{A}$ and hence in $\mathbb{Z}$, so $A = \sqrt{\frac{1}{3}} \begin{pmatrix} 3\alpha & \beta \\ 3\gamma & 3\delta \end{pmatrix} \in SL_2(\mathbb{R}), \alpha, \beta, \gamma, \delta \in \mathbb{Z}$.

Case 2: $c^2 = \frac{1}{3}c'^2 \in \mathbb{Z}$, so $3|c$, so $A = \begin{pmatrix} \alpha & \beta \\ 3\gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$.

■

Proposition 13 *The monodromy group is isomorphic to a subgroup of* $\Gamma_0(3)+$

Proof. This follows from the above lemmas of this section.

■

When we have found the Picard-Fuchs equation we will obtain enough information to find the monodromy group explicitly.

3.2 The Picard-Fuchs equation

3.2.1 The existence of the Picard-Fuchs equation

Proposition 14 *The Picard-Fuchs equation is a third order linear differential equation.*

Proof. We have $H^2_{DR}(X_\lambda, \mathbb{C}) \cong H^2(X_\lambda, \mathbb{C}) \cong \mathbb{C}^{22}$, and

$$H^2(X_\lambda, \mathbb{C}) \supseteq Pic(X_\lambda) \otimes \mathbb{C} \cong \mathbb{C}^{19}.$$

So if ω_λ is a 2-form on X_λ, then

$$[\omega_\lambda], [\partial\omega_\lambda/\partial\lambda], [\partial^2\omega_\lambda/\partial\lambda^2], [\partial^3\omega_\lambda/\partial\lambda^3] \in H^2_{DR}/Pic(X_\lambda) \otimes \mathbb{C}$$

must satisfy a linear relation in $H^2_{DR}/Pic(X_\lambda) \otimes \mathbb{C}$, since this has dimension 22-19=3. So for some C^∞ functions of λ, $f_0, \dots, f_3$, we have

$$f_0\omega_\lambda + \cdots + f_3\partial^3\omega_\lambda/\partial\lambda^3 \in Pic(X_\lambda) \otimes \mathbb{C}.$$

Integrating round a cycle in $Pic(X_\lambda)$ gives zero, this relation becomes a third order differential equation for $\int_\gamma \omega_\lambda$, a period, via differentiating under the integral. (See [GH] for more about DeRham cohomology.) Hence we have a third order differential equation for the periods.

∎

The periods are the solutions of the Picard-Fuchs equation. Now I find a series expansion for one of the periods. From this I can find a differential equation satisfied by this period.

Proposition 15 *There is a period of X_λ of the form*

$$I(\lambda) := \frac{1}{(2\pi i)^3}\int_{|x|=|y|=|z|=1}\frac{\Omega}{\phi-\lambda},$$

where Ω is the 3-form on $\tilde{X}$ lifted from the 3-form $\frac{dx}{x}\wedge\frac{dy}{y}\wedge\frac{dz}{z}$ on X.

Proof. By the Poincaré residue theorem, (see [GH]),

$$I(\lambda) = \frac{1}{(2\pi i)^2}\int_{\Gamma_\lambda} Res_{Z_\lambda}\left(\frac{\Omega}{\phi-\lambda}\right)$$

So $I(\lambda)$ is a period.

∎

Proposition 16 *If $\nu = \frac{1}{\lambda+4}$, then there is a power series expansion for $I(\lambda)$ given by*

$$-\sum \nu^{n+1}a_n,$$

where

$$a_n = \sum_{p+q+r+s=n}\binom{n}{p\,q\,r\,s}^2.$$

Proof. We have

$$I(\lambda) = \frac{1}{(2\pi i)^3} \int_{|x|=|y|=|z|=1} \frac{dx \wedge dy \wedge dz}{xyz((\phi+4)-(\lambda+4))}.$$

Now write this in terms of the X_is; recall,

$$x = X_1/X_2, y = X_2/X_3, z = X_3/X_4,$$

$$\phi + 4 = (X_1 + X_2 + X_3 + X_4)(X_1^{-1} + X_2^{-1} + X_3^{-1} + X_4^{-1}),$$

so

$$\Omega \wedge \frac{dX_4}{X_4} = \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2} \wedge \frac{dX_3}{X_3} \wedge \frac{dX_4}{X_4},$$

and

$$I(\mu) =$$

$$\frac{1}{(2\pi i)^4} \int_{|X_i|=1, 1\leq i\leq 4} \frac{dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4}{X_1X_2X_3X_4(X_1+X_2+X_3+X_4)(X_1^{-1}+X_2^{-1}+X_3^{-1}+X_4^{-1}) - \mu},$$

where $\mu = \lambda + 4$. If $\nu = 1/\mu$ then

$$-(2\pi i)^4 I(\nu) =$$

$$\nu \int \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2} \wedge \frac{dX_3}{X_3} \wedge \frac{dX_4}{X_4} \sum_{n\geq 0} \nu^n (X_1+X_2+X_3+X_4)^n (X_1^{-1}+X_2^{-1}+X_3^{-1}+X_4^{-1})^n$$

$$= -\sum \nu^{n+1} a_n,$$

where a_n =constant term in $(X_1+X_2+X_3+X_4)^n(X_1^{-1}+X_2^{-1}+X_3^{-1}+X_4^{-1})^n$.

So

$$a_n = \sum_{p+q+r+s=n} \binom{n}{p\,q\,r\,s}^2 . \blacksquare$$

The first few terms of this sequence are

$$1, 4, 28, 256, 2716, 31504, 387136, \ldots$$

To find a differential equation satisfied by $I(\lambda)$, we just have to find a recurrence relation for the a_ns. (Proposition 24, page 78).

3.2.2 A recurrence relation

Proposition 17 *If*

$$a_n = \sum_{p+q+r+s=n} \binom{n}{p\,q\,r\,s}^2$$

then there is a recurrence relation

$$n^3 a_n = 2(2n-1)(5n^2-5n+2)a_{n-1} - 64(n-1)^3 a_{n-2}.$$

Proof. In the sum, $n = p+q+r+s$, so $n^m a_n$ can be expressed in terms of sums of the form

$$\sum_{p+q+r+s=n} \binom{n}{p\,q\,r\,s} \mathcal{M}(p,q,r,s),$$

where $\mathcal{M}$ is a monomial. We have

$$na_n = \sum \binom{n}{p\,q\,r\,s}^2 (p+q+r+s) = 4\sum \binom{n}{p\,q\,r\,s}^2 p$$

$$n^2 a_n = \sum \binom{n}{p\,q\,r\,s}^2 (p+q+r+s)^2 = \sum \binom{n}{p\,q\,r\,s}^2 (4p^2+12pq)$$

$$n^3 a_n = \sum \binom{n}{p\,q\,r\,s}^2 (p+q+r+s)^3 = 4\sum \binom{n}{p\,q\,r\,s}^2 (4p^3+36p^2q+24pqr)$$

$$n^5 a_n = \sum \binom{n}{p\,q\,r\,s}^2 (p+q+r+s)^5$$

$$= 4\sum \binom{n}{p\,q\,r\,s}^2 (4p^5 + 60p^4q + 120p^3q^2 + 240p^3qr + 240p^2qpr + 360p^2q^2r).$$

We take four of the monomial sums, (including a_n) and express the remainder in terms of them. Thus we define

$$b_n = \sum_{p+q+r+s=n} \binom{n}{p\,q\,r\,s}^2 p$$

$$c_n = \sum_{p+q+r+s=n} \binom{n}{p\,q\,r\,s}^2 pq$$

$$d_n = \sum_{p+q+r+s=n} \binom{n}{p\,q\,r\,s}^2 pqr$$

Then we have

$$\sum_{p+q+r+s=n} \binom{n}{p\,q\,r\,s}^2 p^2 = \sum_{p=1}^{n} \sum_{q+r+s=n-p} \binom{n}{p\,q\,r\,s}^2 p^2$$

$$= \sum_{p=1}^{n} \sum_{q+r+s=n-p} \left(\frac{n!}{(p-1)!q!r!s!}\right)^2$$

$$= \sum_{p=0}^{n-1} \sum_{q+r+s=n-1-p} \binom{n-1}{p\,q\,r\,s}^2 n^2$$

$$= n^2 a_{n-1}$$

In a similar way (i.e., cancelling squares in the monomial $\mathcal{M}$ with squares in the denominator of $\binom{n}{p-1\,q\,r\,s}^2$), we have the following:

$$\sum_{p+q+r+s=n} \binom{n}{p\,q\,r\,s}^2 p^2 r = n^2 b_{n-1}$$

$$\sum_{p+q+r+s=n} \binom{n}{p\,q\,r\,s}^2 p^2qrs = n^2 d_{n-1}$$

$$\sum_{p+q+r+s=n} \binom{n}{p\,q\,r\,s}^2 p^2q^2r = n^2 \sum \binom{n-1}{p\,q\,r\,s} q^2r = n^2(n-1)^2 b_{n-1}$$

$$\sum_{p+q+r+s=n} \binom{n}{p\,q\,r\,s}^2 p^3 = n^2 \sum \binom{n-1}{p\,q\,r\,s}(p+1) = n^2(b_{n-1}+a_{n-1})$$

$$\sum_{p+q+r+s=n} \binom{n}{p\,q\,r\,s}^2 p^3rs = n^2 \sum \binom{n-1}{p\,q\,r\,s}(p+1)rs = n^2(c_{n-1}+d_{n-1})$$

$$\sum_{p+q+r+s=n} \binom{n}{p\,q\,r\,s}^2 p^3q^2 = n^2 \sum \binom{n-1}{p\,q\,r\,s}(p+1)q^2 = n^2(n-1)^2(b_{n-2}+a_{n-2})$$

$$\sum_{p+q+r+s=n} \binom{n}{p\,q\,r\,s}^2 p^4q = n^2 \sum \binom{n-1}{p\,q\,r\,s}(p+1)^2q = n^2((n-1)^2b_{n-2}+2c_{n-1}+b_{n-1})$$

$$\sum_{p+q+r+s=n} \binom{n}{p\,q\,r\,s}^2 p^5 = n^2 \sum \binom{n-1}{p\,q\,r\,s}(p+1)^3$$
$$= n^2((n-1)^2(b_{n-2}+a_{n-2}) + 3(n-1)^2a_{n-2} + 3b_{n-1} + a_{n-1})$$

substituting the second set of expressions into the first, the $n^i a_n$s ($i = 1,2,3,5$), are expressed in terms of the a_ns, b_ns c_ns, and d_ns. as follows:

$$na_n = 4b_n$$

$$n^2a_n = 4n^2a_{n-1} + 12c_n$$

$$n^3a_n = 40n^2b_{n-1} + 4n^2a_{n-1} + 24d_n$$

$$n^5a_n = n^2(4a_{n-1}+136(n-1)^2a_{n-2}+72b_{n-1}+(n-1)^2554b_{n-2}+360c_{n-1}+480d_{n-1})$$

So, dividing the last by n^2, and using the first three to eliminate the b_ns, c_ns, and d_ns, we obtain the result.

This method works because in the four expansions, of $n = (p+q+r+q), n^2, n^3$, and n^5 in terms of p, q, r, s, there are only three square free monomials (up to permutation of p, q, r, s).

■

3.2.3 A solution to the Picard-Fuchs equation

Proposition 18 *$I(\lambda)$ (as in Proposition 16) satisfies*

$$\mathcal{F}(I(\lambda)) = 0,$$

where

$$\mathcal{F} := \lambda(\lambda+4)(\lambda-12)\frac{d^3}{d\lambda^3}+6(\lambda^2-7\lambda-12)\frac{d^2}{d\lambda^2}+\frac{(7\lambda^2-12\lambda-96)}{(\lambda+4)}\frac{d}{d\lambda}+\frac{\lambda}{\lambda+4}.$$

Proof. From the recurrence relation of Proposition 17, $I(\nu)$ satisfies the differential equation

$$(\Theta-1)^3 - \nu 2(2\Theta-1)(5\Theta^2-5\Theta+2)+\nu^2 64\Theta^3,$$

where

$$\Theta = \nu\frac{d}{d\nu}$$

(see Proposition 24, page 78). After a change of variables, we see that $I(\lambda)$ satisfies the above differential equation, as stated.

■

3.2.4 The Picard-Fuchs equation

Now we show that the differential equation $\mathcal{F}$ of Proposition 18 is the Picard-Fuchs equation. Here I follow [B,P] pages 46 to 48. The argument is the same, but I rewrite the details below.

Lemma 5 *i) $\mathcal{F}$ is irreducible.*

ii) The $\mathbb{C}$-linear space spanned by all branches obtained by analytic continuation of a non-trivial solution of $\mathcal{F}y = 0$ has dimension 3.

Proof. i) This proved in exactly the same way as the lemma on page 46 of [B,P]. If we examine $\mathcal{F}$, we find that it is a Fuchsian differential equation, even with the same set of triples of local exponents as L in [B,P]. In our case, the local exponents are $(0,0,0)$ at $\lambda = -4$, $(0, -\frac{1}{2}, 1)$ at 0 and 12, and $(1,1,1)$ at ∞. (Later on relationships are found between $\mathcal{F}$ and other differential equations which make this less surprising.)

ii) This follows from (i), as in [B,P].

■

Corollary 1 *The Picard-Fuchs equation for the family of K3 surfaces*

$$\phi : X \to B$$

is given by

$$\mathcal{F} := \lambda(\lambda+4)(\lambda-12)y''' + 6(\lambda^2-7\lambda-12)y'' + \frac{(7\lambda^2-12\lambda-96)}{(\lambda+4)}y' + \frac{\lambda}{\lambda+4}y = 0.$$

Proof. This follows from the above lemmas, as for [B,P] Corollary 1.

∎

If Λ is a differential equation, then its symmetric square, $S^2\Lambda$ is a differential equation whose solution space consists of the products of solutions of Λ.

Proposition 19 *if*

$$\Lambda = \frac{d^2}{dz^2} + P\frac{d}{dz} + Q$$

then

$$S^2\Lambda = \frac{d^3}{dz^3} + 3P\frac{d^2}{dz^2} + (4Q + 2P^2 + P')\frac{d}{dz} + (4PQ + 2Q')$$

Proof. [P] page 602.

∎

Proposition 20 *The Picard-Fuchs equation $\mathcal{F}$ (as in Theorem 18) is the symmetric square of the following second order differential equation:*

$$\mathcal{D} = \lambda(\lambda+4)(\lambda-12)y'' + 2(\lambda^2 - 7\lambda - 12)y' + \frac{1}{4}(\lambda-4)y$$

Proof. As explained before (Proposition 10), there is a quadratic relation between a certain choice of roots of this equation, $ab + 3c^2 = 0$, so we can find ζ, ξ with

$$\zeta^2 = a$$

$$-3\xi^2 = b$$

$$\zeta\xi = c$$

Now we can find a differential equation with roots ζ and ξ, and the Picard-Fuchs equation will be its symmetric square.

Working backwards from Proposition 19, we find that the Picard-Fuchs equation is the symmetric square of

$$\mathcal{D} = \lambda(\lambda+4)(\lambda-12)y'' + 2(\lambda^2 - 7\lambda - 12)y' + \frac{1}{4}(\lambda - 4)y$$

The ratio of the roots, $\zeta/\xi\,(= -3c/b)$ gives the period map, $\varphi_\Delta : B \to \mathcal{H}/\Delta$.

■

3.3 The monodromy group.

In this section the solutions of the Picard-Fuchs equation are examined and the monodromy group is hence found.

$\Gamma \subseteq SO(\mathbb{P}(T), \mathbb{Z})$ is the image of $\pi_1(B)$ in $\mathrm{Aut}(T)$, or equivalently in $Aut(Sol(\mathbb{P}(\mathcal{F})))$;

$\Delta \subseteq PGL_2(\mathbb{R})$ is the image of $\pi_1(B)$ in $Aut(Sol(\mathbb{P}(\mathcal{D})))$.

Proposition 21 *The map $\varphi_\Delta : B \to \mathcal{H}/\Delta$ (Equation (3.2), page 37) is given by $\frac{\xi}{\zeta}$, where ξ, ζ is a certain basis for the space of solutions to the differential equation*

$$\lambda(\lambda+4)(\lambda-12)y'' + 2(\lambda^2 - 7\lambda - 12)y' + \frac{1}{4}(\lambda - 4)y = 0.$$

φ_Δ can be extended to

$$\varphi^*_\Delta : \mathbb{P}^1 \to \mathcal{H}^*/\Delta.$$

The singularities of the equation, $0, -4, 12$, *and* ∞ *will be mapped to elliptic fixed points or to cusps by* φ^*_Δ *(see Table 3.1).*

Proof. To see what happens we look at the local solutions at the singular points. For example, at $\lambda = 0$, try the local solution $y = \sum_{n \geq 0} a_n \lambda^{n+r}$. Substituting this in the differential equation, and looking at the coefficient of the lowest power of λ gives the indicial equation for r, which will be a quadratic, as generally we expect to get two independent solutions this way. If the differential equation is

$$(x-a)^2 P(x) y'' + (x-a) Q(x) y' + R(x) y,$$

then the indicial equation at a is

$$r(r-1)P(a) + rQ(a) + R(a) = 0.$$

If r_1, r_2 are solutions, then locally φ_Δ at $\lambda = 0$ will look like

$$\frac{a\lambda^{r_1} \sum a_n \lambda^n + b\lambda^{r_2} \sum a'_n \lambda^n}{c\lambda^{r_2} \sum a'_n \lambda^n + d\lambda^{r_1} \sum a_n \lambda^n} = h(\lambda^{r_1 - r_2}),$$

where h is some holomorphic function. So if $r_1 - r_2 \in \mathbb{Z}$, then either λ is mapped to a cusp, or this is only an apparent singularity; but when $r_1 = r_2$ then there is another solution involving logarithms ([I]§15.31 page 366), so

λ	roots of indicial equation	image of λ
0	$0, \frac{1}{2}$	elliptic point, order 2
-4	0,0	cusp
12	$0, \frac{1}{2}$	elliptic point, order 2
∞	$\frac{1}{2}$, $\frac{1}{2}$	cusp

Table 3.1: Orders of monodromy at singularities

the singularity is not apparent; the universal cover will have infinite order here, and so the point must correspond to a cusp. If $r_1 - r_2 = p/q, p, q \in \mathbb{Z}, (p, q) = 1, g \geq 1$, then λ will be mapped to an elliptic point of order q. (For more details about the indicial equations, see [I].)

The results of inspection of the equation are given in Table 3.1 This gives the result.

■

Note that taking a different basis, $a\xi + b\zeta, c\xi + d\zeta$, corresponds to conjugating the group Δ by the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By Lemma 4, the new basis will also correspond to a basis of the transcendental lattice $T64$, with intersection matrix as in Proposition 7, if and only if $A \in \Gamma_0(3)+$.

There is a formula for the 'area' of the fundamental domain of subgroups in $SL_2(\mathbb{Z})$, which will also give an area of our subgroups :

Proposition 22

$$\frac{1}{2\pi}\int_{\mathcal{H}/\Delta} y^{-2}dxdy = 2g-2+m+\sum_{v=1}^{r}(1-1/e_v)$$

where g is the genus of $\mathcal{H}/\Delta$, m is the number of inequivalent cusps, and $e_1,\ldots,e_r$ are the orders of the inequivalent elliptic points of Δ.

Proof. [Shimura], page 40-44. ■

Proposition 23 *Δ has 2 cusps and 2 elliptic points of order 2, and*

$$[\Gamma_0(3)^+ : \Delta] = 3.$$

Proof. From Lemma 4, $\Delta \subseteq \Gamma_0(3)^+$.

There is a map

$$\varphi^*_\Delta : \mathbb{P}^1 \to \mathcal{H}^*/\Delta,$$

so the genus g of $\mathcal{H}^*/\Delta$ is 0 (Riemann-Hurwitz). We do not know what the order of φ^*_Δ is, so we can as yet only say that Δ has one or two cusps, and one of two elliptic points, since we may have $\varphi^*_\Delta(-12) = \varphi^*_\Delta(0)$ or $\varphi^*_\Delta(-4) = \varphi^*_\Delta(\infty)$.

From Proposition 22, we have the following possibilities:

for $\mathcal{H}/\Delta$

g	0	0	0	0
m	2	2	1	1
r	2	1	2	1
e_1	2	2	2	2
e_2	2	–	2	–
area($\mathcal{H}/\Delta$)	1	$\frac{1}{2}$	0	$-\frac{1}{2}$

For $\mathcal{H}/\Gamma_0(3)^+$

$$\begin{gathered} g = 0 \\ m = 1 \\ r = 2 \\ e_1 = 6 \\ e_2 = 2 \\ \text{so area}(\mathcal{H}/\Gamma_0(3)^+) = 1/3. \end{gathered}$$

This is pictured in Figure 3.4. The ratios of the areas is equal to the index of the subgroups, hence the only possibility is as stated.

■

Although the Hauptmoduls are nicer for the groups $\Gamma_0(n)$, the fundamental domains are easier to consider for $\Gamma^0(n)$, though it is easy to go between these groups by conjugating by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is why I deduce Theorem 2 via the following lemmas.

Lemma 6 $\overline{\Gamma^0(6)}$ *is generated by*

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 7 & -18 \\ 2 & -5 \end{bmatrix}.$$

Proof.

The action of $SL_2(\mathbb{Z})$ on a tree is described in [Se2], §4.2(c), page 35. So $\overline{\Gamma^0(6)}$ acts on the tree of Figure 3.2.

The fundamental domain for $\overline{\Gamma^0(6)}$ is shown in Figure 3.3. From this we find a tree ♣ of representatives of the action, and a subgraph Y, drawn in the figure. On the figure I have written the matrixes mapping extremities of edges of $Y - ♣$ into ♣. There are no stabilisers of the vertices.

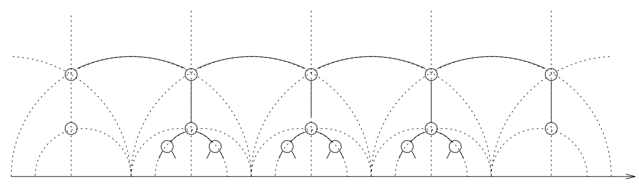


Figure 3.2: Sketch of tree on which $\overline{\Gamma^0(6)}$ acts.

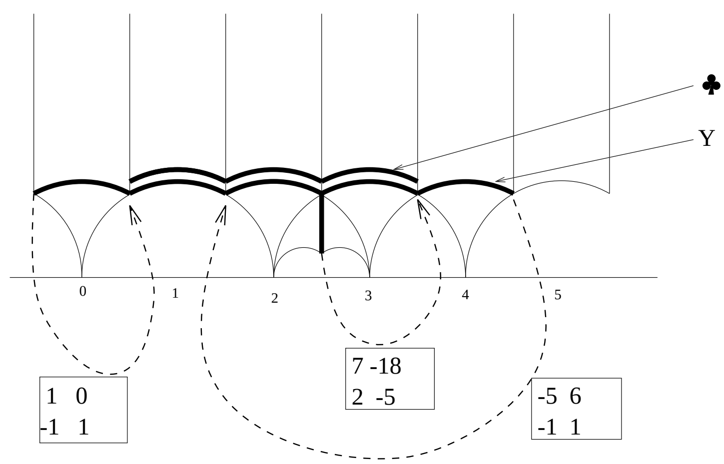


Figure 3.3: The fundamental domain, and trees for $\overline{\Gamma^0(6)}$.

Hence by applying [Se2] Lemma 4, page 34, and noting that

$$\begin{bmatrix} -5 & 6 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix},$$

we have the result.

$\blacksquare$

Lemma 7 *If* ∇ *is a subgroup of index 3 in* $\Gamma^0(3)^+$ *with 2 cusps and 2 elliptic points, then up to conjugation by an element of* $\Gamma^0(3)+$*,* ∇ *has a fundamental domain as in Figure 3.5 B.*

Proof. The fundamental domain for ∇ in $\mathcal{H}$ will be the union of 3 copies of the fundamental domain for $\Gamma^0(3)+$, corresponding to 3 coset representatives for ∇ in $\Gamma^0(3)$. I can be taken as one coset representative, and the others can be chosen so that the union is connected. (See [Rankin] Theorem 2.4.3(ii)) The possibilities for the fundamental domain are shown in Figure 3.5 up to translation. Translates of these correspond to conjugation of ∇ by an element of the form $\begin{bmatrix} 1 & 3n \\ 0 & 1 \end{bmatrix}$, $n \in \mathbb{Z}$. This is an element of $\Gamma^0(3)+$. Regions have been labelled with the corresponding coset representative.

∇ must have two cusps, so the fundamental domain can not be as Figure 3.5 A.

Figures 3.5 B and C, and D and E differ by the action of $\begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} \in \Gamma^0(3)+$.

Now we suppose that the fundamental region is given by B or D. Since ∇ must have two cusps, no element of ∇ maps 0 to ∞. Translates of the fundamental domain under ∇ cover $\mathcal{H}$. Hence there must be a non-trivial map taking the fundamental region to its translate by $z \mapsto z+6$, which maps ∞ to ∞. The only possible element of $\Gamma^0(3)+$ to achieve this is $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$. Hence if the fundamental domain is B or D, $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \in \nabla$. But this means that $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \nabla = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \nabla$. Hence up to conjugation by an element of $\Gamma^0(3)+$, a fundamental domain of ∇ is as in Figure 3.5 B.

■

Lemma 8 *If ∇ is a subgroup of index 3 in $\Gamma^0(3)^+$ with 2 cusps and 2 elliptic points, with a fundamental domain $\boldsymbol{F}$ as in Figure 3.5 B, then $\Gamma^0(6) \subseteq \nabla$.*

Proof. Because translates of $\boldsymbol{F}$ cover $\mathcal{H}$, ∇ must contain some element fixing ∞ and mapping $\frac{-3}{2}$ to $\frac{9}{2}$. The only possibility is $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$. Similarly, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in \nabla$.

∇ has two elliptic points. These must be elliptic points of $\Gamma^0(3)+$. So $3 + i\sqrt{3}$ is an elliptic point of ∇.

A calculation shows that the only non-trivial elements of $\Gamma^0(3)+$ fixing $3+i\sqrt{3}$ are $\sqrt{3}\begin{bmatrix} 1 & -4 \\ 1/3 & -1 \end{bmatrix}$ and its inverse. So these elements are in ∇, and so is

$$\begin{bmatrix} 7 & -18 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1/3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -4 \\ 1/3 & -1 \end{bmatrix}. \tag{3.3}$$

Hence by Lemma 6 we have $\Gamma^0(6) \subseteq \nabla$.

∎

Corollary 2 *Up to conjugation by an element of $\Gamma_0(3)+$,*

$$\Gamma_0(6) \subseteq \Delta,$$

with index 2.

Proof. This is just from conjugation by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ of Lemma 7 and Proposition 23.

∎

τ	stabiliser	$\lambda(\tau)$
0	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	-4
∞	$\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$	∞
$(3+i\sqrt{3})$	$\sqrt{3}\begin{pmatrix} -1 & 4 \\ \frac{1}{3} & 1 \end{pmatrix}$	0
$(3+i\sqrt{3})/2$	$\sqrt{3}\begin{pmatrix} 1 & -2 \\ \frac{2}{3} & -1 \end{pmatrix}$	12

Table 3.2: Stabilisers of the cusps and elliptic points

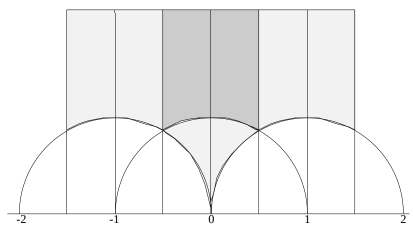
Theorem 2 *Up to conjugation by an element of* $\Gamma_0(3)+$,

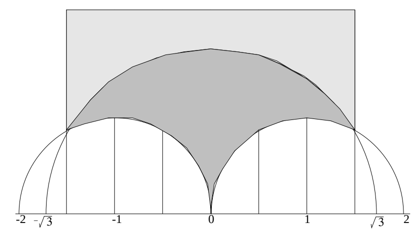
$$\Delta = \Gamma_0(6) + 3$$

Proof.

By Corollary 2 we may assume that $\Gamma_0(6) \subseteq \Delta$, and the index is two, so Δ must be in the normaliser for $\Gamma_0(6)$. The normalising quotient is the four-group, so there are 5 groups of genus zero between $\Gamma_0(6)$ and its normaliser: $\Gamma_0(6)-, \Gamma_0(6)+2, \Gamma_0(6)+3, \Gamma_0(6)+6, \Gamma_0(6)+$. The only one with the right index and the right number of cusps (listed in [C,N], table 2 page 327), is $\Gamma_0(6)+3$.

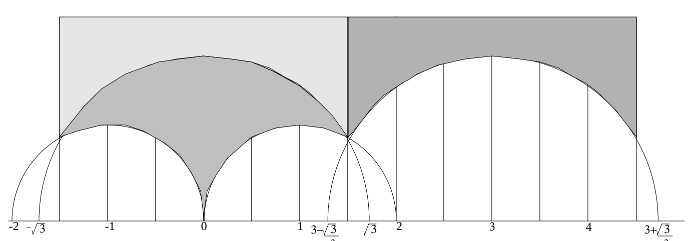
∎





a:The fundamental region for $\Gamma^0(3)$ divided into 4 fundamental regions for $SL_2(\mathbb{Z})$

b:The fundamental region for $\Gamma^0(3)$ divided into 2 fundamental regions for $\Gamma^0(3)^+$



c : Δ, consisting of 3 fundamental regions for $\Gamma^0(3)^+$

Figure 3.4: Fundamental regions

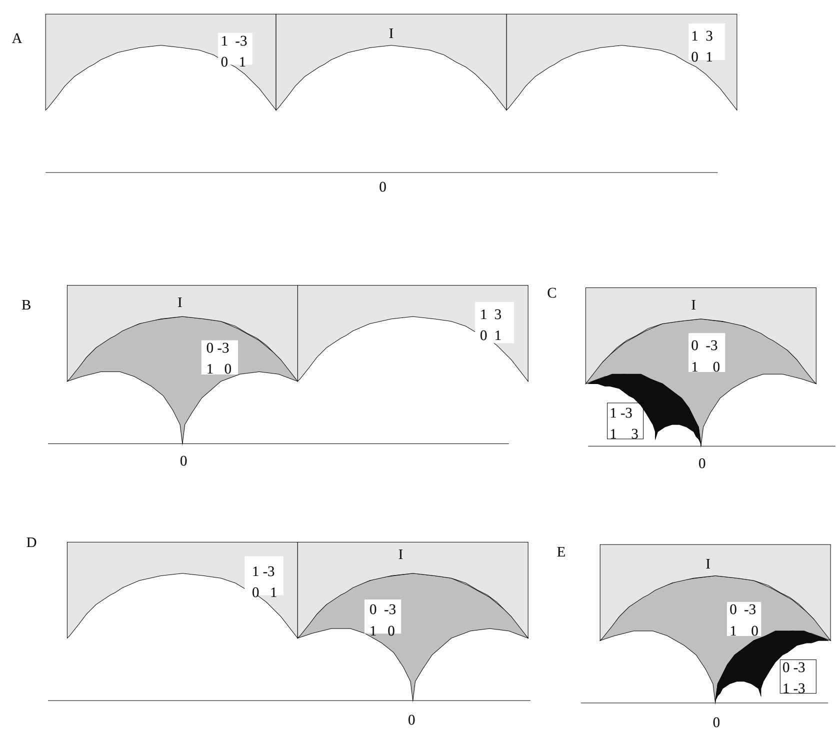


Figure 3.5: Possibilities for the fundamental domain of ∇.

3.4 Explicit solutions to the Picard-Fuchs equation

In this section I shall find the solution space for the Picard-Fuchs equation $\mathcal{F}$. This is done in Subsection 3.4.2 by making a series of transformations, starting from an equation $\mathcal{E}$, with known solutions, and finishing with $\mathcal{F}$. In the following subsection I explain the geometrical interpretation of these equations and the relationships between them. First, in Subsection 3.4.1, the solutions of $\mathcal{E}$ are found.

Throughout this section $s(\tau), V(\tau)$ and $\lambda(\tau)$ are defined by

$$s(\tau) = \frac{\eta(6\tau)^8\eta(\tau)^4}{\eta(2\tau)^8\eta(3\tau)^4}, \tag{3.4}$$

$$V(\tau) = \frac{\eta(2\tau)^6\eta(3\tau)}{\eta(t)^3\eta(6\tau)^2}, \tag{3.5}$$

and

$$\lambda(\tau) = -\left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6 - 4. \tag{3.6}$$

We will need to make use of the following

Lemma 9 *With λ and s as above,*

$$\lambda = -(1-3s)^2/s.$$

The values of $\lambda(\tau)$ at elliptic points and cusps are as in Table 3.2.

Proof.

Line 9 of table 3, page 331, in [C,N],

$$\text{“}6- \qquad 2^8 3^4/1^4 6^8 \qquad 6E(9:6C)\text{”},$$

means that

$$t_{6E} = \frac{\eta(2\tau)^8\eta(3\tau)^4}{\eta(6\tau)^8\eta(\tau)^4} + \text{constant1}$$

is a Hauptmodul for $\Gamma_0(6)$, and that

$$t_{6C} = t_{6E} + 9/t_{6E}, \tag{3.7}$$

where t_{6C} is a Hauptmodul for $\Gamma_0(6)+3$, given by

$$t_{6C} = \left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6 + \text{constant2}.$$

(Line 12 of Table 3, page 331, [C,N].) The constants involved can be determined by a computer calculation, expanding in terms of q, and comparing coefficients. We see that we can take

$$\text{constant1} = 0, \text{constant2} = 10.$$

Then from Equation 3.7, and the relations

$$s = 1/t_{6E}, \qquad \text{and} \qquad \lambda = -t_{6C} + 6,$$

we obtain the required relation.

The q expansion of λ is given by

$$\lambda = -q^{-1} + 2 - 15q + 32q^2 - 87q^3 + 192q^4 - 343q^5 + 672q^6 \dots$$

Here $q = e^{pii\tau/12}$, so we see $\lambda(i\infty) = \infty$. To find the value of λ at 0, we using the relations for η on page 7, and see that

$$\left(\frac{\eta(\frac{1}{\tau})\eta(3\frac{1}{\tau})}{\eta(2\frac{1}{\tau})\eta(6\frac{1}{\tau})}\right)^6 = 2\left(\frac{\eta(r)\eta(3r)}{\eta(2r)\eta(6r)}\right)^{-6},$$

where $r = -\frac{\tau}{6}$, so $\lambda(0) = -4$.

The branch points of $s(\tau)$ must map to the branch points of $\lambda(\tau)$ (i.e., cusps are mapped to cusps). The branch points of $-(1-3s)^2/s$ are at $s = \pm\frac{1}{3}$, and here $-(1-3s)^2/s = 0$ or 12. By a computer estimation we can determine which cusp takes which value.

■

3.4.1 A family of elliptic curves associated to $\Gamma_1(6)$.

In [S,B] §10 and §11, it is shown that the differential equation

$$\mathcal{E} = s(s-1)(9s-1)\frac{d^2}{ds^2} + (27s^2 - 20s + 1)\frac{d}{ds} + (9s-3),$$

is the Picard-Fuchs equation of a certain family $\mathcal{C}$ of elliptic curves,

$$xyz = s(x+y+z)(xy+yz+zx). \tag{3.8}$$

In §14, [S,B] consider B_Γ, the elliptic modular surface associated to $\Gamma \subseteq SL_2(\mathbb{Z})$ (certain conditions are put on which groups Γ are to be considered, which include the group $\Gamma_1(6)$).

B_Γ fibres as $\Phi : B_\Gamma \to \mathcal{H}^*/\Gamma$, with the fibres in Weierstrass form,

$$Y_{t(\tau)} : y^2 + a_1xy + a_3y = x^3 + a_3x^2 + a_4x + a_6,$$

where $t(\tau)$ is a uniformizing parameter for $\mathcal{H}^*/\Gamma$, and $a_i \in \mathbb{C}[t]$.

It is shown that the fibre $Y_{t(\tau)}$ has period lattice $h(\tau)(\mathbb{Z} \oplus \tau\mathbb{Z})$, where $h(\tau)$ is a modular form of weight 1 for Γ. The solution space of the associated Picard-Fuchs equation (with variable $t(\tau)$) is then given by this period lattice tensored by $\mathbb{C}$.

In §14 [S,B], it is shown that the family (3.8) is the elliptic modular family associated to $\Gamma_1(6)$. Hence the Picard-Fuchs equation, $\mathcal{E}$, has solution space

$$h(\tau)(\mathbb{C} \oplus \tau\mathbb{C}),$$

where $h(\tau) \in M_1(\Gamma_1(6))$.

3.4.2 The solutions of the Picard-Fuchs equation

Lemma 10 *The differential equation*

$$\mathcal{E}(v) = s(s-1)(9s-1)\frac{d^2v}{ds^2} + (27s^2 - 20s + 1)\frac{dv}{ds} + (9s-3)v = 0$$

has solution space

$$V(\tau)(\mathbb{C} \oplus \tau\mathbb{C})$$

Proof. [B] page 204, 205 shows that $\mathcal{E}$ has a solution

$$f(x) = \sum_{n=0}^{\infty} v_n x^n, \qquad v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}.$$

It is shown that if $s(\tau)$ is as above, then

$$f(s(\tau)) \in M_1(\Gamma_1(6)).$$

From subsection 3.4.1,

$$sol(\mathcal{E}) = h(\tau)(\mathbb{C} \oplus \tau\mathbb{C}),$$

where $h(\tau) \in M_1(\Gamma_1(6))$.

$f(s(\tau)) \in sol(\mathcal{E}) \cap M_1(\Gamma_1(6)) = h(\tau)\mathbb{C}$, (since τ is not a modular function), so we have

$$sol(\mathcal{E}) = f(s(\tau))(\mathbb{C} \oplus \tau\mathbb{C}).$$

The following lemma shows that $f(s(\tau)) = V(\tau)$, which will finish the proof.

$\blacksquare$

Lemma 11 *If $f(s(\tau))$ is as in Lemma 10, and $V(\tau)$ as in Equation 3.5, then*

$$f(s(\tau)) = V(\tau).$$

Proof. $f(s(\tau)) \in M_1(\Gamma_1(6))$ ([B] page 205), so $f^2(\tau) \in M_2(\Gamma_1(6)) = M_2(\Gamma_0(6))$.

By using Table 3.3, page 74, Lemma 6, page 54 and Equation 3.3, page 57, we see that $V^2(\tau) \in M_2(\Gamma_1(6))$.

[B] states that $M_2(\Gamma_1(6)))$ is spanned by

$$G_2(\tau) - 6G_2(6\tau), \qquad G_2(\tau) - 2G_2(2\tau), \qquad G_2(\tau) - 3G_2(3\tau),$$

where

$$G_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}.$$

The q expansions are given by

$$G_2(\tau) - 6G_2(6\tau) = -5 - 24q - 72q^2 - 96q^3 - 168q^4 - 144q^5 - 144q^6 - 192q^7 \ldots$$

$$G_2(\tau) - 2G_2(2\tau) = -1 - 24q - 24q^2 - 96q^3 - 24q^4 - 144q^5 - 96q^6 - 192q^7 \ldots$$

$$G_2(\tau) - 3G_2(3\tau) = -2 - 24q - 72q^2 - 24q^3 - 168q^4 - 144q^5 - 72q^6 - 192q^7 \ldots$$

So we see that if u and v are in $M_2(\Gamma_0(6))$ and their q-expansions agree to third order, then they must be equal.

The q expansion for $f(s(\tau))$ is

$$1 + 3q + 3q^2 + 3q^3 + 3q^4 + 3q^6 + 6q^7 + 3q^8 \ldots$$

This is the same as the q expansion of $V(\tau)$ to this order, hence the q expansions of their squares are equal to third order, so $f^2(s(\tau)) = V^2(\tau)$, and so we must have $f(s(\tau)) = V(\tau)$.

■

Lemma 12 *The differential equation*

$$\mathcal{L} = 4s(1-s)(1-9s)\frac{d^2y}{ds^2} + 8s^2(9s-5)\frac{dy}{ds} + (9s^2 - 2s + 1)y$$

has solution space

$$s^{\frac{1}{2}}(\tau)V(\tau)(\mathbb{C} \oplus \tau\mathbb{C}).$$

Proof. This is because $\mathcal{L}$ is obtained from $\mathcal{E}$ by making the change of variables $y = s^{\frac{1}{2}} v$

■

Lemma 13 *The differential equation*

$$\mathcal{D} = \lambda(\lambda+4)(\lambda-12)\frac{d^2y}{ds^2} + 2(\lambda^2 - 7\lambda - 12)\frac{dy}{ds} + \frac{1}{4}(\lambda - 4)y$$

has solution space

$$s^{\frac{1}{2}}(\tau)V(\tau)(\mathbb{C} \oplus \tau\mathbb{C})$$

Proof. This is because $\mathcal{D}$ is obtained from $\mathcal{E}$ by a change of variables

$$\lambda = -(1-3s)^2/s.$$

(See Lemma 9.)

■

Lemma 14 *The differential equation*

$$\mathcal{F} = \lambda(\lambda+4)(\lambda-12)\frac{d^3y}{ds^3} + 6(\lambda^2 - 7\lambda - 12)\frac{d^2y}{ds^2} + \frac{(7\lambda^2 - 12\lambda - 96)}{(\lambda+4)}\frac{dy}{ds} + \frac{\lambda}{\lambda+4}y,$$

has solution space

$$G(\tau)(\mathbb{C} \oplus \tau\mathbb{C} \oplus \tau^2\mathbb{C})$$

where

$$G(\tau) = \frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2}.$$

Proof.

$\mathcal{F} = S^2\mathcal{D}$ (Proposition 20), so from Lemma 13,

$$sol(\mathcal{F}) = (sol(\mathcal{D}))^2 = s(\tau)V^2(\tau)(\mathbb{C} \oplus \tau\mathbb{C} \oplus \tau^2\mathbb{C}).$$

$\blacksquare$

3.4.3 A geometrical interpretation

3.4.3.1 Relationship between $\Gamma_0(6)$ and $\Gamma_0(6) + 3$.

[C,N] gives the following Hauptmoduls:

$$\begin{array}{rclcl} s & = & \frac{\eta(6\tau)^8\eta(\tau)^4}{\eta(2\tau)^8\eta(3\tau)^4} & \text{for} & \Gamma_0(6) \\ \lambda & = & -\left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6 - 4 & \text{for} & \Gamma_0(6)+3 \end{array}$$

Since $\lambda = -(1-3s)^2/s$ (Lemma 9), we have a diagram

$$\begin{array}{ccc} \mathcal{H}^*/\Gamma_0(6) & \xrightarrow{2:1} & \mathcal{H}^*/\Gamma_0(6)+3 \\ \uparrow \psi & & \uparrow \varphi_\Delta^* \\ \mathbb{P}^1 \setminus \{0, -1, \frac{1}{9}, \infty\} & \xrightarrow{s \mapsto \frac{-(1-3s)^2}{s}} & \mathbb{P}^1 \setminus \{0, -4, 12, \infty\} \\ \text{if } s = \frac{\eta(6\tau)^8\eta(\tau)^4}{\eta(2\tau)^8\eta(3\tau)^4} & & \text{if } \lambda = -\left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6 - 4 \\ \text{then } \psi(s) = \tau & & \text{then } \varphi_\Delta^*(\lambda) = \tau \end{array}$$

In Subsection 3.4.1, we described the elliptic modular surface $B_{\Gamma_1(6)} = Y$ attached to $\Gamma_1(6)$, which we may also consider to be attached to $\Gamma_0(6)$, since

$\overline{\Gamma_0(6)} = \overline{\Gamma_1(6)}$:

$$\begin{array}{ccc} Y & \supset & Y_x \\ \Big\downarrow f & & \downarrow \\ \mathcal{H}^*/\Gamma_0(6) & \ni & x \end{array}$$

Y_x, the fibre over x is a smooth elliptic curve, except over the cusps.

We consider a second family of elliptic curves over $\Gamma_0(6)$, defined by taking the fibre over s to be $Y_{\frac{1}{9s}}$, i.e., we just make a base change. Call the Picard-Fuchs equation of this family $i^*\mathcal{E}$.

Lemma 15 *$i^*\mathcal{E}$ is given by*

$$s(s-1)(9s-1)\frac{d^2}{ds^2} + (9s^2-1)\frac{d}{ds} + \frac{(1-3s)}{s}$$

Proof. This is just obtained from $\mathcal{E}$ by the change of variables $s \to \frac{1}{9s}$.

■

Lemma 16

$$sol(i^*\mathcal{E}) = s(\tau)V(\tau)(\mathbb{C} \oplus \tau\mathbb{C})$$

where s and V are as above.

Proof. Given the equation $\mathcal{E}$, we can construct an equation $\mathcal{E}'$, such that for any solution v of $\mathcal{E}$, sv is a solution for $\mathcal{E}'$. $\mathcal{E}'$ turns out to be equal to $i^*\mathcal{E}$ defined in Lemma 15. Hence the result.

■

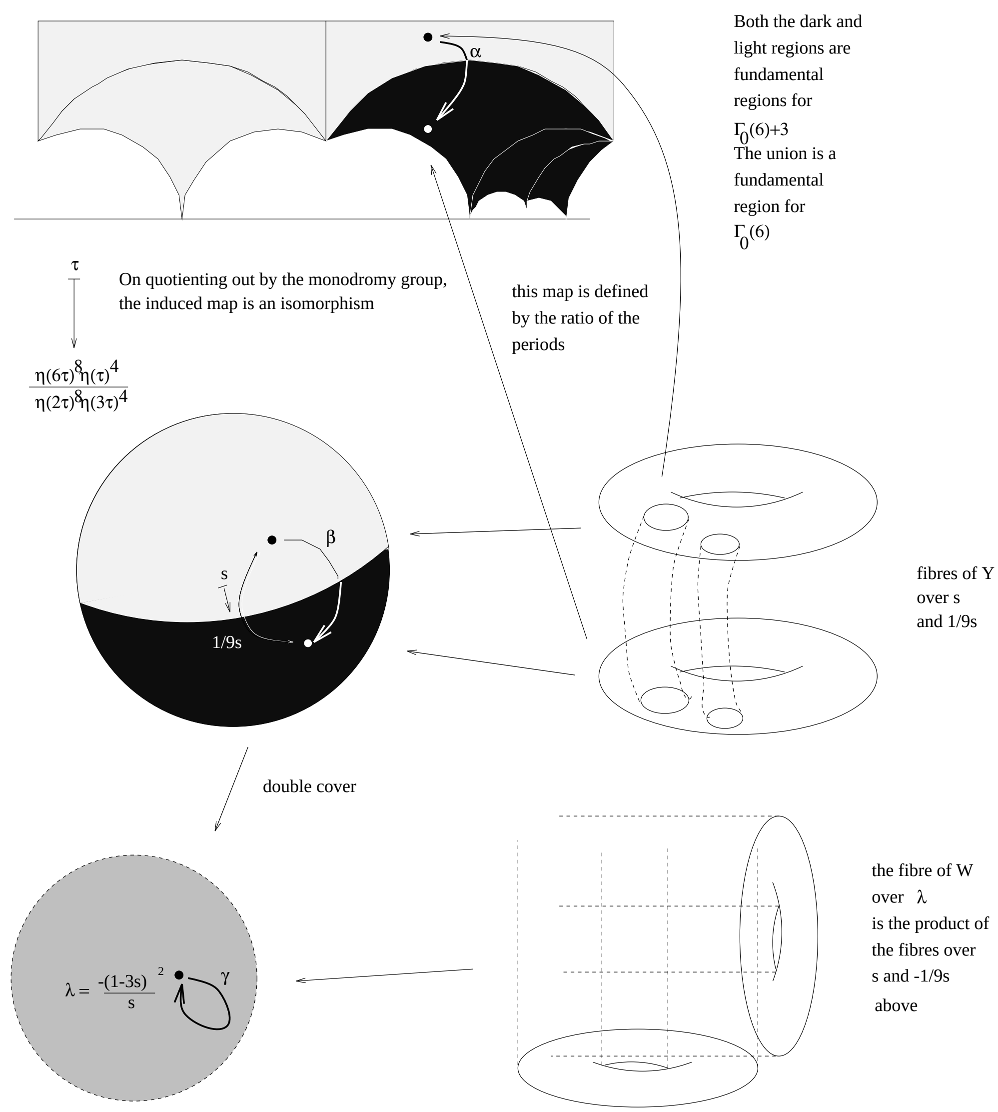


Figure 3.6: Construction of the family W

3.4.3.2 A pencil of Abelian surfaces over $\Gamma_0(6)+3$.

In this section I follow the method in [P]§5.3.3 to construct a family W of Abelian surfaces over $\mathcal{H}^*/\Gamma_0(6)+3$:

$$\begin{array}{ccc} W & \supset & W_x = Y_x \times Y_{\gamma x} \\ \big\downarrow & & \downarrow \\ \mathcal{H}^*/\Gamma_0(6)+3 & \ni & x \end{array}$$

were $f: Y \to \mathcal{H}^*/\Gamma_0(6)$ is the family of elliptic curves referred to in Subsection 3.4.1, with fibres Y_x. Figure 3.6 illustrates the construction I now describe. $f: Y \to \mathcal{H}^*/\Gamma_0(6)$ pulls back under the involution $i: \tau \mapsto \gamma\tau$ to give a family $f': Y' \to \mathcal{H}^*/\Gamma_0(6)$:

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ \mathcal{H}^*/\Gamma_0(6) \arrow[r, "i"] \arrow[d, "s"] & \mathcal{H}^*/\Gamma_0(6) \arrow[d, "s"] \\ \mathbb{P}^1 \arrow[r, "s \mapsto 1/9s"] & \mathbb{P}^1 \end{tikzcd}$$

Here i is defined by the action of the matrix $\gamma = \sqrt{3}\begin{pmatrix} -1 & -\frac{1}{3} \\ 4 & 1 \end{pmatrix}$ on $\mathcal{H}$ which becomes $s \to \frac{1}{9s}$ on $\mathbb{P}^1$. This involution commutes with the double cover map; i.e., there is a commutative diagram:

$$\begin{tikzcd} \mathbb{P}^1 \arrow[d, "s \mapsto \frac{1}{9s}"'] \arrow[dr, "s \to -\frac{(1-3s)^2}{s}"] & \\ \mathbb{P}^1 \arrow[r, "s \to -\frac{(1-3s)^2}{s}"'] & \mathbb{P}^1 \end{tikzcd}$$

We take the fibre product of f, f' over $\mathcal{H}^*/\Gamma_0(6)$ to get the family $f' \times f : A \to \mathcal{H}^*/\Gamma_0(6)$:

$$\begin{tikzcd}
 & A \arrow[dl] \arrow[dr] & \\
Y' \arrow[dr, "f'"] & & Y \arrow[dl, "f"] \\
 & \mathcal{H}^*/\Gamma_0(6) &
\end{tikzcd}$$

The fibre over $x \in \mathcal{H}^*/\Gamma_0(6)$ is the surface $Y_x \times Y_{\gamma x}$. Interchanging the factors, we have an isomorphism

$$\begin{array}{cccc}
\tilde{i}: & A & \to & A \\
\tilde{i}_x & : Y_x \times Y_{\gamma x} & \to & Y_{\gamma x} \times Y_x \\
 & (x, y) & \to & (y, x)
\end{array}$$

which lifts i:

$$\begin{array}{ccc}
A & \xrightarrow{\tilde{i}} & A \\
\downarrow{\scriptstyle f' \times f} & & \downarrow{\scriptstyle f' \times f} \\
\mathcal{H}^*/\Gamma_0(6) & \xrightarrow{i} & \mathcal{H}^*/\Gamma_0(6)
\end{array}$$

and the quotient by $\tilde{i}$ is a family $\frac{f' \times f}{\tilde{i}} : W \to \mathcal{H}^*/\Gamma_0(6) + 3$, since i is the image of $\sqrt{3}\begin{pmatrix} -1 & -\frac{1}{3} \\ 4 & 1 \end{pmatrix}$ on $\mathcal{H}$, so

$$\left(\mathcal{H}/\Gamma_0(6)\right)/i = \mathcal{H}/\left\langle \Gamma_0(6), \sqrt{3}\begin{pmatrix} -1 & -\frac{1}{3} \\ 4 & 1 \end{pmatrix} \right\rangle = \mathcal{H}/\left(\Gamma_0(6) + 3\right).$$

Lemma 17 *The monodromy of the family* $\frac{f' \times f}{\tilde{i}} : W \to \mathcal{H}^*/(\Gamma_0(6) + 3)$ *is* $\Gamma_0(6) + 3$

Proof.

The Künneth Formulas give

$$H_*(M \times N, \mathbb{Q}) \cong H_*(M, \mathbb{Q}) \otimes H_*(N, \mathbb{Q}),$$

and

$$H^*(M \times N, \Omega^*_{M \times N}) \cong H^*(M, \Omega^*_M) \otimes H^*(N, \Omega^*_N),$$

for two complex manifolds M and N. (Pages 58 and 103 to 105 of [H])

So, from this, we have that if the differential (1,0)-form on Y_s is ω_s, so that on $Y_{\frac{-1}{9s}}$ is $\omega_{\frac{-1}{9s}}$, then the differential (2,0)-form on $W_\lambda = Y_s \times Y_{\frac{-1}{9s}}$ is $p_1^*(\omega_s) \otimes p_2^*(\omega_{\frac{-1}{9s}})$, where p_1 and p_2 are the projections.

Let $g_{s1} \in H_2(Y_s, \mathbb{Z})$ and $g_{s2} \in H_2(Y_{\frac{-1}{9s}}, \mathbb{Z})$

The space of periods of W_λ is spanned by periods of the form

$$\int_{p_1^* g_{s_i} \times p_2^* g_{\frac{-1}{9s}j}} p_1^*(\omega_s) \otimes p_2^*(\omega_{\frac{-1}{9s}}) = \int_{g_{s_i}} \omega_s \times \int_{g_{\frac{-1}{9s}j}} \omega_{\frac{-1}{9s}}.$$

Let $\mathcal{G}$ be the Picard-Fuchs equation for the family $\frac{f' \times f}{i} : W \to \mathcal{H}^*/(\Gamma_0(6)+3)$ is $\Gamma_0(6)+3$.

Then from the above

$$sol(\mathcal{G}) = sol(\mathcal{E}) sol(\imath^* \mathcal{E}),$$

and G must be equal to F, and hence has the same monodromy.

■

	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\6&1\end{pmatrix}$	$\begin{pmatrix}-1&-\frac{1}{3}\\4&1\end{pmatrix}$
$\eta(\tau)$	$e^{\pi i/12}\eta(\tau)$	$\sqrt{(6\tau+1)}e^{\pi i/2}\eta(\tau)$	$-\sqrt{3(4\tau+1)}e^{-1\pi i/12}\eta(3\tau)$
$\eta(3\tau)$	$e^{3\pi i/12}\eta(3\tau)$	$\sqrt{(6\tau+1)}e^{-2\pi i/12}\eta(3\tau)$	$-\sqrt{(4\tau+1)}e^{-5\pi i/12}\eta(\tau)$
$\eta(2\tau)$	$e^{2\pi i/12}\eta(2\tau)$	$\sqrt{(6\tau+1)}e^{-3\pi i/12}\eta(2\tau)$	$\sqrt{3(4\tau+1)}e^{4\pi i/12}\eta(6\tau)$
$\eta(6\tau)$	$e^{6\pi i/12}\eta(6\tau)$	$\sqrt{(6\tau+1)}e^{-\pi i/12}\eta(6\tau)$	$-\sqrt{(2\tau+1)}e^{-2\pi i/12}\eta(2\tau)$

This table shows the action of some matrices on some functions. These matrices generate $\overline{\Gamma_0(6)+3}$ (Lemmas 6 and 7)

Table 3.3: The action of some matrices.

3.4.4 A family of Kummer surfaces.

Define a family $D \to \mathcal{H}^*/\Gamma_0(6)+3$ of Kummer surfaces as follows. Take the family of Abelian surfaces defined as $W \to \mathcal{H}^*/\Gamma_0(6)+3$ in Subsection 3.4.3.2. Choose a smoothly varying group law, so that the quotient by the involution followed by blowing up of the singularities gives a pencil of Kummer K3 surfaces.

Conjecture: The families of K3 surfaces $\phi : X_{A_3} \to \mathcal{H}^*/\Gamma_0(6)+3$ is isomorphic to the family $D \to \mathcal{H}^*/\Gamma_0(6)+3$, and so X_λ is a Kummer surface.

Justification: This should follow from the fact that the Picard-Fuchs equations of the two families are the same, using the Torelli theorem for K3 surfaces [P-S,S].

3.5 A combinatorial application

The a_n are related to something purely combinatorial. Consider a random walk on the root lattice A_3. At each stage we have equal probability of moving in any of the 12 directions $e_i - e_j$. See Figure 2.1.

Set

$$b_n = \text{ the probability of getting back to the origin in } n \text{ steps.}$$

Set

$$\Phi = (X_1 + X_2 + X_3 + X_4)(X_1^{-1} + X_2^{-1} + X_3^{-1} + X_4^{-1}).$$

Define the linear function $\deg_0$ on $\mathbb{C}[X_1, \ldots, X_4, X_1^{-1}, \ldots, X_4^{-1}]$ by $\deg_0(f) =$ constant term of f.

Lemma 18

$$12^n b_n = deg_0 \left(\left(\sum_{i \neq j} X_i X_j^{-1} \right)^n \right) = deg_0\ ((\Phi - 4)^n).$$

Proof. After n steps, the position in the lattice is determined by the sequence of directions taken:

$$e_{1_i} - e_{1_j}, e_{2_i} - e_{2_j}, \ldots, e_{n_i} - e_{n_j}$$

The final position will be

$$e_{1_i} - e_{1_j} + e_{2_i} - e_{2_j} + \cdots + e_{n_i} - e_{n_j}.$$

These correspond to sequences of $X_iX_j^{-1}$, and the sum of the $e_i - E_j$ corresponds to their product. The product represents a path which returns to the origin after n steps only if it is constant. So the total number of paths that get back to the origin after n steps is the constant term of Φ, and the total number of paths of length n is 12^n. Hence the result.

■

Lemma 19

$$(-3)^n b_n = n! \sum_{m=0}^{n} \frac{(-4)^{-m} a_m}{m!} \frac{1}{(n-m)!}$$

Proof. Recall

$$a_n = \deg_0 \Phi$$

From the above

$$\begin{array}{rcc} 12^n b_n & = & \deg_0 ((\Phi - 4)^n) \\ & = & \deg_0 ((-4)^n \sum \left(\frac{\Phi}{-4}\right)^m \binom{n}{m} \\ & = & \deg_0 ((-4)^n \sum (-4)^{-n} \binom{n}{m} \deg_0 (\Phi^n)) \end{array}$$

Hence result.

■

Let

$$d_n = \frac{(-3)^n b_n}{n!}$$

and

$$e_n = \frac{(-4)^{-m} a_m}{m!}$$

so that

$$d_n = \sum_{m=0}^{n} e_n \frac{1}{(n-m)!}.$$

The aim is to find a recurrence relation for the b_ns. The recurrence relation for the $a_n s$ gives one for the e_ns, and then for the d_ns, and finally for the b_ns.

Recurrence Relation 1 *Recall Lemma 17; the a_n satisfy*

$$n^3 a_n - 2(2n-1)(5n^2-5n+2)a_{n-1} + 64(n-1)^3 a_{n-2} = 0. \tag{3.9}$$

Recurrence Relation 2 *the e_n satisfy*

$$2n^4 e_n + (2n-1)(5n^2-5n+2)e_{n-1} + 8(n-1)^2 e_{n-2} = 0,$$

Proof. Rewriting Equation 3.9, we have:

$$n^3 n(n-1)\frac{a_n}{n(n-1)} - 2(2n-1)(5n^2-5n+2)(n-1)\frac{a_{n-1}}{(n-1)} + 64(n-1)^3 a_{n-2} = 0.$$

Dividing this by $(n-2)!$ gives

$$n^3 n(n-1)\frac{a_n}{n!} - 2(2n-1)(5n^2-5n+2)(n-1)\frac{a_{n-1}}{(n-1)!} + 64(n-1)^3\frac{a_{n-2}}{(n-2)!} = 0,$$

which can be rewritten as

$$\frac{n^3 n(n-1)}{(-4)^{-2}}\frac{(-4)^{-2}a_n}{n!} - \frac{2(2n-1)(5n^2-5n+2)(n-1)}{(-4)^{-1}}\frac{(-4)^{-1}a_{n-1}}{(n-1)!}$$
$$+64(n-1)^3\frac{a_{n-2}}{(n-2)!} = 0.$$

So the e_n satisfy

$$16n^3n(n-1)e_n - 8(2n-1)(5n^2-5n+2)e_{n-1} + 64(n-1)^3e_{n-2} = 0,$$

which is rewritten to give the result.

■

Proposition 24 *If $\Upsilon(v) = \sum a_n v^n$, then the a_ns satisfy a recurrence relation*

$$a_nF_0(n) + a_{n-1}F_1(n) + \cdots + a_{n-m}F_m(n)$$

if and only if $\Upsilon(v)$ satisfies a differential equation

$$F_0(\Theta) + vF_1(\Theta+1) + \cdots + v^mF_m(\Theta+m),$$

where $\Theta = v\frac{d}{dv}$, and F_i is a polynomial.

Proof. This follows from the fact that $\Theta v^n = nv^n$.

■

Recurrence Relation 3

$$-2n^4d_n - n(n-1)(2n-1)d_{n-1}$$

$$+2(5n^2-10n+6)d_{n-2} - 3(2n-3)d_{n-3} = 0$$

Proof. So if we let $\mathcal{E}(v) = \sum e_nv^n$, then $\mathcal{E}(v)$ satisfies

$$2\Theta^4 + v(2\Theta+1)(5\Theta^2+5\Theta+2) + 8v^2(\Theta+1)^2 = 0.$$

Rewrite this as

$$2\Theta^4 + 10v\Theta^3 + v\Theta^2(15 + 8v) + v\Theta(9 + 16v) + v(2 + 8v).$$

Now $\mathcal{D}(v) := \sum d_n v^n = e^v \mathcal{E}(v)$, so

$$\mathcal{E} = e^{-v}\mathcal{D}$$

$$\Theta\mathcal{E} = e^{-v}(-v\mathcal{D} + \Theta\mathcal{D})$$

$$\Theta^2\mathcal{E} = e^{-v}(v(v-1)\mathcal{D} - 2v\Theta\mathcal{D} + \Theta^2\mathcal{D})$$

$$\Theta^3\mathcal{E} = e^{-v}(-v(v^2 - 3v + 1)\mathcal{D} + 3v(v-1)\Theta\mathcal{D} - 3v\Theta^2\mathcal{D} + \Theta^3\mathcal{D})$$

$$\Theta^4\mathcal{E} = e^{-v}(v(v^3 - 6v^2 + 7v - 1)\mathcal{D} - 4v(v^2 - 3v + 1)\Theta\mathcal{D} + 6v(v-1)\Theta^2\mathcal{D} - 4v\Theta^3\mathcal{D} + \theta^4\mathcal{D})$$

substitute these into the equation for $\mathcal{E}$ to get a differential equation satisfied by $\mathcal{D}$:

$$2\Theta^4 + (2\Theta^3 + 3\Theta^2 + \Theta)v + (-10\Theta^2 - 20\Theta - 12)v^2 + (6\Theta + 9)v^3$$

so we have the above recurrence relation for the d_ns.

∎

Theorem 3

$$0 = 3n^3 b_n - \frac{1}{2}n(n-1)(2n-1)b_{n-1}$$
$$-\frac{1}{3}(n-1)(5n^2 - 10n + 6)b_{n-2} - \frac{1}{6}(n-2)(n-1)(2n-3)b_{n-3}$$

Proof. Similar to the method in getting from a_ns to e_ns.

∎

3.5.0.1 The generating Function

The generating function for the b_ns is given by

$$\mathcal{B}(v) = \sum_{n\geq 0} v^n b_n.$$

Theorem 4 *If*

$$v(\tau) = \frac{(\eta(2\tau)\eta(6\tau))^6}{12(\eta(\tau)\eta(3\tau))^6 - 4(\eta(2\tau)\eta(6\tau))^6}$$

then

$$\mathcal{B}(v(\tau)) = \frac{(\eta(\tau)\eta(3\tau))^4}{(\eta(2\tau)\eta(6\tau))^2}.$$

Proof. Define

$$\mathcal{A}(v) = \sum_{n\geq 0} v^{n+1} a_n.$$

If

$$v(\tau) = -\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^6$$

then

$$\mathcal{A}(v(\tau)) = -\frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2}.$$

This is because by Propositions 15 and 16, $\mathcal{A}(v) = I(\lambda)$ where $\lambda + 4 = -1/v$. $I(\lambda)$ is a solution of the Picard-Fuchs equation $\mathcal{F}$ (Proposition 18), which has solution space

$$G(\tau)(\mathbb{C} \oplus \tau\mathbb{C} \oplus \tau^2\mathbb{C})$$

where

$$G(\tau) = \frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2},$$

by Lemma 14. Hence $\mathcal{A}(v(\tau)) = G(\tau)(a + b\tau + c\tau^2)$ for some $a, b, c \in \mathbb{C}$. A computer estimation tells us that $a = -1, b = c = 0$.

We have

$$\mathcal{A}(v) = v\mathcal{B}\left(\frac{\frac{v}{12}}{1 - 4\frac{v}{12}}\right),$$

or

$$\mathcal{B}(v) = \frac{1 + 4v}{12v}\mathcal{A}\left(\frac{12v}{1 + 4v}\right).$$

Hence result.

■

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