Hypergeometric functions and Configuration Spaces

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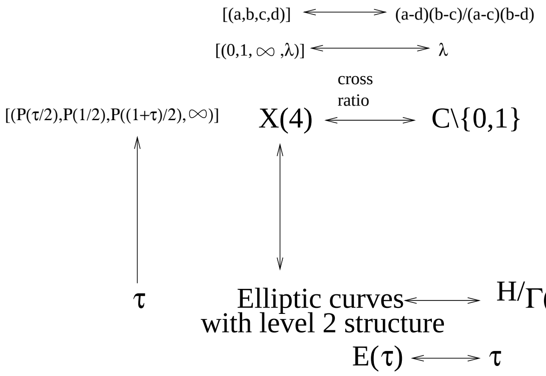
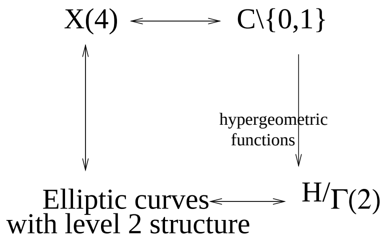
Lecture 8

Now we look at hypergeometric functions. What we have got so far is shown in the picture below:



We know that the diagonal, from the elliptic curves to $\mathbb{C}$ is given by the j invariant, and we know that that must be surjective, just because we can make a map to $\mathbb{C}*$, and then the image mulct be the whole thing. But this does not tell us what the image is. The hypergeometric functions allow us to complete the diagram, and find an inverse:



Since every point of $\mathbb{H}/\Gamma(2)$ is an equivalence class of points of $\mathbb{H}$, we might like to find a multi-valued function. Anyway, there are various ways of trying to find such a function, what we do is find an appropriate differential equation, and the solution of that will be the function we're looking for.

Next time we will see how the hypergeometric equation is related to finding a certain integral. But now we define the hypergeometric function first, and then find the differential equation, and then show that the solution of the differential equation has the required "monodromy", ie, show that it does map from $\mathbb{C} - \{0, 1\}$ to $\mathbb{H}/\Gamma(2)$.

1. definition of the hypergeometric series

For $-c \notin \mathbb{N}$, define the **hypergeometric series** by

$$F(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(d, n)} x^n,$$

where

$$(a, n) = a(a + 1)(a + 2) \cdots (a + (n - 1)).$$

Properties of $F(a, b, c; x)$

• If a or b is negative, $F(a,b,c;z)$ is a polynomial.
• By the ratio test, $F(a,b,c;z)$ has radius of convergence 1, so this sum gives a function holomorphic on the unit disc.
• $F(a,b,c;z) = F(b,a,c;z)$
It will turn out that the analytic continuation of $F(1/2,1/2,1;z)$ will give an inverse of λ, and of $F(1/12,5/12,1)$ will give an inverse of j.

2. The hypergeometric differential equation

Now we are going to find a differential equation satisfied by $F(a,b,c;z)$. We use the differential operator D given by

$$D := x\frac{d}{dx}.$$

The differential operator D satisfies a very nice property:

$$Dx^n = nx^n$$

Because of this we have the following useful result:

PROPOSITION 2.1. *Let a_n, $n \geq 0$, be a sequence of numbers, m some integer, and let $P_i(x)$ be polynomials in x for $i = 0...m$. Then a_n satisfy a recurrence relation:*

$$P_0(n)a_n + P_1(n)a_{n-1} + P_2(n)a_{n-2} + \cdots + P_{m-1}(n)a_{n-m+1} + P_1(n)a_{n-m} = 0 > \forall n$$

if and only if the formal power series

$$f(x) := \sum_0^\infty a_n x^n$$

satisfies the differential equation

$$P_0(D) + zP_1(D+1) + z^2P_2(D+2) + \cdots + z^mP_m(D+m)$$

where $D = z\frac{d}{dz}$ and F_i are polynomials.

$$P(D+) + xP(D+) + x^P(D+) + x^P(D+) + x^P(D+) + x^P(D+) = 0$$

PROOF. Exercise. (Note, formal just means we don't care if if diverges or not. This is just an algebraic exercise.) $\square$

A) If f is a function with a Taylor series expansion

$$f(z) := \sum_{n=0}^\infty a_n z^n$$

show that the a_n satisfy a recurrence relation

$$a_nF_0(n) + a_{n-1}F_1(n) + \cdots + a_{n-m}F_m(n)$$

if and only if $f(z)$ satisfies a differential equation
Example
So, to find a differential equation for $F(a,b,c,d)$, all we have to do is find a recurrence relation for the terms

$$A(n) = \frac{(a,n)(b,n)}{(c,n)(1,n)}$$

And that comes from the fact that

$$\frac{A(n+1)}{A(n)} = \frac{(a+n)(b+n)}{(c+n)(1+n)},$$

which gives us a two term recurrence relation:

$$(c+n)(1+n)A(n+1) = (a+n)(b+n)A(n).$$

And so we'll get an order two differential equation in D, and by substituting xd/dx for D, we'll end up with the following:

PROPOSITION 2.2. *The hypergeometric series $F(a,b,c;x)$ is a solution of the differential equation:*

$$E(a,b,c) := x(1-x)\frac{d^2}{dx^2} + (c-(a+b+1)x)\frac{d}{dx} - ab = 0$$

PROOF. Applying the above results, we can see that the coefficients $A(n)$ satisfy

$$A(n)(D-1+c)D - A(n-1)(n-1+a)(n-1+b)$$

so $F(a,b,c;x)$ must satisfy

$$(D-1+c)D - x(D+a)(D+b)$$

Rewrite this in terms of $\frac{d}{dx}$ to get the above expression. $\square$

For lots of stuff about differential equations, see the book on differential equations by Ince.

3. The hypergeometric differential equation, and it's monodromy

The differential equation $E(a,b,c)$, defined above, is called the hypergeometric differential equation. We know that $F(a,b,c;x)$ is one local solution about 0. But given such a differential equation, we can find local solutions about any point x_0, (not equal to $0,1,\infty$), just by writing down

$$f(z) = \sum_{n=o}^{\infty} b_n(z-x_0)^n,$$

substituting this into the differential equation; this will give a recurrence relation that the b_n must satisfy, so that will allow the b_n to be found. The solution for b_n is not unique. What we'll get is that there is a two dimensional vector space of local solutions about z_0.

Anyway, we can try and glue together these local solutions to get something that's defined on all $\mathbb{C}$. But of course, you can't really expect this to work; you won't get a single valued function on $\mathbb{C}$. But when you extend f along some path γ, you still keep getting a function that satisfies the hypergeometric differential equation, so it doesn't give you much freedom of movement. How the function changes is described by the **monodromy**

4. The fundamental group

Before we go any further, we better make sure everyone remembers the definition of the fundamental group. This is the group of loops which we are going to extend functions along.

A path in a topological space X, which starts at a point x_0 and ends at a point x_1, $(x_0, x_1 \in X)$, is a continuous function

$$\gamma : [0,1] \to \mathbb{C}\{0,1\}$$

with

$$\gamma(0) = x_0,\ \gamma(1) = x_1$$

A **loop** based at x_0 is a path with $x_1 = x_0$, so it starts and ends at the same point.

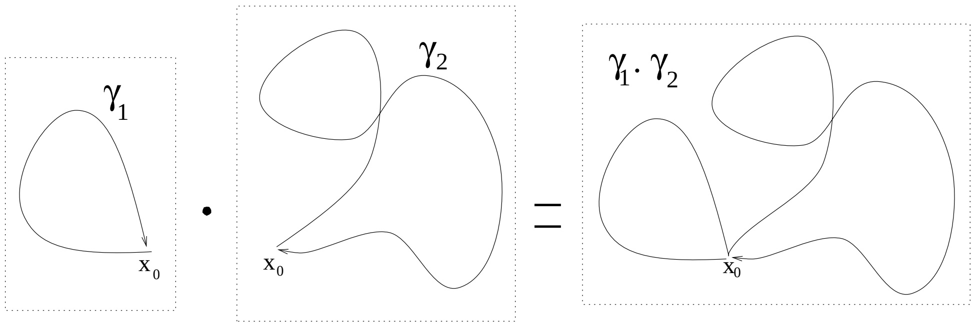
In our case, we're interested in $X = \mathbb{C} - \{0,1\}$.

The collection of loops in $\mathbb{C} - \{0,1\}$, based at x_0, up to homotopy, forms a group, called $\pi_1(\mathbb{C} - \{0,1\}, x_0)$. Composition in this group is just defined by adjoining two paths; ie, for γ_1 and γ_2, define $\gamma_1.\gamma_2$ by

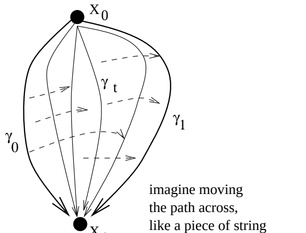
$$\gamma_1.\gamma_2(t) = \gamma_1(2t)\ t \in [0, 0.5]$$

$$\gamma_1.\gamma_2(t) = \gamma_2(2t-1)\ t \in [0.5, 1]$$

So we go round each loop twice as fast, so we have time to go round both of them.



Homotopy means that if you can deform one path to another in $\mathbb{C} - \{0,1\}$, you consider them to be the same. Deform just means moving one continuously to the other, keeping the end points fixed, so the line can not wiggle away. Here's the picture:



Here's the formal definition:

Paths γ_0 and γ_1 in X are homotopic if and only if there is a function

$$T : [0,1] \times [0,1] \to X$$

$$T(0,t) = \gamma_0(t)$$

$$T(1,t) = \gamma_1(t)$$

$$T(t,0) = x_0,\ T(t,1) = x_1$$

So for each t, $T(t,*) =: \gamma_t$ is a path from x_0 to x_1, and as t changes, the path γ_t changes from γ_0 to γ_1 in a continuous way.

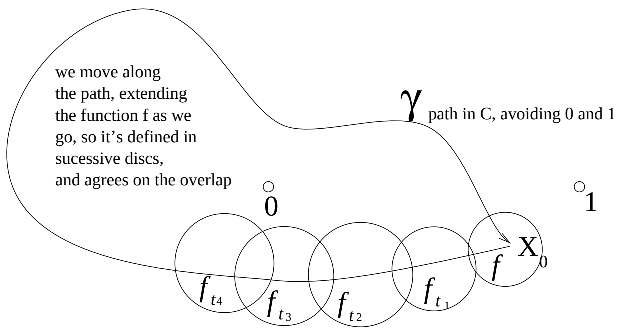
This definition is given for paths. For loops just set $x_1 = x_0$.

It's not too hard to check that with these definitions, everything is well defined, and we do have a group, with an identity (given by the equivalence class of the path that sends all of $[0,1]$ to x_0), and inverses (given by going round a loop backwards). For more details about the fundamental group see eg book by Maunder on Algebraic topology.

5. The monodromy group

Now we know what the fundamental group is, we can define the monodromy group.

As we move along γ, we start with a solution to $E(a,b,c;z)$, say $f = au_1 + bu_2$, defined in a neighborhood of x_0. Now for each $t \in [0,1]$, we can define f_t to be defined in a neighborhood B_t of t, so that these functions agree on the intersection of neighborhoods. So there's a unique way to do this. When we get all the way round to f_1, we must have that f_1 is some linear combination of u_1 and u_2, but there's no reason why it should be the same function as f.



We will get a transformation:

$$\gamma_* \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = M(\gamma) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where $M(\gamma)$ is some matrix in $GL(2,\mathbb{C})$, the group of nonsingular two by two matrices. It's a non singular matrix, because if a function was locally zero, then it would have to be zero everywhere, so there's no way you can transform a non zero function to a zero one.

So this gives us a map

$$\pi_1(\mathbb{C}\{0,1\} \to GL(2,\mathbb{C})$$

This is called the **monodromy representation** of the differential equation.

It's simple to check this is well defined on homotopy classes, and is a group homomorphism.

Changing the basis of local solutions will have the effect of conjugating the matrix, as will changing the base point. The image of this map is called the **monodromy group**. If we quotient out by scalers, we get the **projective monodromy group** in $PGL(2,\mathbb{C})$ The conjugacy class of these groups does not depend on base point or basis, so it is determined uniquely by the differential equation.

6. The solutions of $E(a,b,c)$ around the origin

We know that $F(a,b,c;z)$ is a solution of $E(a,b,c)$ around the origin, and that there should be a two dimensional vector space of solutions around the origin. Now we look for another solution to $E(a,b,c)$ so that we can define all the solutions near 0.

To find another solution, note that

$$Dx^s = x^s(s+D)$$

Eg.

$$Dx^2 f = x\frac{d}{dx}x^2 f = x(x^2\frac{d}{dx}f + f2x) = x^2(2 + x\frac{d}{dx}) = x^2(2+D).$$

Above, we saw that $E(a,b,c)$ can also be written as

$$(D-1+c)D - x(D+a)(D+b)$$

Now, what happens if we apply $E(a,b,c)x^r$ to a function? We get

$$((D-1+c)D - x(D+a)(D+b))x^r = (D-1+c)Dx^r - x(D+a)(D+b)x^r$$

$$= (D-1+c)x^r(r+D) - x(D+a)x^r(r+D+b) = x^r(r+D-1+c)(r+D) - xx^r(r+D+a)(r+D+b)$$

$$= x^r((r+D-1+c)(r+D) - x(r+D+a)(r+D+b))$$

If $r = 1-c$, the coefficients of x^0 is divisible by D, and is of the form $E(a',b',c')$. We have:

$$((D-1+c)D - x(D+a)(D+b))x^{1-c}$$

$$= x^{1-c}((1-c+D-1+c)(1-c+D) - x(1-c+D+a)(1-c+D+b))$$

$$= E(a+1-c, b+1-c, 2-c)$$

A solution of $E(a+1-c, b+1-c, 2-c)$ is $F(a+1-c, b+1-c, 2-c, x)$. So since $((D-1+c)D - x(D+a)(D+b))x^{1-c} = E(a+1-c, b+1-c, 2-c)$, we have

$$E(a,b,c)x^{1-c}F(a+1-c, b+1-c, 2-c, x) =$$

$$((D-1+c)D - x(D+a)(D+b))x^{1-c}F(a+1-c, b+1-c, 2-c, x)$$

$$= E(a+1-c, b+1-c, 2-c)F(a+1-c, b+1-c, 2-c, x) = 0,$$

ie, $x^{1-c}F(a+1-c, b+1-c, 2-c, x)$ is another solution of $E(a,b,c)$ around the origin.

So the set of all solutions of $E(a,b,c)$ in a neighbourhood of the origin is

$$F(a,c,b;x)\mathbb{C} + x^{1-c}F(a+1-c, b+1-c, 2-c; x)\mathbb{C}.$$

But these functions are only defined near zero; we would like to find an analytic continuation of $F(a,b,c;z)$ to the whole of $\mathbb{C}$.

7. Reminder about branching

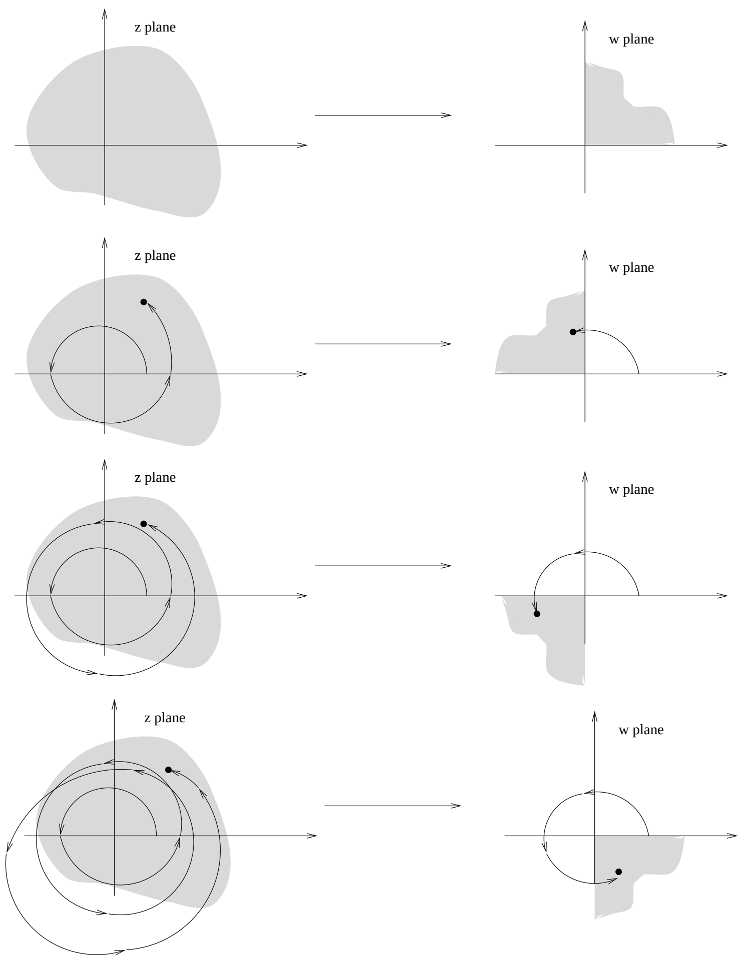
We have a vector space of solutions near 0 now we want to extend. Actually, the solution $x^{1-c}F(a+1-c,b+1-c,2-c,x)$ will branch - you go round the origin, and when you get back, you're at a different function. This is because of the x^{1-c} generally this happens for functions of the form z^a.

In case you forgot about branching of functions, here's a simple example, for the function $x \mapsto x^{\frac{1}{4}}$:

We could define a branch in a infinite number of ways, but here are four convenient branches:

$$\begin{array}{llcll}
\text{Branch one:} & b_1(z) & = & r^{\frac{1}{4}}e^{i\theta/4} & \text{where } z = re^{i\theta},\ r,\theta \in \mathbb{R},\ \theta \in [0,2\pi) \\
\text{Branch two:} & b_2(z) & = & r^{\frac{1}{4}}e^{i\theta/4} & \text{where } z = re^{i\theta},\ r,\theta \in \mathbb{R},\ \theta \in [2\pi,4\pi) \\
\text{Branch three:} & b_3(z) & = & r^{\frac{1}{4}}e^{i\theta/4} & \text{where } z = re^{i\theta},\ r,\theta \in \mathbb{R},\ \theta \in [4\pi,6\pi) \\
\text{Branch four:} & b_4(z) & = & r^{\frac{1}{4}}e^{i\theta/4} & \text{where } z = re^{i\theta},\ r,\theta \in \mathbb{R},\ \theta \in [6\pi,8\pi)
\end{array}$$

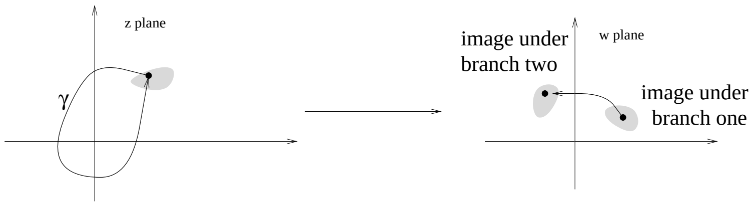
And here are pictures of the images of the upper half plane under these four branches:



Also in these diagrams, we show the image of a path, which we start off by evaluating using branch b_1, and just keep going.

In this case, the space of branches is spanned by one function, b_1. The effect of going round the origin once is to multiply by i:

$$\gamma : b_1 \mapsto b_2 = i b_1$$



Note that $\frac{d}{dz}(z^{\frac{1}{4}}) = \frac{1}{4}z^{\frac{1}{4}}/z$ so $z^{\frac{1}{4}}$ is a solution of an order one differential equation:

$$4z\frac{d}{dz} - 1 = 0$$

Ie if

$$(4z\frac{d}{dz} - 1)f = 4z\frac{df}{dz} - f = 0$$

then f is in the one dimensional space of solutions,

$$b_1\mathbb{C}.$$

The fundamental group is generated by the path γ which goes once round 0, as shown in above diagram. So

$$\pi_1(\mathbb{C}) - \{0\} = < \gamma >$$

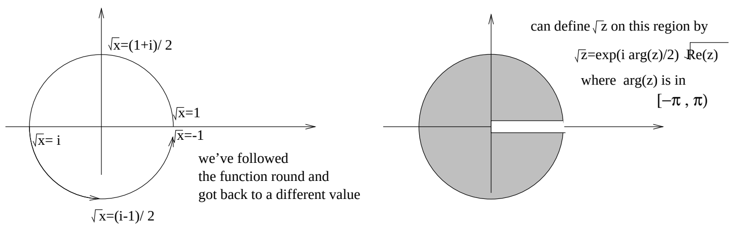
The image of the monodromy group will be a subgroup of the 1×1 matrices in $\mathbb{C}$, ie, just an element of $\mathbb{C}$:

$$\pi_1(\mathbb{C} - \{0\}) \to \mathbb{C}$$

$$\gamma^n \mapsto i^n$$

So in this case, the image of the monodromy group is isomorphic to the cyclic group of order 4.

Another example, $\sqrt{z}$, will behave in a similar way. If we want to define a function, we have to leave out some line; it's not possible to define the function on all the unit disc:

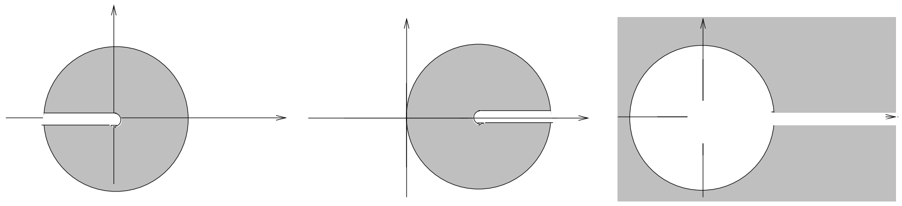


8. Solutions of $E(a, b, c)$ near 1 and ∞

To make a single valued function, we have to fix a branch and a suitable domain, as in the diagrams.

We have solutions for $E(a, b, c)$ near 0 which converge and are well defined on the same disc as in the above picture.

Next we consider a transformation to find solutions around 1 and ∞:



F(a,b,c;z)
z^{1-c} F(a+1-c, b+1-c, 2 - c; z)
defined near z=0

F(a,b,a+b+1; 1-z)
$(1-z)^{c-a-b}$ F(c-a,c-b,c+1-a-b; 1+z)
defined near z=1

$(-z)^{-a}$F(a,1+a-c,a+b+1; 1/z)
$(-z)^{-b}$F(b,1+b-c,b-a+1; 1/z)
defined near infinity

I'll just write out the method for getting a solution near 1.

We have that $F(a,b,c;x)$ satisfies

$$x(1-x)\frac{d^2}{dx^2}+(c-(a+b+1)x)\frac{d}{dx}-ab=0$$

By making a change of variables $t=1-x$, we have

$$(t(1-t)\frac{d^2}{dt^2}+(c-(a+b+1)t)\frac{d}{dt}-ab)F(a,b,c;t)=0$$

But $\frac{d}{dt}=-\frac{d}{dx}$, so we get

$$((1-x)(1-1+x)\frac{d^2}{dt^2}+(c-(a+b+1)(1-x))\frac{d}{dt}-ab)F(a,b,c;1-x)=0$$

$$((1-x)x\frac{d^2}{dt^2}-(c-a-b-1-(a+b+1)x))\frac{d}{dt}-ab)F(a,b,c;1-x)=0$$

And so replacing c by $a+b+1-c$ in the above, we get

$$((1-x)x\frac{d^2}{dt^2}+(c-(a+b+1)x))\frac{d}{dt}-ab)F(a,b,a+b+c;1-x)=0$$

This was defined for $|t|<1$, ie $|x-1|<1$.

Similarly, we define solutions around ∞, by making the transformation $t=1/x$, and get functions as in the above diagram.

near 0	$f_{01}:=F(a,b,c;x)$ $f_{02}:=x^{1-c}F(a+1-c,b+1-c,2-c;x)$
near 1	$f_{11}:=F(a,b,a+b+1-c;1-x)$ $f_{12}:=(1-x)^{c-a-b}F(c-a,c-b,c+1-a-b;1-x)$
near ∞	$f_{\infty 1}:=\left(\frac{-1}{x}\right)^a F(a,1+a-c,a-b+1;\frac{1}{x})$ $\left(\frac{-1}{x}\right)^b F(b,1+b-c,b-a+1;\frac{1}{x})$

The solutions must span the same space of solutions of the differential equation, so there are matrices relating them:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} f_{11} \\ f_{12} \end{pmatrix}=\begin{pmatrix} f_{01} \\ f_{02} \end{pmatrix}, \quad \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} f_{11} \\ f_{12} \end{pmatrix}=\begin{pmatrix} f_{\infty 1} \\ f_{\infty 2} \end{pmatrix}$$

These are called the **connection matrices**. We can use these to find a function well defined on all the upper half plane, or all the lower half plane, which is a solution to the hypergeometric differential equation.

lecture 9

9. Schwarz triangles

We will define a map from the upper half plane $\mathcal{H}$ using the solutions of the hypergeometric differential equation.

We have that on the domains

$$D_0 = \{z \in \mathbb{C} \mid |z| \leq 1 z \notin [-1, 0)\}\,,$$

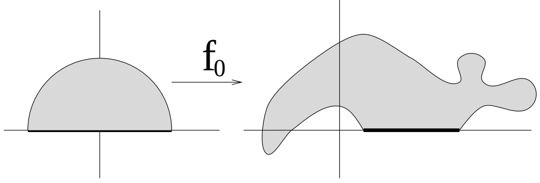
$$D_1 = \{z \in \mathbb{C} \mid |1 - z| \leq 1 z \notin (0, 1]\}\,,$$

$$D_\infty = \{z \in \mathbb{C} \mid |z| \geq 1 z \notin [1, \infty)\}\,,$$

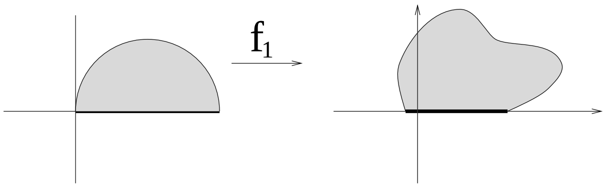
which are the domains pictured in the above diagram, we can define branches of the functions f_{ij}. Define the branches as in the above table, always taking $z = re^{i\theta}$, with $\theta \in [-\pi, pi]$.

First we will just define a function on the upper half plane. We define functions from D_i to $\mathbb{P}^1$ for $i = 0, 1, \infty$. The diagrams show the image, where $\mathbb{P}^1 - [1 : 0]$ is drawn as $\mathbb{A}^1$, with $[x : y] \to x/y$.

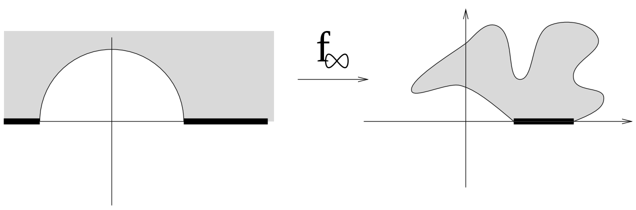
$$f_0 : D_0 \cap \mathcal{H} \to \mathbb{P}^1$$
$$z \mapsto [f_{01} : f_{02}]$$



$$f_1 : D_1 \cap \mathcal{H} \to \mathbb{P}^1$$
$$z \mapsto [f_{11} : f_{12}]$$



$$f_\infty : D_\infty \cap \mathcal{H} \to \mathbb{P}^1$$
$$z \mapsto [f_{\infty 1} : f_{\infty 2}]$$



As depicted in the diagrams, since each function f_{ij} is real valued for real z, from the choice of branch, so the image of a real intereval is a real interval.

These functions do not match up, but by using the connection matrices, we have relationships between these functions, so that we can define a function that is well defined on $\mathcal{H}$. Suppose the connection matrices are A_1 and A_∞, so that

$$A_1 \begin{pmatrix} f_{11} \\ f_{12} \end{pmatrix} = \begin{pmatrix} f_{01} \\ f_{02} \end{pmatrix}, \quad A_\infty \begin{pmatrix} f_{\infty 1} \\ f_{\infty 2} \end{pmatrix} = \begin{pmatrix} f_{01} \\ f_{02} \end{pmatrix}$$

We define the **Schwarz map** f by

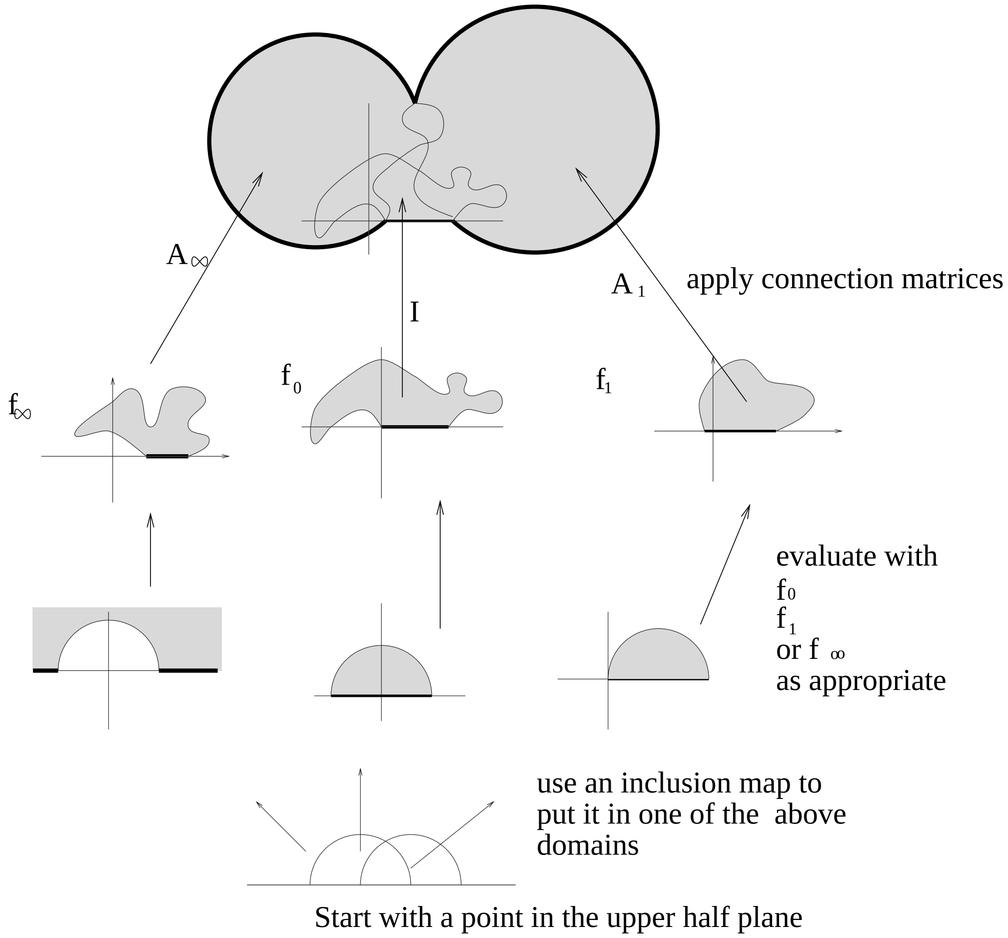
$$f : \mathcal{H} \to \mathbb{P}^1$$

$$f(z) = [f_{01} : f_{02}] \text{ for } z \in D_0$$

$$f(z) = [f_{11} : f_{12}] \text{ for } z \in D_1$$

$$f(z) = [f_{\infty 1} : f_{\infty 2}] \text{ for } z \in D_\infty$$

Then these do give a well defined function:



Since the maps A_i are mobius transformations, they will map the segments of the real line to segments of a circle, as shown in the diagram.

Hence the image of the upper half plane $\mathcal{H}$ under f is a triangle. Note, by "triangle" I mean something bounded by three lines, where "line" means straight line or circle, since on the projective plane, these are the same thing.

The above diagram of the image of $\mathcal{H}$ under f is not accurate. We could find the vertices of the triangle under f explicitly, which I'll do here just to show it's possible:

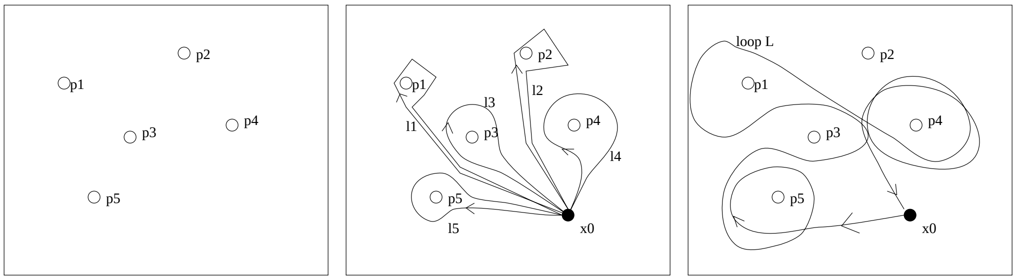
pig

Now we want to find what is the image of the lower half plane, and also what is the monodromy, as when we continue round a path, we'll get to different branches of f, which will map the upper half plane to different triangles. Before going on to say what happens globally, we'll look at the local monodromy.

10. Local monodromy

The monodromy group tells us how analytically continuing going along any path will change the solution of our differential equation.

If we remove points $p_1, p_2 ... p_n$ from the complex plane, then the monodromy group will be generated by loops round each of these points:



Complex plane with 5 points removed

the homotopy classes of these loops generate the fundamental group of loops based at x0

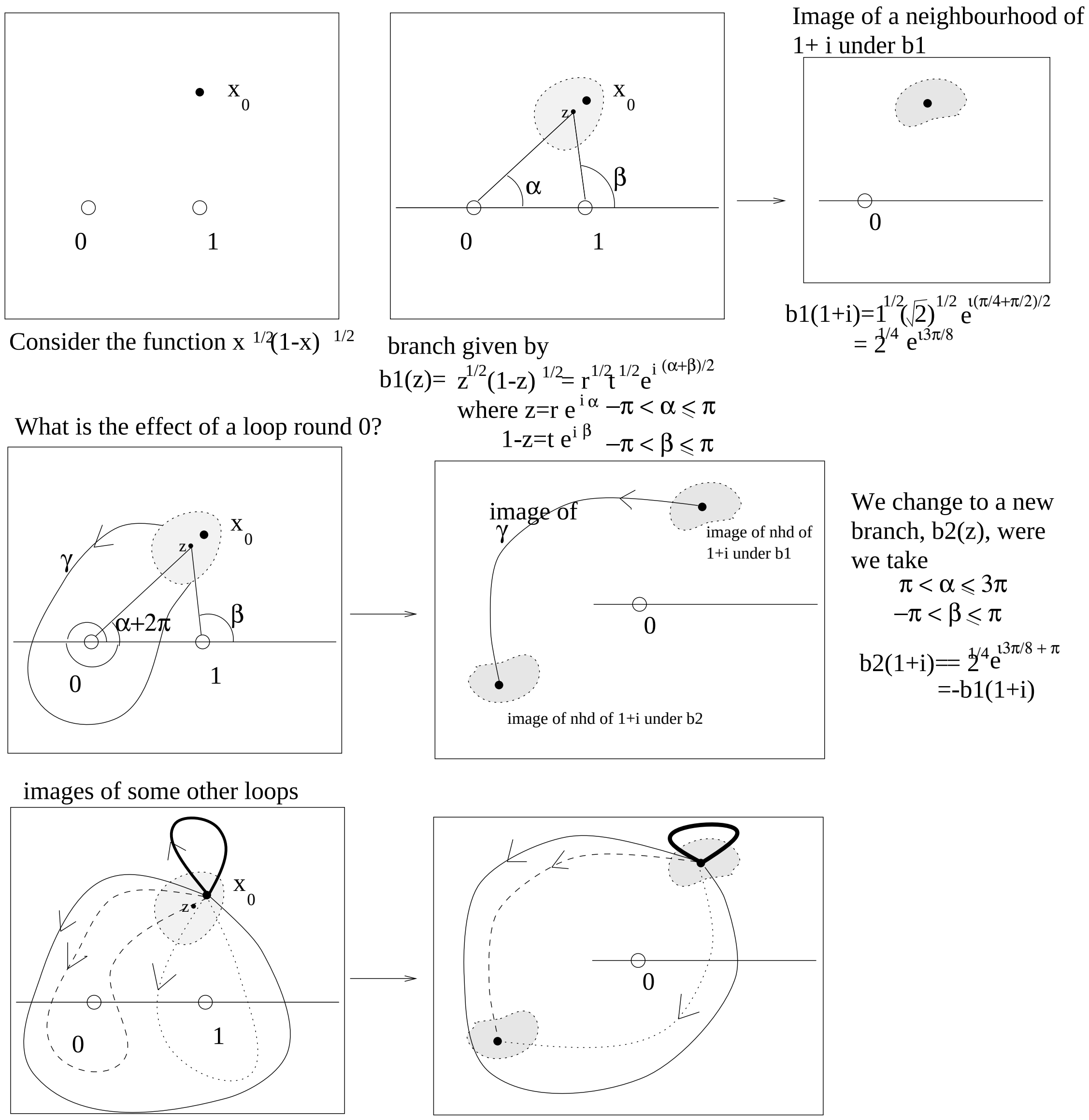
example: [L]= 2[l5] + [l1] - 2[l4]

So the monodromy group will be generated by the images of these loops, which will be linear maps on the space of solutions of the differential equations.

Hence it will be useful to find the images of these loops.

We can get some information by working locally; if we have a solution to the differential equation near a point p_1, say $\sum a_n(z-p_1)^n$, we can look at how this changes as we go round a loop based near p_1. Then to find the effect of the loop based at x_0 which goes round p_1, (call this loop l_1), we just have to conjugate by some matrix corresponding to getting from x_0 to be near p_1. The conjugation will change the entries, but it won't change the order of the matrix, so even if you don't bother finding the exact matrix, you do end up knowing it's order. We will see later how this information will help us towards finding the monodromy group.

The following diagram shows a very simple example. The fundamantal group in this case is generated by two loops, one round 0, and one round 1. The effect of either one is to multiply the function by -1:



The monodromy group is hence the cyclic group

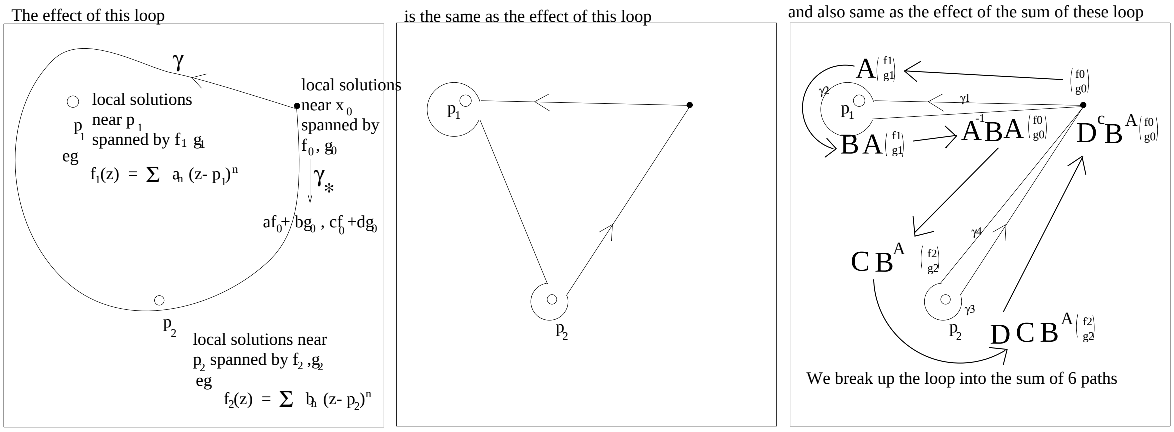
$$C_2.$$

In general, we have matrices of larger size, and we don't expect to get an abelian group.

Note that in this case, the differential equation that these branches come from is

$$2x(1-x)\frac{d}{dx} - 1.$$

In general, we will be able to find the orders by looking at expansions of the solution about the singular points. The following diagram shows the idea of what we want to do:



The path γ round the points p_1 and p_2 as in the diagram can be broken down:

$$[\gamma] = [\gamma_1] + [\gamma_2] - [\gamma_1] + [\gamma_3] + [\gamma_4] - [\gamma_5].$$

Note that although I've written the sum using "+", the group is still not abelian, so $[\gamma_1]$ and $-[\gamma_1]$ do not cancel.

Each of these paths will give us a matrix, eg, γ_1 corresponds to a matrix A, which tells us how going from x_0 to p_1 along γ_1 brings us from f_0 to a certain linear combination of f_1 and g_1, and so on, as indicated in the diagram. To restate:

$$\begin{array}{rccc}
\gamma_{1*}: & \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} & \mapsto & A\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \\
\gamma_{2*}: & \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} & \mapsto & B\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \\
-\gamma_{1*}: & \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} & \mapsto & A^{-1}\begin{pmatrix} f_0 \\ g_0 \end{pmatrix} \\
\gamma_{3*}: & \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} & \mapsto & C\begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \\
\gamma_{4*}: & \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} & \mapsto & D\begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \\
-\gamma_{3*}: & \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} & \mapsto & C^{-1}\begin{pmatrix} f_0 \\ g_0 \end{pmatrix}
\end{array}$$

So if p_1 and p_2 are the only singular points, the monodromy group will be generated by B^A and D^C. (my notation is $B^A = A^{-1}BA$.)

So all we have to do to find the monodromy group is find A, B, C, D. To find A and C will not be so nice; we will have to write down expansions for both local solutions, eg:

$$f_0(z) = \sum_{n\geq 0} u_n(z - x_0)^n$$

$$f_1(z) = \sum_{n} v_n(z - p_1)^n$$

$$g_1(z) = \sum_{n} w_n(z - p_2)^n$$

Then we want to write

$$\sum_{n\geq 0} u_n(z-x_0)^n = \alpha \sum_n v_n(z-p_1)^n + \beta \sum_n w_n(z-p_2)^n,$$

by expanding everything out and finding α and β. This means you have to explicitly find all the expressions.

But to find B and D is very easy, there's a formula for how to do that from the differential equation, called the indicial equation. I won't write that down here, but just show a simple example.

Suppose we have two solutions around p_1:

$$f_1(z) = \sum_{n\geq 0} v_n(z-p_1)^n$$

$$g_1(z) = (z-p_2)^m \sum_{n\geq 0} w_n(z-p_2)^n$$

where m is some integer; so this is the form of the solutions of the hypergeometric differential equation.

Then suppose we start at a branch where we take $z = re^{i\theta}$ with $\theta \in (-\pi, \pi)$, so that

$$g_1(z) = r^m e^{im\theta} \sum_{n\geq 0} w_n(z-p_2)^n$$

Then after going round p_1 once, we now have to take $\theta \in (\pi, 3\pi)$, so that we get to a new branch:

$$g_1(z) = r^m e^{im\theta+2\pi im} \sum_{n\geq 0} w_n(z-p_2)^n$$

Since f_1 does not involve any fractional powers of z, it does not change when we go round p_1. The monodromy about p_1 is given by

$$\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\pi m} \end{pmatrix} \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}$$

If we choose a different basis of functions, we'll get a different matrix, but it will still have the same order. We will assume that m is not an integer (so that the solutions f_1 and g_1 are definitely independent); then if m is a rational, the matrix will have finite order; otherwise it will have infinite order.

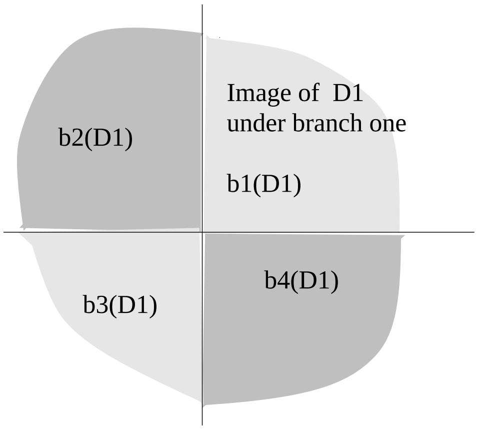
Suppose $m = \frac{1}{4}$. Then the matrix describing the monodromy is

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

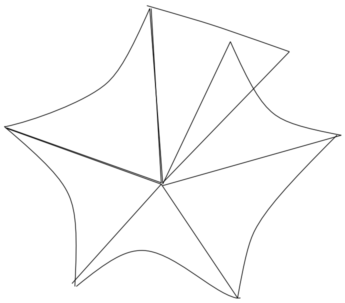
which is of order 4 We can consider the function

$$z \mapsto g_1(z)/f_1(z)$$

Assuming the coeffiecients of the Taylor series expansion are real, and that for convenience $p_2 = 0$, we get that the first branch maps from D_1 (it's domain of definition) into the first quadrant, the second to the second quadrant, and so on:



In this case, the boundaries of the images match up nicely. In general we will get that images of the domain under successive branches of the function do not match up:



In any case, since the real axis near $p_2 (= 0)$ gets mapped sucessively to lines at angles πm to the real axis, all these images have the same angle at 0.

11. Image of The upper half plane under f

We can now apply the above work to find the angles of the triangle that is the image of upper half plane under the function f.

Recall that we defined f by

$$f : \mathcal{H} \to \mathbb{P}^1$$

$$f(z) = [f_{01} : f_{02}] \text{ for } z \in D_0$$

$$f(z) = [f_{11} : f_{12}] \text{ for } z \in D_1$$

$$f(z) = [f_{\infty 1} : f_{\infty 2}] \text{ for } z \in D_\infty$$

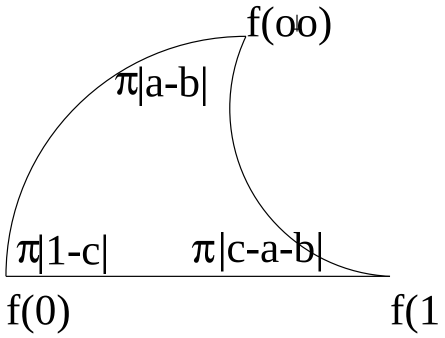
and that

near 0	$f_{01} := F(a, b, c; x)$ $f_{02} := x^{1-c} F(a + 1 - c, b + 1 - c, 2 - c; x)$
near 1	$f_{11} := F(a, b, a + b + 1 - c; 1 - x)$ $f_{12} := (1 - x)^{c-a-b} F(c - a, c - b, c + 1 - a - b; 1 - x)$
near ∞	$f_{\infty 1} := \left(\frac{-1}{x}\right)^a F(a, 1 + a - c, a - b + 1; \frac{1}{x})$ $\left(\frac{-1}{x}\right)^b F(b, 1 + b - c, b - a + 1; \frac{1}{x})$

So the image of a disc round 0 under f will have an angle $2\pi(1-c)$ at 0. The boundaries are the images of the positive real axis. If we want to find the image of the upper half plane, then the boundaries will be the images of the positive real axis, and the negative real axis. Since the negative real axis is at angle π to the positive real axis, (for our usual way of measuring angles for the first branch), it will get sent to a line at an angle $\pi|1-c|$ to the real axis.

In the same way, we can find the other angles of the triangle. We use the fact that the mobius transformations A_1 and A_∞ preserve angles.

$$\begin{array}{rc} \text{angle at } 0 & \pi|1-c| \\ \text{angle at } 1 & \pi|c-a-b| \\ \text{angle at } \infty & \pi|a-b| \end{array}$$



We don't care too much exactly what the triangle is; if we changed our functions by some element of $PSL_2(\mathbb{C})$, ie, take a different basis for the local solutions of the differential equations, then this will have the effect of a mobius transformation, which can take the three corners to any other three points we want (since $X(3)$ has only one point). All that will be preserved is the angles, so that's all that's important.

12. Image of the lower half plane

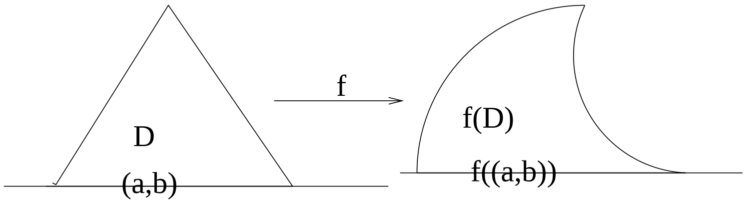
We will use the **Schwarz's reflection principle** to find an extension of f to the lower half plane. This is the following proposition:

PROPOSITION 12.1. *Let f be a holomorphic function defined on a domain D, which is in the upper half plane, and includes the real line segment (a,b) as part of it's boundary, and such that we can extend f continuously to (a,b), and then $f(t)$ is real for $t \in (a,b)$. Then we can extend f to $D \cup \overline{D} \cup (a,b)$ by*

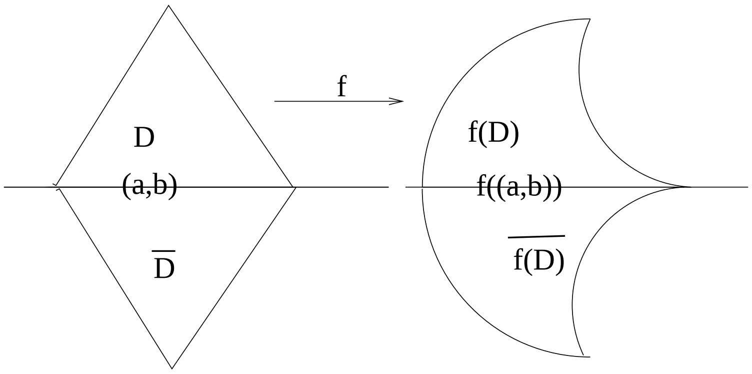
$$f(z) = \overline{f(\overline{z})}\ z \in \overline{D} := \{\overline{z} | z \in D\}$$

The extended function is holomorphic and has image $f(D) \cup f((a,b)) \cup \overline{f(D)}$.

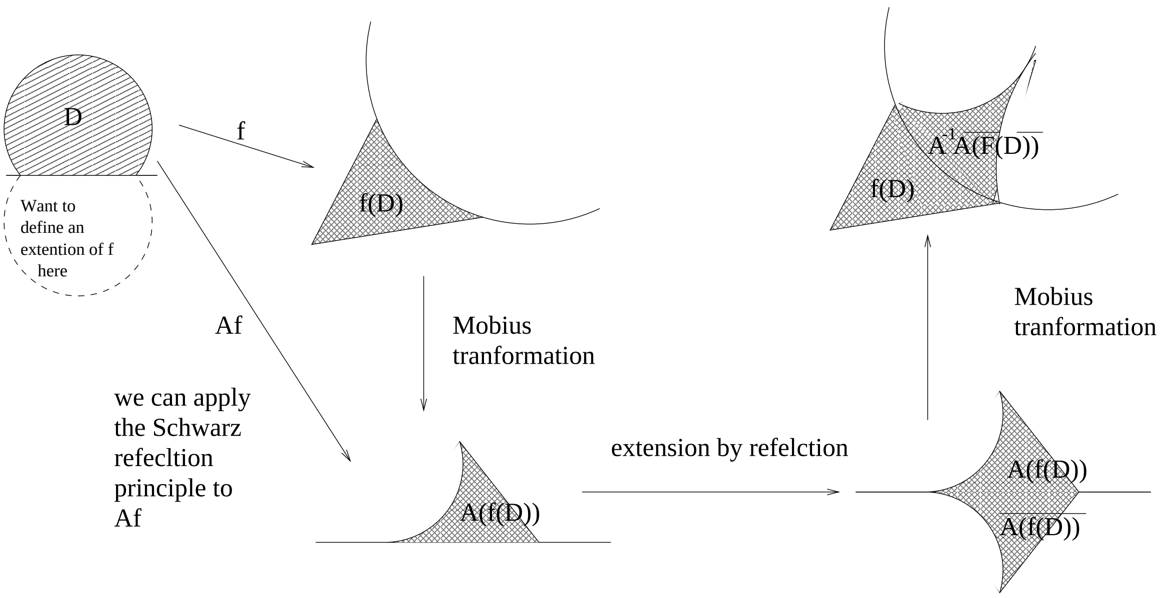
PROOF. See text book $\square$



if f maps the domain D to f(D) as above, we can extend:



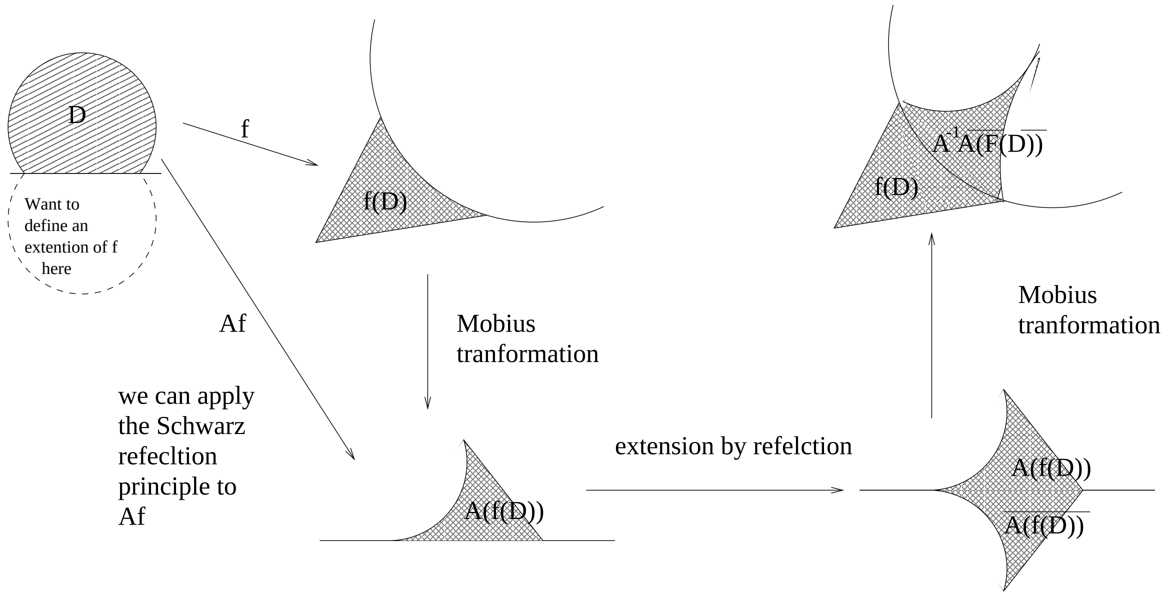
A corollary is that if we have a function that maps a segment of a circle to a segement of a cicle, we can use a mobius transformation to translate the circle to the real axis, then reflect, so that we can extend the function by refecting in the circle segement:



See text book for a precise statement.

13. The image of the complex plane under the Schwarz map

Now we see that we can define f on the lower half plane by reflection in the intervals $(-\infty, 0)$, $(0,1)$ and $(1, \infty)$. We can only add at most any one of these intervals while still keeping that the image of the real line segment we add is real. So we get different ways of extending f depending on which of these intervals we cross when we make an anylitic extension along a path.



So that the images meet up nicely, we want to choose our angles nicely, which means putting some conditions on a, b, c. If we want angles $\pi/p, \pi/q, \pi/r$, for $p, q, r \in \mathbb{Z}$, we can require that

$$|1-c| = \frac{1}{p}$$

$$|c-a-b| = \frac{1}{q}$$

$$|a-b| = \frac{1}{r}$$

We can now make a checkerboard pattern by colouring images of the upperhalf plane under the Schwarz map blue, and images of the lower half plane orange.

We will call the monodromy group

$$\Gamma(p, q, r),$$

and this will be the group of symmetries generated by an even number of reflections, as we have to preserve the colour, ie, and image of the upper half plane is always an image of the upper half plane, which is always orange, no matter how we take our loop we analytically continue along. Note that an even number of reflections will correspond to a mobius transformation. We have an isomorphism:

$$\mathbb{C} - \{0, 1\} \to (\mathbb{P}^1 - \{\text{vertices of the triangles}\})/\Gamma(p, q, r)$$

We have three possible cases:

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$$

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$$

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$$

13.1. Case $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$. In this case, we will get something equivalent to a triangulation of the surface of the sphere. There are only 4 possible cases:

p	q	r	Name of group
2	2	n	Dihedral
2	3	3	Tetrahedral
2	3	4	Octahedral
2	3	5	Icosahedral

In these cases a finite number of triangles covers the image; the monodromy group is finite. See text book for pictures.

13.2. Case $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$. In this case what we get is a tessellation of the plane by triangles. The projective monodromy group is a discrete subgroup of the group of Euclidean motions of the complex plane ($z\mapsto ax+b$, non singular linear map.) In this case the monodromy group is called a **crystallographic group**. There are 4 possibilities:

p	q	r	Name of group
2	2	∞	ruler
2	3	6	Hexagonal lattice
2	4	4	Square lattice
3	3	3	equilateral triangle lattice

13.3. Case $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$. There are infinitely many possibilities here; namely everything not in the above two cases. In these cases we get a tessellation of the hyperbolic plane by triangles.

There are two cases we are interested in, which give the same picture as the pattern obtained by translates of the fundamental region for j and λ. If we try to look for a hypergeometric function which will give triangles with the same angles as for the fundamental domains for $\Gamma(2)$ and $SL_2(\mathbb{Z})$, we get:

$$(a,b,c)\mapsto(p,q,r)$$

$$(\frac{1}{2},\frac{1}{2},1)\mapsto(\infty,\infty,\infty)$$

$$(\frac{1}{12},\frac{5}{12},1)\mapsto(\infty,2,3)$$

We will later see a proper proof of the following proposition:

PROPOSITION 13.1. *The Scwarz map of the hypergeometric equation* $E(1/2,1/2,1)$ *gives an isomorphism*

$$X(4)\cong\mathbb{C}-\{0,1\}\to\mathbb{H}/\Gamma(2)$$

which is the inverse of the λ invariant.

The Scwarz map of the hypergeometric equation $E(1/12, 5/12, 1)$ *gives an isomorphism*

$$X\{4\} \cong \mathbb{C} \to \mathbb{H}/PSL_2(\mathbb{Z})$$

which is the inverse of the j *invariant.*

So finally we have shown how to get from $X(4)$ to a quotient of the upper half plane via elliptic curves, to the complex plane via the λ and j invariants, and now we've come back, via the Schwarz map.