Hypergeometric functions and Configuration Spaces

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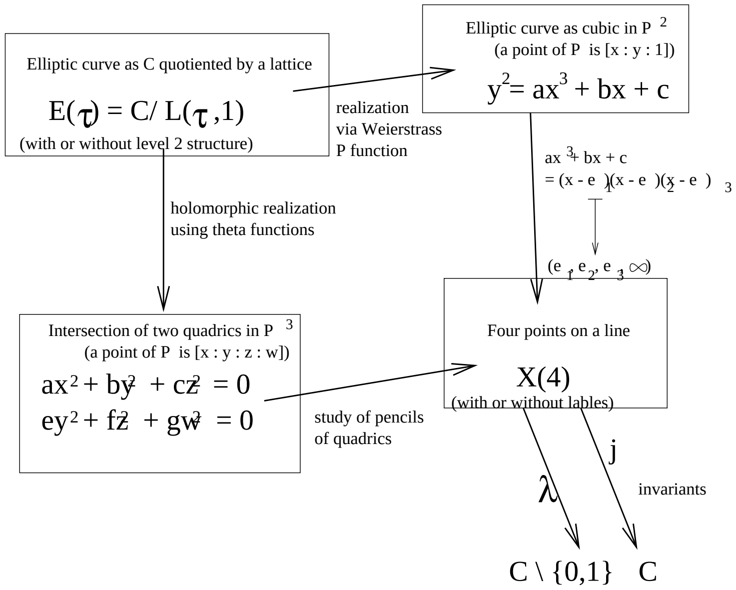
Lecture 5

We will give a realization for an elliptic curve $E(\tau)$, as an algebraic curve, and show how this gives us four points on the line, and how the space of isomorphism classes of elliptic curves is the same as the space of isomorphism classes of four points on $\mathbb{P}^1$.

Then we can use the λ and j functions which we used to get a realization for the space of sets of four points on $\mathbb{P}^1$, to get a realization of the space of isomorphism classes of elliptic curves.

Remember how we didn't like the λ function as a realization of $X(4)$, since for one thing, it had poles? Well, there is the same problem here; and the way to solve it will be to use **theta function**. We will also find it useful to consider pencils of quadrics.

Here's a picture of what we're going to do



1. Realization of an elliptic curve $E(\tau)$ by the Weierstrass $\wp$ function

An elliptic curve $E(\tau)$ is given by

$$\mathbb{C}/L(\tau, 1),$$

so a realization means a map

$$F : \mathbb{C} \to S,$$

where S is some nice space we can understand, and with $F(x) = F(y) \iff x \sim y$.

The way we do this is find a map

$$F : \mathbb{C} \to \mathbb{C}^2,$$

where F is an analytic map, and the image will turn out to be an algebraic curve in $\mathbb{C}^2$. We will have to have $F = (f, g)$ where f, g are analytic functions on $\mathbb{C}$, which are invariant under the lattice. Such functions are called **elliptic functions**.

We will look at some properties that elliptic functions must satisfy, and then construct one, which is called the **Weierstrass** $\wp$ function. We'll show that $\wp'$ is also an elliptic function, and that we can take $(\wp, \wp')$ to be the realization. We can find a relationship between $\wp$ and $\wp'$, and this tells us what the image of the map is.

1.1. elliptic functions.

DEFINITION 1.1. An **elliptic function** with respect to a lattice $L(\omega_1, \omega_2)$, is a function f which is meromorphic on $\mathbb{C}$, and invariant under the action of the lattice on $\mathbb{C}$, ie

$$f(u+v) = f(u)\, \forall v \in L(\omega_1, \omega_2)\, \forall u \in \mathbb{C}.$$

We also say that f is periodic, with periods ω_1 and ω_2.

Note, **meromorphic** means analytic with only isolated poles. A meromorphic function has a Lorenz expansion about any point. Eg, about a point a, for $|z-a|$ sufficiently small, we have

$$f(z) = \sum_{n=-N}^{\infty} a_n (z-a)^n,$$

for some coefficients $a_n \in \mathbb{C}$. If the sum starts from $-N < 0$, we say we have a pole of order (or multiplicity) N. If it starts from $N > 0$, we say we have a zero of order (or multiplicity) N. If we start from $n = -1$, we say we have a simple pole, and the residue is a_{-1}.

We have the following facts about elliptic functions:

• A holomorphic elliptic function is constant

• There is no holomorphic elliptic function with only one simple pole in the fundamental domain.

• If f is an elliptic function, the we have

$$\#\{\text{ zeros}\} = \#\{\text{poles}\}$$

where the zeros and poles are counted with multiplicity.

See text book for a proof. Actually, we will apply these ideas over and over again in getting results about elliptic functions, so it's a good idea to look them up.

1.2. The Weierstrass $\wp$ function. We have given a few facts about properties elliptic functions must satisfy; but we need to construct one. Obviously, something along the lines of

$$f(z) = \sum_{v \in L} h(v+z)$$

for some function h, would be a good choice, since then if we add $v \in L$ to z, we're just going to shift the sum, and we'll easily get $f(z+v) = f(z)$ for $v \in L$. We'd need to choose $h(z)$ so that the function will converge, eg, something like

$$f_n(z) := \sum_{v \in L} \frac{1}{(v+z)^n}.$$

As long as $n > 2$, this will converge.

The function f_2 does not converge, but we can define:

$$\wp(z) := \frac{1}{u^2} \sum_{v \in L} \left(\frac{1}{(v+z)^2} - \frac{1}{\omega^2} \right)$$

This does converge to give a meromorphic function, and satisfies $\wp'(u) = -2f_3(u)$. It is called the **Weierstrass $\wp$ function**.

We will use $\wp$ and $\wp'$ to get a map

$$\phi : E(\tau) \to \mathbb{C}$$
$$z \mapsto (\wp(z), \wp'(z))$$

This will give a realization of the elliptic curve. To do this, we need to do the following:

- Is ϕ well defined? Yes, since $\wp$ and $\wp'$ are elliptic functions.
- Show that ϕ is injective.
- What is the image of ϕ?

Now we answer the latter two points.

1.3. What is the image of ϕ. Now we find a relation between $\wp(z)$ and $\wp'(z)$, namely

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

where

$$g_2 := 60 \sum_{\omega \in L - \{0\}} \frac{1}{\omega^4}$$
$$g_3 := 140 \sum_{\omega \in L - \{0\}} \frac{1}{\omega^6}$$

So ϕ maps to a curve C given by

$$C : y^2 = 4x^3 - g_2x - g_3$$

To prove that the above relationship between $\wp$ and $\wp'$ is satisfied, we write Lorenz expansions:

$$\wp(u) = \frac{1}{u^2} + a_2u^2 + a_1u^4 + \cdots$$
$$\wp'(u) = \frac{1}{u^2} + 2a_2u + 4a_1u^3 + \cdots$$

So we just need to concoct some polynomial in terms of $\wp$ and $\wp'$ that has no poles, and then it must be constant, and so we're done. It's easy to find such a function; you just take monomials in $\wp$ and $\wp'$ and write some linear equations in terms of the coefficients a_i that must be satisfied in order for the coefficients of u^n to all vanish for $n < 0$. We end up with the above equation. g_2 and g_3 are obtained by finding expressions for a_2, a_4.

PROPOSITION 1.2. *The map ϕ is injective.*

PROOF. This is from using the fact that $\wp$ is even, $\wp'$ is odd, and from considering the number of zeros of $\wp(z) - \wp(u_1)$ and $\wp'(z) - \wp'(u_1)$. See text book for details. □

Next we take the "completion" of ϕ. The functions $\wp$ and $\wp'$ are infinite at 0, so ϕ is not defined. But we can define a map

$$z \mapsto [z^3\wp(z) : z^3\wp'(z) : z^3] \in \mathbb{P}^2$$

and since the order of the zero at 0 is 2 for $\wp$, and 3 for $\wp'$, at 0, this map will have value $[0 : 1 : 0]$. So now we can take ϕ to be defined on all of $E(\tau)$, and this is a non constant map between compact Riemann surfaces, so it must be onto. So we have:

Theorem 1.3. *The map*

$$\phi : E(\tau) \to \mathbb{P}^2$$
$$\phi : z \mapsto [z^3\wp(z) : z^3\wp'(z) : z^3]$$

gives a realization of the elliptic curve $E(\tau)$

Remark 1.4. It turns out that any elliptic function is equal to

$$R_1(\wp) + \wp' R_2(\wp)$$

where R_1 and R_2 are rational functions. See Singerman and Jones "complex variables" page 99 for a proof.

2. Elliptic curves as double covers of the line

Now we have shown that $E(\tau)$ is isomorphic to the curve

$$y^2 = 4x^3 - g_2x - g_3.$$

This is a double cover of $\mathbb{P}^1$, branched at the roots of the right hand side cubic, and at ∞. The roots are

$$e_1 := \wp(\tau/2), e_2 := \wp(1/2), e_3 := \wp((\tau+1)/2)$$

With these values of e_1, e_2, e_3, we have the following:

Theorem 2.1. *The space of isomorphism classes of elliptic curves with level two structure is isomorphic to $X(4)$, via the map:*

$$E(\tau) \mapsto (e_1, e_2, e_3, \infty)$$

Theorem 2.2. *The space of isomorphism classes of elliptic curves is isomorphic to $X\{4\}$, via the map:*

$$E(\tau) \mapsto \{e_1, e_2, e_3, \infty\}$$

2.1. Note about the realization ϕ. Note that any cubic in $\mathbb{P}^2$ can be expressed in **Weierstrass normal form**, that is, in the form

$$y^2 = 4x^3 - g_2x - g_3.$$

Note, an important structure on elliptic curves is the **group law.** It's obvious what this is on the elliptic curve as $\mathbb{C}/L(\tau)$; just addition as usual. We can show $z_1 + z_2 + z_3 \in L(\tau)$ if and only if $\phi(z_1)$,$\phi(z_2)$,$\phi(z_3)$ are all co-linear in $\mathbb{P}^2$. This will allow us to define a group law in a natural way on elliptic curves in $\mathbb{P}^2$.

3. The Lambda function, Realization for elliptic curves with level 2 structure

From an elliptic curve $E(\tau)$, we get a point (e_1, e_2, e_3, ∞), and then we can take the cross ratio, λ. We have

Theorem 3.1. *There is an isomorphism given by*

$$\lambda : \mathbb{H}/\Gamma(2) \to \mathbb{C}\{0, 1, \} \cong X(4)$$
$$\tau \mapsto \frac{e_1(\tau) - e_2(\tau)}{e_1(\tau) - e_3(\tau)}$$

4. The j invariant for elliptic curves

THEOREM 4.1. *There is an isomorphism given by*

$$J : \mathbb{H}/SL(2,\mathbb{Z}) \to \mathbb{C} \cong X\{4\}$$

$$\tau \mapsto \frac{g_2^3}{g_2^3 - 27g_3^2}$$

The expression $\frac{g_2^3}{g_2^3-27g_3^2}$ comes from expressing $j(\lambda)$ in terms of e_i, and then using the fact that

$$4x^3 - g_2x - g_3 = 4(x-e_1)(x-e_2)(e-e_3)$$

to express e_i in terms of g_i to get the result.

This function gives the J invariant for elliptic curves. Two elliptic curves are isomorphic if and only if they have the same J invariant.

LECTURE 6

5. A holomorphic realization

We want to find a map which is holomorphic. It's not possible to have a non constant holomorphic map f on $\mathbb{C}$; so what we do is look for a function

$$[f_1 : f_2 : f_3 : f_4],$$

where the f_i all change in the same way under the action of the lattice, so that their ratio remains constant. These functions will be called **theta functions**. So this will give us a map from the space of elliptic curves into $\mathbb{P}^3$. Then we have the problem of working out what the image will be. We can show that the image is the intersection of two quadtics. So, we've got a realization of the elliptic curve, and the image is the intersection of two quadrics in $\mathbb{P}^3$. When we used the Weierstrass $\wp$ function, we got a cubic, and from the cubic, we got four points on a line. So, we could get from our quadric to the cubic, via the elliptic curve, and end up with four points:

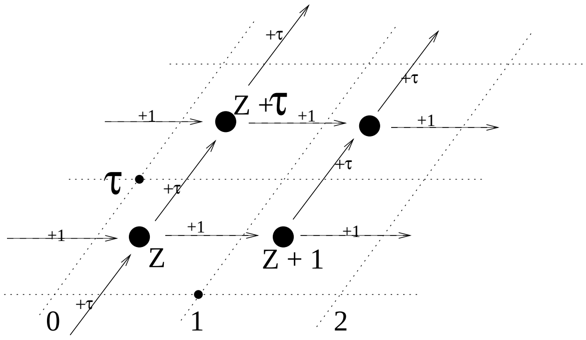
intersection of two quadric in $\mathbb{P}^3$ $\mapsto$ elliptic curve $\mapsto$ cubic $\mapsto$ four points on a line

Now, is there a way to go directly from the intersection of the two quadrics to the four points? Yes there is, and this is what the last section explains. First we consider the case of quadratics in $\mathbb{P}^2$, and then we use what we learn there to look at the case in $\mathbb{P}^3$.

6. Theta functions

We define a function $\theta_{00}(z,\tau) = \theta(z,\tau)$, which satisfies:
- $\theta(z+1,\tau) = \theta(z,\tau)$
- $\theta(-z,\tau) = \theta(z,\tau)$
- $\theta(z+\tau,\tau) = exp(-\pi i(\tau+2z))\theta(z,\tau)$

$z \mapsto z+1$ and $z \mapsto z+\tau$ generate the action of the lattice $L(\tau,1)$, and so we want a function that does not change when we translate by 1 or τ:



With τ fixed, $\theta(z,\tau)$ is almost invariant under the action of this lattice, and we construct functions $\theta_{01}(x,\tau), \theta_{10}(x,\tau), \theta_{11}(x,\tau)$ which behave in the same way under the action of the lattice. Then we can define a function

$$\mathbb{C} \to \mathbb{P}^3$$

$$z \mapsto [\theta_{00}(2z,\tau) : \theta_{01}(2z,\tau) : \theta_{10}(2z,\tau) : \theta_{11}(2z,\tau)]$$

So, what we have to do now is

- define θ,
- prove that θ satisfies the invariance properties given,
- define $\theta_{01}, \theta_{10}, \theta_{11}$,
- find out what the image is by finding relationships between the functions $\theta_{i,j}$,
- show that this map defines an embedding, so that the image will give a realization of $E(\tau)$

So that will tell us that our map really is a nice realization. In the next section, we will work out what the image is.

6.1. Definition and properties of θ. Define

$$\theta(z,\tau) := \sum_{n=-\infty}^{\infty} exp\left\{\pi i(n^2\tau + 2nz)\right\},\ z \in \mathbb{C},\ \tau \in \mathbb{H}$$

LEMMA 6.1. $\theta(z+1,\tau) = \theta(z,\tau)$

PROOF. This is obvious, since each term of the sum is invariant under $z \mapsto z+1$ □

LEMMA 6.2. $\theta(-z,\tau) = \theta(z,\tau)$

PROOF. This is clear, since the map $z \mapsto -z$ will just swap the n and $-n$ terms of the sum. □

LEMMA 6.3. $\theta(z+\tau,\tau) = exp\left\{-\pi i(\tau + 2z)\right\}\theta(z,\tau)$

PROOF. Suppose, for fixed τ, we write $a_n(z) = n^2\tau + 2nz$, so that

$$\theta(z,\tau) = \sum_{n=-\infty}^{\infty} exp(2\pi i a_n(z))$$

We want to know the affect of $z \mapsto z + \tau$

$$\begin{aligned} a_n(z+1) &= n^2\tau + 2n(z+\tau) &&= n^2\tau + 2n\tau + 2nz \\ &&&= ((n+1)^2 - 1)\tau + 2nz \\ &&&= (n+1)^2\tau - \tau + 2(n+1)z - 2z \\ &&&= (n+1)^2\tau + 2(n+1)z - \tau - 2z \quad = \quad a_{n+1}(z) - (\tau + 2z) \end{aligned}$$

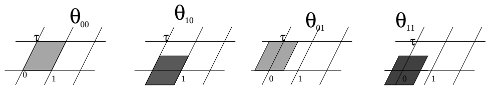
So the affect of this translation on θ is just to multiply by $exp\{2\pi i(2z+\tau)\}$. $\square$

7. The functions $\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}$

We define the functions

$$\begin{aligned} \theta_{00}(z,\tau) &= & &\theta(z, & &\tau) \\ \theta_{01}(z,\tau) &= & &\theta(z+\tfrac{1}{2}, & &\tau) \\ \theta_{10}(z,\tau) &= & exp\left\{\tfrac{\pi i\tau}{4} + \pi i z\right\} &\theta(z+\tfrac{\tau}{2}, & &\tau) \\ \theta_{11}(z,\tau) &= & exp\left\{\tfrac{\pi i\tau}{4} + \pi i(z+\tfrac{1}{2})\right\} &\theta(z+\tfrac{\tau}{2}+\tfrac{1}{2}, & &\tau) \end{aligned}$$

These functions are obtained by shifting the values of θ, (and multiplication by some function) and the following diagram is supposed to be a visual representation of this:



7.1. transfomation of $\theta_{ij}(z,\tau)$.

The following table shows how $z \mapsto z+1$ and $z \mapsto z+\tau$ affect the functions θ_{ij}:

	$z \mapsto z+1$	$z \mapsto z+\tau$
$\theta_{00}(z,\tau)$	$\theta_{00}(z,\tau)$	$exp(-\pi i(\tau+2z))\theta_{00}(z,\tau)$
$\theta_{01}(z,\tau) = \theta(z+\frac{1}{2},\tau)$	$\theta_{01}(z,\tau)$	$-exp(-\pi i(\tau+2z))\theta_{01}(z,\tau)$
$\theta_{10}(z,\tau) = exp\left\{\frac{\pi i\tau}{4} + \pi i z\right\}\theta(z+\frac{\tau}{2},\tau)$	$-\theta_{10}(z,\tau)$	$exp(-\pi i(\tau+2z))\theta_{10}(z,\tau)$
$\theta_{11}(z,\tau) = exp\left\{\frac{\pi i\tau}{4} + \pi i(z+\frac{1}{2})\right\}\theta(z+\frac{\tau}{2}+\frac{1}{2},\tau)$	$-\theta_{11}(z,\tau)$	$-exp(-\pi i(\tau+2z))\theta_{11}(z,\tau)$

So these functions do not vary in quite the same way. But we will have

	$z \mapsto z+2$	$z \mapsto z+2\tau$
$\theta_{ij}(z,\tau)$	$\theta_{ij}(z,\tau)$	$exp(-4\pi i(\tau+z))\theta_{ij}(z,\tau)$

Which means that the functions $\theta_{ij}(2z,\tau)$ all vary in the same way under $z \mapsto z+1$ and $z \mapsto z+\tau$.

We now define a map ϕ_2 by

$$\phi_2 : E(\tau) \to \mathbb{C}$$

$$\phi_2(z) : z \mapsto [\theta_{00}(2z,\tau) : \theta_{01}(2z,\tau) : \theta_{10}(2z,\tau) : \theta_{11}(2z,\tau)]$$

8. Image of the ϕ_2

To find the image of the map $z \mapsto [\theta]$, we need to find relationships between the $\theta_{i,j}$

Let

$$\theta ij(z) = \theta_{ij}(z,\tau)$$

$$\theta_{00} = \theta_{00}(0,\tau)$$

$$\theta_{01} = \theta_{01}(0,\tau)$$

$$\theta_{10} = \theta_{10}(0,\tau)$$

$$\theta_{11} = \theta_{11}(0, \tau)$$

Then we have the following result:

Proposition 8.1. *We have the relations*

$$\theta_{00}^2\theta_{00}(z)^2 - \theta_{01}^2\theta_{01}(z)^2 - \theta_{10}^2\theta_{10}(z)^2 = 0$$
$$\theta_{10}^2\theta_{01}(z)^2 - \theta_{01}^2\theta_{10}^2 - \theta_{00}^2\theta_{11}^2(z) = 0$$

Proof. To prove the first of these, we need to show that

$$\frac{\theta_{00}^2\theta_{00}(z)^2 - \theta_{01}^2\theta_{01}(z)^2}{\theta_{10}^2\theta_{10}(z)^2} = 1.$$

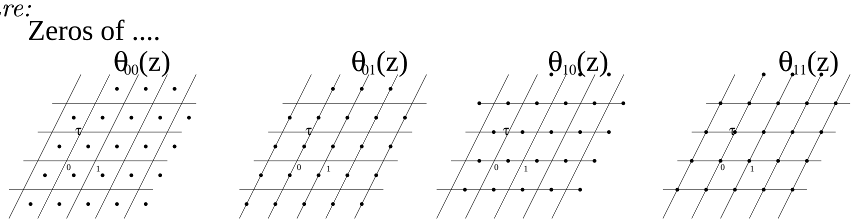
Since the numerator and denominator have the same way of transforming under the action of $L(\tau, 1)$, the quotient will be an elliptic function for $L(\tau, 1)$. We want to show it's constant; we can do that if we can show it has at most one simple pole in the fundamental domain.

We defined θ to be holomorphic, so this can only have poles when the denominator is zero. So we want to know about the zeros of θ. We'll use the following lemma, and that will give the result. The second relation is proved similarly. $\square$

Proposition 8.2. *The zeros $\theta_{ij}(z)$ are simple zeros at the following points:*

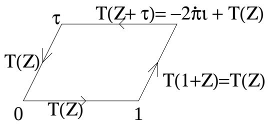
$$\begin{array}{lcl}
\theta_{00}(z) = 0 & \iff & z \in \mathbb{Z} + \tau\mathbb{Z} + \frac{1+\tau}{2} \\
\theta_{01}(z) = 0 & \iff & z \in \mathbb{Z} + \tau\mathbb{Z} + \frac{\tau}{2} \\
\theta_{10}(z) = 0 & \iff & z \in \mathbb{Z} + \tau\mathbb{Z} + \frac{1}{2} \\
\theta_{11}(z) = 0 & \iff & z \in \mathbb{Z} + \tau\mathbb{Z}
\end{array}$$

Picture:



Proof. Because of the definition of $\theta_{i,j}(z)$ in terms of $\theta(z)$, we only need to do the work for $\theta_{00}(z)$.

There are no poles in a fundamental region F and we can choose F so that it avoids zeros. So integrating $\theta'(z)/\theta(z)$ round the boundary of F will give the number of zeros times $-2\pi i$, counted according to multiplicity. Using the transformation properties:



gives us that the integral is $-2\pi i$. Hence we only have one simple zero. Where is it? We can show that $\theta_{11}(-z) = -\theta_{11}(z)$, so $\theta_{11}(0) = -\theta_{11}(0) = 0$, so this means the zero of $\theta_{11}(0)$ is at 0, and this tells us where all the other functions have zeros too. To show that $\theta_{11}(z)$ is odd, note that $\theta_{11}(z)$ is a sum of exponentials of $a_{n+\frac{1}{2}}(z+\frac{1}{2})$, where a_n is as above. This transforms as

$$a_{n+\frac{1}{2}}(-z + \frac{1}{2}) = -a_{-n-1+\frac{1}{2}}(z)$$

so we get the result. $\square$

Hence, if C is given by the intersection of two quadrics as follows:

$$C : \left\{ \begin{array}{lllllllll} F_1 & := & \theta_{00}^2 x^2 & - & \theta_{01}^2 y^2 & - & \theta_{10}^2 z^2 & & & = & 0 \\ F_2 & := & & & \theta_{10}^2 y^2 & - & \theta_{01}^2 z^2 & - & \theta_{00}^2 w^2 & = & 0 \end{array} \right.$$

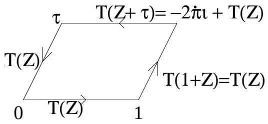
then

PROPOSITION 8.3. *The map $\phi : E(\tau) \to C$ defines a realization of the elliptic curve $E(\tau)$.*

PROOF. Injectivity: Suppose that $\phi_2(x) = \phi_2(y)$ for x, y both in a fundamental domain F. So we'll also have $\phi_2(x + \frac{1}{2}) = \phi_2(y + \frac{1}{2})$. Then choose w different from $x, y, x + \frac{1}{2}, y + \frac{1}{2}$, and find a, b, c, d so that the function

$$f(z) = a\theta_{00}(2z) + b\theta_{01}(2z) + c\theta_{10}(2z) + d\theta_{11}(2z)$$

has five zeros, at $x, y, x + \frac{1}{2}, y + \frac{1}{2}, w$. (if these points are not distinct, consider $x + \frac{\tau}{2}, y + \frac{\tau}{2}$ instead of $x + \frac{1}{2}, y + \frac{1}{2}$.)



But this is impossible, since f is holomorphic, and $f(z+1) = f(z)$, and $f(z+\tau) = exp(-4\pi i(\tau + 2z)f(z)$, so $f'/f(z+1) = f'/f(z)$, and $f'/f(z+\tau) = -8\pi i f'/f(z)$, so when we integrate round the boundary of F, we should get 4, which means only 4 zeros. This is a contradiction.

Subjectivity: The image is a Riemann surface. We have a continuous map from a compact space to a compact space; it's image must be a point or the whole thing. Since it's not a point, it must be the whole thing, and so ϕ_2 is surjective. $\square$

Note: Since the ratios of the θ functions are elliptic functions for the lattice, we should be able to write the map ϕ_2 in terms of ϕ, and so get a map directly from the intersection of the two quadrics to the elliptic curve.

REMARK 8.4. Since all elliptic functions can be expressed in terms of $\wp$ and $\wp'$, and θ_{ij}/θ_{kl} are elliptic functions, we should be able to express the theta function map in terms of the Weierstrass $\wp$ function and it's dertivative, and so relate the intersection of the two quadrics to the cubic.

The following table gives the zeros of the theta functions in the fundamantal parallelogram:

θ_{11}	$0,$	$\frac{1}{2},$	$\frac{\tau}{2},$	$\frac{\tau+1}{2}$	*odd*
θ_{01}	$\frac{1}{4},$	$\frac{1}{2},$	$\frac{\tau}{2},$	$\frac{\tau+1}{2}$	*even*
θ_{10}	$\frac{\tau}{4},$	$\frac{3}{4},$	$\frac{2\tau+1}{4},$	$\frac{2\tau+3}{4}$	*even*
θ_{00}	$\frac{\tau+1}{4},$	$\frac{3\tau+1}{4},$	$\frac{\tau+3}{4},$	$\frac{3\tau+3}{4}$	*even*

Hence the functions θ_{01}/θ_{11}, θ_{10}/θ_{11}, θ_{00}/θ_{11}, are odd, and have four poles, three of which are at zeros of $\wp'$. If two functions have zeros and poles of the same orders at the same points, they are equal up to a constant, so we have

$$\theta_{01}/\theta_{11} = \frac{(\wp(z) - \wp\frac{1}{4}))(\wp(z) - \wp(\frac{1}{2}))(\wp(z) - \wp(\frac{\tau}{2}))(\wp(z) - \wp(\frac{\tau+1}{2})}{\wp'}$$

$$\theta_{10}/\theta_{11} = \frac{(\wp(z)-\wp(\frac{\tau}{4}))(\wp(z)-\wp(\frac{3}{4}))(\wp(z)-\wp(\frac{2\tau+1}{4}))(\wp(z)-\wp(\frac{2\tau+3}{4})}{\wp'}$$
$$\theta_{00}/\theta_{11} = \frac{(\wp(z)-\wp(\frac{\tau+1}{4}))(\wp(z)-\wp(\frac{3\tau+1}{4}))(\wp(z)-\wp(\frac{\tau+3}{4}))(\wp(z)-\wp(\frac{3\tau+3}{4})}{\wp'}$$

9. How to relate the image of the theta function map to $X\{4\}$

We can get from the elliptic curve to a point of $X(4)$ via the cubic equation. But how can we go directly from the intersection of two quadrics to a point of $X(4)$?

10. conics in $\mathbb{P}^2$

A quadric in $\mathbb{P}^3$ is a surface given by an equation of degree 2:

$a_1x^2 + a_2y^2 + a_3z^2 + a_4w^2 + a_5xy + a_6xz + a_7xw + a_8yz + a_9yw + a_{10}zw$

Reminder: In $\mathbb{P}^3$, equations must be homogeneous, or else they won't make sense, since only ratios are defined in $\mathbb{P}^3$. eg, x^2+y has no meaning. x^2+y^2 has no meaning as a function, we would need something like $(x^2+y^2)/z^2$. But the zeros of x^2+y^2 are well defined, since for $\lambda \neq 0$, $x^2+y^2=0 \iff (\lambda x)^2+(\lambda y)^2=0$. If we take homogeneous coorinates, $[x:y:z:w]$, then the equation is as above. For inhomogeneous coordinates, set $w=1$, so (x,y,z) corresponds to $[x:y:z:1]$, and the equation in ordinary three space is:

$$a_1x^2 + a_2y^2 + a_3z^2 + a_4 + a_5xy + a_6xz + a_7x + a_8yz + a_9y + a_{10}z.$$

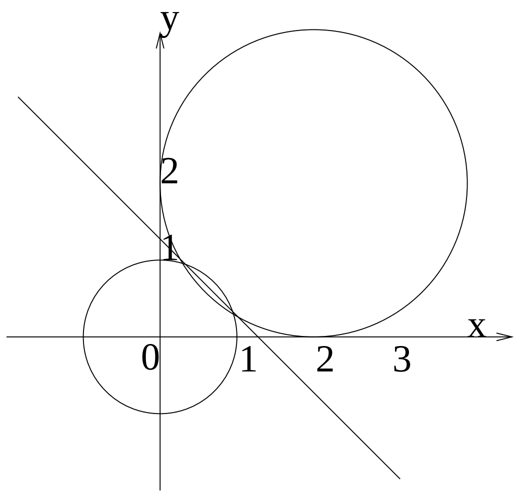
Before thinking about quadrics in $\mathbb{P}^3$, lets look at conics in $\mathbb{P}^2$. A conic in $\mathbb{P}^2$ is given by an equation of the form:

$$a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz$$

We are interested in the intersection of two quadrics in $\mathbb{P}^3$. The analogous question is about the interesection of two conics in $\mathbb{P}^2$.

question: What is the equation of the line through the points of intersection of the following two conics?

$$x^2+y^2=z^2$$
$$(x-2z)^2+(y-2z)^2=4z^2$$



Note, in inhomogeneous coordinates, these are just:

$$x^2 + y^2 = 1$$
$$(x-2)^2 + (y-2)^2 = 4$$

ie., the two circles in the diagram. We could solve by substitution, but the easiest way to solve is as follows:

Rewrite as

$$f_1 := x^2 + y^2 - z^2, \text{ and } f_2 := x^2 + y^2 - 2xz - 2yz + 4z^2$$

We want to find a line through points $[x : y : z]$ with

$$f_1([x : y : z]) = f_2([x : y : z]) = 0$$

If $f_1([x : y : z]) = f_2([x : y : z]) = 0$, then for any $a, b \in K$ (K whatever field we are working in, eg $\mathbb{C}$), we'll also get

$$af_1([x : y : z]) + bf_2([x : y : z]) = 0$$

So, taking $a = 1, b = -1$, we get at a point of intersection:

$$\begin{array}{cccccccccc} & & x^2 & + & y^2 & & & & - & z^2 \\ - & (& x^2 & + & y^2 & - & 4xz & - & 4yz & + & 4z^2) \\ \hline & & & & & & 4xz & + & 4yz & -5z^2 \end{array}$$

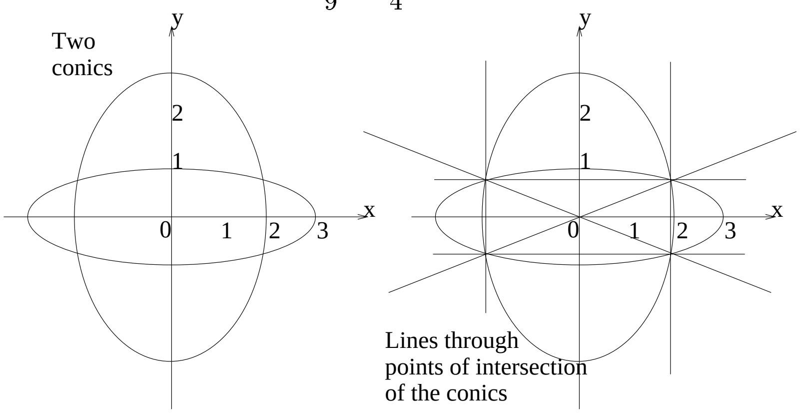
So for the points of intersection, $z(4x + 4y - 5z) = 0$, so in particular, the line

$$4x + 4y = 5z$$

must pass though both points of intersection, and is the equation of the line shown in the above diagram.

Question 2: What are the lines through the points of interesection of the the following two conics:

$$x^2 + \frac{y^2}{9} = z^2$$
$$\frac{x^2}{9} + \frac{y^2}{4} = z^2$$



To solve this, we can do the same thing, subtracting certain multiples of the first equation from the second, and factoring the resulting equation:

$$\begin{array}{ccccccccc} & 9(& x^2 & + & \frac{y^2}{9} & - & z^2) & & = 0 \\ - & 4(& \frac{x^2}{9} & + & \frac{y^2}{4} & - & z^2) & & = 0 \\ \hline & & \frac{77}{9}x^2 & & & - & 5z^2 & = \frac{1}{9}(\sqrt{77}x + 3\sqrt{5}z)(\sqrt{77}x - 3\sqrt{5}z) & = 0 \end{array}$$

$$\begin{array}{rccccll} & x^2 & + & \frac{y^2}{9} & - & z^2 & = 0 \\ -\ 9(& \frac{x^2}{9} & + & \frac{y^2}{4} & - & z^2) & = 0 \\ \hline - & \frac{77}{9.4}y^2 & & & + & 8z^2 \quad = -\frac{1}{36}(\sqrt{77}y+6\sqrt{8}z)(\sqrt{77}y-6\sqrt{8}z) & = 0 \end{array}$$

$$\begin{array}{rccccll} & x^2 & + & \frac{y^2}{9} & - & z^2 & = 0 \\ -\ (& \frac{x^2}{9} & + & \frac{y^2}{4} & - & z^2) & = 0 \\ \hline & \frac{8}{9}x^2 & - & \frac{5}{36}y^2 & & \quad = \frac{1}{36}(2\sqrt{8}x+\sqrt{5}z)(2\sqrt{8}x-\sqrt{5}z) & = 0 \end{array}$$

So we get three pairs of lines:

$$(\sqrt{77}x = -3\sqrt{5}z \text{ and } \sqrt{77}x = 3\sqrt{5}z)$$
$$\sqrt{77}y = -6\sqrt{8}z \text{ and } \sqrt{77}y = 6\sqrt{8}z)$$
$$2\sqrt{8}x = -\sqrt{5}z \text{ and } 2\sqrt{8}x = \sqrt{5}z)$$

Question 3: Why is there a difference in the above two cases?

In the first case above, there was only one line (two if you count the line $z = 0$, which can't be seen in the above diagram, since the diagram is where $z = 1$, and the line $z = 0$ is "at infinity"); in the second case there are six lines. The first case has imaginary solutions, so you can't see them in the diagram. If we work over the complex numbers, and over the projective plane, there will be no difference. However, for the pencil of conics generated by x^2 and yz, ie conics of the form $ax^2 + byz$, there are only two degenerate solutions, $[a : b] = [0 : 1]$ and $[1 : 0]$. In this case the problem is degenerate; there are only three points of intersection, and $[1 : 0]$ is a double solution.

To understand how to find all the solutions in general, think in terms of matrices. Eg, for the first question, we can write the equations as:

$$x^2 + y^2 - z^2 = (x, y, z)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and

$$x^2 + y^2 + 4z^2 - 4zx - 4zy = (x, y, z)\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Then the calculation becomes:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 2 & 2 & -5 \end{pmatrix}$$

Generally we can do this, ie, we have two conics f_1, f_2, and we can find matrices A_1, A_2, so that

$$f_1(x, y, z) = (x, y, z)A_1\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad f_2(x, y, z) = (x, y, z)A_2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Suppose the points of intersection are a, b, c, d. We'll get a pair of lines through a, b, c, d by finding $[r : s] \in \mathbb{P}^1$ such that

$$rf_1 + sf_2$$

factors into the product of two linear equations, ie, the set of zeros of this equation will be two lines. Note, we take $[r:s]$ in $\mathbb{P}^1$ because (r,s) will give the same pair of lines as $(ar:as)$ for any $a \in K$. Then we use the following result:

PROPOSITION 10.1. *If f_1 and f_2 correspond to A_1 and A_2 as above, then for $[r:s] \in \mathbb{P}^1$,*

$$rf_1 + sf_2 \text{ factors} \iff det(rA_1 + sA_2) = 0$$

PROOF. Exercise. □

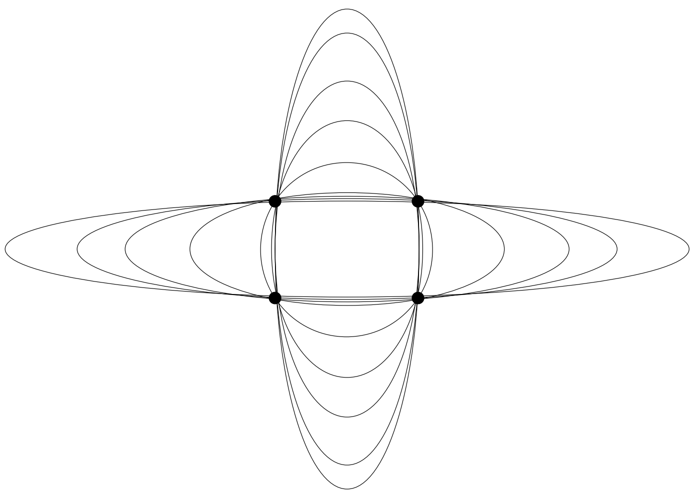
Next, note that $det(rA_1 + sA_2) = 0$ is a cubic in $[r:s]$, so there are in general exactly three solutions $[r:s]$ to $det(rA_1 + sA_2) = 0$, so these three pairs $[r_1:s_1]$, $[r_2:s_2]$, $[r_3:s_3]$ give us our three lines. In the case where there is a double root, there we will not see so many lines. (Note, when I say "$det(rA_1 + sA_2) = 0$ is a cubic in $[r:s]$", I mean since it's homgeneous, we can divide by either r or s so it's a cubic in either $\frac{r}{s}$ or $\frac{s}{r}$.)

Question 4: how is this related to configuration spaces?

First, we're really only interested in the intersection points of the two conics. And the intersection points could be given by any number of pairs of quadrics.

Eg, for the following four points:

we could choose any two conics from the following pencil of conics to define the four points:

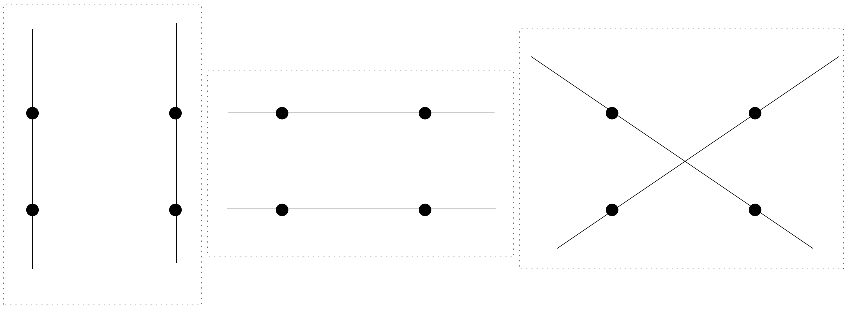


If f_1, f_2 are two members of the family, the **pencil** of conics generated by f_1 and f_2 is the collection of conics

$$rf_1 + sf_2, \ [r:s] \in \mathbb{P}^1.$$

It's called a pencil since it's parameterized by $\mathbb{P}^1$, which is $1-$dimensional, as is a pencil.

The solutions will be as in the following diagram:



These are the degenerate members of the family, ie, the members where the corresponding matrix has zero determinant.

These solutions are given by three points in $\mathbb{P}^1$, $[r_1 : s_{]}], [r_2 : s_2], [r_3 : s_3]$, corresponding to $r_i f_1 + s_i f_2$ for $i = 1, 2, 3$. This depends on the choice of f_1 and f_2.

We could have choosen g_1 g_2 We'll have

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = A \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

for some non-degenerate matrix A, and then in terms of g_1 and g_2, the solutions will be given by $A[r_i : s_i]$ for $i = 1, 2, 3$.

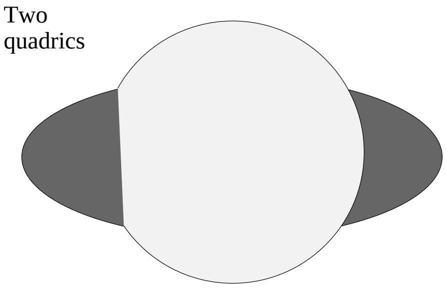
So the solution to the problem is given by three points on $\mathbb{P}^1$; and which three points is only determined up to a projective transformation. ie, we have an element of $X\{3\}$. But $X\{3\}$ is a single point, so there's only really one possibility.

10.1. intersection of two quadrics in $\mathbb{P}^3$. In the case of quadrics in $\mathbb{P}^2$, we will get a point in $X(3)$. For a dimension higher, we'll get a point in $X\{4\}$.

Take $\mathbb{P}^3$ with homogeneous coordinates $[x : y : z : w]$. Suppose C is given in $\mathbb{P}^3$ by

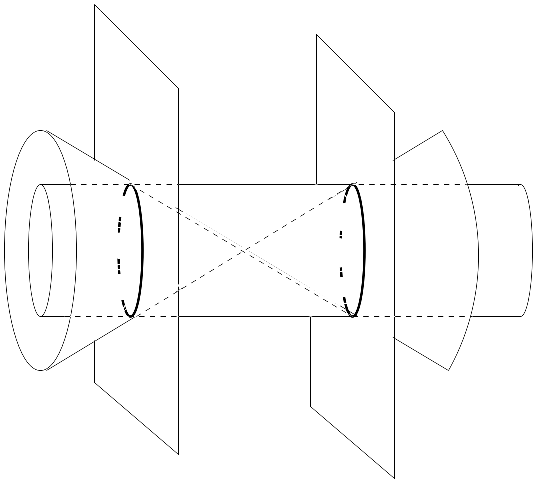
$$C : \begin{cases} f_1 & := & a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{44}w^2 + \frac{1}{2}a_{12}xy + \frac{1}{2}a_{13}zy + \frac{1}{2}a_{23}yz + \cdots \\ f_2 & := & b_{11}x^2 + b_{22}y^2 + b_{33}z^2 + b_{44}w^2 + \frac{1}{2}b_{12}xy + \frac{1}{2}b_{13}zy + \frac{1}{2}b_{23}yz + \cdots \end{cases}$$

We're given two quartics, f_1, f_2:



These have a certain curve as intersection:

We want to know "What are all the degenerate quadric surfaces passing through the intersection of f_1 and f_2?"



Note that though C is given by f_1 and f_2, it will also be defined by

$$\begin{aligned} af_1 + bf_2 &= 0 \\ cf_1 + df_2 &= 0 \end{aligned}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is any non singular matrix. So C is defined by the **pencil** of quadrics:

$$f(r, s) := rf_1 + sf_2, \; [r : s] \in \mathbb{P}^1$$

So we want to know about what are the degenerate quadric surfaces through all members of the family, which is defined by C.

A more convenient way to write the quadrics is as

$$f_1 = (x, y, z, w) \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Any quadric surface is given by a martix M:

$$(x, y, z, w)M(x, y, z, w) = 0$$

A quadric surface will look like an ellipsoid, or a sphere, or something hyperbolic, etc. In the case that $det(M) = 0$, we say that the corresponding quadric surface is **degenerate**. This gives us things like cones, and pairs of planes.

So if we write A for the matrix corresponding to f_1, and B for the matrix corresponding to f_2, then

$$f(r, s) := (x, y, z)(rA + sB)(x, y, z)^t$$

The degenerate quadrics are the members of this family with

$$det(rA + sB) = 0$$

This is a quartic in $[r : s]$, so it has four roots, ie, four points of $\mathbb{P}^1$. What exactly the roots are depends on what choice of f_1 f_2 was made for the basis of the pencil of quadrics; a change of basis will mean a projective transformation of the points.

Hence the pencil of quadrics determines a point in $X(4)$ We get the following theorem:

THEOREM 10.2. *Let $\{C\}$ be the pencil of quadrics generated by two quadric surfaces such that there are four distinct degenerate members of the pencil. $PGL(4)$ acts on this space, and there is an isomorphism:*

$$\{C\}/PGL(4) \to X\{4\}$$

PROOF. Exercise. See text book. $\square$

So now we have a way to get from an elliptic curve to a point of $X\{4\}$ which only involves holomorphic functions.