Hypergeometric functions and Configuration Spaces

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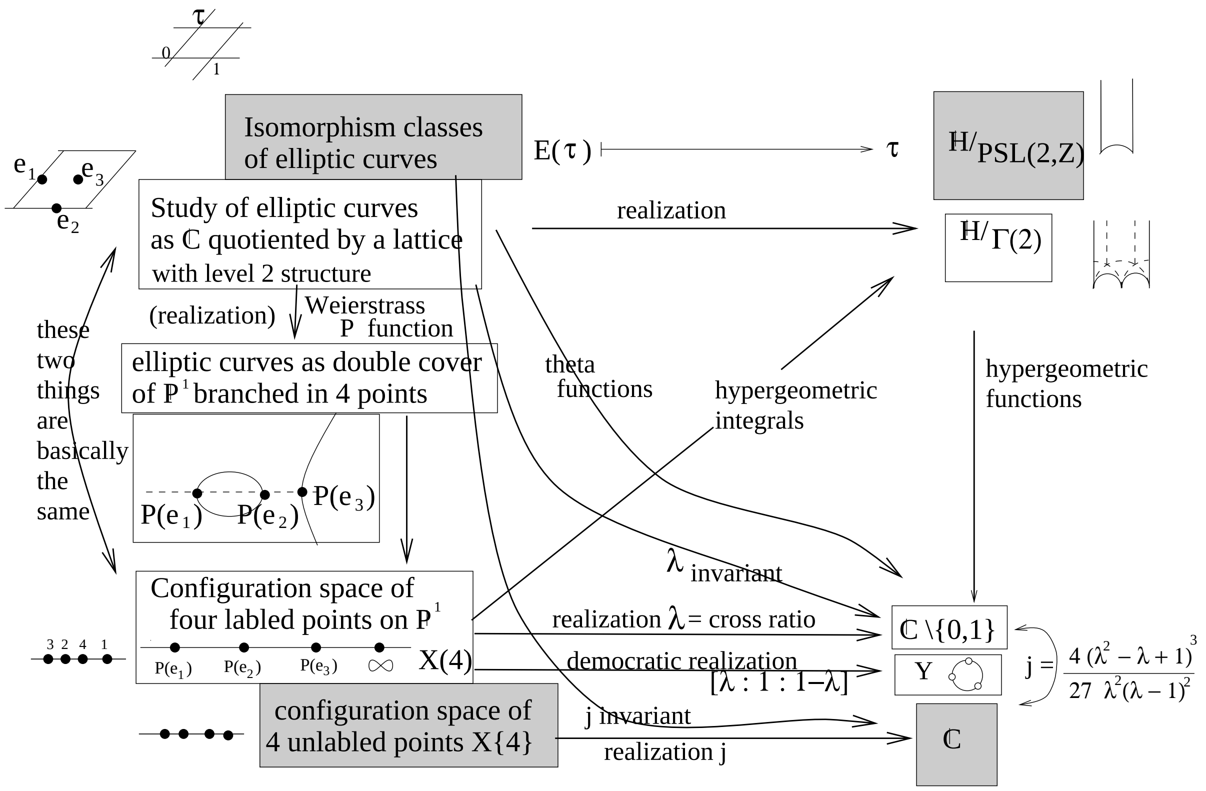
Lecture 4

So far we have introduced the idea of the configuration space of four points on the line, $X(4)$. We have also briefly given a definition of elliptic curves. These two ideas, and several others will be tied together. The following diagram shows connections which we will investigate and explain over the next few weeks. After we've done this, we will try and see what happens for other configuration spaces, eg $X(2,5)$ and $X(3,6)$.



On the Left hand side, we have various objects, and on the right hand side we have "moduli spaces" for these objects, so the things on the right are a level higher in the universe than the things on the left. A point of a thing on the left is just a point. A point of a thing on the right represents a thing on the left. You might almost think of them as "meta" objects; but they have the same kind of structure as the things on the left. Really weird, eh.

In this lecture we do the following:

- Go over the definition of an elliptic curve, and elliptic curves with level two structure.
- Show what isomorphism of elliptic curves means, ie, when are $E(\tau)$ and $E(\tau')$ isomorphic ?
- Show how the space of elliptic curves is isomorphic to the upper half plane quotiented by the action of $PSL(2,\mathbb{Z})$.

1. Elliptic Curves

We define an **elliptic curve** $E(\omega_1, \omega_2)$ by

$$E(\omega_1, \omega_2) := \mathbb{C}/L(\omega_1, \omega_2),$$

where the lattice $L(\omega_1, \omega_2)$ is given by

$$L(\omega_1, \omega_2) := \omega_1 \mathbb{Z} \omega_2 \mathbb{Z},$$

and the equivalence relation on $\mathbb{C}$ is given by

$$x \sim y \iff x - y \in L(\omega_1, \omega_2).$$

We also assume the condition $\Im(\omega_1/\omega_2)$ is positive. This assumption is made because if the imaginary part was zero, we would get one line, not a lattice, and if $\Im(\omega_1/\omega_2)$ is negative, we can just swap the order of ω_1 and ω_2, because $\Im(\omega_2/\omega_1)$ will be positive if $\Im(\omega_1/\omega_2)$ is negative.

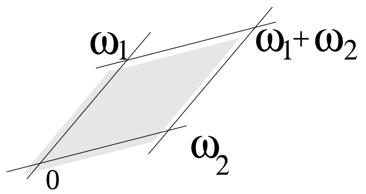
2. Fundamental parallelogram

The parallelogram F with vertices at $0, \omega_1, \omega_2, \omega_1 + \omega_2$, and any displacement of this parallelogram by any number $a \in \mathbb{C}$ is called a **fundamental parallelogram** for the elliptic curve:

Technical point:

The point of the fundamental parallelogram is the next proposition. To get it to work properly, we have to include the right amount of boundary; eg, $0 \sim \omega_1 \sim \omega_2 \sim \omega_1 + \omega_2$, so we only want to include one of these four boundary points. The fundamental parallelogram will be given by

$$F := \{\nu\omega_1 + \mu\omega_2 \in \mathbb{C} | \nu, \mu \in [0, 1)\},$$



or any displacement of this:

$$a + F = \{a + \nu\omega_1 + \mu\omega_2 \in \mathbb{C} | \nu, \mu \in [0, 1)\}.$$

Any point in $E(\omega_1, \omega_2)$ is represented by a unique point x in the fundamental parallelogram F; ie, we have

PROPOSITION 2.1. *With F as above, for all $y \in \mathbb{C}$ there is a unique point $x \in F$ such that*

$$[y] = [x].$$

PROOF. Exercise ◻

3. Structure of an elliptic curve

Note that the elliptic curve has a complex analytic structure, and the structure of an Abelian group.

• **complex analytic structure:** This means any neighbourhood of a point is isomorphic to $\mathbb{C}$, ie it locally looks like $\mathbb{C}$; pretty obvious really, since it is constructed from $\mathbb{C}$.

• **Abelian group structure:** Since we can add together any two points of $\mathbb{C}$, we can also try and add together any two points of $E(\omega_1, \omega_2)$; this will give a group structure on the elliptic curve. We just need to check this addition is well defined, ie:

LEMMA 3.1. *given x, y, z, w in $\mathbb{C}$, we have*

$$[x] = [y], [z] = [w] \Rightarrow [x + z] = [y + w].$$

PROOF. Exercise □

It is also easy to show that $[x] + [y] = [y] + [x]$, $[x] + [-x] = [0]$, $[x] + [0] = [x]$, so we have an identity, $[0]$, inverses, and it's Abelian.

4. Isomorphism

Given two elliptic curves, $E(\omega_1, \omega_2)$, and $E(\omega_1', \omega_2')$, we want to say they are isomorphic if and only if they have both of the above structures isomorphic. Ie:

DEFINITION 4.1. The curves $E(\omega_1, \omega_2)$ and $E(\omega_1', \omega_2')$ are isomorphic if and only if there is a map

$$f : E(\omega_1, \omega_2) \to E(\omega_1', \omega_2')$$

such that f is induced by an analytic map $\overline{f} : \mathbb{C} \to \mathbb{C}$, with $f([x]) = [\overline{f}(x)]$ for all $x \in \mathbb{C}$, and

$$f([x] + [y]) = f([x]) + f([y])$$

for all $x, y \in \mathbb{C}$.

We want to study what different kinds of elliptic curves we can get, so the first thing to do is try and find out what "isomorphic" means, ie, an easy way to see if two elliptic curves are isomorphic or not. Then we will be able to look at the space of all isomorphism classes of elliptic curves, and consider the geometry of that space. (Which is like asking "how do we deform one elliptic curve to get to another?")

There are basically two ways to get an isomorphism of elliptic curves. One is an equality, just got by generating the lattice in a different way; the other is by multiplication. These are given by the following two lemmas:

LEMMA 4.2. *If $L(\omega_1, \omega_2) = L(\omega_1', \omega_2')$, then*

$$E(\omega_1, \omega_2) = E(\omega_1', \omega_2').$$

We have this equality of the lattices if and only if there is an element $A \in SL(2, \mathbb{Z})$ with

$$A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}$$

PROOF. Since

$$\omega_1 \in \omega_1\mathbb{Z} + \omega_2\mathbb{Z} = \omega_1'\mathbb{Z} + \omega_2'\mathbb{Z},$$

we can express ω_1 as a linear combination of ω_1' and ω_2' with coefficients in $\mathbb{Z}$. Similarly for ω_2. This gives the matrix A relating ω_1, ω_2, to ω_1', ω_2'. We can also express ω_1', ω_2' in terms of ω_1, ω_2, which means that the matrix A is invertible, so since it's coefficients are in $\mathbb{Z}$, it has determinant ± 1. To show the determinant is 1, remember that we enforce the condition $\Im(\omega_1/\omega_2) > 0$, and show that

$$\Im(\omega_1'/\omega_2') = \frac{|\omega_2|^2}{|c\omega_1 + d\omega_2|^2} det(A) \Im(\omega_1/\omega_2),$$

where $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $\square$

PROPOSITION 4.3. *For any $a \in \mathbb{C}$, there is an isomorphism*

$$E(\omega_1, \omega_2) \to E(a\omega_1, a\omega_1)$$

given by

$$x \mapsto ax$$

PROOF. Exercise $\square$

From the above proposition, we have

$$E(\omega_1, \omega_2) \cong E(\omega_1/\omega_2, 1),$$

So any elliptic curve can be written as

$$E(\tau) := E(\tau, 1).$$

So now, given any points τ_1 and τ_2 in $\mathbb{C}$, we want to know whether are $E(\tau_1)$ and $E(\tau_2)$ isomorphic.

We can apply the above two maps. Take α any number in $\mathbb{C}$, and take $A \in SL_2(\mathbb{Z})$. Apply as follows:

$$\begin{pmatrix} \tau \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \alpha\tau \\ \alpha \end{pmatrix} \mapsto A \begin{pmatrix} \alpha\tau \\ \alpha \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha\tau \\ \alpha \end{pmatrix} = \begin{pmatrix} a\alpha\tau + b\alpha \\ c\alpha\tau + d\alpha \end{pmatrix}$$

So we have

$$\begin{array}{ccccccccc} E(\tau, 1) & \cong & E(\alpha\tau, \alpha) & = & E(a\alpha\tau + b\alpha, c\alpha\tau + d\alpha) & \cong & E(\frac{a\alpha\tau + b\alpha}{c\alpha\tau + d\alpha}, 1) & = & E(\frac{a\tau + b}{c\tau + d}, 1) \\ z & \mapsto & [\alpha z] & = & [\alpha z] & \mapsto & [\frac{\alpha z}{c\alpha\tau + d\alpha}] & = & [\frac{z}{c\tau + d}] \end{array}$$

So if $\tau_2 = A\tau_1$, then $E(\tau_1) \cong E(\tau_2)$. Is this the only way two elliptic curves can be isomorphic? Yes:

THEOREM 4.4.

$$E(\tau_1) \cong E(\tau_2) \iff \exists A \in SL_2(\mathbb{Z}) s.t A\tau_1 = \tau_2$$

PROOF. We already have one direction. Suppose $E(\tau_1) \cong E(\tau_2)$, so we have a map

$$f : \mathbb{C} \to \mathbb{C}$$

We can assume that $f(0) = 0$, since if it doesn't, we can apply an element of $SL_2(\mathbb{Z})$ to make sure it does. We can also ensure that the fundamental domains are mapped to each other; then the map has only one zero, and no poles. It's analytic, so it has a Taylor series expansion, which must be just multiplication by a constant. So we're done. $\square$

5. Moduli space of elliptic curves

The moduli space of elliptic curves means a space X, for which each point of x naturally corresponds to an elliptic curve, and all elliptic curves are represented by a unique point of X. From the above work, we know that an elliptic curve is given by $E(\tau)$, so it's represented by a point τ of $\mathbb{C}$ up to multiplication by elements of $SL_2(\mathbb{Z})$. Just to recap, $SL_2(\mathbb{Z})$, the special linear group, means 2×2 matrices with coefficients in $\mathbb{Z}$, and determinant 1. We also write $SL(2, \mathbb{Z})$ to mean the same thing. This group acts on $\mathbb{C}$ via:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b\tau}{c\tau + d\tau}$$

Note that $-A$ will have the same affect as A. To get a group of transformations of $\mathbb{C}$ such that each transformation is different, we quotient out by $\pm I$, and write

$$PSL_2(\mathbb{Z}) := SL_2(\mathbb{Z}) / \{\pm I\}.$$

This is the projective special linear group; sometimes I might write $\overline{A}$ to mean $[A] = \{A, -A\} \in PSL_2)(\mathbb{Z})$ for some $A \in SL_2\mathbb{Z}$, but a lot of the time, we just write A as usual.

From the previous work, we have the following theorem:

THEOREM 5.1. *The space of isomorphism classes of elliptic curves is given by*

$$\mathbb{H}/PLS(2, \mathbb{Z}),$$

where $\mathbb{H}$ *is the upper half plane,* $\tau \in \mathbb{C}, \Im(\tau) > 0$

6. Understanding the space of elliptic curves: Fundamental domain

Ah, now we have another quotient space to try and understand. We want to get a realization of this - actually, we've not even got a realization of the elliptic curve yet, but we did get a fundamental parallelogram to help see what was happening, and we can do the same thing here.

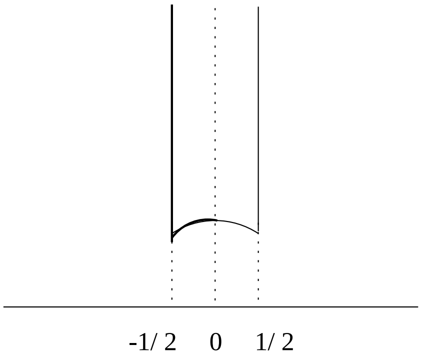
It will be useful to consider not just the action of $PSL_2(\mathbb{Z})$ on $\mathbb{H}$, but also subgroups of $PSL_2(\mathbb{Z})$.

DEFINITION 6.1. For a subgroup Γ of $PSL_2(\mathbb{Z})$, a fundamental domain for the action of Γ on $\mathbb{H}$ is a connected set F satisfying

$$\forall x \in \mathbb{H} \;\; \exists!\, y \in F \; s.t \; \exists g \in \Gamma \text{ with } gx = y$$

So a point $[x]$ of $\mathbb{H}/\Gamma$ is represented by a unique point y in F.

PROPOSITION 6.2. *A fundamental domain for $\mathbb{H}/PSL_2\mathbb{Z})$ is given by the following set:*



$$F := \left\{ z \in \mathbb{H} | -\frac{1}{2} \leq \Re(z) < \frac{1}{2}, |z| \geq 1, \textit{and } \Re(z) > 0 \Rightarrow |z| > 1 \right\}$$

So we can think of the moduli space of elliptic curves as being constructed by rolling up F so the right and left sides are glued together, and so the bottom is glued up by reflection. This is not a proper realization. Also what we get like this is not compact - we are missing the point at infinity. Later we will work out what this corresponds to, and compactify the space, just like the space of four points on a line got compactified by adding in some degenerate configuration.

7. level two structure

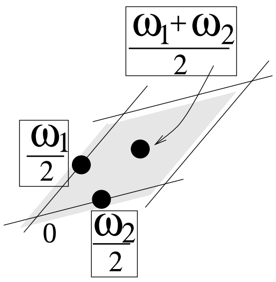
In working out how to realise this space, it will be very useful to work out how elliptic curves are related to four points on $\mathbb{P}^1$. But remember, there were two moduli spaces we got then, one for when the points were labeled, and one when we didn't care about their labels or order. So, just as we put the extra structure of "labels" onto the four points, we can put an extra structure on to an elliptic curve, which will exactly correspond to the labeling structure. This is called a "level two structure".

DEFINITION 7.1. An elliptic curve with level two structure is an elliptic curve together with three distinct points a, b, c of order two:

$$(E(\omega_1, \omega_2), a, b, c)$$

It's easy to see that the three points of order two are

$$[\omega_1/2], [\omega_2/2], [(\omega_1 + \omega_2)/2].$$



DEFINITION 7.2. An isomorphism of level two structures $(E(\omega_1,\omega_2),a,b,c)$ and $(E(\omega_1',\omega_2'),a',b',c')$ is an isomorphism $f : E(\omega_1,\omega_2) \to E(\omega_1',\omega_2')$ such that $f(a) = a', f(b) = b', f(c) = c'$.

Now, since there are six ways of ordering three points, we expect that there are six times as many elliptic curves with level two structure as there are elliptic curves, since for an elliptic curve E, the elliptic curves with level two structure:

$$(E,a,b,c),(E,a,c,b),(E,b,c,a),(E,b,a,c),(E,c,a,b),(E,c,b,a)$$

are all different (in general), but they all give the same elliptic curve.

8. Moduli space for elliptic curves with level two structure

We want to know when are $(E(\omega_1,\omega_2),(\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_1+\omega_2}{2}$ and $(E(\omega_1',\omega_2'),\frac{\omega_1'}{2},\frac{\omega_2'}{2},\frac{\omega_1'+\omega_2'}{2}$ isomorphic?

We can tackle this from two points of view. We could ask "What isomorphism of E does each permutation of the order two points a,b,c correspond to?"

We get that the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has effect:

$$\left[\frac{\tau}{2}\right] \mapsto \left[\frac{\tau/2}{\tau}\right] = \left[\frac{1}{2}\right]$$

$$\left[\frac{1}{2}\right] \mapsto \left[\frac{1/2}{\tau}\right] = \left[\frac{1}{2\tau} - \frac{1}{\tau}\right] = \left[\frac{1}{2}\frac{-1}{\tau}\right]$$

$$\left[\frac{\tau+1}{2}\right] \mapsto \left[\frac{(\tau+1)/2}{\tau}\right] = \left[\frac{1}{2} + \frac{1}{2\tau}\right] = \left[\frac{1}{2}\left(1 + \frac{-1}{\tau}\right)\right]$$

So it maps (E,a,b,c) to (E,b,a,c). And similarly we could work out how to make the other swaps.

But we are still lacking understanding of what's going on. (Ie, this way of thinking does not give "if and only if" conditions.) A more straight forwards way is the following:

PROPOSITION 8.1. *There is an isomorphism:*

$$(E(\tau),\frac{\tau}{2},\frac{1}{2},\frac{1+\tau}{2}) \cong (E(\tau'),\frac{\tau'}{2},\frac{1}{2},\frac{1+\tau'}{2})$$

if and only if there is a matrix $A \in PSL_2(\mathbb{Z})$ such that

$$A\tau = \tau',$$

and

$$A = I \ mod\ 2$$

ie, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $a \equiv d \equiv 1$ mod 2 and $c \equiv b \equiv 0$ mod 2.

PROOF. Well, we know that the part about having to have a matrix A in $SL_2(\mathbb{Z})$. Where do the remaining conditions come from? The map defined by A is given by

$$f : [z] \mapsto [\frac{z}{c\tau + d}]$$

Note that

$$[0] = \left[d\left(\frac{a\tau+b}{c\tau+d}\right)-b\right] = \left[\frac{da\tau+db-bc\tau-bd}{c\tau+d}\right] = \left[\frac{(ad-bc)\tau}{c\tau+d}\right] = f([\tau]) = f([0])$$

so $f[0]=[0]$, which should be reassuring. We need $f[\frac{\tau}{2}] = [\frac{\tau'}{2}]$. We have

$$f[\frac{\tau}{2}] = \left[\frac{\tau/2}{c\tau+d}\right] = \left[\frac{1}{2}\frac{(ad-bc)\tau}{c\tau+d}\right] = \left[\frac{1}{2}\left(d\frac{(a\tau+b)\tau}{c\tau+d}-b\right)\right] = [\frac{1}{2}(d\tau'-b)]$$

and so we want

$$[\frac{1}{2}(d\tau'-b)] = [\frac{\tau'}{2}].$$

Ie, we need,

$$\frac{1}{2}(d\tau'-b) - \frac{\tau'}{2} \in \mathbb{Z}\tau' + \mathbb{Z}$$

Ie

$$\frac{d-1}{2}\tau' + \frac{b}{2} \in \mathbb{Z}\tau' + \mathbb{Z}.$$

Since τ' and 1 are linearly independent over $\mathbb{R}$, we need both coefficients $\frac{d-1}{2}$ and $\frac{b}{2}$ to be in $\mathbb{Z}$, ie $d-1 \equiv b \equiv 0 \bmod 2$.

We can look at the images of the other points of order two similarly to get the other conditions on A. □

So we define a group

$$\Gamma(2) := \{A \in SL_2(\mathbb{Z}) | A = I_2 \bmod 2\}$$

And we have the following:

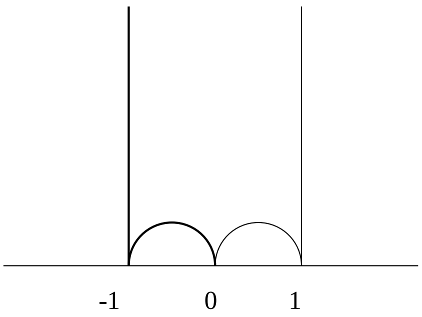
Theorem 8.2. *The space of isomorphism classes of elliptic curves with level two structure is isomorphic to*

$$\mathbb{H}/\Gamma(2)$$

9. Fundamental domain for elliptic curves with level two structure

We want to represent $\mathbb{H}/\Gamma(2)$ in the same way as we did for $\mathbb{H}/PSL_2(\mathbb{Z})$ by finding a fundamental domain F_2, such that every point x in $\mathbb{H}/\Gamma(2)$ is represented by a unique point y in F_2. We have the following theorem:

Theorem 9.1. *A fundamental domain F_2 for $\mathbb{H}/\Gamma(2)$ is given by*

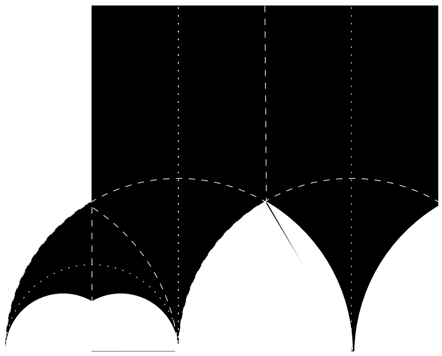


$$F_2 := \left\{z \in \mathbb{H} | -1 \le \Re(z) < 1, |z+\frac{1}{2}| \ge \frac{1}{2}, \textit{and } |z-\frac{1}{2}| > \frac{1}{2}\right\}$$

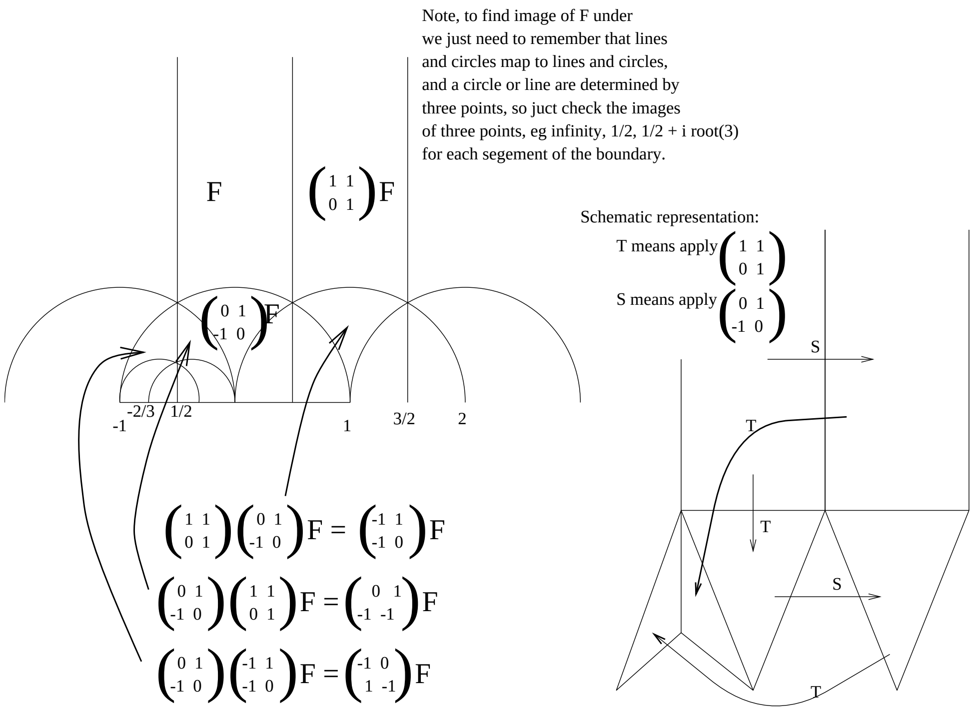
Proof. Given that we have a proof above of what F, the fundamental domain for $\mathbb{H}/SL_2\mathbb{Z}$ is, we can use that to get a proof here, by using the fact that we have a set of coset representative, eg:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

(which we refer to as r_i, i=1...64 for short). This a set of representatives of the cosets of $\Gamma(2)$ in $PSL_2(\mathbb{Z})$, and that the following domain:



is got by taking a union of r_iF for $i = 1...6$:



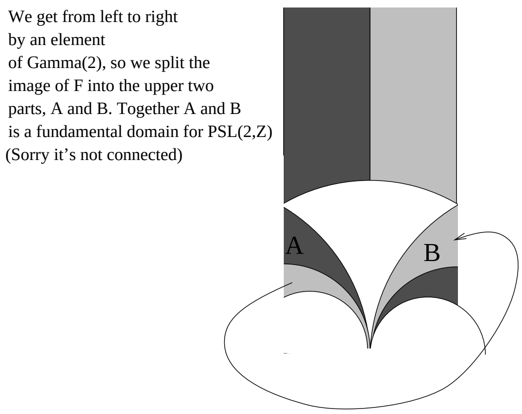
(Note, to show that this is a set of coset reps, probably the best thing to do is to get an isomorphism from $SL_2(\mathbb{Z})/\Gamma(2)$ to $SL_2(\mathbb{F}_2)$, the space of 2×2 matrices with coefficients in $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$, and show that these representatives give us representatives of each element of the group $SL_2(\mathbb{Z})/\Gamma(2)$. The group structure will make life easier.)

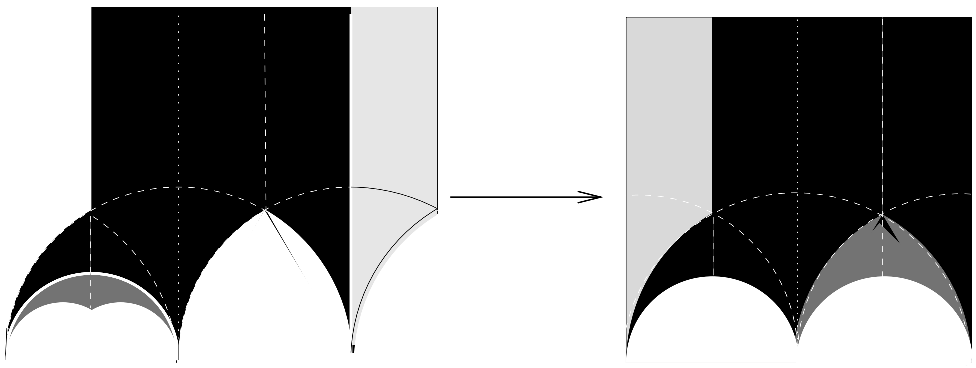
Then for $x \in \mathbb{H}$ we know that there is some $g \in SL_2(\mathbb{Z})$ with $gx \in F$. Let the representative of the cosets be called r_i for $i = 1...6$. Then even if g is not in $\Gamma(2)$, we have that $g \in r_i\Gamma(2)$ for some r_i, and then

$$gx \in F$$
$$r_i^{-1}gx \in r_i^{-1}F$$

And now $r_i^{-1}F$ is part of F_2, and $r_i^{-1}g \in \Gamma(2)$, so we've moved x into F_2 by using an element of $\Gamma(2)$, so we're done.

Actually, you might note we're not quite done - there are r_i^{-1}s, when we want r_is; but this is not a problem, since we can show r_i and r_i^{-1} are in the same coset; also the two pictures are not quite the same, but we can cut one of the domains for $\Gamma(2)$ up and shuffle it round to get the other easily. Actually, we have to cut F up a bit too, and move bits round. It's all fairly straight forwards really:





take fundamental domain apart and glue together
in a different way; it's still a fundamental domain

We also have to show that no two points in F_2 are equivalent under the action of $\Gamma(2)$. If the two points are both in the same r_iF, then they can't be equivalent under the action of $SL_2(\mathbb{Z})$, so they certainly can't be equivalent under the action of $\Gamma(2) \sup SL_2(\mathbb{Z})$. For $y \in r_iF$, there is a unique way to get from y to an $x \in F$, and this is via an element of $r_i\Gamma(2)$. (since if there were two ways, this would give us two equivalent points of F). Since it's unique, that means we can't do it with an element in $\Gamma(2)$. In a similar way, any point $y \in r_iF$ and $x \in r_jF$ can't be the same under $\Gamma(2)$, so F_2 really is a domain. □

Interesting Question

An interesting question which was raised in the class was:

If the space of isomorphism classes of four unordered points on $\mathbb{P}^1$ is parameterised by $\mathbb{H}/PSL_2\mathbb{Z}$, and the space of isomorphism classes of four ordered points on $\mathbb{P}^1$ is parameterised by $\mathbb{H}/\Gamma(2)$, then what does the space $\mathbb{H}/\Gamma_0(2)$ parameterize, where $\Gamma_0(2)$ is the group of matrices in $SL_2(\mathbb{Z})$ of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cong \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod 2.$$

(Actually, we've not really shown the first two results yet; we will do that next lecture.)

What's next?

In the next class we will see how to realize elliptic curves, and their moduli space, and show how this is the same thing as four points on the line, and how labeling of the points corresponds to the level two structure of the elliptic curves. We will get λ and j invariants for elliptic curves.