Coin Rearrangement Problems

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Abstract

We analyze a family of 1-player games, involving the movement of coins from one configuration to another. Moves are restricted in that a coin can only be placed in a location that is adjacent to at least two other coins. The goal of this paper is to specify exactly which of these games are solvable. We examine two boards, based on equilateral triangles or squares, and also consider the problem on a general graph.

1 Introduction

Let start with an example. Consider the two configurations of "coins" shown in Figure 1; the idea is to get from the position on the left to that on the right, by moving coins so that any coin can be moved to a position where it will touch at least two other coins.

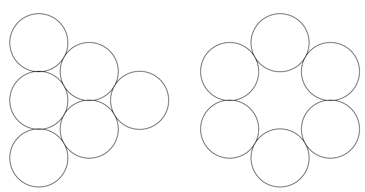


Figure 1: Puzzle: Get from the first configuration to the second

Here is another example, with the same rules:

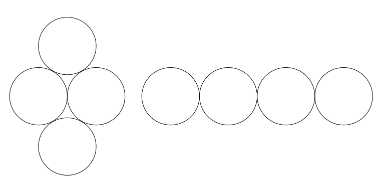
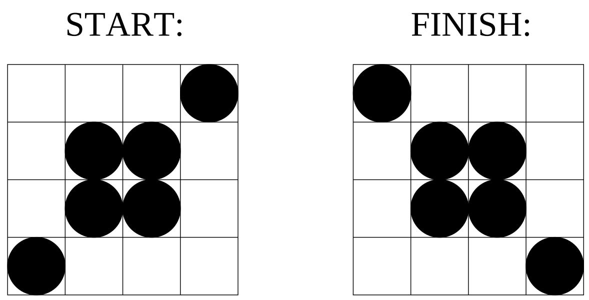


Figure 2: Can you get from start to finish, in only 4 moves?

This is an example of a certain kind of puzzle, that has appeared in various places, for example in [WW], [L2], and many examples in [B]. In [L1], the 24 possible ways of solving the puzzle in

Figure1 are enumerated. Sometimes extra rules are given, eg, coins must slide, and can not be lifted, or coins must be slided so that at all times they touch another coin. Many variants are possible.

We can generalise to consider puzzles starting and ending in other positions, and played on other grids, and other graphs, for example:



In this example, valid moves are to positions next to at least two coins.

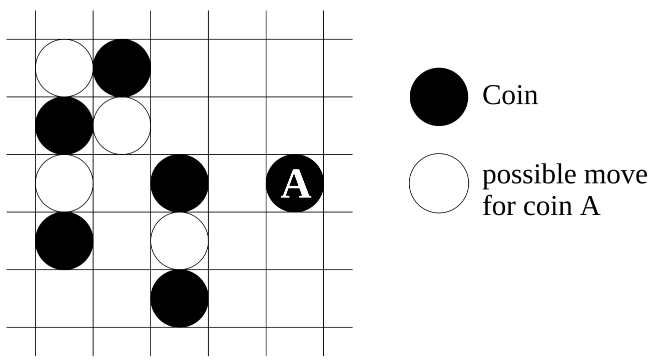


Figure 3: Valid moves

[Motivation of the problem, and some examples and references from the literature.]

[Outline of the following sections.]

2 General Theory

This section provides some general concepts that we will find useful for analyzing the games. We begin in Section 2.1 with a formal problem statement and definitions of terminology. Section 2.2 presents the notion of the span of a configuration, which essentially specifies the reachable positions given an unlimited supply of extra coins.

2.1 Problem Statement

Let us consider a formal framework for this family of games. A game is played with a collection of coins, on an undirected graph G called the *board*. Let P denote the set of *positions*, that is, vertices of G. A *configuration* is a subset of P. The *size* of a configuration is its cardinality, that is, the number of coins that it uses.

A play consists of an initial configuration, and a sequence of valid moves and resulting configurations. A *move* is a pair (p, q) of positions. It is *valid* if p is occupied and q is unoccupied; here a position is *occupied* if it is an element of the current configuration. Depending on the problem, we will place additional restrictions on the validity of moves. For example, the *k-adjacency restriction* disallows a move (p, q) when q has less than k neighboring coins.

If a move (p, q) is applied to a configuration A, the *result* is the configuration $A' = A - p + q$, that is, A with element p replaced by q. A configuration A is *transferable* into B if there is a sequence of valid moves that brings A to B. The main goal of this paper is to determine whether a given configuration is transferable into another given configuration.

2.2 Span

The *span* of a configuration A is defined recursively as follows. Let $(p_1, q_1), \ldots, (p_n, q_n)$ be all the valid moves from A. If $n = 0$, the span of A is just A itself. Otherwise, it is the span of $A \cup \{q_1, \ldots, q_n\}$. If this process never terminates, the span is defined to be the limit, which exists since it is a countable union of finite sets.

The span of a configuration lists all the positions you could reach if you had an unlimited number of coins to place. Hence, if A is transferable into B, then $B \subseteq \operatorname{span} A$, because you can only do less with fewer coins. Because of this and the definition of span, $\operatorname{span} B \subseteq \operatorname{span} A$; in other words, valid moves can never cause the span of the current configuration to increase.

Suppose we employ the k-adjacency restriction. Then a *chain* is a sequence of positions, such that the distance between two consecutive positions is at most k. We will use chains as basic "units" for creating a desired span.

For each type of board, the basic approach of our analysis is as follows. First, we give some basic properties of the span. Then we determine the meaning of the span in terms of transferability.

3 Square Grid

This section considers games on the square grid (or lattice) $\mathbf{Z} \times \mathbf{Z}$ with the 2-adjacency restriction. [Though we may generalize to k-adjacency and $\mathbf{Z^d}$.] We characterize the transferability of configurations by showing that, other than the span not being able to increase, essentially anything is possible.

All of our transformations will exploit two extra coins. They are *extra* in that their positions are irrelevant in terms of the destination configuration. However, their starting positions can be specified and are not arbitrary. This idea will be formalized and justified below.

[Justification of why two extra coins are necessary (zero and one extra coins are boring).]

3.1 Definitions

In this section we give definitions as they are defined for the square grid puzzles.

A **coin** is an element of $\mathbf{Z} \times \mathbf{Z}$.

A **position** for a coin is also an element of $\mathbf{Z} \times \mathbf{Z}$.

To distinguish between coins and positions for coins, we denote the first by $[a, b]$, and the second by (a, b).

The **distance** between two coins $[a_1, b_1]$ and $[a_2, b_2]$ is given by $d([a_1, b_1], [a_2, b_2]) = |a_1 - a_2| + |b_1 - b_2|$.

Coins $[a_1, b_1]$, $[a_2, b_2]$ are adjacent iff $d([a_1, b_1], [a_2, b_2]) = 1$. Adjacency for positions of coins is defined similarly. Figure 4 shows the four positions adjacent to coin A.

A **configuration** is a finite set of coins, ie, a finite subset of $\mathbf{Z} \times \mathbf{Z}$.

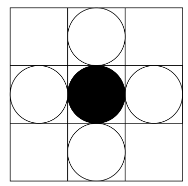


Figure 4: Adjacent positions

For a configuration P, define $v_1(P)$, the set of valid positions for P, by

$$v_1(P) := \{c \in \mathbf{Z} \times \mathbf{Z} \setminus \mathbf{P} | \exists \mathbf{c_2}, \mathbf{c_3} \in \mathbf{P}, \mathbf{c_2} \neq \mathbf{c_3}, \mathbf{d}(\mathbf{c}, \mathbf{c_2}) = \mathbf{d}(\mathbf{c}, \mathbf{c_3}) = \mathbf{1}\}$$

Recursively define $v_n(P)$ by

$$v_n(P) := v_1(v_{n-1}(P)),$$

for $n > 1$, and define $v_0(P) = P$.

The **span** of a configuration P is given by

$$\operatorname{span} P := \cup_{i \geq 0} v_i(P).$$

For example, Figure 5 shows the span of a certain configuration, with the numbers indicating the elements of the set $v_i(P)$.

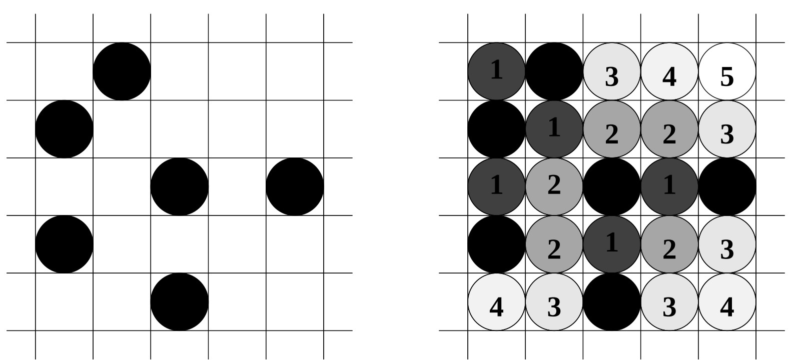


Figure 5: Span

There is a **valid move** from configurations P_1 to configuration P_2 if there are coins $c_1 \neq c_2$ such that $P_1 \setminus \{c_1\} = P_2 \setminus \{c_2\}$, and $c_2 \in v_1(P_1) \setminus \{c_1\}$.

By **rectangle** we mean a configuarion

$$R_{(a_0,b_0),(a_1,b_1)} = \{[a,b] | a_0 \leq a \leq a_1, b_0 \leq b \leq b_1\}$$

The distance between two rectangles R_1, R_2 is given by

$$d(R_1, R_2) = inf_{c_1 \in R_1, c_2 \in R_2}(d(c_1, c_2))$$

The **game-graph G**, is a directed graph, where there is a vertex v_P for every subset P of $\mathbf{Z} \times \mathbf{Z}$ and an edge $e_{(P,Q)}$ from v_P to v_Q exactly when there is a valid move from P to Q.

A **sequence of moves** is a finite directed subtree of $\mathbf{G}$, such that exactly two vertices have degree 1, and all the edges point in the same direction, ie., a subgraph of the form

$$v_{P_1}, e_{(P_1,P_2)}, v_{P_2}, e_{(P_2,P_3)}, v_{P_3}, e_{(P_3,P_4)}, v_{P_4}, \dots v_{P_{n-1}}, e_{(P_{n-1},P_n)}, v_{P_n}.$$

A position P is **transferable** to a position Q if there is a sequence of moves starting at P and finishing at Q.

A puzzle is specified by giving two configurations P and Q, and asking whether P is transferable to Q. We are going to answer the question of exactly which puzzles can be solved in Theorem 3.1. We will also give bounds on the number of moves required, ie, the distance between P and Q in $\mathbf{G}$.

Lemma 3.1 *For any configuration A,* $\operatorname{span} A$ *is a disjoint union of rectangles, which are distance at least* 3 *from each other.*

Proof. Since a single coin is a rectangle, $\operatorname{span} A$ is certainly a disjoin union of rectangles. We also want to show that each connected component is a rectangle, and disjoint connected components are distance 3 or more apart.

Suppose C is a connected component of $\operatorname{span} A$. Define $(a_0, b_0), (a_1, b_1)$ by

$$a_0 = min\,\{a_i|[a_i, b_i] \in \operatorname{span} A\}\,,\; b_0 = min\,\{b_i|[a_i, b_i] \in \operatorname{span} A\}$$

$$a_1 = max\,\{a_i|[a_i, b_i] \in \operatorname{span} A\}\,,\; b_1 = max\,\{b_i|[a_i, b_i] \in \operatorname{span} A\}$$

We may assume that $a_0 \neq a_1$ and $b_0 \neq b_1$, since in these cases $\operatorname{span} A$ is a rectangle, and the result is easy. We claim that $C = R_{(a_0,b_0),(a_1,b_1)}$. Suppose not, then there is a position $(a, b) \in R_{(a_0,b_0),(a_1,b_1)} \setminus C$ with one of the positions adjecent to (a, b) in C. We can follow round the boundary of C from this position until we reach a position in $R_{(a_0,b_0),(a_1,b_1)} \setminus C$ with two neighbours in C. But then this position would also be in C. Hence we have a contradiction, and thus the connected components are rectangles. Clearly any two disconnected rectangles must be at least distance three apart, or else a coin could be added, contrdicting that we have added all possible coins in constructing $\operatorname{span} A$. ■

The more coins we have in a rectangle, the easier it is to reach configurations, for example, Figure 6 shows a configuration of 13 coins in a 4×4 square; clearly is it possible to get from any configuration of 13 coins on this grid to any other configuration, in less than 3 moves.

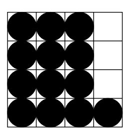


Figure 6: Nearly Full

Since the fewer coins, the more difficult and interesting, we will prove theorems about how to get between "minimal" positions, and use this to answer the question of transferability in general.

Given a rectangle R, we define a **minimal spanning set** for R to be a configuration P such that $\operatorname{span} P = R$ and for all $c \in P$, $\operatorname{span}(P \setminus C) \neq \operatorname{span} P$. Figure 7 shows some examples of minimal spanning sets.

Lemma 3.2 *For a rectangle R, if A is a minimal spanning set for R, then*

$$\left\lceil \frac{n}{2} \right\rceil \leq |A| \leq \left\lceil \frac{2}{3} n \right\rceil,$$

Where $n = max\,\{d(a, b)|a, b \in R\}$.

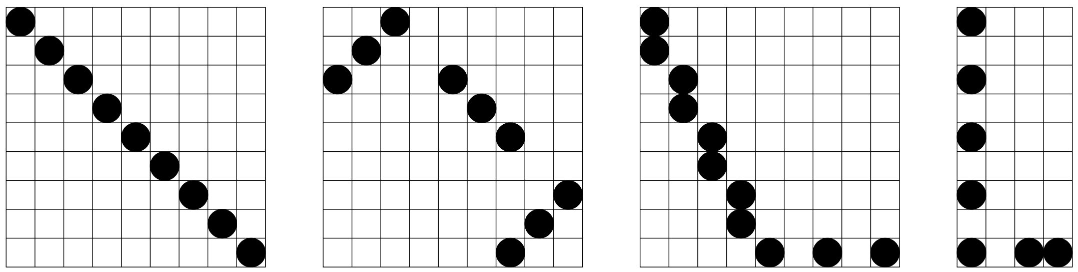


Figure 7: Minimal spanning sets.

We will be able to prove this lemma after obtaining some more results.

Note, these bounds are attained, as exemlified in the examples of Figure 7.

Because of the following lemma, it is not very interesting to play with mimimal spanning configurations.

Lemma 3.3 *If A is a minimal spanning set, and if P is a configuration obtained by making one valid move from A, then* $\operatorname{span} P$ *is strictly less than* $\operatorname{span} A$*.*

Lemma 3.4 *If A is a configuration such that there is a coin $a \in A$ with $A \setminus \{a\}$ is a minimal spanning set, and if P is a configuration obtained by moving any coin other than a, then* $\operatorname{span} P$ *is strictly less than* $\operatorname{span} A$*.*

An example of the result of this lemma is shown in Figure 8.

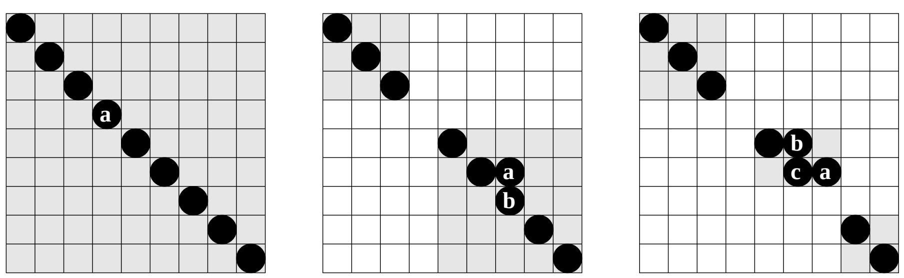


Figure 8: Moving any coin from the first position, eg a, reduces the span of the configuration. Moving a second coin b further reduces the span. But then c can be moved without reducing the span. Note that in this example we still need to reduce the span further to get to positions in which more than coins a, b and c move. In this figure, the shaded areas represent the span of the sonfiguration.

This means that we can never reach a postion B with $\operatorname{span} B = \operatorname{span} A$ from a minimal spanning set A.

[also we will prove this for the general graph - email Erik sent - should we put that result earlier and just refer to it here, or put it in a later section on results for the general graph?]

We now introduce the idea of "extra" coins.

An **extra** coin of a configraution P is a coin c such that $c \in P$, $\operatorname{span} P \setminus \{c\} = \operatorname{span} P$, and c is a valid position in $\operatorname{span} P \setminus \{c\}$.

A **configuration with** k **extra coins** is a configuration P such there is a sequence of coins $c_0, c_1, \ldots c_{k-1}$ with and c_i an extra coin for P_i for $0 \leq i \leq k-1$, where $P_0 := P$, and for $1 \leq i \leq k-1$, $P_i := P_{i-1} \setminus \{c_{i-1}\}$.

If we start with a minimal configuration A, such that $\operatorname{span} A$ is connected, then after one move from A, we will be at a configuration B with $\operatorname{span} B$ consisting of 2 connected components, R_1 and R_2. Consider the coin b which was moved; then $A \setminus \{b\} = A_1 \sqcup A_2$, such that $\operatorname{span} A_1 = R_1, \operatorname{span} A_2 = R_2$. We can assume that $B = B_1 \sqcup B_2$, with $B_1 = A_1$ and $B_2 = A_2 \cup \{b\}$. Then A_1 and A_2 are minimal spanning sets, and the coin b can be moved between the two sets without reducing $\operatorname{span} B$; however, moving any other coin will further reduce the span of the game.

This is illustrated in figure 8.

Suppose we are aiming to transfer from configruation A to configruation B, with $|A| = |B|$, and $\operatorname{span} B \subset \operatorname{span} A$, via a sequence of moves $A_0 = A, A_1, A_2, \ldots A_n = B$. Then we have $\operatorname{span} A_{i+1} \leq \operatorname{span} A_i$, so at some stage $\operatorname{span} A_k = \operatorname{span} B$, and from thence on, we are playing with a fixed number of connected components, say $\operatorname{span} B = R_1 \sqcup R_2 \ldots R_m$. Suppose coins $c_1, \ldots c_l$ can be removed from A_k so that $A_k \setminus \{c_1, c_2, \ldots c_l\}$ is a minimal spaning set for $\operatorname{span} B$. Let $A_k \setminus \{c_1, c_2, \ldots c_m\} \cap R_j = A_{k,j}$. Now we have reduced our game to a disjoint union of m games with extra coins, that is, the problem of tranferabilty from $A_{k,j}$ to $B \cap R_j$, with up to l extra coins allowed.

Hence we may restrict ourselves to considering games where the starting and ending positions have the same span, and where there are exactly 2 extra coins, ie., our starting configurations A is such that there are a coins c, d with $\operatorname{span} A \setminus \{c, d\} = \operatorname{span} A$.

Theorem 3.1 *Given two configurations A and B, A is transferable into B (using two extra coins) if and only if $|A| = |B|$, and* $\operatorname{span} B \subseteq \operatorname{span} A$.

We showed the "only if" part of this theorem in Section 2.2. We will show the "if" part by first transfering A into a chain with the same span, transfering that into another chain with the span of B, and then transfering into B. These transformations are valid by the following lemmas.

First we make a few more definitions.

A **diagonal** of a rectangle $R_{(a,b)}$ is a chain $C = \{c_1, \ldots c_n\}$ starting at a, finishing at b, such that C is a minimal spanning set, and $d(c_i, c_{i+1}) = 2$. Note, a rectangle $R_{(0,0),(a,b)}$ has a diagonal if and only if $a + b$ is even. In this case, there are precisely

$$2\Re\left(\sum_{k+j=a} \binom{a+b}{j} \binom{a+b}{k} \omega^j\right)$$

diagonals, where ω is a primative third root of unity. [can proving this with generating functions, but probably not important enough to include proof - I just wanted to find something to count!]

In the case that $a + b$ is odd, we may define **odd diagonals** to be minimal spanning chains such that there are exactly $\frac{a+b+1}{2}$ coins in the chain, and there is just one pair of adjacent coins. For $a + b$ odd, the diagonal $R_{(0,0),(a,b)}$ has exactly ? odd diagonals.

Lemma 3.5 *Given 2 extra coins, a minimal spanning chain C for a rectangle R, starting and ending at opposite corners of R, is transferable to a diagonal (or an odd diagonal), plus $|C| - \frac{d(R)}{2}$ extra coins. (Here $d(R)$ is the maximum distance of any two postions in R.*

Proof. This result can be proved by induction on the size of the rectangle. In fact we will aim to achieve a particular kind of diagonal, as shown in the examples of Figure 9. We assume for now that C starts and ends at a and b, with a and b opposite corners of the rectangle. Figure 10 shows the induction step; we find some point where we can divide the rectangle in two; apply induction, and then use the extra coins to make the diagonal into the required form, and finally when we have the straight part along the bottom of the rectangle, we can remove extraneous coins.

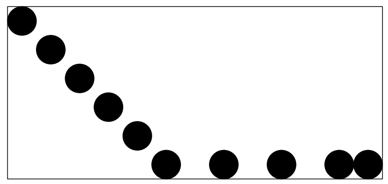


Figure 9: Example of a nice diagonal.

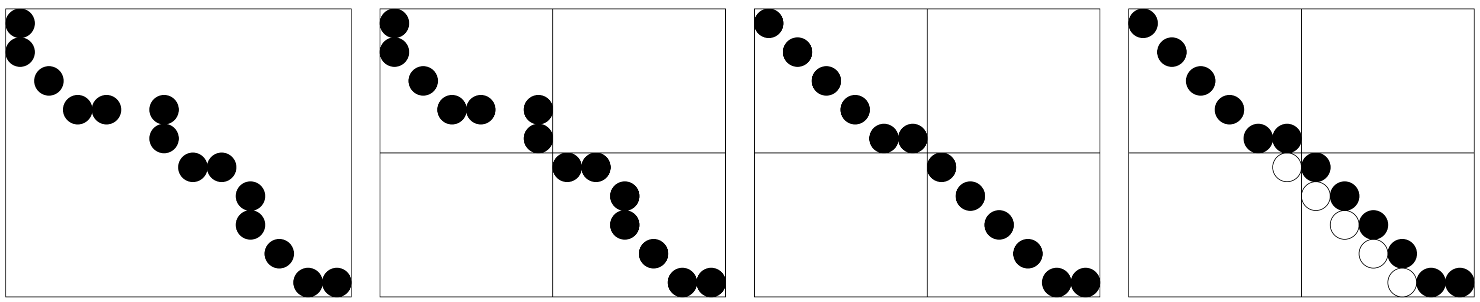


Figure 10: Induction step.

■

Lemma 3.6 *Any two diagonals A and B of a rectangle R, starting and ending at the same points, are intertransferable.*

Proof. It is easy to see how to step through a sequence of all the diagonals of a rectangle, which start and end at the same point. Hence result. [Should counting be used to make this more explicit?]. ■

Lemma 3.7 *Any two spanning chains in a rectangle are intertransferable.*

Proof. This can be achieved by transfering from the chains to diagonals in a reversible way, and then transfering between diagonals using the above two results on diagonals. Note that if the chain does not start and end at opposite corners of the rectangle, we break the board up into smaller rectangles, such that in the restriction to each rectangle this condition is satisfied, and then apply the above results on diagonals and chains. ■

Lemma 3.8 *If A and B are two diagonals between opposite corners of a rectangle, then A and B are intertranserable.*

Proof. [Need to explicitly write down that algorithm for reversing diagonals. ■

Lemma 3.9 *Given any configuration A, (with $\operatorname{span} A$ connected), there exists a chain B such that A and B are inter-transferable using two extra coins.*

Proof. Let R_i be a collection of minimal rectangles which contain the connected components of A. The idea is to rearange the coins in each rectangle, so that the connected components can be joined together, and so increase in size, and use some kind of induction. ∎

3.2 Reordering Coins

Let us consider labeled coins. To support this in our model, we must change the definition of a configuration to be a one-to-one function from C to P, where C is the set of coins.

Theorem 3.2 *Any two configurations A and B with the same images are inter-transferable using two extra coins.*

4 Triangular Grid

Lemma 4.1 *If A is a configuration with a valid move, then* $\operatorname{span} A = P$. *Otherwise,* $\operatorname{span} A = A$.

Theorem 4.1 *If A and B are configurations both with valid moves, then A is transferable into B using one extra coin.*

[Is 1 the correct value? When are no extra coins necessary?]

5 Conclusion

References

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