The L-series of certain rigid Calabi-Yau threefolds

H. A. Verrill

Abstract. The L-series of some pencils of Calabi-Yau type varieties are investigated. Specifically, we show how certain rigid Calabi–Yau threefolds should be modular in the sense that the L–series is given by a modular form of weight four.

1. Introduction

A smooth projective variety $\mathcal{Z}$ of dimension n is called a *Calabi–Yau type* variety if $H^i(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) = 0$ for every i, $0 < i < n$, and $K_{\mathcal{Z}} \equiv 0$, i.e., the canonical bundle is trivial.

Calabi-Yau type varieties are of interest to physicists due to their relation to string theory, mirror symmetry, among others, and are of interest to number theorist because they are a certain generalization of elliptic curves.

Here we are most interested in the number theoretic aspects. Many results which are well understood for elliptic curves are still unsolved problems for higher dimensional cases. We are interested in investigating these higher dimensional analogues.

For instance, the Langlands program predicts that a certain class of Calabi-Yau threefolds should be "modular" in the sense that the L-series are given by weight four modular forms.

The varieties described in this paper are Calabi–Yau type varieties with a fibration of Calabi–Yau type varieties of one dimension lower. Specifically, we will consider Calabi–Yau type varieties $\mathcal{Z}_R$ constructed from the root lattices R. We will show that Calabi-Yau threefolds $\mathcal{Z}_{A_3}$ and $\mathcal{Z}_{A_1\times A_1\times A_1}$ are rigid (i.e., $h^{1,2} = 1$) and that they should be modular.

An additional, also very interesting reason for looking at these particular examples is their relationship with the Monster group. For instance, the mirror maps associated to our Calabi–Yau type varieties of dimension ≤ 3 are related to Thompson series arising from the representation theory of the Monster group. The monster group is still not well understood. It is hoped that this work and future extensions of this work will help explain relationships between "Monstrous Moonshine" ([CN]) and these Calabi-Yau type varieties.

1991 *Mathematics Subject Classification.* 11G40, 14J32.
supported by Professor Noriko Yui through her NSERC Research grant.

The construction of the varieties in this paper was partly given in [V1]. A slight modification of this construction is given here to ensure that not only the fibers, but also the ambient space is a Calabi-Yau type variety.

The construction starts with a root lattice R of dimension n, and uses a toric geometry construction to produce a pencil of $n-1$ dimensional Calabi-Yau type varieties, $\mathcal{Z}_R$. To illustrate the construction, look at 2-dimensional root lattices, for example, the lattice $A_1 \times A_1 \cong \mathbb{Z} \times \mathbb{Z}$. We obtain a pencil of elliptic curves.

In fact, it was shown by Beukers, Peters and Stienstra (see, [SB], [PS] and [B]) that the pencils constructed from various root lattices (e.g., $A_1 \times A_1$, A_2 and $A_1 \times A_1 \times A_1$) coincide with pencils of elliptic curves and K3-surfaces studied in relation to Apery's recurrence relation, and to the formal Brauer groups of K3 surfaces. Furthermore, they calculated the the Picard-Fuchs equation, monodromy group, and the L-series for them. The monodromy group turns out to be one of groups related to the monster group ([CN]).

In this paper I will determine the L-series of the pencil derived from the A_3 root lattice, and collect all these results together, and describe some relationships between the different cases, with the aim of shedding light on a larger picture, of which this is a small part.

The results are summarized in the following table:

	Variety			
	$\mathcal{Z}_{A_1\times A_1}$	$\mathcal{Z}_{A_2}$	$\mathcal{Z}_{A_1\times A_1\times A_1}$	$\mathcal{Z}_{A_3}$
fibre	elliptic curve	elliptic curve	K3 surface	K3 surface
monodromy group	$\overline{16}$	$\overline{12}$	$\overline{12\|2+6}$	$\overline{12+12}$
periods	$16^4 4^4/8^6$	$12^4 2^2/6^2 4^4$	$1^6 6^6/2^6 3^6$	$2^4 6^4/1^2 3^2$
L-series	4^6	$2^3 6^3$	$2^4 4^4$	$1^2 2^2 3^2 6^2$
v_n	$\binom{n}{\frac{1}{2}n}$	$\sum_{p+q+r=n} \binom{n}{p,q,r}^2$	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$	$\sum_{p+q+r+s=n} \binom{n}{p,q,r,s}^2$
Reference	$[SB]$	$[SB]$	$[PS][B][V2]$	$[V1],[V2]$

Here we ought to explain the notations and symbols in the table.

• In row for the periods and L–series, the notation a^α is shorthand for $\eta(a\tau)^\alpha$, as used in [CN]. Here $\eta(\tau)$ is the Dedekind eta function given by $\eta(\tau) = q^{\frac{1}{24}} \prod(1-q^n)$, where $q = e^{2i\pi\tau}$.

• In row three, the monodromy group is given up to isomorphism. The notation used in this row is also as given in [CN], where "$a+b$" is written to mean the group generated by $\Gamma_0(a)$ together with certain Fricke involutions, and "$\overline{a+b}$" means we have quotiented by $\pm I$.

• The row for the periods gives a modular form $f(\tau)$ such that the period lattice is given by

$$f(\tau)(\mathbb{Z} \times \tau\mathbb{Z} \times \tau^2\mathbb{Z}),$$

for $\mathcal{Z}_{A_1\times A_1\times A_1}$ and $\mathcal{Z}_{A_3}$, and $f(\tau)(\mathbb{Z} \times \tau\mathbb{Z})$ for $\mathcal{Z}_{A_1\times A_1}$ and $\mathcal{Z}_{A_2}$.

It is expected that there will be a certain congruence relation between the coefficients of an expansion of the period, and the coefficients of the L-series. This work verifies that this result holds in these cases.

• The v_n in the table is the coefficient of the t expansion of the period f. These numbers are related to the coefficients of the q expansion of L–series. In the appendix numerical data is given to illustrate this result.

2. Pencils of varieties defined by root lattices

The general reference for toric varieties is Danilov [D].

In [V1], I have described a toric construction of a pencil of n dimensional varieties, $\mathcal{X}_R$, from an n dimensional root lattice, R. Here we will show how to obtain from $\mathcal{X}_R$ a Calabi–Yau type variety, $\mathcal{Z}_R$, of dimension n.

We know from [V1] that if R is of type A_n, or products of such root lattices, then the fibers are Calabi-Yau type varieties.

For the root lattice A_n, $\mathcal{X}_{A_n}$, is given in local coordinates by:

$$(1+x_1+x_1x_2+\cdots+x_1\ldots x_n)(1+x_n+x_{n-1}x_n+\cdots+x_1\ldots x_n) = (\lambda+n+1)x_1\ldots x_n$$

The variety $\mathcal{X}_{A_n}$ is toric, and hence rational, and so can not be a Calabi-Yau type variety, since a toric variety X will have canonical class equivalent to the divisor $X \setminus T$. Here T is the open torus $\mathbb{C}^{*n}$ in X, and X is the closure of T. (This follows from the basic properties of toric varieties; see [D].) We can produce a variety with trivial canonical class by taking a double cover of $\mathcal{X}_{A_n}$. We will describe the construction for $R = A_3$ in the following theorem:

THEOREM 2.1. *For $R = A_3$, there is a Calabi-Yau threefold $\mathcal{Z}_R$, which is a double cover of the variety $\mathcal{X}_R$, and is given by the pull back in the following diagram:*

$$\begin{array}{ccc} \mathcal{Z}_{A_3} & \xrightarrow{\pi} & \mathcal{X}_{A_3} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{t\mapsto\lambda=\frac{(t-1)^2}{t}} & \mathbb{P}^1 \end{array}$$

PROOF. By the adjunction formula, we have

$$K_{\mathcal{Z}_{A_3}} = \pi^* K_{\mathcal{Z}_{A_3}} + \tilde{B},$$

where $\tilde{B} = \pi^{-1}B \subset \mathcal{X}$.

We have $\lambda = \frac{(t-1)^2}{t}$, so $t = \frac{2+\lambda\pm\sqrt{\lambda(\lambda+4)}}{2}$, and B is given when $t = 0$ or $t = -4$ so,

$$B = X_0 + X_{-4}$$

Hence

$$\tilde{B} = \tilde{X}_0 + \tilde{X}_{-4}.$$

Since X_0 and X_{-4} are both fibers of the fibration of $\mathcal{Z}_{A_3}$, we have $X_0 \sim X_{-4}$, and also $\tilde{X}_0 \sim \tilde{X}_{-4}$. Hence $\tilde{B} \sim 2\tilde{X}_0 \sim \pi^* X_0$, so $K_{\mathcal{Z}_{A_3}} = \pi^*(-X_\infty) + \pi^* X_0 = 0$

We also have $h^1 = 1$ since $\mathcal{Z}_{A_3}$ is connected. Hence $\mathcal{Z}_{A_3}$ is a Calabi-Yau three-fold. □

REMARK 2.2. The above proof can also be used to construct $\mathcal{Z}_{A_n}$, for higher dimensional lattices A_n. The double cover should be chosen so that the Calabi-Yau variety obtained is nice. It is not clear what nice means, or how this should be done in general. The above choice for $\mathcal{Z}_{A_3}$ is made so that the Picard lattice has large enough dimension that the Calabi-Yau three fold is rigid. This choice is determined by the set of generators found for the Picard lattice for the general fibre. See [V1] proposition 2.

3. The $A_1 \times A_1$ and A_2 cases, and the $A_1 \times A_1 \times A_1$ case:

In the case of $A_1 \times A_1$ and A_2, the double covers are chosen so that the pencils $\mathcal{Z}_{A_1 \times A_1}$ and $\mathcal{Z}_{A_2}$ are isomorphic to the pencils $\mathcal{A}'$ and $\mathcal{C}$ of [SB], respectively. So the following results for $A_1 \times A_1$ and A_2 come from [SB].

For the case of $A_1 \times A_1 \times A_1$, the double cover is chosen so that the pencils $\mathcal{Z}_{A_1 \times A_1 \times A_1}$ is isomorphic the the pencil $\mathcal{Z}$ which is described in [PS], which is also studied in [SB] and [B].

The variety $\mathcal{X}_{A_1{}^3}$ is the same as that in [V2], where I give further details of those results, and more detail of how to obtain the L-series from the variety.

We shall briefly summarize the results for these root lattices.

A: The $A_1 \times A_1$ case

THEOREM 3.1. *The pencil of elliptic curves, $\mathcal{Z}_{A_1 \times A_1}$ has monodromy group isomorphic to $\overline{16}$.*

PROOF. It is shown in [SB] § 14, that the isomorphic family $\mathcal{A}'$ is an elliptic modular surface associated with the group $\Gamma_0(16) \cap \Gamma_1(4)$. When we quotient out by $\pm I$, this is the same as $\overline{16}$. ☐

THEOREM 3.2. *The L-series of $\mathcal{Z}_{A_1 \times A_1}$ is given by $\eta(4\tau)^4$*

PROOF. See [SB] Theorem 14.2 ☐

B: The A_2 case

THEOREM 3.3. *The pencil of elliptic curves $\mathcal{Z}_{A_2}$ has monodromy group isomorphic to $\overline{12}$.*

PROOF. See [SB] § 14. ☐

THEOREM 3.4. *The L-series of $\mathcal{Z}_{A_2}$ is given by $\eta(2\tau)^3\eta(6\tau)^3$.*

PROOF. See [SB] Theorem 14.2 ☐

C: The $A_1 \times A_1 \times A_1$ case

THEOREM 3.5. *The monodromy group of the pencil of K3 surfaces, $\mathcal{Z}_{A_1 \times A_1 \times A_1}$ is isomorphic to $\overline{12|2+6}$.*

PROOF. See [PS] Theorem 6. ☐

THEOREM 3.6. *The Calabi-Yau three-fold $\mathcal{Z}_{A_1 \times A_1 \times A_1}$ has L-series given by $\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$.*

PROOF. See [B], [V2] ☐

4. The A_3 case

Now we consider X_{A_3}, a Calabi-Yau threefold, defined over $\mathbb{Q}$, locally given by

$$X_{A_3} : (1+x+xy+xyz)(1+z+zy+xyz)t = (t+1)^2 xyz,$$

plus some blowing up. In [V2] a detailed description of the construction of this variety is given. Below, we describe the resolution of the singularities of this variety.

In [V1] and [V2] a relationship between $\mathcal{Z}_{A_3}$ and a pencil of elliptic curves is described, which gives some justification for why we expect $\mathcal{Z}$ to be modular, that is, to give rise to a modular form when the L-series is taken. Below, we give further details of this relationship.

A: The construction

First we recall the construction of Calabi–Yau type threefolds that we are to investigate.

THEOREM 4.1. *If $\mathcal{E}$ is the Picard-Fuchs equation of $\mathcal{X}_{A_2}$ and $\mathcal{F}$ is the Picard-Fuchs equation of $\mathcal{X}_{A_3}$, then up to a change of variables, $\mathcal{F}$ is the symmetric square of $\mathcal{E}$.*

PROOF. See [V1] Theorem 3. □

From this we expect that the pencil $\mathcal{X}_{A_3}$ and $\mathcal{X}_{A_2}$ will be related, and we now determine this relationship more explicitly.

Let Y be the modular curve for the group $\Gamma_0(6)$. So Y has fibration $f : Y \to \mathcal{H}/\Gamma_0(6)$, with fibre over $t \in \mathcal{H}/\Gamma_0(6)$ an elliptic curve E_t. (Here $\mathcal{H}$ is the upper half plane). Let $\sigma = \sqrt{3}\begin{pmatrix} -1 & -\frac{1}{3} \\ 4 & 1 \end{pmatrix}$. Also let σ denote the corresponding involution of $\mathcal{H}/\Gamma_0(6)$, $(t \mapsto \sigma t.)$ Let Z_0 be obtained by taking a pull back of f and $f\sigma$. Next we quotient out by the lift of the map $\mathcal{H}/\Gamma_0(6) \to \mathcal{H}/<\Gamma_0(6), \sigma>$, to get Z_1. This is pictured below:

$$\begin{array}{ccccc} & & Z_0 & \longrightarrow & Z_1 \\ & \swarrow & \downarrow & & \\ Y & & Y & & \downarrow \\ \downarrow_f & \searrow & \downarrow_f & & \\ \mathcal{H}/\Gamma_0(6) & \xrightarrow{\sigma} & \mathcal{H}/\Gamma_0(6) & \xrightarrow{\text{quotient by } \sigma} & \mathcal{H}/<\Gamma_0(6), \sigma> \end{array}$$

Now the fibre of Z_1 over $t \in \mathcal{H}/<\Gamma_0(6), \sigma>$ is $E_t \times E_{\sigma t}$.

LEMMA 4.2. *For each fibre, we have a picture:*

$$\begin{array}{ccccc} E_t \times E_{\sigma t} & & & & X_t \\ & i \searrow & & \swarrow j & \\ & & S_t & & \end{array}$$

Here i is a 2:1 map obtained by quotienting by the Kummer involution, j is a 2:1 map obtained by quotienting by the Nikulin involution. S_t is a Kummer surface.

PROOF. Suppose that $H_1(E_\tau, \mathbb{Z})$ is generated by α_1 and β_1, and that $H_1(E_{\sigma\tau}, \mathbb{Z})$ is generated by α_2 and β_2. Then we have

$$E_\tau, E_{\sigma\tau} \in NS(E_t \times E_{\sigma t}),$$

and

$$\gamma = a[\alpha_1 \otimes \alpha_2] + b[\alpha_1 \otimes \beta_2] + c[\beta_1 \otimes \alpha_2] + d[\beta_1 \otimes \beta_2] \in NS(E_\tau, E_{\sigma\tau})$$

if and only if

$$a\tau\sigma\tau + b\tau + c\sigma\tau + d = 0.$$

We know that σ is given by the matrix

$$\begin{pmatrix} -1 & -\frac{1}{3} \\ 4 & 1 \end{pmatrix},$$

so

$$\gamma = 12[\alpha_1 \otimes \alpha_2] + 3[\alpha_1 \otimes \beta_2] + 3[\beta_1 \otimes \alpha_2] + [\beta_1 \otimes \beta_2] \in NS(E_\tau, E_{\sigma\tau}).$$

So $\gamma^2 = 6$, and the intersection matrix of $NS(E_\tau, E_{\sigma\tau})$ is given by

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

This is the same as the intersection matrix of X_t by proposition 3 of [V1]. Hence by Theorem 6.3 of [M] we have the required relationship between X_t and $E_t \times E_{\sigma t}$. □

Replacing each fibre $E_t \times E_{\sigma t}$ by a K3 surface K_t, obtained by taking the Kummer involution and then the inverse of the Nikulin involution, we get a variety Z_3, with a K3 fibration $Z_3 \to \mathcal{H}/ < \Gamma_0(6), \sigma >$. Next make a change of base: let $\tau = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}$. Then $< \Gamma_0(6), \sigma > = < G, \tau >$, where G is an index two subgroup of $< \Gamma_0(6), \sigma >$, given by $\Gamma_0(12) + 12$ conjugated by $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$, where

$$\Gamma_0(12) + 12 := \left\{ \begin{pmatrix} a & b \\ 12c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | a, b, c, d \in \mathbb{Z} \right\}$$

$$\cup \left\{ \frac{1}{\sqrt{12}} \begin{pmatrix} 12a & b \\ 12c & 12d \end{pmatrix} \in SL_2(\mathbb{R}) | a, b, c, d \in \mathbb{Z} \right\}$$

Let τ also denote the corresponding involution of G. Then X is the variety obtained by the change of base:

$$\begin{array}{ccc} X & \longrightarrow & Z_3 \\ \downarrow & & \downarrow \\ \mathcal{H}/G & \longrightarrow & \mathcal{H}/ < \Gamma_0(6), \sigma > \end{array}$$

THEOREM 4.3. *$\mathcal{Z}_{A_3}$ is isomorphic to X.*

PROOF. The Picard-Fuchs equations in both cases are the same, by Theorem 4.1, as are the Picard lattices, by Lemma 4.2, so we have the result by the Torrelli theorem. □

REMARK 4.4. This relationship between the pencil of K3 surfaces $\mathcal{X}_{A_3}$ and the pencil of elliptic curves $\mathcal{A}_{A_2}$ is predicted by Theorem 7.6 [Do], where it is shown how M_n polarized pencils of K3 surfaces will be birationally isomorphic to some pencil of Kummer surfaces. those which we are considering, are obtained from elliptic curves in the same manner as in this example.

Note that the relationship between the Picard-Fuchs equations of $\mathcal{X}_{A_3}$ and $\mathcal{X}_{A_2}$ gives rise to the following interesting combinatorial relationship:

THEOREM 4.5. *if*

$$a_n = \sum_{p+q+r+s=n} \binom{n}{p,q,r,s}^2$$

and

$$v_n = \sum_{p+q+r=n} \binom{n}{p,q,r}^2$$

then

$$\sum_{l+k+m=n} (-1)^l 9^m a_l \binom{l+k}{k}\binom{l+m}{m} = \sum_{s=0}^{n} v_s v_{n-s}$$

PROOF. This result is obtained by writing out local expansions for the solutions of the Picard-Fuchs equations, and relating the relationship between the Picard-Fuchs equations to the relationship between the coefficients. □

REMARK 4.6. It would be very interesting if there was a counting argument or some other method to prove this, which could be used to generalize this result to find the relationship between $\mathcal{Z}_{A_n}$ and $\mathcal{Z}_{A_{n+1}}$ in general. A concrete geometrical relationship between the fibers is give below, in lemma 4.7, but it is not clear how this is related to the relationship of Theorem 4.3

B: The fibration of X

There is a fibration of $\mathcal{Z}_{A_3}$ with fibre X_t a K3 surfaces, given by:

$$X_t : (1+x+xy+xyz)(1+z+zy+zyx)t = (t+1)^2 xyz.$$

X_t is a smooth K3 surface for general t. In terms of the second construction, X_t is given by the K3 surface K_t above. We also have the following relationship between the fibers of $\mathcal{X}_{A_n}$ and $\mathcal{X}_{A_{n-1}}$

LEMMA 4.7. *There is a double cover,*

$$\mathcal{X}_{A_n,\lambda} \xrightarrow{2:1} \mathcal{X}_{A_{n-1}},$$

branched along $\mathcal{X}_{A_{n-1},((\lambda+n+1)^{\frac{1}{2}}+1)^2}$ *and* $\mathcal{X}_{A_{n-1},((\lambda+n+1)^{\frac{1}{2}}-1)^2}$.

PROOF. From the point of view of the toric construction, this is because of the root lattice A_{n-1} embedding in the root lattice A_n. However, we will consider the local equations to show this is true.

We rewrite the equation for $\mathcal{X}_{A_n}$ as

$$(1+x_1A)(B+x_1C) = x_1LC,$$

where

$$A = x_2 + x_2x_3 + \cdots + x_2x_3\ldots x_n, B = 1 + x_n + x_{n-1}x_n + \cdots + x_2x_3\ldots x_n,$$
$$C = x_2x_3\ldots x_n, L = \lambda + n + 1.$$

So, we have a quadratic in x_1, with solution:

$$\frac{-AB + (\lambda+n)C \pm \sqrt{\left(AB - \left(1-\sqrt{L}\right)^2 C\right)\left(AB - \left(1+\sqrt{L}\right)^2 C\right)}}{2AC}$$

Now

$$AB - (1 \pm \sqrt{\lambda+n+1})^2 C$$

is the local equation of $\mathcal{X}_{A_{n-1},(1\pm\sqrt{\lambda+n+1})^2}$, so locally we have a double cover branched along the required curve. From the toric construction, we can see that this extends globally, so we have the result. □

COROLLARY 4.8. *The fibre X_λ of $\mathcal{Z}_{A_3}$ is an elliptic K3 surface*

PROOF. If we take a line which passes though the point of intersection of $\mathcal{X}_{A_{n-1},(1-(\lambda+n+1)^{\frac{1}{2}})^2}$ and $\mathcal{X}_{A_{n-1},(1+(\lambda+n+1)^{\frac{1}{2}})^2}$, then the lift of this line to $X_{n,\lambda}$ will be an elliptic curve, and the set of all lines through this point will give a fibration of $X_{n,\lambda}$. The fibres of $\mathcal{Z}_{A_3}$ and $\mathcal{X}_{A_3}$ are the same. □

The Picard number $\rho(X_t)$ of X_t is 19 for general t. A spanning set for the Picard group is the collection of lines locally given by $(y=0,x=-1)$,$(y=0,z=-1)$,$(z=0,x(y+1)=-1)$,$(x=0,z(y+1)=-1)$, and $(z=-1,x=-t)$, $(z=-1,x=-\frac{1}{t})$, for each set of local co-ordinates. All these lines will also be defined over finite fields.

The monodromy group for this pencil is the group G defined above.

REMARK 4.9. In the appendix a description of the singular fibres is given.

C: The periods and monodromy of X

In the case of elliptic curves, the expansion of the canonical differential told us about the coefficients of the L-series. Here the period should tell us about the coefficients of the L-series. For more details of the calculation refer to [V1].

Let ω_t be the unique differential $2,0$ form on X_t. A period is given by

$$\int_\gamma \omega_t,$$

where γ is a loop in $B := \mathbb{P}^1 \setminus \{t \text{ with } X_t \text{ singular}\}$.

LEMMA 4.10. *There is a period which has an expansion in terms of the local co-ordinate t:*

$$\sum_{n\geq 1} b_n t^n,$$

where

$$b_n = (-1)^{n+1}\sum_{m=0}^{n-1}\sum_{p+q+r+s=m}(-1)^m\binom{n+m}{2m+1}\left(\frac{m!}{p!q!r!s!}\right)^2.$$

The b_n are also described as follows:

t is given by the Hauptmodul for the group G:

$$\frac{\eta(q^3)^4\eta(q^{12})^8\eta(q^2)^{12}}{\eta(q)^4\eta(q^4)^8\eta(q^6)^{12}}.$$

A period is given by a modular form of weight 2 for G:

$$\frac{(\eta(q^2)\eta(q^6))^4}{(\eta(q)\eta(q^3))^2}$$

and so

$$\frac{(\eta(q^2)\eta(q^6))^4}{(\eta(q)\eta(q^3))^2} = \sum b_n\left(\frac{\eta(q^3)^4\eta(q^{12})^8\eta(q^2)^{12}}{\eta(q)^4\eta(q^4)^8\eta(q^6)^{12}}\right)^n.$$

PROOF. See [V2] □

LEMMA 4.11. *The Hodge diamond of X is as follows:*

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 50 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 50 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

PROOF. See [V2]. □

COROLLARY 4.12. *X is a rigid Calabi–Yau threefold defined over $\mathbb{Q}$.*

D: The L–series

Now I shall describe how the L-series of the Calabi-Yau threefold is defined. As before, we have zeta functions for X:

$$Z_p(X,s) = \frac{P_1(t)P_3(t)P_5(t)}{P_0(t)P_2(t)P_4(t)P_6(t)},$$

where $P_i(t) = det(1 - tF_p^*|_{H^i})$, so $degP_i = h^i$. This holds for $p \neq 2, 3$.

E: The Frobenius action

We want to find the action of the Frobenius on each cohomology group, $H^0, H^1, \ldots H^6$. By duality, we only need to consider H^0, H^1, H^2, H^3. We have $H^1 = 0$, and $H^0 = 1$, so we know the action here. The action on H^2 will give a 2-dimensional representation. We can determine the trace for this if we can determine the action for H^4, and then use the Lefscetz fixed point theorem, as below.

LEMMA 4.13. *If 3 is a square mod p, the action of the Frobenius on H^4 is trivial. If 3 is not a square mod p, the action is given by the matrix*

$$\left(\begin{array}{c|c} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0 \\ \hline 0 & p^2 I_{40} \end{array}\right),$$

where I_{40} is the 40×40 identity matrix. If 3 is a square mod p,

$$P_4(t) = (1 - p^2 t)^{50}, \text{and } P_2(t) = (1 - pt)^{50},$$

and if 3 is not a square mod p,

$$P_4(t) = (1-p^2t)^{48}(1-p^2\sqrt{-1}t)(1+p^2\sqrt{-1}t), \text{and} P_2(t) = (1-pt)^{48}(1-p\sqrt{-1}t)(1+p\sqrt{-1}t).$$

PROOF. By lemma 4.11, $h^4 = 50$. The Picard group is spanned by the components of the singular fibers of $\mathcal{Z}_{A_3}$, and the general fibre, and the sections coming from the Picard group of the general fibre. The only part of this which is not always defined over $\mathbb{Z}$ are the singular surfaces X_α and X_β, where $\alpha, \beta = 7 \pm 4\sqrt{3}$ are the roots of $(t-1)^2 - 12t = 0$. When 3 is a square, these will be fixed by the action of the Frobenius, but if 3 is not a square, they will be interchanged. Hence result. The polynomial P_2 is obtained by duality. □

We have:

$$Z_p(X,s) = \frac{P_3(t)}{(1-t)P_2(t)P_4(t)(1-p^3t)}.$$

Here

$$P_3(t) = det(1 - tF_p^*|_{H^3}) = 1 + a_p t + p^3 t^2.$$

So

$$a_p = TrF_p|_{H^3}.$$

The L-series is given by $L(X,s) := \prod_p \frac{1}{P_{p,3}(p^{-s})}$. The Lefschetz fixed point theorem says that

$$|X(\mathbb{F}_p)| = \sum (-1)^i TrF_p|_{H^i}.$$

For $i \neq 3$, the action on H^i is as given in lemma 4.13, so we have for 3 a square mod p,

$$a_p = 1 - 0 + 50p + 50p^2 - 0 + p^3 - |X(\mathbb{F}_p)|,$$

and for 3 not a square mod p,

$$a_p = 1 - 0 + 48p + 48p^2 - 0 + p^3 - |X(\mathbb{F}_p)|.$$

So to get an estimate for the L-series, all we have to do is count points on X over some finite fields.

We also know that the L-series is expected to be a modular form of weight 4. We also hope to get information about the L-series from looking at the expansion of a period, just as the L-series for the elliptic curve was related to the expansion of the canonical differential.

Now we use similar methods to those in [B] to determine what the L-series should be.

LEMMA 4.14. *Let p be a prime not dividing 6, and*

$$\omega(t) = \sum_0^\infty b_n t^{n-1} dt$$

a differential form, with $b_n \in \mathbb{Z}_p$, the p-adic integers. Let $t(q) = \sum_1^\infty \gamma_n q^n$, with $\gamma_n \in \mathbb{Z}_p$, and suppose $\omega(t(q)) = \sum_1^\infty \gamma_n q^{n-1} dq$. Then if there are $\delta_p, \beta_p \in \mathbb{Z}_p$, with $p|\beta_p$, and

$$b_{mp^r} - \delta_p b_{mp^{r-1}} + \beta_p b_{mp^{r-2}} \equiv 0(\ mod\ p^r), \forall m, r \in \mathbb{N}, (1)$$

then

$$\gamma_{mp^r} - \delta_p \gamma_{mp^{r-1}} + \beta_p \gamma_{mp^{r-2}} \equiv 0(\ mod\ p^r), \forall m, r \in \mathbb{N}. (2)$$

If α_1 is a p-adic unit, then conversely, (2) implies (1).

PROOF. Proposition 3 of [B]. □

LEMMA 4.15.

$$q(\sum_{n=1}^\infty b_n t(q)^{n-1})\frac{dt(q)}{dq} = (\eta(q)\eta(q^2)\eta(q^3)\eta(q^6))^2 + 8(\eta(q^2)\eta(q^4)\eta(q^6)\eta(q^{12}))^2$$

PROOF. We know that

$$\sum_{n\geq 1} b_n t(q)^n = \frac{(\eta(q^2)\eta(q^6))^4}{(\eta(q)\eta(q^3))^2}$$

from the working in lemma 4.10 Also $q = e^{2\pi i\tau}$, so that

$$q\sum_{n=1}^\infty b_n t(q)^{n-1}\frac{dt(q)}{dq} = \frac{1}{t}\frac{(\eta(q^2)\eta(q^6))^4}{(\eta(q)\eta(q^3))^2}\frac{1}{2\pi i}\frac{dt}{d\tau}$$

Here, t is a modular form of weight 0 for $\Gamma_0(6)$, $\frac{(\eta(q^2)\eta(q^6))^4}{(\eta(q)\eta(q^3))^2}$ is a modular form of weight 2 for $\Gamma_0(6)$, and $\frac{dt}{d\tau}$ is a modular form of weight 2 for $\Gamma_0(6)$. Hence the whole thing is a modular form of weight 4. Now, The space of modular forms of weight 4 for $\Gamma_0(6)$, has dimension 2. One element is $(\eta(q)\eta(q^2)\eta(q^3)\eta(q^6))^2$, and $(\eta(q^2)\eta(q^4)\eta(q^6)\eta(q^{12}))^2$ is also an element. By comparing $q-$expansions, we can show we must have the relationship given.

□

LEMMA 4.16. *Let γ_n and $\tilde{\gamma}_n$ be given by the $q-$expansions:*

$$\sum_{n\geq 1}\gamma_n q^n = (\eta(q)\eta(q^2)\eta(q^3)\eta(q^6))^2$$

$$\sum_{n\geq 1}\tilde{\gamma}_n q^n = (\eta(q)\eta(q^2)\eta(q^3)\eta(q^6))^2 + 8(\eta(q^2)\eta(q^4)\eta(q^6)\eta(q^{12}))^2$$

Then for any odd prime p,

$$\tilde{\gamma}_{mp^r} - \gamma_p\tilde{\gamma}_{mp^{r-1}} + p^3\tilde{\gamma}_{mp^{r-2}} = 0 \ mod \ mp^r$$

PROOF. Because the γ_n come from a cusp form of weight 4 for $\Gamma_0(6)$, there is an Euler product:

$$\sum_{n\geq 1}\frac{\gamma_n}{n^s} = \prod_{p\neq 2,3}\frac{1}{1-\gamma_p p^{-s}+p^3p^{-2s}}$$

We have that

$$\sum_{n\geq 1}\frac{\tilde{\gamma}_n}{n^s} = \sum_{n\geq 1}\frac{\gamma_n + 8\gamma_{n/2}}{n^s}$$

$$\sum_{n\geq 1}\frac{\gamma_n}{n^s} + \frac{8}{2^s}\sum_{n\geq 1}\frac{\gamma_n}{n^s}$$

$$= (1-2^{3-s})\prod_{p\neq 3}\frac{1}{1-\gamma_p p^{-s}+p^3p^{-2s}}$$

Hence for any odd prime q,

$$(1-\gamma_q q^{-s}+q^3q^{-2s})\sum_{n\geq 1}\frac{\tilde{\gamma}_n}{n^s} = (1-2^{3-s})\prod_{p\neq q}\frac{1}{1-\gamma_p p^{-s}+p^3p^{-2s}}$$

So, since the right hand side has zero coefficient for n^{-s}, when $q|n$, the same is true for the left hand side, which is given by:

$$(1-\gamma_q q^{-s}+q^3q^{-2s})\sum_{n\geq 1}\frac{\tilde{\gamma}_n}{n^s} = \sum_{n\geq 1}\frac{\tilde{\gamma}_n}{n^s} - \sum_{n\geq 1}\gamma_q\frac{\tilde{\gamma}_n}{(nq)^s} + \sum_{n\geq 1}q^3\frac{\tilde{\gamma}_n}{(nq)^s}.$$

The component of mq^r, for $m, r\geq 1$ is given by

$$\tilde{\gamma}_{mq^r} - \gamma_q\tilde{\gamma}_{mq^{r-1}} + q^3\tilde{\gamma}_{mq^{r-2}}$$

Hence result. □

THEOREM 4.17. *If*

$$(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2 = \sum\gamma_n q^n$$

and

$$b_n = (-1)^{n+1}\sum_{m=0}^{n-1}\sum_{p+q+r+s=m}(-1)^m\binom{n+m}{2m+1}\left(\frac{m!}{p!q!r!s!}\right)^2$$

then for p prime $\neq 2, 3$,

$$b_p \equiv \gamma_p \ mod\ p.$$

PROOF. By lemma 4.16,

$$\tilde{\gamma}_{mp^r} - \gamma_p \tilde{\gamma}_{mp^{r-1}} + p^3 \tilde{\gamma}_{mp^{r-2}} = 0 \bmod mp^r.$$

Then lemma 4.15 shows how $\tilde{\gamma}_p$ and b_p are related via a differential form, and so lemma 4.14 can be applied to obtain that

$$b_{mp^r} - \gamma_p b_{mp^{r-1}} + p^3 b_{mp^{r-2}} = 0 \bmod mp^r.$$

We have $b_1 = 1$, so when $m = r = 1$ we get

$$b_p \equiv \gamma_p \mathrm{mod} p.$$

$\square$

Finally we state the following conjecture of the modularity of our Calabi–Yau threefold.

CONJECTURE 4.18. Let X be the Calabi–Yau threefold constructed from the root lattice A_3. Then X is rigid, and its L–series is given by a certain twist of the modular form $(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$.

Because of [S2], and the relation between the solution of the Picard-Fuchs equation and the modular form $(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$, we expect that the L-series should be a twist of $(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$. Note that this is a modular form of weight 4 and level 6. The bad primes of X are 2 and 3, since for 2 and 3 the above analysis breaks down (eg, the singular fibers X_0 and X_{12} will coincide). So it is to be expected that the level will be 6.

I have not yet found the exact relation between the L-series and this modular form. A calculation using a computer to find the first few coefficients of the L-series gives the following for coefficients a_p:

p	a_p
5	265
7	502
11	1390
13	2155
17	4468
19	6112
23	10777

These numbers were calculated by constructing the non singular variety $\mathcal{Z}_{A_3}$ explicitly. Details of this construction are given in the following section.

5. Some computations

The following calculations were made using Maple and Pari.

The following table gives some coefficients b_n, and γ_n, and shows their values modulo n.

See Theorem 4.17 for a proof that $b_p \equiv \gamma_p \bmod p$ for primes p.

n	b_n	$b_n(n)$	γ_n	$\gamma_n(\text{mod}n)$
1	1	0	1	0
2	-2	0	-2	0
3	15	0	-3	0
4	-124	0	4	0
5	1181	1	6	1
6	-12126	0	6	0
7	131059	5	-16	5
8	-1469192	0	-8	0
9	16926849	0	9	0
10	-199207162	8	-12	8
11	2384562863	1	12	1
12	-28942705860	0	-12	0
13	355374207109	-1	38	-1
14	-4406289445862	4	32	4
15	55092364316715	0	-18	-3
16	-693832000063984	0	16	0
17	8793599395657913	10	-126	10
18	-112073616945714690	0	-18	0
19	1435468902675042391	1	20	1
20	-18467585515930466444	-4	24	4

We have the following q-expansions:

$$(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2 = q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + 6q^6 - 16q^7 - 8q^8$$
$$+9q^9 - 12q^{10} + 12q^{11} - 12q^{12} + 38q^{13} + 32q^{14} - 18q^{15} + 16q^{16} + O(q^{17}).$$

This is a modular form of weight 4 for $\Gamma_0(6)$.

$$t(q) = \frac{\eta(q^3)^4\eta(q^{12})^8\eta(q^2)^{12}}{\eta(q)^4\eta(q^4)^8\eta(q^6)^{12}}$$
$$= q + 4q^2 + 2q^3 - 12q^4 - 17q^5 + 12q^6 + 44q^7 + 20q^8 - 49q^9$$
$$-72q^{10} - 12q^{11} + 60q^{12} + 101q^{13} + 96q^{14} - 48q^{15} - 300q^{16} + O(q^{17})$$

This is a modular form of weight 0 for $\Gamma_0(12) + 12$.

$$\sum_{n\geq 1} b_n t(q)^n = \frac{(\eta(q^2)\eta(q^6))^4}{(\eta(q)\eta(q^3))^2} = q + 2q^2 + q^3 + 4q^4 + 6q^5 + 2q^6 + 8q^7 + 8q^8 + q^9$$
$$+12q^{10} + 12q^{11} + 4q^{12} + 14q^{13} + 16q^{14} + 6q^{15} + 16q^{16} + O(q^{17})$$

This is a modular function of weight 2 for $\Gamma_0(6)$.

6. Singular fibres and resolution of singularities

6.1. Singular fibres of $\mathcal{X}_{A_3}$. To resolve the singularities of $\mathcal{Z}_0$, we need to know about the singularities of the fibres, which are at $\lambda = 0, 12, -4, \infty$. This description is also used to calculate the Picard lattice of $\mathcal{Z}_{A_3}$.

Table 6.1 describes the singular fibres, before any double cover of resolution has been made, ie, this describes the closure of the inverse image, $X_\lambda = \overline{\phi^{-1}(\lambda)}$, of

λ	description of fibre	singularities
0	K3 surface with 4 isolated singularities	$(1,\pm1,-1),(-1,\pm1,1)$
12	K3 surface with 1 isolated singularity	$(1,1,1)$
-4	Two surfaces intersecting in three lines	$(-1,r,-1),(r,-r^{-1},r),(r,-1,r^{-1})$
∞	14 surfaces, 8 are copies of $\mathbb{P}^2$ blown up in three points, 6 are copies of $\mathbb{P}^1\times\mathbb{P}^1$.	locally, the x,y and z axes. In total 36 lines.

TABLE 1. Description of singular fibres

the map from the toric variety $\mathcal{X}_0$ to $\mathbb{P}^1$ given by $(x,y,z)\mapsto((1+x+xy+xyz)(1+z+zy+zyx)-4)/xyz$.

After taking the double cover, and finding the resolution, as described in the following section, for general t, the fibre X_t is a smooth K3 surface, and the singular fibres are as follows:

$\lambda=\frac{(t-1)^2}{t}=12$: X_t has an isolated singularity, at $(x,y,z)=(1,1,1)$.

$\lambda=\frac{(t-1)^2}{t}=-4$: X_t has 5 components. V_{-4} has two components, the surfaces $1+x+xy+xyz=0$, and $1+z+zy+zyx=0$, which intersect in the three lines $(x=z=-1)$, $(x=z,xy=-1)$, and $xz=y=-1$. In passing from V_{-4} to X_{-1}, these three lines are blown up, so there are 5 components.

$\lambda=\frac{(t-1)^2}{t}=\infty$: X_t has 14 components. Locally these are given by $x=0,y=0$ and $z=0$. The completion of $x=0$ in X is a copy of $\mathbb{P}^2$ blown up in three points. Similarly for the completion of $z=0$. For $y=0$, we obtain $\mathbb{P}^1\times\mathbb{P}^1$.

Note that in the fibration of the toric variety from which $\mathcal{Z}_{A_3}$ is constructed, the fibre over $\lambda=0$ is also singular, with singularities at $(x,y,z)=(1,-1,-1)$, $(-1,-1,1)$, $(-1,1,1)$, and $(1,1,-1)$. However, in the process of resolving the singularities to obtain $\mathcal{Z}_{A_3}$, the singularities of this fibre are also resolved, to give a smooth K3 surface over $t=1$.

For all these singularities, except those from X_{12}, the coefficients are in $\mathbb{Z}$, and hence for any p, the field of definition of is $\mathbb{F}_p$. For the singular fibres Z_α and Z_β, we have α,β are the roots of $t^2-14t+1$. If $\left(\frac{3}{p}\right)=1$, then the field of definition of these singularities is $\mathbb{F}_p$. Otherwise it is $\mathbb{F}_p^2$.

6.2. Resolution of singularities. Away from singularities, $\mathcal{Z}_{A_3}$ is locally given by

$$\left\{(x,y,z,t)\in\mathbb{A}^4_{\mathbb{F}}|(1+x+xy+xyz)(1+z+zy+zyx)=(\lambda+4)xyz\right\},$$

where $\mathbb{F}$ is the field we are interested in. Since the equation has all coefficients in $\mathbb{Z}$, this is well defined for all fields.

This three-fold is a double cover of the threefold $\mathcal{X}_{A_3}$, which is locally given by

$$\left\{(x,y,z,t)\in\mathbb{A}^4_{\mathbb{F}}|(1+x+xy+xyz)(1+z+zy+zyx)t=(1+t)^2xyz\right\}.$$

This variety is obtained from a toric variety.

Note, these are local equations, but all the local pieces have the same equation, so it is enough to consider how to resolve this affine piece.

The Table 2 gives a sequence of maps which are used to obtain the variety $\mathcal{Z}_{A_3}$

variety	notes	coordinates of singularity					
		x	y	z	λ/μ	t/s	r_1/r_2
$\mathcal{X}_0$	smooth toric variety						
$\downarrow$	blow up base locus						
$\mathcal{X}_1$	30 singular points	-1	0	-1	-4		$(*6)$
		-1	0	0	∞		$(*12)$
$\downarrow$	take double cover	0	0	-1	∞		$(*12)$
$\mathcal{Z}_1$	singularities over branch points	-1	0	-1	-1		$(*6)$
		-1	0	0	∞	∞	$(*12)$
		0	0	-1	∞	∞	$(*12)$
		-1	0	0	∞	0	$(*12)$
		0	0	-1	∞	0	$(*12)$
		1	-1	-1	0	1	
$\downarrow$	blow up component of Z_{-1}	-1	1	1	0	1	
		-1	-1	1	0	1	
		1	1	-1	0	1	
		-1	r	-1	-4	-1	
		r	$-r^{-1}$	r	-4	-1	
		r	-1	r^{-1}	-4	-1	
$\mathcal{Z}_2$	16 singularities	1	-1	-1	0	1	
		-1	1	1	0	1	
		-1	-1	1	0	1	
		1	1	-1	0	1	
$\downarrow$	blow up surface $y = r_1 = t + 1 = 0$	-1	0	-1	-4	-1	0
		-1	1	-1	-4	-1	± 1
		-1	-1	-1	-4	-1	± 1
		1	-1	1	-4	-1	± 1
$\mathcal{Z}_3$	10 singularities	1	-1	-1	0	1	
		-1	1	1	0	1	
		-1	-1	1	0	1	
		1	1	-1	0	1	
$\downarrow$	blow up $y + 1 = xz - 1 = t + 1 = 0$	-1	1	-1	-4	-1	± 1
		-1	-1	-1	-4	-1	± 1
		1	-1	1	-4	-1	± 1
$\mathcal{Z}_4$	6 singularities	1	-1	-1	0	1	
		-1	1	1	0	1	
		-1	-1	1	0	1	
$\downarrow$	blow up certain surface	1	1	-1	0	1	
		-1	1	-1	-4	-1	± 1
$\mathcal{Z}_5$	4 singularities	1	-1	-1	0	1	
		-1	1	1	0	1	
$\downarrow$	blow up $(x + 1)(z + 1) = (x + tz) = 0$	-1	-1	1	0	1	
		1	1	-1	0	1	
$\mathcal{Z}_6$	smooth variety						

TABLE 2. Resolution of singularities

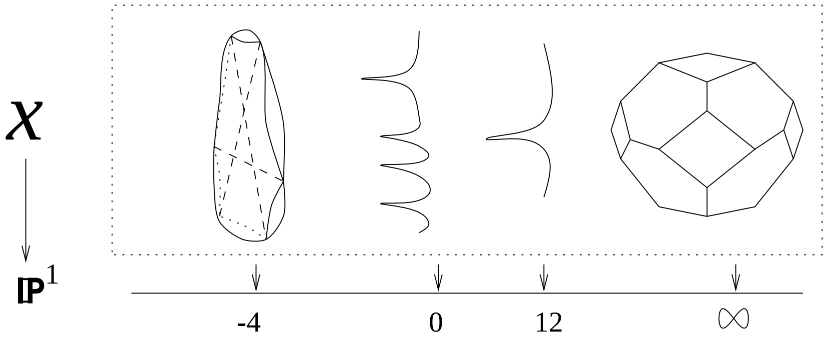

FIGURE 1. Fibration of $\mathcal{X}_1$

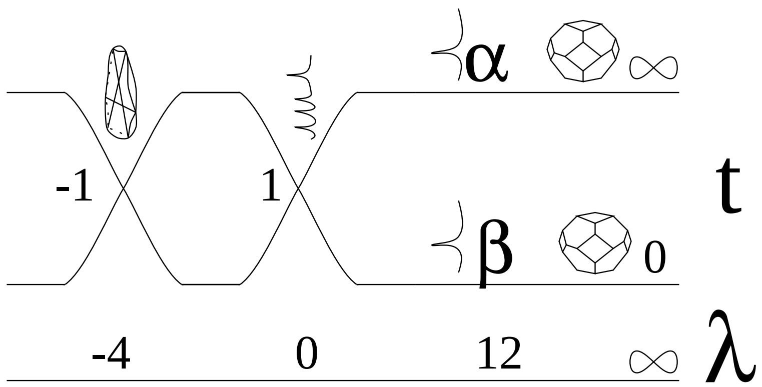

FIGURE 2. Double cover

6.3. attempt to visualize $\mathcal{X}_{A_3}$. Figure 1 gives a representation of the fibration from $\mathcal{X}_1$ to $\mathbb{P}^1$. Only the singular fibres are drawn. These fibres are described in table 6.1. In the process of resolving singularities of the three fold, the fibres will also be altered.

Figure 2 represents the double cover $\lambda = \frac{(t-1)^2}{t}$. The variety $\mathcal{Z}_{A_3}$ will have 6 singular fibres.

6.4. Note for singularities in characteristic p. In characteristic 2, the singular fibres X_0, X_{-4}, X_{12} all coincide, and in characteristic 3, the fibres X_0 and X_{12} coincide; but all other characteristics, the above gives all the singular fibres.

In the following, for char $\neq 2, 3$, all the singularities are defined and distinct, since the coefficients are all either ± 1 or 0, and so the following describes a resolution of singularities regardless of the characteristic.

When a point is blown up, we gain a copy of $\mathbb{P}^1$, ie, p points. In going from $\mathcal{Z}_2$ to $\mathcal{Z}_3$ we blow up three lines. These lines are given by algebraic equations, with coefficients ± 1 or 0, and give three distinct rational lines, which intersect in three points, in any characteristic $\neq 2, 3$.

7. Detailed description of resolution of singularities

7.1. $\mathcal{X}_0$**.** The variety $\mathcal{X}_0$ is a smooth toric variety with local variables x, y, z.

7.2. $\mathcal{X}_1$**.** The variety $\mathcal{X}_1$ is a sub-variety of $\mathcal{X}_0 \times \mathbb{P}^1$ given by

$$(1+x+xy+xyz)(1+z+xy+xyz)\mu = (\lambda+4\mu)xyz,$$

where λ and μ are the coordinates of the copy of $\mathbb{P}^1$.

The singularities are described in table 1 above.

7.3. $\mathcal{Z}_1$**.** $\mathcal{Z}_1$ is obtained by taking a double cover of $\mathcal{Z}_0$. In $\mathcal{X}_0 \times \mathbb{P}^1$, it is given by

$$(1+x+xy+xyz)(1+z+xy+xyz)ts = (t+s)^2 xyz.$$

Here t and s are the coordinates of the copy of $\mathbb{P}^1$.

7.4. $\mathcal{Z}_2$**.** The variety $\mathcal{Z}_2$ is obtained from $\mathcal{Z}_1$ by blowing up the variety which is the intersection of the two surfaces composing the fibre Z_{-1}. The equations are given in $\mathcal{X}_0 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by

$$r_1 t(1+x+xy+xyz) = r_2(t+s)y$$
$$r_2 s(1+z+zy+xyz) = r_1(t+s)xz$$

Here r_1, r_2 are the coordinates of the copy of $\mathbb{P}^1$ added.

7.5. $\mathcal{Z}_3$**.** We resolve the singularities at $(x,y,z)(s:t)(r_1:r_2) = (-1,0,-1)(1:-1)(0:1)$ and equivalent points.

This variety is defined in $\mathcal{X}_0 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with variables $(x,y,z)(s:t)(r_1:r_2)(u_1:u_2)$ by equations:

$$r_1 u_1 = u_2 r_2 y$$
$$t(1+x+xy+xyz)u_2 = u_1(t+s)$$
$$r_2 s(1+z+zy+xyz) = r_1(t+s)xz$$

This will only change the variety at the given point, adding a copy of $\mathbb{P}^1$.

7.6. $\mathcal{Z}_4$**.** We resolve the singularities at $(x,y,z)(s:t)(r_1:r_2)(u_1:u_2) = (\pm1,-1,\pm1)(1:-1)(r_1:r_2)(\pm1:1), x = z$.

This variety is defined in $\mathcal{X}_0 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ with variables $(x,y,z)(s:t)(r_1:r_2)(u_1:u_2)(w_1:w_2:w_3)$ by requiring that in addition to previous equations,

$$(s(y+1):s(xz-1):t+s) = (w_1:w_2:w_3)$$

So we have equations:

$$r_1 u_1 = u_2 r_2 y$$
$$t((1+x)w_1 + yw_2)u_2 = u_1 s w_3$$
$$r_2 s((z+1)w_1 + yw_2) = r_1 w_3 xz$$
$$w_1(1-xz) = w_2(1+y)$$
$$w_1(t+s) = w_3 s(1+y)$$
$$w_2(t+s) = w_3 s(1-xz)$$

This just adds a line at the four points we're interested in; elsewhere it is 1-1.

7.7. $\mathcal{Z}_5$**.** Next we resolve the two singularities at

$$(x,y,z)(s,t)(u_1:u_2) = (-1,1,-1)(1:-1)(\pm 1:1)$$

This can be done in the same way as for the other singularities over the intersection of the three lines of intersection of the two surfaces of Z_{-1}. This is because all these points can be permuted by permuting the indices of the variables X_i which are used to translate from one affine piece to another.

7.8. $\mathcal{Z}_6$**.** There are singularities at

$$(x,y,z)(t:s) = (1,-1,-1)(1:1), (-1,1,1)(1:1), (-1,-1,1)(1:1), (1,1,-1)(1:1)$$

Locally analytically, these singularities look like $ab = cd$, which can be resolved by blowing $a = c = 0$. So we resolve these singularities by blowing up a surface in the three fold passing through them.

We can forget about the variables u_i, r_i. The three fold can also be given locally by

$$(x+1)(xy+1)(zy+1)(x+1)t - y(tx+z)(x+tz) = 0$$

since the left hand side is equal to

$$(1+x+xy+xyz)(1+z+zy+zyx)t - (t+1)^2 xyz$$

Hence the surface $(x+1)(z+1) = (x+tz) = 0$ is in the three fold. As given, this is singular, but the line of singularities $x = z = t = -1$ has been blown up by the above sequence of maps, so the strict transform of this surface in $\mathcal{Z}_5$ is smooth, and passes through the four singular points of Z_{-1}, hence when we blow it up, we resolve these singularities.

The above equations are given in terms of the local variables, but this process does resolve the singularities globally.

7.9. Counting points. From adding up the number of points that are added each time in the above process, we have that number of points of the variety $\mathcal{Z}$ is $N + 3p^2 + 64p$. Here N is the number of points on the variety before the resolution.

We break N into two parts,

$$N_0 = \#\{\text{points on} \mathcal{Z}_1 \text{ with } xyz = 0\}$$

$$N_1 = \#\{\text{points on} \mathcal{Z}_1 \text{ with } xyz \neq 0\}$$

$$= \#\left\{(x,y,z,t) \in (\mathbb{F}_p^*)^4 | (1+x+xy+xyz)(1+z+zy+zyx)t = (1+t)^2 xyz\right\}.$$

The value of N_1 is found for small p using a computer.

The value of N_0 is given as follows.

For each fibre apart from Z_0 and Z_∞, the number of points on Z_t with $xyz = 0$ is given by B :=the number of points on the base locus. Z_0 and Z_∞ have the same number of points, C say. So we have

$$N_0 = 2C + (p-1)B.$$

The Figure 7.9 show how B and C are found.

We have

$$C = 2(7p^2 + 4p + 1)$$

$$B = 20p - 10$$

Hence the total number of points on $\mathcal{Z}$ is

$$N_1 + 4(7p^2 + 4p + 1) + (p-1)(20p - 10) + 3p^2 + 64p$$

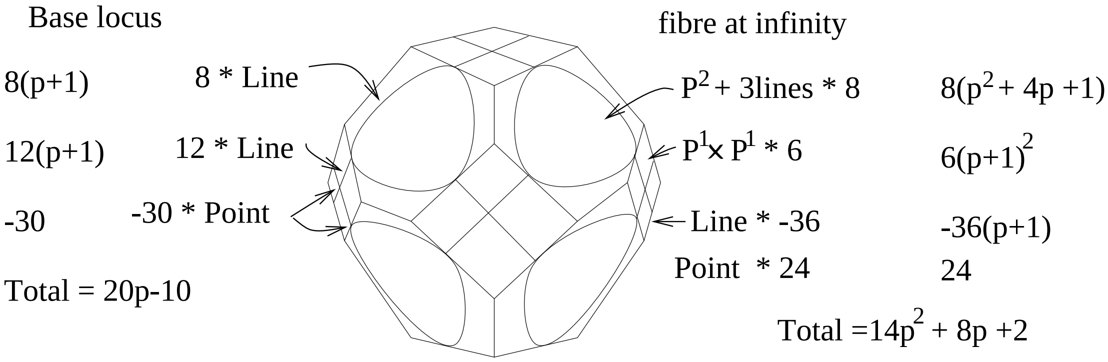

FIGURE 3. Number of points on base locus and Z_∞

$$= N_1 + 51p^2 - 14p + 78$$

A computer program gives the following values for N_0, N, and hence the values of a_p:

p	N_0	N	a_p
5	18	1301	265
7	51	2530	502
11	183	6278	1390
13	264	8779	2155
17	555	15134	4468
19	765	18988	6112
23	1152	27887	10777

Acknowledgements

I am grateful to Noriko Yui for her advice and encouragement, and continuing support.

References

[B] Beukers, F.: Another congruence for the Apéry numbers, Journal of Number Theory 25, (1987) 201–210.
[B2] Beukers, F.: Une formule explicite dans la théorie des courbes elliptiques, preprint, Leiden 1984.
[CN] Conway, J. H., Norton, S.P.: Monstrous moonshine. Bull. London Math. Soc. 11, (1979) 308–340.
[D] Danilov, V. I.: The geometry of toric varieties. Russian Math. Surveys 33, (1978) 97–154.
[Do] Dolgachev, I. V.: Mirror symmetry for lattice polarized $K3$ surfaces. Preprint, alg-geom/9502005.
[M] Morrison, D. R.: On K3 surfaces with large Picard number. Invent. Math. 75, (1984) 105–121.
[PS] Peters, C., Stienstra, J.: A pencil of K3-surfaces related to Apery's recurrence for $\zeta(3)$ and Fermi surfaces for potential zero. In: Barth, W.-P., Lange, H. (Eds) Arithmetic of Complex Manifolds, Proceedings, Erlangen 1988 (Lect. Notes Math., vol. 1399, pp. 110–127) Berlin Heidelberg New York: Springer 1989..
[S] Silverman, J.: The arithmetic of elliptic curves, New York: Springer-Verlag, 1996.
[SB] Stienstra, J. and Beukers, F.: On the Picard-Fuchs equation and the formal Brauer group of certain elliptic K3 surfaces. Math. Ann. 271, (1985) 269–304.

[S2] Stienstra, J.: Formal groups and congruences for L-functions. Amer. J. Math. 109 (1987), no. 6, 1111–1127.

[V1] Verrill, H.: Root lattices and pencils of varieties, Kyoto Journal of mathematics, Volume 36. Number 2, July 1996, 421–446.

[V2] Verrill, H.: Arithmetic of a certain Calabi-Yau threefold, proceedings of the 5th conference of the CNTA. 1997

Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada, K7L 3N6

E-mail address: `helena@pws.mast.queensu.ca`