Arithmetic of a certain Calabi-Yau threefold

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Abstract. This is a study of a certain Calabi-Yau threefold, and its L-series, which from preliminary calculations appears to be a modular form of weight 4 and level 6.

1. Introduction

This work arose from considering varieties obtained from root lattices [V]. For some root lattices R, we can obtain pencils of Calabi-Yau type varieties and we can also make the ambient space into a Calabi-Yau type variety, $\mathcal{Z}_R$. Several families of elliptic curves and K3 surfaces can be obtained in this way, not to mention higher dimensional families. In [V] the Picard-Fuchs equations for some of these pencils are given. See [V] for details of these calculations.

Since the equations defining the varieties have coefficients in $\mathbb{Z}$, we can also look at the arithmetic of the situation. I.e., we can ask how many integer solutions there are to the equations. This leads to the study of the L-series. The problem is to determine the L-series of the pencil. The L-series is expected to be related to the Picard-Fuchs equations, as in the cases considered in [S2].

Strong evidence that the L-series for $\mathcal{Z}_{A_3}$ is $(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$, is given by Theorem 3.5 of this paper, which shows that there is a congruence relation between the coefficients of the L-series, and the coefficients of an expansion for a period. Here η is the Dedekind eta function, given by $\eta(\tau) = q^{\frac{1}{24}}\prod(1-q^n)$, where $q = e^{2i\pi\tau}$. The results of [S2] suggest that there should be such a relationship for the L-series, and [H] should show that we will obtain a modular form for the L-series, so given the congruence relation, this should determine the L-series uniquely.

Several examples of pencils of elliptic curves and K3 surfaces can be viewed as arising from this construction. (See [V].) So this is a good place to look for more examples, and eventually it is hoped that some general results for these pencils of varieties arising from root lattices can be found. In the case of A_n root lattices and products of this type of lattice, we get pencils of Calabi-Yau type varieties. In the $A_1 \times A_1 \times A_1$ and A_3 cases, the pencils of K3 surfaces are related to pencils of elliptic curves, which suggests that we should expect these Calabi-Yau three folds to be modular, that is, the Mellin transform of the L-series is a modular form. There is reason to expect that this happens more generally for pencils from other lattices of this type.

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We first consider the $A_1 \times A_1 \times A_1$ case, and then discuss the A_3 case. This is a preliminary report. Full details can be found in [V2].

2. The $A_1 \times A_1 \times A_1$ case:

The pencil of K3 surfaces studied by Peters in [PS] is denoted $\mathcal{X}_{A_1^3}$ in [V]. See [PS] for details of defining equations etc. The result of this section is perhaps implicit in the papers [SB], [PS], [B], but I do not know of the details written up anywhere, so I include them here, as motivation for the A_3 case. However, the A_3 case is harder, since for the A_3 case not all the singularities will be defined over $\mathbb{Z}$.

2.1. Structure of $X_{A_1^3}$. In [PS], $\overline{\mathcal{X}}$ is a compactification of the affine threefold $\mathcal{X}$ given by

$$\xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1} + \xi_3 + \xi_3^{-1} + \xi_0 + \xi_0^{-1} = 0.$$

The singularities are all defined over $\mathbb{Z}$, and are resolved in all odd characteristics by blowing up the surfaces of the form $\xi_0 + \xi_1 = 0$, $\xi_3 + \xi_4 = 0$. I shall write $\mathcal{Z}_{A_1^3}$ to denote the three-fold obtained after resolving the singularities, and Z_t to denote the fiber at $\xi_0 = t$. This fibre is given by equations with coefficients in $\mathbb{Z}$.

LEMMA 2.1. *We have that*

$$\chi(\mathcal{Z}_{A_1^3}) = 140, \;\; \rho(\mathcal{Z}_{A_1^3}) = 70,$$

and the Hodge diamond is as follows:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 70 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 70 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

PROOF. We have

$$\rho(\mathcal{Z}_{A_1^3}) = \rho(\text{general fibre}) + 1 + \sum_{Z_s}(n(Z_s) - 1),$$

where $n(Z_s)$ is the number of components of Z_t. Here $n(Z_s) = 1$, except for Z_∞, and Z_0, which have 26 components. The general fibre has $\rho = 19$ (by [PS]), so

$$\rho(\mathcal{Z}_{A_1^3}) = 19 + 1 + 26 - 1 + 26 - 1 = 70 = h^{1,2}.$$

Next,

$$\chi(X) = \chi(\mathbb{P}^1) \times \chi(\text{K3 surface}) + \sum_{Z_t \text{ a singular fibre}} (\chi(Z_t) - \chi(\text{general fibre}))$$

$$= 2 \times 24 + 4 \times (23 - 24) + 2 \times (72 - 24) = 140.$$

We have $h^0 = 1$ and $h^1 = 0$ because $\mathcal{Z}_{A_1^3}$ is simply connected, and $h^4 = h^2 = 70$, because $\rho = 70$, and all the Hodge $h^{1,1}-$cycles are algebraic. Since $\chi(X) = \sum(-1)^i h^i$, we have $h^3 = 2(70 - 0 + 1) - 140 = 2$. The rest will follow by the symmetry of the Hodge diamond. □

Let

$$N_p := \text{the total number of points on } \mathcal{Z}_{A_1^3}(\mathbb{F}_p)$$

and let N_p^* be the number of solutions of

$$\xi_0 + \xi_0^{-1} + \xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1} + \xi_3 + \xi_3^{-1} = 0$$

in $\mathbb{F}_p$ with $\xi_0\xi_1\xi_2\xi_3 \neq 0$.

LEMMA 2.2. *The L-series of $X_{A_1^3}$ is given by $\sum_{n\geq 1} \frac{a_n}{n^s}$, where for p prime (and $p \neq 2,3$), a_p is even, and given by $a_p = p^3 - 2p^2 - 7 - N_p^*$.*

PROOF. By analyzing what is added by the resolution of the singularities, and using the Lefschetz fixed point theorem, we obtain the expression for a_n. By pairing up fibers and using this expression, we get the parity. □

THEOREM 2.3. *The Mellin transform of the L-series of $X_{A_1^3}$ is equal to*

$$(\eta(2\tau)\eta(4\tau))^4,$$

which is a cusp form for $\Gamma_0(8)$.

PROOF. The cusp form $(\eta(2\tau)\eta(4\tau))^4$ has weight 4 and conductor 8. The q expansion of $(\eta(2\tau)\eta(4\tau))^4$ is

$$q - 4q^3 - 2q^5 + 24q^7 - 11q^9 - 44q^{11} + 22q^{13} + 8q^{15} + O(q^{17})$$

The coefficients of $(\eta(2\tau)\eta(4\tau))^4$ agree with the coefficients of the L-series calculated above, as far as calculated by computer. Use Livnés [L] results to show that the fact that these modular forms agree to high enough degree, and have even coefficients means that they are equal. □

It is shown in [PS] that an expansion of a period for the family $\mathcal{Z}_{A_1^3}$ is

$$F(\xi_0) = \sum_{n=0}^{\infty} v_n \xi^{2n+1}$$

where

$$v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}.$$

The following result relates the period to the L-series:

THEOREM 2.4. *$a_n \equiv v_{\frac{n-1}{2}}$ mod n.*

PROOF. See [B] □

3. The A_3 case:

Now we would like to repeat all the above analysis for the A_3 case, though the A_3 case is more complex, as the action of the Frobenius automorphism will not be trivial. We consider $\mathcal{Z}_{A_3}$, a Calabi-Yau threefold, defined over $\mathbb{Q}$, locally given by

$$\mathcal{Z}_{A_3} : (1 + x + xy + xyz)(1 + z + zy + xyz)t = (t + 1)^2 xyz,$$

plus some blowing up. So far this work is not completed, but Theorem 3.5, and the ideas of [S2] suggest that the modular form we expect to find is:

$$(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2.$$

3.1. Construction of the Calabi-Yau threefold $\mathcal{Z}_{A_3}$. There are several ways of thinking about the Calabi-Yau threefold under consideration. First it can be described by a toric construction; second it can be described by taking $\mathbb{P}^3$, and applying a sequence of blow ups and a change of base (it is also considered as a fibre space). Also it can be viewed as coming from a kind of fibre product of a pencil of elliptic curves.

The toric construction begins with a root lattice A_3. See [V] for details. I'll describe the other two ways of thinking about $\mathcal{Z}_{A_3}$ in more detail.

3.1.1. *Construction of $\mathcal{Z}_{A_3}$ starting from $\mathbb{P}^3$.* Here $\mathcal{Z}_{A_3}$ will be defined over $\mathbb{Q}$. Start with $\mathbb{P}^3$, with homogeneous co-ordinates X_1,X_2,X_3, X_4. Blow up the points $(0:0:0:1),(1:0:0:0),(0:1:0:0),(0:0:1:0)$, and then blow up the strict transforms of the 6 lines between these points, to obtain a variety U. Now we take local co-ordinates of an affine piece, $x = X_1/X_2, y = X_2/X_3, z = X_3/X_4$. U is covered by 24 such affine pieces, obtained from each other by permuting the indices of the X_i. Next blow up the locus given locally by $(1+x+xy+xyz)(1+z+zy+xyz) = xyz = 0$, to obtain V. Now V has a fibration $\phi : V \to \mathbb{P}^1$, given by $\phi(x,y,z) = \frac{(1+x+xy+xyz)(1+z+zy+zyx)}{xyz} - 4$. The general fibre $V_\lambda = \phi^{-1}(\lambda)$ is a smooth K3 surface. Next make a change of base, corresponding to the map $\alpha : \mathbb{P}^1 \to \mathbb{P}^1$ given by $t \mapsto \frac{(t-1)^2}{t}$, to obtain W, as the pull back of ϕ and α two maps. Finally, $\mathcal{Z}_{A_3}$ is obtained by finding a resolution W, and ϕ gives rise to a fibration $\psi : \mathcal{Z}_{A_3} \to \mathbb{P}^1$, with fibre X_t isomorphic to $V_{\frac{(t-1)^2}{t}}$ for general t. Note that U can also be described by a toric construction, and the V_λ are toric hyper-surfaces. Then we can show we have K3 surfaces by using the standard form of the canonical form for toric varieties, and the adjunction formula, as the general fibre is equivalent to the fibre at infinity, which is equivalent to $-K$ for K the canonical divisor. The following diagram shows the sequence of maps from $\mathbb{P}^3$ to $\mathcal{Z}_{A_3}$.

$$\begin{array}{ccccccccc}
\mathcal{Z}_{A_3} & \xrightarrow{\text{resolution}} & W & \xrightarrow{\text{pull back}} & V & \xrightarrow{\text{blow up base locus}} & U & \xrightarrow{\text{blow up}} & \mathbb{P}^3 \\
\downarrow\psi & & \downarrow & & \downarrow\phi & & & & \\
\mathbb{P}^1 & = & \mathbb{P}^1 & \xrightarrow{\text{double cover}} & \mathbb{P}^1 & & & &
\end{array}$$

3.1.2. *Relationship of $\mathcal{Z}_{A_3}$ to a pencil of elliptic curves.* In this section, the construction of $\mathcal{Z}_{A_3}$ is over $\mathbb{C}$. This part shows a relationship between $\mathcal{Z}_{A_3}$ and a pencil of elliptic curves, and this gives justification for why we expect $\mathcal{Z}_{A_3}$ to be modular, that is, to give rise to a modular form.

Let Y be the modular curve for the group $\Gamma_0(6)$. So Y has fibration $f : Y \to \mathcal{H}/\Gamma_0(6)$, with fibre over $t \in \mathcal{H}/\Gamma_0(6)$ an elliptic curve E_t. (Here $\mathcal{H}$ is the upper half plane). Let $\sigma = \sqrt{3}\begin{pmatrix} -1 & -\frac{1}{3} \\ 4 & 1 \end{pmatrix}$. Also let σ denote the corresponding involution, $\sigma : t \mapsto \sigma t$, of $\mathcal{H}/\Gamma_0(6)$. Let Z_0 be obtained by taking a pull back of f and $f\sigma$. Next we quotient out by the lift of the map $\mathcal{H}/\Gamma_0(6) \to \mathcal{H}/<\Gamma_0(6),\sigma>$, to get Z_1. This is pictured below:

$$\begin{array}{ccccc}
 & & Z_0 & \longrightarrow & Z_1 \\
 & \swarrow & \downarrow & & \\
Y & & Y & & \downarrow \\
\downarrow f & \searrow & \downarrow f & & \\
\mathcal{H}/\Gamma_0(6) & \xrightarrow{\sigma} & \mathcal{H}/\Gamma_0(6) & \xrightarrow{\text{quotient by}\sigma} & \mathcal{H}/<\Gamma_0(6),\sigma>
\end{array}$$

Now the fibre of Z_1 over $t \in \mathcal{H}/ < \Gamma_0(6), \sigma >$ is $E_t \times E_{\sigma t}$.

For each fibre, we have a picture:

$$\begin{array}{ccccc} E_t \times E_{\sigma t} & & & & K_t \\ & i \searrow & & \swarrow j & \\ & & S_t & & \end{array}$$

Here i is a 2:1 map obtained by quotienting by the Kummer involution, j is a 2:1 map obtained by quotienting by the Nikulin involution. S_t is a Kummer surface, and K_t a K3 surface. See [M].

Replacing each fibre $E_t \times E_{\sigma t}$ by K_t, we get a variety Z_3, with a K3 fibration $Z_3 \to \mathcal{H}/ < \Gamma_0(6), \sigma >$. Next make a change of base: let $\tau = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}$. Then $< \Gamma_0(6), \sigma > = < G, \tau >$, where G is an index two subgroup of $< \Gamma_0(6), \sigma >$, given by $\Gamma_0(12) + 12$ conjugated by $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$, where

$$\Gamma_0(12) + 12 := \left\{ \begin{pmatrix} a & b \\ 12c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | a, b, c, d \in \mathbb{Z} \right\}$$

$$\cup \left\{ \frac{1}{\sqrt{12}} \begin{pmatrix} 12a & b \\ 12c & 12d \end{pmatrix} \in SL_2(\mathbb{R}) | a, b, c, d \in \mathbb{Z} \right\}$$

Let τ also denote the corresponding involution of G. Then X is the variety obtained by the change of base:

$$\begin{array}{ccc} X & \longrightarrow & Z_3 \\ \downarrow & & \downarrow \\ \mathcal{H}/G & \longrightarrow & \mathcal{H}/ < \Gamma_0(6), \sigma > \end{array}$$

To show that X is birational to $\mathcal{Z}_{A_3}$, one shows that the Picard-Fuchs equation in both cases are the same, as are the Picard lattices, and use the Torrelli theorem. See [V] for the Picard-Fuchs equations.

3.2. The fibration of $\mathcal{Z}_{A_3}$. In terms of the first construction, $\mathcal{Z}_{A_3}$ has a fibration given by $\phi : \mathcal{Z}_{A_3} \to \mathbb{P}^1$, with fibre X_t a K3 surfaces, given by:

$$X_t : (1 + x + xy + xyz)(1 + z + zy + zyx)t = (t + 1)^2 xyz.$$

X_t is a smooth K3 surface for general t. The Picard number $\rho(X_t)$ of X_t is 19 for general t, by [V] Proposition 2. The monodromy group for this pencil is the group G defined above.

3.2.1. *The singular fibers.* For general t, the fibre X_t is a smooth K3 surface. The singular fibers are as follows:

$\lambda = \frac{(t-1)^2}{t} = 12$: X_t has an isolated singularity, at $(x, y, z) = (1, 1, 1)$. So whether these singularities appear over $\mathbb{F}_p$ will depend on the solutions of $(t-1)^2 = 12t$ over $\mathbb{F}_p$.

$\lambda = \frac{(t-1)^2}{t} = -4$: X_t has 5 components, which are all defined over $\mathbb{Z}$, and hence over $\mathbb{F}_p$.

$\lambda = \frac{(t-1)^2}{t} = \infty$: X_t has 14 components, also all defined over $\mathbb{Z}$ and $\mathbb{F}_p$. See [V] Proposition 1.

3.3. The L-series. First we will need to know the Hodge numbers, the Euler characteristic $\chi(\mathcal{Z}_{A_3})$, and Picard number $\rho(\mathcal{Z}_{A_3})$ for the calculation of the L-series.

LEMMA 3.1. *The Euler characteristic and the maximum Picard number are given by*

$$\chi(\mathcal{Z}_{A_3}) = 100, \ \overline{\rho(\mathcal{Z}_{A_3})} = 50.$$

The Hodge diamond of $\mathcal{Z}_{A_3}$ *is as follows:*

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 50 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 50 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

We have zeta functions for $\mathcal{Z}_{A_3}$:

$$Z_p(\mathcal{Z}_{A_3}, s) = \frac{P_1(t)P_3(t)P_5(t)}{P_0(t)P_2(t)P_4(t)P_6(t)},$$

where $P_i(t) = det(1 - tF_p^*|_{H^i})$, so $degP_i = h^i$. This holds for $p \neq 2, 3$.

When $(t-1)^2 = 12t$ has two solutions in $\mathbb{F}_p$, the action of the Frobenius on the cycles in H^4, is trivial, and

$$Z_p(\mathcal{Z}_{A_3}, s) = \frac{P_3(t)}{(1-t)(1-pt)^{50}(1-p^2t)^{50}(1-p^3t)}.$$

Here

$$P_3(t) = det(1 - tF_p^*|_{H^3}) = 1 + a_pt + p^3t^2,$$

and by Lefschetz,

$$a_p = TrF_p|_{H^3} = 1 - 0 + 50p + 50p^2 - 0 + p^3 - |\mathcal{Z}_{A_3}(\mathbb{F}_p)|.$$

However, this will not always be the case. Further analysis is necessary to determine what happens for other primes, though we can see immediately that we will have

$$P_4(t) = Q(t)(1-p^2t)^{48},$$

for some quadratic $Q(t)$.

The L-series is expected to be a modular form of weight 4, and should be related to the expansion of a period, as in [S2]. The following lemmas show that $(\eta(q)\eta(q^2)\eta(q^3)\eta(q^6))^2$ satisfies the expected relation.

LEMMA 3.2. *There is a period which has an expansion in terms of the local co-ordinate* t*:*

$$\sum_{n\geq 1} b_n t^n,$$

where

$$b_n = (-1)^{n+1}\sum_{m=0}^{n-1}\sum_{p+q+r+s=m}(-1)^m\binom{n+m}{2m+1}\left(\frac{m!}{p!q!r!s!}\right)^2.$$

The b_n *also satisfy*

$$\frac{(\eta(q^2)\eta(q^6))^4}{(\eta(q)\eta(q^3))^2} = \sum b_n\left(\frac{\eta(q^3)^4\eta(q^{12})^8\eta(q^2)^{12}}{\eta(q)^4\eta(q^4)^8\eta(q^6)^{12}}\right)^n.$$

Now we use similar methods to those in [B] to prove the following, which determine what the L-series should be.

Lemma 3.3. *If*

$$t = \frac{\eta(q^3)^4\eta(q^{12})^8\eta(q^2)^{12}}{\eta(q)^4\eta(q^4)^8\eta(q^6)^{12}},$$

then

$$q(\sum_{n=1}^{\infty} b_n t(q)^{n-1})\frac{dt(q)}{dq} = (\eta(q)\eta(q^2)\eta(q^3)\eta(q^6))^2 + 8(\eta(q^2)\eta(q^4)\eta(q^6)\eta(q^{12}))^2$$

Lemma 3.4. *Let* γ_n *and* $\tilde{\gamma}_n$ *be given by the* $q-$*expansions:*

$$\sum_{n\geq 1}\gamma_n q^n = (\eta(q)\eta(q^2)\eta(q^3)\eta(q^6))^2$$

$$\sum_{n\geq 1}\tilde{\gamma}_n q^n = (\eta(q)\eta(q^2)\eta(q^3)\eta(q^6))^2 + 8(\eta(q^2)\eta(q^4)\eta(q^6)\eta(q^{12}))^2$$

Then for any odd prime p,

$$\tilde{\gamma}_{mp^r} - \gamma_p\tilde{\gamma}_{mp^{r-1}} + p^3\tilde{\gamma}_{mp^{r-2}} = 0 \ mod\ mp^r$$

Thus, applying Proposition 3 of [B], we have the following

Theorem 3.5. *If*

$$(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2 = \sum \gamma_n q^n$$

and

$$b_n = (-1)^{n+1}\sum_{m=0}^{n-1}\sum_{p+q+r+s=m}(-1)^m\binom{n+m}{2m+1}\left(\frac{m!}{p!q!r!s!}\right)^2$$

then for p prime,

$$b_p \equiv \gamma_p \ mod\ p.$$

Theorem 3.5 describes how $(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$ is related to the Picard-Fuchs equation of $\mathcal{Z}_{A_3}$. In [S2], we see how for certain varieties, the L-series is related to the differential form, by this kind of congruence relation. We hope that the result of [S2] should hold more generally, and also apply to $\mathcal{Z}_{A_3}$. Then the Mellin transform of the L-series for $\mathcal{Z}_{A_3}$ would have pth coefficient equal (mod p) to the pth coefficient in the $q-$expansion of $(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$. Then since these are both modular forms of weight 4 and level 6, they must be equal. Hence from the preceding, we have the following:

Conjecture 3.6. *The L-series of* $\mathcal{Z}_{A_3}$ *is* $(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$.

Note that this is a modular form of weight 4 and level 6. The bad primes of $\mathcal{Z}_{A_3}$ are 2 and 3, since for 2 and 3 the above analysis of singular fibers breaks down (e.g., the singular fibers X_0 and X_{12} will coincide). So it is to be expected that the level will be 6.

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