Néron models, Ed Schaefer, Santa Clara University

Let us say that we have an abelian variety, (like an elliptic curve) over a p-adic field. We want to talk about reduction. We can describe the abelian variety by integral equations, but what we have may not be a smooth object over the integer ring because the reduction may be singular. So we want to "enlarge it" so that we have the same abelian variety over the p-adic field and the reduction is a non-singular group variety. This is called the Néron model. We will see how it is defined and some applications.

First let us review some necessary concepts from algebraic geometry. The spectrum of a ring is the collection of prime ideals and a sheaf structure.

Example. $\mathrm{spec}(\mathbf{Z}_5) = \{(0), (5)\}$. We often call the elements points. We will not need to discuss the sheaf structure.

A scheme is locally the spectrum of a ring. To the variety $E : ZY^2 = X^3 + Z^3/\mathbf{Q}_5$ (the projective closure of $y^2 = x^3 + 1$) we associate a scheme. We have $\mathrm{spec}\mathbf{Q}_5[x, y]/(y^2 - x^3 - 1)$ and $\mathrm{spec}\mathbf{Q}_5[\mathcal{X}, \mathcal{Z}]/(\mathcal{Z} - \mathcal{X}^3 - \mathcal{Z}^3)$ glued together. These come from letting $x = X/Z, y = Y/Z$ and $\mathcal{X} = X/Y, \mathcal{Z} = Z/Y$, respectively. Some sample elements of this scheme are $(x - 2, y - 3) = (\mathcal{X} - 2\mathcal{Z}, 1 - 3\mathcal{Z})$, $(x - 1, y^2 - 2)$, $(\mathcal{X}, \mathcal{Z})$ (corresponding to the point at ∞ for $y^2 = x^3 + 1$), and (0). There is one for each Galois-orbit of points on the curve over $\overline{\mathbf{Q}_5}$, plus (0), the generic point.

What do we mean by a scheme over a ring? There are ring homomorphisms $\mathbf{Z}_5 \to \mathbf{Z}_5[x, y]/(y^2 - x^3 - 1)$ and $\mathbf{Z}_5 \to \mathbf{Z}_5[\mathcal{X}, \mathcal{Z}]/(\mathcal{Z} - \mathcal{X}^3 - \mathcal{Z}^3)$. Let $\mathcal{E}$ be union of the spectra, glued as above. The homomorphisms induce a morphism $f : \mathcal{E} \to \mathrm{spec}\mathbf{Z}_5$. Recall that the inverse image of a prime ideal via a ring homomorphism is a prime ideal. Thus we get a map on spectra in the other direction. A morphism of schemes is induced from ring homomorphisms in the other direction. We write $\mathcal{E}/\mathbf{Z}_5$ to describe this situation. Sample points of $\mathcal{E}$ are $(x - 2, y - 3)$ and $(x - 2, y - 3, 5)$.

The set $f^{-1}((5))$ consists of ideals containing 5; it is called the special fiber. It is closed and equal to $\tilde{E}/\mathbf{F}_5$ (i.e. the scheme associated to that variety).

The set $f^{-1}((0))$ consists of ideals not containing 5; it is called the generic fiber. It is open and equal to $E/\mathbf{Q}_5$ (i.e. the scheme associated to that variety).

$\mathcal{E}$: 

$\downarrow$ f $\qquad\qquad \downarrow \qquad\qquad \downarrow$

Spec $\mathbf{Z}_5$: $\qquad (0) \qquad\qquad (5)$

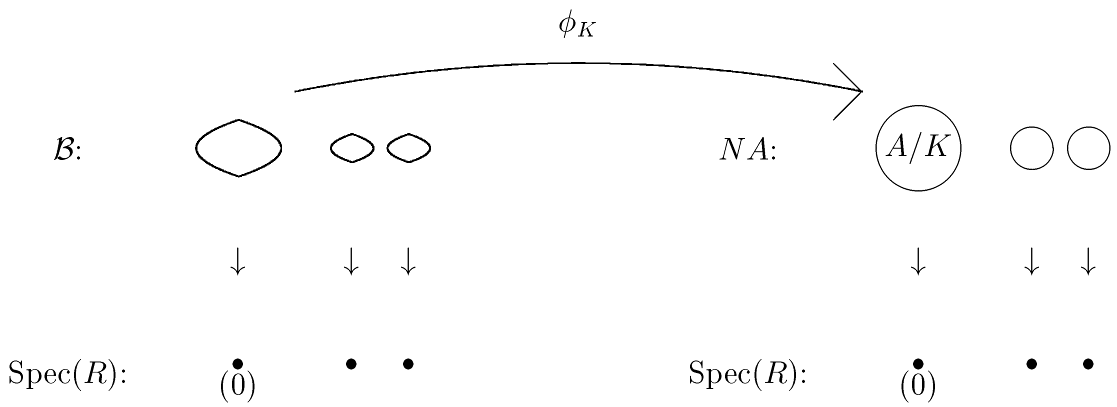
A section ι, of f, is a morphism $\iota : \mathrm{spec}\mathbf{Z}_5 \to \mathcal{E}$ so $f\iota = \mathrm{id}$. One homomorphism sends $x, y \mapsto 2, 3$ (we'll just look on an affine subset). This gives the section sending $(0), (5) \mapsto (x - 2, y - 3), (x - 2, y - 3, 5)$. All sections arise in this way. Therefore the sections as a set are equal to $E(\mathbf{Q}_5)$.

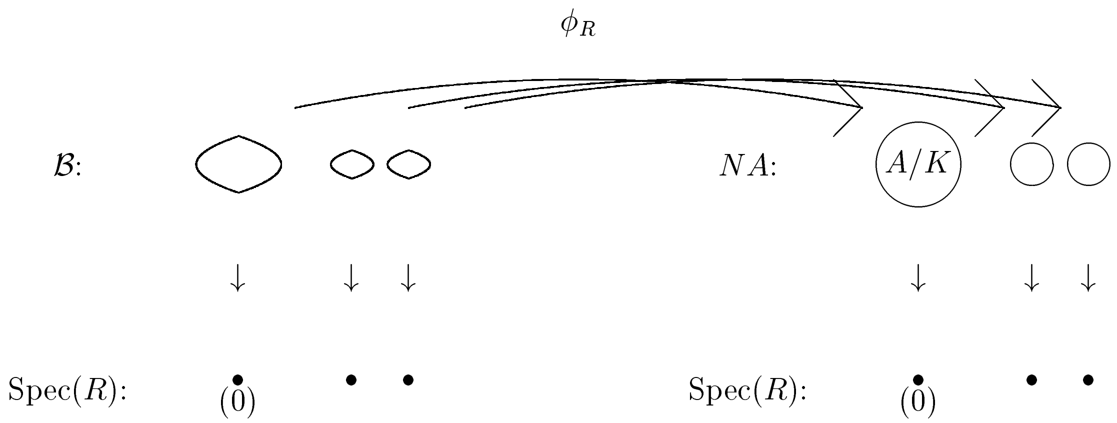
The important property of a group scheme S/R, is that the fibers are group varieties over the appropriate fields. In the above example we had elliptic curves over $\mathbf{Q}_5$ and $\mathbf{F}_5$. Sections from R to S, written $S(R)$, form a group. In the above example we have $\mathcal{E}(\mathbf{Z}_5) \cong E(\mathbf{Q}_5)$.

Intuitive "definition": A morphism $S \to \operatorname{spec}(R)$ is smooth if the fibers are non-singular.

Definition: Let R be a Dedekind domain with fraction field K and let A/K be an abelian variety. A Néron model for A/K is a smooth group scheme NA/R whose generic fiber is A/K and which satisifies the following universal property:

Let $\mathcal{B}/R$ be a smooth R-scheme with generic fiber B/K and let $\phi_K : B/K \to A/K$ be a morphism, defined over K. Then there exists a unique R-morphism $\phi_R : \mathcal{B}/R \to NA/R$ extending ϕ_K.





Theorem.

1. Néron models exist, and are unique up to unique isomorphism.

(The second part follows from the universal property, letting $\mathcal{B}$ be another Néron model).

2. $NA(R) \cong A(K)$. Proof. Let $\mathcal{B} = \operatorname{spec}(R)$; then $B = \operatorname{spec}(K)$. We have $\{\operatorname{spec}(K) \to A/K\} \cong A(K)$. By the universal property, each extends uniquely to a section

$$\{\operatorname{spec}(R) \to NA/R\} \stackrel{\text{defn}}{=} NA(R)$$

End proof.

3. Let K'/K be a finite unramified extension. Let R' be the integral closure of R in K'. Let NA/R be the Néron model for A/K. Then $NA \times_R R'$ is the Néron model for A/K'.

4. Each fiber is an extension of a finite group by an algebraic group.

Elliptic curves and Jacobians. We get the Néron model of an elliptic curve using the minimal proper regular model.

Intuitive "definition": A morphism is proper if its fibers are (an algebraic analogy of) compact and Hausdorff.

Proper implies complete fibers.

Theorem. Let R be a Dedekind domain, and C/K be a non-singular projective curve of genus greater than 0. There is a proper regular model $\mathcal{C}^{\min}/R$, for C/K such that if $\mathcal{C}/R$ is another proper regular model for C/K and we fix an isomorphism of the generic fiber of $\mathcal{C}$ to that of $\mathcal{C}^{\min}$ (which is C/K by definition of model) then the induced R-birational map $\mathcal{C} \to \mathcal{C}^{\min}$ is an R-morphism.

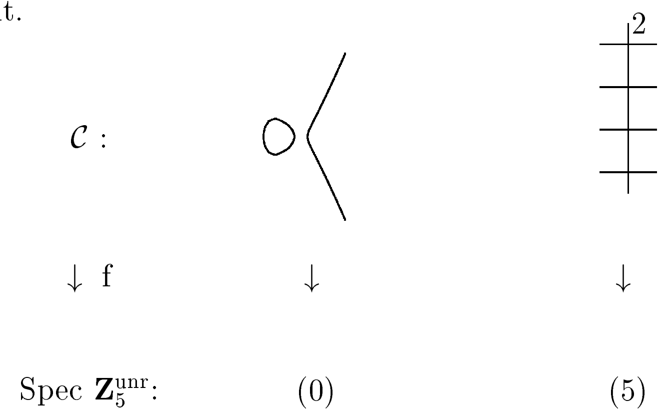
$\mathcal{C}^{\min}/R$ is called the minimal proper regular model for C/K. It is unique up to unique R-isomorphism.

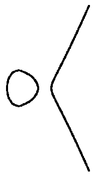
Theorem (Castelnuovo). Let R be a discrete valuation ring with separably closed residue class field k. From a proper, regular model of a curve over R, you can get the minimal one by blowing down exceptional components.

This gives existence. We have algorithms for finding the minimal proper regular model for elliptic curves (Tate) and genus 2 curves when $\mathrm{char}(k) \neq 2$ (Liu).

How do we get the Néron model for an elliptic curve? Let R be a Dedekind domain, and E/K be an elliptic curve. Let $\mathcal{C}/R$ be the minimal proper regular model for E/K. NE/R can be obtained by removing all of the singularities from the special fibers. We write this $NE/R = \mathcal{C}_{\mathrm{sm}}/R$.
Let us give some evidence that this is plausible. Regularity of $\mathcal{C}$ insures $\mathcal{C}_{\mathrm{sm}}(R) = \mathcal{C}(R)$ (so you can not lift singular points of the reduction to a section). Properness ensures $\mathcal{C}(R) = E(K)$ (complete fibers ensure that all K-rational points are reached by sections). So $E(K) = NE(R)$. Smoothness and minimality are used to show that NE/R is a group scheme.

Example. $E_5 : ZY^2 = X^3 - 125Z^3/\mathbf{Z}_5^{\mathrm{unr}}$. We blow this up and get a minimal proper regular model $\mathcal{C}/\mathbf{Z}_5^{\mathrm{unr}}$. The special fiber has a double component (shown vertically) so we must remove it.



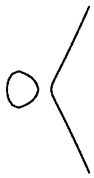
NE_5 :  —— —— —— ——

$= \mathcal{C}_{\text{sm}}$

$\downarrow$ f $\qquad \downarrow \qquad \downarrow$

Spec $\mathbf{Z}_5^{\text{unr}}$: (0) (5)

Filtration. Let R be the integer ring in a p-adic field (by that we mean the completion of a number field or a maximal unramified extension thereof), K its field of fractions and k its reduction. Let A be an abelian variety over K. When we remove the non-identity components of the special fiber of NA we get NA^0, an open subgroup scheme, called the identity component of NA.

NE_5^0 :  ——

$\downarrow$ f $\qquad \downarrow \qquad \downarrow$

Spec $\mathbf{Z}_5^{\text{unr}}$: (0) (5)

We have $NA^0(R) \subset NA(R) \cong A(K)$. Define $A^0(K) \cong NA^0(R)$. The group $A(K)/A^0(K)$ is called the component group. Its size is known as the Tamagawa number.

Fact: If C has good reduction, then $NJ/R = NJ^0/R$, (so the reduction of J is an abelian variety over k).

Let $\tilde{A} = NA \times_R k$ denote the special fiber of NA. $\tilde{A}$ is a group variety over k called the reduction of A. Define $A^1(K) = \ker : A(K) \to \tilde{A}(k)$; it is called the kernel of reduction. It is a pro-p group isomorphic to a formal group. It contains a subgroup isomorphic to $\mathcal{O}_K^{\dim(A)}$ of finite index.

What does this have to do with minimal Weierstrass equations?. Let E be given by a minimal Weierstrass equation. Then $E^0(K)$ is the points with non-singular reduction and $E^1(K)$ is the kernel of reduction for that equation. We have

$$E^0(K)/E^1(K) \cong \tilde{E}^0(k) \cong E_{ns}(k)$$

where the middle term is the identity component of the special fiber and the last term is the set of non-singular k-rational points of the curve described by the minimal Weierstrass equation.
Fact: If E has good reduction and we describe it by a minimal Weierstrass equation, then $E/R = NE/R$.

Let us describe the filtration for $E_5/\mathbf{Q}_5^{\mathrm{unr}}$. We can use the algorithm from the Tamagawa lecture to find that $E_5(\mathbf{Q}_5^{\mathrm{unr}})/E_5^0(\mathbf{Q}_5^{\mathrm{unr}}) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. Representatives (using the affine equation $y^2 = x^3 - 125$) are given by the set of 2-torsion points: $(5,0), (5\zeta_3, 0), (5\zeta_3^2, 0), \mathcal{O}$. In addition, we have $E_5^0(\mathbf{Q}_5^{\mathrm{unr}})/E_5^1(\mathbf{Q}_5^{\mathrm{unr}}) \cong E_{5,ns}(\overline{\mathbf{F}_5}) \cong \overline{\mathbf{F}_5}^+$. Note that the special fiber of NE_5^0 is $\mathbf{P}^1(\overline{\mathbf{F}_5})$ minus a point, which is $\overline{\mathbf{F}_5}^+$. Lastly $E_5^1(\mathbf{Q}_5^{\mathrm{unr}})$ is a pro-5 formal group which is isomorphic to $\mathbf{Z}_5^{\mathrm{unr}}$.