Euler Systems

Karl Rubin

Author address:

Department of Mathematics
Stanford University
Stanford, CA 94305-2125
USA

E-mail address: rubin@math.stanford.edu

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Introduction

History. In 1986, Francisco Thaine [**Th**] introduced a remarkable method for bounding ideal class groups of real abelian extensions of $\mathbf{Q}$. Namely, if F is such a field, he used cyclotomic units in fields $F(\boldsymbol{\mu}_\ell)$, for a large class of rational primes ℓ, to construct explicitly a large collection of principal ideals of F. His construction produced enough principal ideals to bound the exponent of the different Galois-eigencomponents of the ideal class group of F, in terms of the cyclotomic units of F. Although Thaine's results were already known as a corollary of Iwasawa's "main conjecture", proved by Mazur and Wiles [**MW**], Thaine's proof was very much simpler. The author [**Ru1**] was able to apply Thaine's method essentially unchanged to bound ideal class groups of abelian extensions of imaginary quadratic fields in terms of elliptic units, with important consequences for the arithmetic of elliptic curves with complex multiplication.

Shortly after this, Victor Kolyvagin [**Ko1**] discovered (independently of Thaine) a similar remarkable method, in his case to bound the Selmer group of an elliptic curve. Suppose E is a modular[1] elliptic curve over $\mathbf{Q}$, with sign $+1$ in the functional equation of its L-function. Kolyvagin's method used Heegner points on E over anticyclotomic extensions of prime conductor of an imaginary quadratic field K (in place of cyclotomic units in abelian extensions of $\mathbf{Q}$) to construct cohomology classes over K (in place of principal ideals). He used these cohomology classes, along with duality theorems from Galois cohomology, to bound the exponent of the Selmer group of E over $\mathbf{Q}$. The overall structure of his proof was very similar to that of Thaine.

Inspired by Thaine's work and his own, Kolyvagin then made another fundamental advance. In his paper [**Ko2**] he introduced what he called "Euler systems." In Thaine's setting (the Euler system of cyclotomic units)

[1] C. Breuil, B. Conrad, F. Diamond, and R. Taylor have recently announced that they have succeeded in extending the methods of Wiles to prove that every elliptic curve over $\mathbf{Q}$ is modular. Given this result, one can remove the assumption that E is modular, both here and throughout the discussion of elliptic curves in §3.5.

Kolyvagin showed how to use cyclotomic units in fields $F(\boldsymbol{\mu}_r)$, for a large class of integers r (no longer just primes), to bound the *orders* of the different Galois-eigencomponents of the ideal class group of F, rather than just their exponents. Similarly, by using a larger collection of Heegner points in the situation described above, Kolyvagin was able to give a bound for the order of the Selmer group of E. Thanks to the theorem of Gross and Zagier [**GZ**], which links Heegner points with the L-function of E, Kolyvagin's bound is closely related to the order predicted by the Birch and Swinnerton-Dyer conjecture.

This book. This book describes a general theory of Euler systems for p-adic representations. We start with a finite-dimensional p-adic representation T of the Galois group of a number field K. (Thaine's situation is the case where $K = \mathbf{Q}$ and T is $\varprojlim \boldsymbol{\mu}_{p^n}$ twisted by an even Dirichlet character; in Kolyvagin's case T is the Tate module of a modular elliptic curve.) We define an Euler system for T to be a collection of cohomology classes $\mathbf{c}_F \in H^1(F, T)$, for a family of abelian extensions F of K, with a relation between $\mathbf{c}_{F'}$ and $\mathbf{c}_F$ whenever $F \subset F'$.

Our main results show how the existence of an Euler system leads to bounds on the sizes of Selmer groups attached to the Galois module $\mathrm{Hom}(T, \boldsymbol{\mu}_{p^\infty})$, bounds which depend only on the given Euler system. The proofs of these theorems in our general setting parallel closely (with some additional complications) Kolyvagin's original proof. Results similar to ours have recently been obtained independently by Kato [**Ka2**] and Perrin-Riou [**PR5**].

What we do *not* do here is construct new Euler systems. This is the deepest and most difficult part of the theory. Since Kolyvagin's introduction of the concept of an Euler system there have been very few new Euler systems found, but each has been extremely important. Kato [**Ka3**] has constructed a new Euler system for a modular elliptic curve over $\mathbf{Q}$, very different from Kolyvagin's system of Heegner points (see §3.5). Flach [**Fl**] has used a collection of cohomology classes (but not a complete Euler system) to bound the exponent but not the order of the Selmer group of the symmetric square of a modular elliptic curve.

One common feature of all the Euler systems mentioned above is that they are closely related to special values of L-functions. An important benefit of this connection is that the bounds on Selmer groups that come out of the Euler system machinery are then linked to L-values. Such bounds provide evidence for the Bloch-Kato conjectures [**BK**], which predict the orders of these Selmer groups in terms of L-values.

Our definition of Euler system says nothing about L-values. If there is an Euler system for T then there is a whole family of them (for example, the collection of Euler system cohomology classes is a $\mathbf{Z}_p$-module, as well as a $\mathrm{Gal}(\bar{K}/K)$-module). If one multiplies an Euler system by p, one gets a new Euler system but a worse bound on the associated Selmer groups. The philosophy underlying this book, although it is explicitly discussed only in Chapter 8, is that under certain circumstances, not only should there exist an Euler system for T, but there should exist a "best possible" Euler system, which will be related to (and contain all the information in) the p-adic L-function attached to T.

A remark about generality. It is difficult to formulate the "most general" definition of an Euler system, and we do not attempt to do this here. The difficulty is partly due to the fact that the number of examples on which to base a generalization is quite small. In the end, we choose a definition which does not cover the case of Kolyvagin's Heegner points, because to use a more inclusive definition would introduce too many difficulties. (In Chapter 9 we discuss possible modifications of our definition, including one which does include the case of Heegner points.) On the other hand, we do allow the base field K to be an arbitrary number field, instead of requiring $K = \mathbf{Q}$. Although this adds a layer of notation to all proofs, it does not significantly increase the difficulty. A reader wishing to restrict to the simplest (and most interesting) case $K = \mathbf{Q}$ should feel free to do so.

Organization. In Chapter 1 we introduce the local and global cohomology groups, and state the duality theorems, which will be required to state and prove our main results. Chapter 2 contains the definition of an Euler system, followed by the statements of our main theorems bounding the Selmer group of $\mathrm{Hom}(T, \boldsymbol{\mu}_{p^\infty})$ over the base field K (§2.2) and over $\mathbf{Z}_p^d$-extensions K_∞ of K (§2.3).

Chapter 3 contains some concrete applications of the theorems of Chapter 2. We apply those theorems to three different Euler systems. The first is constructed from cyclotomic units, and is used to study ideal class groups of real abelian fields (§3.2). The second is constructed from Stickelberger elements, and is used to study the minus part of ideal class groups of abelian fields (§3.4). The third is constructed by Kato from Beilinson elements in the K-theory of modular curves, and is used to study the Selmer groups of modular elliptic curves (§3.5).

The proofs of the theorems of Chapter 2 are given in Chapters 4 through 7. In Chapter 4 we give Kolyvagin's "derivative" construction,

taking the Euler system cohomology classes defined over abelian extensions of K and using them to produce cohomology classes over K itself. We then analyze the localizations of these derived classes, information which is crucial to the proofs of our main theorems. In Chapter 5 we bound the Selmer group over K by using the derived classes of Chapter 4 and global duality. Bounding the Selmer group over K_∞ is similar but more difficult; this is accomplished in Chapter 7 after a digression in Chapter 6 which is used to reduce the proof to a simpler setting.

In Chapter 8 we discuss the conjectural connection between Euler systems and p-adic L-functions. This connection relies heavily on conjectures of Perrin-Riou [**PR4**]. Assuming a strong version of Perrin-Riou's conjectures, and subject to some hypotheses on the representation T, we show that there is an Euler system for T which is closely related to the p-adic L-function.

Chapter 9 discusses possible variants of our definition of Euler systems.

Finally, there is some material which is used in the text, but which is outside our main themes. Rather than interrupt the exposition with this material, we include it in four appendices (A–D).

Notation

If F is a field, $\bar{F}$ will denote a fixed separable closure of F and

$$G_F \;=\; \mathrm{Gal}(\bar{F}/F).$$

(All fields we deal with will be perfect, so we may as well assume that $\bar{F}$ is an algebraic closure of F.) Also, F^{ab} will denote the maximal abelian extension of F, and if F is a local field F^{ur} will denote the maximal unramified extension of F. If F is a global field and Σ is a set of places of F, then F_Σ will denote the maximal extension of F which is unramified outside Σ. If $K \subset F$ is an extension of fields, we will write $K \subset_{\mathrm{f}} F$ to indicate that $[F : K]$ is finite.

If F is a field and B is a G_F-module, $F(B)$ will denote the fixed field of the kernel of the map $G_F \to \mathrm{Aut}(B)$, i.e., the smallest extension of F whose absolute Galois group acts trivially on B.

If $\mathcal{O}$ is a ring and B is an $\mathcal{O}$-module then $\mathrm{Ann}_{\mathcal{O}}(B) \subset \mathcal{O}$ will denote the annihilator of B in $\mathcal{O}$. If $M \in \mathcal{O}$ then B_M will denote the kernel of multiplication by M on B, and similarly if M is an ideal of $\mathcal{O}$. If B is a free $\mathcal{O}$-module, τ is an $\mathcal{O}$-linear endomorphism of B, and x is an indeterminate, we will write

$$P(\tau|B; x) \;=\; \det(1 - \tau x|B) \;\in\; \mathcal{O}[x],$$

the determinant of $1 - \tau x$ acting on B. *Caution:* If σ is, for example, a Galois automorphism which acts on B, then $P(\tau|B;\sigma)$ means $P(\tau|B;x)$ evaluated at $x = \sigma$, and *not* $\det(1 - \tau\sigma|B)$.

The Galois module of n-th roots of unity will be denoted by $\boldsymbol{\mu}_n$. If p is a fixed rational prime and F is a field of characteristic different from p, the cyclotomic character

$$\varepsilon_{\mathrm{cyc}} : G_F \longrightarrow \mathbf{Z}_p^{\times}$$

is the character giving the action of G_F on $\boldsymbol{\mu}_{p^\infty}$, and the Teichmüller character $\omega : G_F \to (\mathbf{Z}_p^{\times})_{\mathrm{tors}}$ is the character giving the action of G_F on $\boldsymbol{\mu}_p$ (if p is odd) or on $\boldsymbol{\mu}_4$ (if $p = 2$). Hence ω has order at most $p - 1$ or 2, respectively (with equality if $F = \mathbf{Q}$) and $\langle\varepsilon\rangle = \omega^{-1}\varepsilon_{\mathrm{cyc}}$ takes values in $1 + p\mathbf{Z}_p$ (resp. $1 + 4\mathbf{Z}_2$).

If B is an abelian group, B_{div} will denote the maximal divisible subgroup of B. If p is a fixed rational prime, we define the p-adic completion of B to be the double dual

$$B\hat{} = \mathrm{Hom}(\mathrm{Hom}(B, \mathbf{Q}_p/\mathbf{Z}_p), \mathbf{Q}_p/\mathbf{Z}_p)$$

(where Hom always denotes continuous homomorphisms if the groups involved comes with natural topologies). For example, if B is a $\mathbf{Z}_p$-module then $B\hat{} = B$; if B is a finitely generated abelian group then $B\hat{} = B \otimes_{\mathbf{Z}} \mathbf{Z}_p$. In general, $B\hat{}$ is a $\mathbf{Z}_p$-module and there is a canonical map from B to $B\hat{}$. If τ is an endomorphism of B then we will often write $B^{\tau=0}$ for the kernel of τ, write $B^{\tau=1}$ for the subgroup fixed by τ, etc.

If $\{A_i : i \in I\}$ is an inverse system, we will denote by $\{a_i\}_i$ (or sometimes simply $\{a_i\}$) an element of $\varprojlim A_i$.

Finally, if S is a set, then $|S|$ will denote the cardinality of S.

Most of these notations will be recalled when they first occur.

CHAPTER 1

Galois Cohomology of p-adic Representations

In this chapter we introduce our basic objects of study: p-adic Galois representations, their cohomology groups, and especially Selmer groups.

We begin by recalling basic facts about cohomology groups associated to p-adic representations, material which is mostly well-known but included here for completeness.

A Selmer group is a subgroup of a global cohomology group determined by "local conditions". In §1.3 we discuss these local conditions, which are defined in terms of special subgroups of the local cohomology groups. In §1.4 we state without proof the results we need concerning the Tate pairing on local cohomology groups, and we study how our special subgroups behave with respect to this pairing.

In §1.5 and §1.6 we define Selmer groups and give the basic examples of ideal class groups and Selmer groups of elliptic curves and abelian varieties. Then in §1.7, using Poitou-Tate global duality and the local orthogonality results from §1.4, we derive our main tool (Theorem 1.7.3) for bounding the size of Selmer groups.

1.1. p-adic Representations

Definition 1.1.1. Suppose K is a field, p is a rational prime, and $\mathcal{O}$ is the ring of integers of a finite extension Φ of $\mathbf{Q}_p$. A *p-adic representation* of $G_K = \mathrm{Gal}(\bar{K}/K)$, with coefficients in $\mathcal{O}$, is a free $\mathcal{O}$-module T of finite rank with a continuous $\mathcal{O}$-linear action of G_K.

Let $\mathbf{D}$ denote the divisible module $\Phi/\mathcal{O}$. Attached to a p-adic representation T we define

$$\begin{aligned} V &= T \otimes_{\mathcal{O}} \Phi, \\ W &= V/T = T \otimes_{\mathcal{O}} \mathbf{D}, \\ W_M &= M^{-1}T/T \subset W \quad \text{for nonzero } M \in \mathcal{O}, \end{aligned}$$

so W_M is the M-torsion in W. Note that T determines V and W, and W determines $T = \varprojlim W_M$ and V, but in general there may be different $\mathcal{O}$-modules T giving rise to the same vector space V.

Example 1.1.2. Suppose $\rho : G_K \to \mathcal{O}^\times$ is a character (continuous, but not necessarily of finite order). Then we can take $T = \mathcal{O}_\rho$, where $\mathcal{O}_\rho$ is a free rank-one $\mathcal{O}$-module on which G_K acts via ρ. Clearly every one-dimensional representation arises in this way. When ρ is the trivial character we get $T \cong \mathcal{O}$, and when $\mathcal{O} = \mathbf{Z}_p$ and ρ is the cyclotomic character

$$\varepsilon_{\text{cyc}} : G_K \longrightarrow \text{Aut}(\boldsymbol{\mu}_{p^\infty}) \xrightarrow{\sim} \mathbf{Z}_p^\times$$

we get

$$\begin{aligned} T &\cong \mathbf{Z}_p(1) = \varprojlim_n \boldsymbol{\mu}_{p^n}, \\ V &\cong \mathbf{Q}_p(1) = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \varprojlim_n \boldsymbol{\mu}_{p^n}, \\ W &\cong (\mathbf{Q}_p/\mathbf{Z}_p)(1) = \boldsymbol{\mu}_{p^\infty}. \end{aligned}$$

For general $\mathcal{O}$ we also write $\mathcal{O}(1) = \mathcal{O} \otimes \mathbf{Z}_p(1)$, write $\Phi(1) = \Phi \otimes \mathbf{Q}_p(1)$, and write $\mathbf{D}(1) = \mathbf{D} \otimes \mathbf{Z}_p(1)$.

Definition 1.1.3. If T is a p-adic representation of G_K then so is the *dual representation*

$$T^* = \text{Hom}_{\mathcal{O}}(T, \mathcal{O}(1)).$$

We will also write

$$\begin{aligned} V^* &= \text{Hom}_{\mathcal{O}}(V, \Phi(1)) = \text{Hom}_{\mathcal{O}}(T, \Phi(1)) = T^* \otimes_{\mathcal{O}} \Phi, \\ W^* &= V^*/T^* = \text{Hom}_{\mathcal{O}}(T, \mathbf{D}(1)). \end{aligned}$$

Example 1.1.4. If $\rho : G_K \to \mathcal{O}^\times$ is a continuous character as in Example 1.1.2, and $T = \mathcal{O}_\rho$, then $T^* = \mathcal{O}_{\rho^{-1}\varepsilon_{\text{cyc}}}$.

Example 1.1.5. Suppose A is an abelian variety defined over K, and p is a prime different from the characteristic of K. We can take $\mathcal{O}$ to be $\mathbf{Z}_p$ and T to be the p-adic Tate module of A defined by

$$T_p(A) = \varprojlim_n A_{p^n}$$

where A_{p^n} denotes the p^n-torsion in $A(\bar{K})$. Then $\text{rank}_{\mathbf{Z}_p} T = 2\dim(A)$. If A and A' are isogenous, their Tate modules $T = T_p(A)$ and $T' = T_p(A')$ need not be isomorphic (as G_K-modules), but the corresponding $\mathbf{Q}_p$-vector spaces V and V' are isomorphic.

If the endomorphism algebra of A over K contains the ring of integers $\mathcal{O}_F$ of a number field F, and $\mathfrak{p}$ is a prime of F above p, we can also take $\Phi = F_{\mathfrak{p}}$, the completion of F at $\mathfrak{p}$, and

$$T = T_{\mathfrak{p}}(A) = \varprojlim_n A_{\mathfrak{p}^n}$$

which has rank $2\dim(A)/[F:\mathbf{Q}]$ over the ring of integers $\mathcal{O}$ of Φ. If A is an elliptic curve with complex multiplication by $F \subset K$, this is another important source of one-dimensional representations.

1.2. Galois Cohomology

Suppose K is a field. If B is a commutative topological group with a continuous action of G_K, then we have the continuous cohomology groups

$$H^i(K,B) = H^i(G_K,B).$$

If further the action of G_K factors through the Galois group $\mathrm{Gal}(K'/K)$ for some extension K' of K, we also write $H^i(K'/K,B) = H^i(\mathrm{Gal}(K'/K),B)$. See Appendix B for the basic facts which we will need about continuous cohomology groups.

Example 1.2.1. We have

$$\begin{aligned} H^1(K,\mathbf{Q}_p/\mathbf{Z}_p) &= \mathrm{Hom}(G_K,\mathbf{Q}_p/\mathbf{Z}_p), \\ H^1(K,\mathbf{Z}_p) &= \mathrm{Hom}(G_K,\mathbf{Z}_p). \end{aligned}$$

By Kummer theory and Proposition B.2.3, respectively,

$$\begin{aligned} H^1(K,\boldsymbol{\mu}_{p^\infty}) &= K^\times \otimes (\mathbf{Q}_p/\mathbf{Z}_p), \\ H^1(K,\mathbf{Z}_p(1)) &= \varprojlim_n H^1(K,\boldsymbol{\mu}_{p^n}) = \varprojlim_n K^\times/(K^\times)^{p^n} = K^\times \hat{\otimes} \mathbf{Z}_p, \end{aligned}$$

where $\hat{\otimes}$ denotes the (p-adically) completed tensor product.

Suppose T is a p-adic representation of G_K with coefficients in $\mathcal{O}$ as in §1.1, and $M \in \mathcal{O}$ is nonzero. Recall that $V = T \otimes \Phi$ and $W = V/T$. We will frequently make use of the following exact sequences and commutative diagram.

$$0 \longrightarrow W_M \longrightarrow W \xrightarrow{M} W \longrightarrow 0 \tag{1.1}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \xrightarrow{M} & T & \xrightarrow{M^{-1}} & W_M & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle M^{-1}} & & \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0. \end{array} \tag{1.2}$$

Lemma 1.2.2. *Suppose $M \in \mathcal{O}$ is nonzero.*

(i) *The exact sequence* (1.1) *induces an exact sequence*

$$0 \longrightarrow W^{G_K}/MW^{G_K} \longrightarrow H^1(K,W_M) \longrightarrow H^1(K,W)_M \longrightarrow 0.$$

(ii) *The bottom row of* (1.2) *induces an exact sequence*

$$V^{G_K} \longrightarrow W^{G_K} \longrightarrow H^1(K,T)_{\text{tors}} \longrightarrow 0.$$

(iii) *The kernel of the map* $H^1(K,T) \to H^1(K,W)$ *induced by the composition* $T \twoheadrightarrow T/MT \xrightarrow{\sim} W_M \hookrightarrow W$ *is*

$$MH^1(K,T) + H^1(K,T)_{\text{tors}}.$$

Proof. Assertion (i) is clear, and so is (ii) once we show that the kernel of the natural map $H^1(K,T) \to H^1(K,V)$ is $H^1(K,T)_{\text{tors}}$. But this is immediate from Proposition B.2.4, which says that the map $H^1(K,T) \to H^1(K,V)$ induces an isomorphism $H^1(K,V) \cong H^1(K,T) \otimes \mathbf{Q}_p$.

The diagram (1.2) induces a commutative diagram with exact rows

$$\begin{array}{ccccc} H^1(K,T) & \xrightarrow{M} & H^1(K,T) & \longrightarrow & H^1(K,W_M) \\ \| & & \downarrow{\scriptstyle \phi_1} & & \downarrow \\ H^1(K,T) & \xrightarrow{\phi_2} & H^1(K,V) & \xrightarrow{\phi_3} & H^1(K,W) \end{array}$$

where ϕ_1 is induced by $M^{-1} : T \to V$. Since

$$\ker(\phi_3) \;=\; \phi_2(H^1(K,T)) \;=\; \phi_1(MH^1(K,T)),$$

we see that

$$\ker(\phi_3 \circ \phi_1) \;=\; MH^1(K,T) + \ker(\phi_1) \;=\; MH^1(K,T) + H^1(K,T)_{\text{tors}},$$

which proves (iii). □

1.3. Local Cohomology Groups

Unramified local cohomology. Suppose for this section that K is a finite extension of $\mathbf{Q}_\ell$ for some ℓ, where we allow ℓ to be either a rational prime or ∞ (in which case $\mathbf{Q}_\ell = \mathbf{R}$). Let $\mathcal{I}$ denote the inertia subgroup of G_K (so $\mathcal{I} = G_K$ if K is archimedean), let $K^{\text{ur}} = \bar{K}^{\mathcal{I}}$ be the maximal unramified extension of K, and if K is nonarchimedean let $\text{Fr} \in \text{Gal}(K^{\text{ur}}/K)$ denote the Frobenius automorphism.

Definition 1.3.1. Suppose B is a G_K-module. We say that B is unramified if $\mathcal{I}$ acts trivially on B. We define the subgroup of unramified cohomology classes $H^1_{\text{ur}}(K,B) \subset H^1(K,B)$ by

$$H^1_{\text{ur}}(K,B) \;=\; \ker\bigl(H^1(K,B) \longrightarrow H^1(\mathcal{I},B)\bigr).$$

Note that if T is as in §1.1, then

$$T \text{ is unramified} \iff V \text{ is unramified} \iff W \text{ is unramified},$$

and if K is nonarchimedean of residue characteristic different from p, then this is equivalent to T^*, V^*, and/or W^* being unramified.

Lemma 1.3.2. *Suppose B is a G_K-module which is either a finitely generated $\mathbf{Z}_p$-module, or a finite-dimensional $\mathbf{Q}_p$-vector space, or a discrete torsion $\mathbf{Z}_p$-module.*

(i) *If K is nonarchimedean then*

$$H^1_{\mathrm{ur}}(K,B) \cong H^1(K^{\mathrm{ur}}/K, B^{\mathcal{I}}) \cong B^{\mathcal{I}}/(\mathrm{Fr}-1)B^{\mathcal{I}}.$$

(ii) *If K is nonarchimedean of residue characteristic different from p, then*

$$H^1(K,B)/H^1_{\mathrm{ur}}(K,B) \cong H^1(\mathcal{I},B)^{\mathrm{Fr}=1}.$$

(iii) *If K is archimedean then $H^1_{\mathrm{ur}}(K,B) = 0$.*

Proof. Assertion (iii) and the first isomorphism of (i) follow from the inflation-restriction exact sequence (Proposition B.2.5(i)). The second isomorphism of (i) (induced by the map on cocycles $c \mapsto c(\mathrm{Fr})$) is Lemma B.2.8.

The hypotheses on B guarantee (see Propositions B.2.5(ii) and B.2.7) that we have an inflation-restriction exact sequence

$$0 \to H^1(K^{\mathrm{ur}}/K, B^{\mathcal{I}}) \to H^1(K,B) \to H^1(\mathcal{I},B)^{\mathrm{Fr}=1} \to H^2(K^{\mathrm{ur}}/K, B^{\mathcal{I}}).$$

If K is nonarchimedean then $\mathrm{Gal}(K^{\mathrm{ur}}/K)$ has cohomological dimension one, so $H^2(K^{\mathrm{ur}}/K, B^{\mathcal{I}}) = 0$. This proves (ii). □

Corollary 1.3.3. *Suppose K is nonarchimedean of residue characteristic different from p, and V is a $\mathbf{Q}_p[G_K]$-module which has finite dimension as a $\mathbf{Q}_p$-vector space. Then*

(i) $\dim_{\mathbf{Q}_p}(H^1_{\mathrm{ur}}(K,V)) = \dim_{\mathbf{Q}_p}(V^{G_K})$,

(ii) $\dim_{\mathbf{Q}_p}(H^1(K,V)/H^1_{\mathrm{ur}}(K,V)) = \dim_{\mathbf{Q}_p}(H^2(K,V))$.

Proof. Using Lemma 1.3.2(i) we have an exact sequence

$$0 \longrightarrow V^{G_K} \longrightarrow V^{\mathcal{I}} \xrightarrow{\mathrm{Fr}-1} V^{\mathcal{I}} \longrightarrow H^1_{\mathrm{ur}}(K,V) \longrightarrow 0$$

which proves (i).

Since the residue characteristic of K is different from p, the inertia group $\mathcal{I}$ has a unique maximal p-divisible subgroup $\mathcal{I}'$, and $\mathcal{I}/\mathcal{I}' \cong \mathbf{Z}_p$ (see [**Fr**] §8 Corollary 3). Thus both $\mathcal{I}$ and $\mathrm{Gal}(K^{\mathrm{ur}}/K)$ have p-cohomological dimension one. It follows that

$$H^m(K^{\mathrm{ur}}/K, H^n(\mathcal{I},V)) = 0$$

if $m > 1$ or $n > 1$. Therefore the Hochschild-Serre spectral sequence (Propositions B.2.5(ii) and B.2.7) shows that

$$H^1(K^{\mathrm{ur}}/K, H^1(\mathcal{I},V)) = H^2(K,V).$$

On the other hand, Lemma 1.3.2 shows that

$$H^1(K^{\mathrm{ur}}/K, H^1(\mathcal{I}, V)) \cong H^1(\mathcal{I}, V)/(\mathrm{Fr}-1)H^1(\mathcal{I}, V),$$
$$H^1(K, V)/H^1_{\mathrm{ur}}(K, V) \cong H^1(\mathcal{I}, V)^{\mathrm{Fr}=1}.$$

Thus there is an exact sequence

$$0 \to H^1(K, V)/H^1_{\mathrm{ur}}(K, V) \to H^1(\mathcal{I}, V) \xrightarrow{\mathrm{Fr}-1} H^1(\mathcal{I}, V) \to H^2(K, V) \to 0.$$

Since $\dim_{\mathbf{Q}_p}(H^1(\mathcal{I}, V))$ is finite (Proposition B.2.7(iii)), this proves (ii). □

Special subgroups. As above, let K be a finite extension of $\mathbf{Q}_\ell$ with $\ell \leq \infty$. Let T be a p-adic representation of G_K, let $V = T \otimes \Phi$, and let $W = V/T$ as in §1.1. Following many authors (for example Bloch and Kato [**BK**] §3, Fontaine and Perrin-Riou [**FPR**] §I.3.3, or Greenberg [**Gr2**]) we define special subgroups $H^1_f(K, \cdot)$ of certain cohomology groups $H^1(K, \cdot)$. We assume first that $\ell \neq p$ and $\ell \neq \infty$, and we discuss the other cases in Remarks 1.3.6 and 1.3.7 below.

Definition 1.3.4. Suppose $\ell \neq p$ and $\ell \neq \infty$. Define the *finite* part of $H^1(K, V)$ by

$$H^1_f(K, V) = H^1_{\mathrm{ur}}(K, V).$$

Define $H^1_f(K, T) \subset H^1(K, T)$ and $H^1_f(K, W) \subset H^1(K, W)$ to be the inverse image and image, respectively, of $H^1_f(K, V)$ under the natural maps

$$H^1(K, T) \longrightarrow H^1(K, V) \longrightarrow H^1(K, W).$$

For every $M \in \mathcal{O}$ define $H^1_f(K, W_M) \subset H^1(K, W_M)$ to be the inverse image of $H^1_f(K, W)$ under the natural map $H^1(K, W_M) \to H^1(K, W)$.

Finally, for V, T, W, or W_M define the *singular* quotient of $H^1(K, \cdot)$ by

$$H^1_s(K, \cdot) = H^1(K, \cdot)/H^1_f(K, \cdot),$$

so there are exact sequences

$$0 \longrightarrow H^1_f(K, \cdot) \longrightarrow H^1(K, \cdot) \longrightarrow H^1_s(K, \cdot) \longrightarrow 0.$$

If A is a $\mathbf{Z}_p$-module let A_{div} denote its maximal divisible subgroup.

Lemma 1.3.5. *Suppose T is as above, $\ell \neq p$, and $\ell \neq \infty$. Then:*

(i) $H^1_f(K, W) = H^1_{\mathrm{ur}}(K, W)_{\mathrm{div}}$.

(ii) $H^1_{\mathrm{ur}}(K, T) \subset H^1_f(K, T)$ *with finite index and* $H^1_s(K, T)$ *is torsion-free.*

(iii) *Writing* $\mathcal{W} = W^{\mathcal{I}}/(W^{\mathcal{I}})_{\mathrm{div}}$, *there are natural isomorphisms*

$$H^1_{\mathrm{ur}}(K, W)/H^1_f(K, W) \xrightarrow{\sim} \mathcal{W}/(\mathrm{Fr}-1)\mathcal{W}$$

and

$$H^1_f(K,T)/H^1_{\mathrm{ur}}(K,T) \xrightarrow{\sim} \mathcal{W}^{\mathrm{Fr}=1}.$$

(iv) *If T is unramified then*

$$H^1_f(K,T) = H^1_{\mathrm{ur}}(K,T) \text{ and } H^1_f(K,W) = H^1_{\mathrm{ur}}(K,W).$$

Proof. It is immediate from the definitions that $H^1_f(K,W)$ is divisible and $H^1_s(K,T)$ is torsion-free. The commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1_{\mathrm{ur}}(K,T) & \longrightarrow & H^1(K,T) & \longrightarrow & H^1(\mathcal{I},T) \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1_f(K,V) & \longrightarrow & H^1(K,V) & \longrightarrow & H^1(\mathcal{I},V) \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1_{\mathrm{ur}}(K,W) & \longrightarrow & H^1(K,W) & \longrightarrow & H^1(\mathcal{I},W)
\end{array}$$

shows that $H^1_f(K,W) \subset H^1_{\mathrm{ur}}(K,W)$ and $H^1_{\mathrm{ur}}(K,T) \subset H^1_f(K,T)$. The rest of assertions (i) and (ii) will follow once we prove (iii), since $W^{\mathcal{I}}/(W^{\mathcal{I}})_{\mathrm{div}}$ is finite.

Note that the image of $V^{\mathcal{I}}$ in $W^{\mathcal{I}}$ is $(W^{\mathcal{I}})_{\mathrm{div}}$. Taking $\mathcal{I}$-cohomology and then $\mathrm{Gal}(K^{\mathrm{ur}}/K)$-invariants of the exact sequence $0 \to T \to V \to W \to 0$ gives an exact sequence

$$0 \longrightarrow (W^{\mathcal{I}}/(W^{\mathcal{I}})_{\mathrm{div}})^{\mathrm{Fr}=1} \longrightarrow H^1(\mathcal{I},T)^{\mathrm{Fr}=1} \longrightarrow H^1(\mathcal{I},V)^{\mathrm{Fr}=1}.$$

Therefore using Lemma 1.3.2 we have

$$\begin{aligned}
H^1_f(K,T)/H^1_{\mathrm{ur}}(K,T) & \\
&= \ker(H^1(K,T)/H^1_{\mathrm{ur}}(K,T) \to H^1(K,V)/H^1_{\mathrm{ur}}(K,V)) \\
&= \ker(H^1(\mathcal{I},T)^{\mathrm{Fr}=1} \to H^1(\mathcal{I},V)^{\mathrm{Fr}=1}) \\
&= (W^{\mathcal{I}}/(W^{\mathcal{I}})_{\mathrm{div}})^{\mathrm{Fr}=1},
\end{aligned}$$

$$\begin{aligned}
H^1_{\mathrm{ur}}(K,W)/H^1_f(K,W) &= \mathrm{coker}(H^1_{\mathrm{ur}}(K,V) \to H^1_{\mathrm{ur}}(K,W)) \\
&= \mathrm{coker}(V^{\mathcal{I}}/(\mathrm{Fr}-1)V^{\mathcal{I}} \to W^{\mathcal{I}}/(\mathrm{Fr}-1)W^{\mathcal{I}}) \\
&= W^{\mathcal{I}}/((W^{\mathcal{I}})_{\mathrm{div}} + (\mathrm{Fr}-1)W^{\mathcal{I}}).
\end{aligned}$$

This proves (iii). If T is unramified then $W^{\mathcal{I}} = W$ is divisible, so (iv) is immediate from (iii). □

Remark 1.3.6. When the residue characteristic ℓ is equal to p, the choice of a subspace $H^1_f(K,V)$ is much more subtle. Fortunately, for the purpose of working with Euler systems it is not essential to make such a

choice. However, to understand fully the arithmetic significance of the Selmer groups we will define in §1.5, and to get the most out of the applications of Euler systems in Chapter 3, it is necessary to choose a subspace $H^1_f(K,V)$ in the more difficult case $\ell = p$.

In this case, Bloch and Kato define $H^1_f(K,V)$ using the ring B_{cris} defined by Fontaine ([**BK**] §3). Namely, they define

$$H^1_f(K,V) \;=\; \ker\left(H^1(K,V) \longrightarrow H^1(K, V\otimes B_{\text{cris}})\right).$$

For our purposes we will allow an *arbitrary* special subspace of $H^1(K,V)$, which we will still denote by $H^1_f(K,V)$. This notation is not as bad as it may seem: in our applications we will always choose a subspace $H^1_f(K,V)$ which is the same as the one defined by Bloch and Kato, but we need not (and will not) prove they are the same. One could also choose, for example, $H^1_f(K,V) = 0$ or $H^1_f(K,V) = H^1(K,V)$.

Once $H^1_f(K,V)$ is chosen, we define the groups $H^1_f(K,T)$, $H^1_f(K,W)$, and $H^1_f(K,W_M)$ in terms of $H^1_f(K,V)$ exactly as in Definition 1.3.4.

Remark 1.3.7. If $K = \mathbf{R}$ or $\mathbf{C}$ then $H^1(K,V) = 0$, so $H^1_f(K,V) = 0$ and proceeding as above we are led to define

$$\begin{aligned} H^1_f(K,W) &= 0, \\ H^1_f(K,T) &= H^1(K,T), \\ H^1_f(K,W_M) &= \ker(H^1(K,W_M) \to H^1(K,W)) \;=\; W^{G_K}/MW^{G_K}. \end{aligned}$$

Note that all of these groups are zero unless $K = \mathbf{R}$ and $p = 2$.

Lemma 1.3.8. *Suppose $M \in \mathcal{O}$ is nonzero. Then:*

(i) *The submodule $H^1_f(K,W_M)$ is the image of $H^1_f(K,T)$ under the map $H^1(K,T) \longrightarrow H^1(K,W_M)$ induced by $T \twoheadrightarrow M^{-1}T/T = W_M$.*

(ii) *If T is unramified, $\ell \neq p$, and $\ell \neq \infty$ then*

$$H^1_f(K,W_M) \;=\; H^1_{\text{ur}}(K,W_M).$$

Proof. The diagram (1.2) gives rise to a commutative diagram with exact rows

$$\begin{array}{ccccccc} H^1(K,T) & \xrightarrow{M} & H^1(K,T) & \longrightarrow & H^1(K,W_M) & \longrightarrow & H^2(K,T) \\ \| & & \downarrow{\scriptstyle M^{-1}} & & \downarrow & & \| \\ H^1(K,T) & \longrightarrow & H^1(K,V) & \longrightarrow & H^1(K,W) & \longrightarrow & H^2(K,T). \end{array} \tag{1.3}$$

It is immediate from this diagram and the definitions that the image of $H^1_f(K,T)$ is contained in $H^1_f(K,W_M)$.

Suppose $c_{W_M} \in H^1_f(K, W_M)$. Then the image of c_{W_M} in $H^1(K, W)$ is the image of some $c_V \in H^1_f(K, V)$. Thus (1.3) shows that c_{W_M} is the image of some $c_T \in H^1(K, T)$, and the image of c_T in $H^1(K, V)$ differs from c_V by an element c' of $H^1(K, T)$. Therefore $c_T - Mc' \in H^1_f(K, T)$ and $c_T - Mc'$ maps to c_{W_M}. This shows that $H^1_f(K, W_M)$ is contained in the image of $H^1_f(K, T)$, and completes the proof of (i).

If T is unramified, $\ell \neq p$, and $\ell \neq \infty$ then

$$H^1_f(K, W_M) = \text{image}(H^1_f(K, T)) \;=\; \text{image}(H^1_{\text{ur}}(K, T)) \;\subset\; H^1_{\text{ur}}(K, W_M)$$

by (i) and Lemma 1.3.5(iv). Similarly, if $\iota_M : H^1(K, W_M) \to H^1(K, W)$ is the natural map then Lemma 1.3.5(iv) shows that

$$H^1_f(K, W_M) \;=\; \iota_M^{-1}(H^1_f(K, W)) \;=\; \iota_M^{-1}(H^1_{\text{ur}}(K, W)) \;\supset\; H^1_{\text{ur}}(K, W_M)$$

which proves (ii). □

Remark 1.3.9. We can view W_M either as a subgroup of W or as a quotient of T. Lemma 1.3.8(i) says that it makes no difference whether we define $H^1_f(K, W_M)$ as the inverse image of $H^1_f(K, W)$ (as we did) or as the image of $H^1_f(K, T)$.

Corollary 1.3.10. *There are natural horizontal exact sequences and vertical isomorphisms*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_f(K, W) & \longrightarrow & H^1(K, W) & \longrightarrow & H^1_s(K, W) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \varinjlim_M H^1_f(K, W_M) & \longrightarrow & \varinjlim_M H^1(K, W_M) & \longrightarrow & \varinjlim_M H^1_s(K, W_M) & \longrightarrow & 0 \\
 & & & & & & & & \\
0 & \longrightarrow & H^1_f(K, T) & \longrightarrow & H^1(K, T) & \longrightarrow & H^1_s(K, T) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \varprojlim_M H^1_f(K, W_M) & \longrightarrow & \varprojlim_M H^1(K, W_M) & \longrightarrow & \varprojlim_M H^1_s(K, W_M) & \longrightarrow & 0
\end{array}$$

Proof. The groups inside the inverse limits are finite (Proposition B.2.7(ii)), so the horizontal exact sequences are clear.

The isomorphism $H^1(K, W) = \varinjlim H^1(K, W_M)$ is a basic fact from Galois cohomology, and the isomorphism $H^1_f(K, W) = \varinjlim H^1_f(K, W_M)$ follows immediately from the definition of $H^1_f(K, W_M)$. The isomorphism $H^1_s(K, W) = \varinjlim H^1_s(K, W_M)$ now follows.

The second set of isomorphisms is similar, except that to handle the inverse limits we use Proposition B.2.3 for the center and Lemma 1.3.8(i) for the left. □

1.4. Local Duality

Suppose again that K is a finite extension of $\mathbf{Q}_\ell$ for some prime $\ell \leq \infty$, and T is a p-adic representation of G_K.

Theorem 1.4.1 (Local duality). *Suppose that either K is nonarchimedean and $0 \leq i \leq 2$, or K is archimedean and $i = 1$. Then the cup product and the local invariant map induce perfect pairings*

$$\begin{aligned} H^i(K,V) \times H^{2-i}(K,V^*) &\longrightarrow H^2(K,\Phi(1)) \xrightarrow{\sim} \Phi \\ H^i(K,W_M) \times H^{2-i}(K,W_M^*) &\longrightarrow H^2(K,\mathcal{O}(1)/M\mathcal{O}(1)) \xrightarrow{\sim} \mathcal{O}/M\mathcal{O} \\ H^i(K,T) \times H^{2-i}(K,W^*) &\longrightarrow H^2(K,\mathbf{D}(1)) \xrightarrow{\sim} \mathbf{D}. \end{aligned}$$

Proof. See for example [**Mi**] Corollary I.2.3 or [**Se2**] §II.5.2 (and use Propositions B.2.3 and B.2.4). □

Without fear of confusion, we will denote all of the pairings of Theorem 1.4.1 by $\langle\ ,\ \rangle_K$.

Proposition 1.4.2. *Suppose that either K is archimedean, or K is nonarchimedean of residue characteristic $\ell \neq p$. Then $H^1_f(K,V)$ and $H^1_f(K,V^*)$ are orthogonal complements under the pairing $\langle\ ,\ \rangle_K$.*

Proof. If K is archimedean then all the groups are zero, so there is nothing to prove.

Suppose that K is nonarchimedean of residue characteristic $\ell \neq p$. The pairing

$$\langle\ ,\ \rangle_K : H^1_f(K,V) \times H^1_f(K,V^*) \longrightarrow \Phi$$

factors through $H^2(K^{\mathrm{ur}}/K,\Phi(1))$, which is 0 since $\mathrm{Gal}(K^{\mathrm{ur}}/K)$ has cohomological dimension 1. Thus $H^1_f(K,V)$ and $H^1_f(K,V^*)$ are orthogonal. Further, Corollary 1.3.3(i), local duality (Theorem 1.4.1), and Corollary 1.3.3(ii), respectively, give the three equalities

$$\begin{aligned} \dim_\Phi(H^1_f(K,V^*)) &= \dim_\Phi(H^0(K,V^*)) = \dim_\Phi(H^2(K,V)) \\ &= \dim_\Phi(H^1(K,V)) - \dim_\Phi(H^1_f(K,V)), \end{aligned}$$

so $H^1_f(K,V)$ and $H^1_f(K,V^*)$ are orthogonal complements. □

Proposition 1.4.3. *Suppose that either*

(a) *K is archimedean,*
(b) *K is nonarchimedean of residue characteristic $\ell \neq p$, or*
(c) *K is nonarchimedean of residue characteristic $\ell = p$ and we choose subspaces $H^1_f(K,V)$ and $H^1_f(K,V^*)$ which are orthogonal complements under the pairing $\langle\ ,\ \rangle_K$.*

Then under the pairings $\langle\ ,\ \rangle_K$*,*

(i) $H^1_f(K,T)$ *and* $H^1_f(K,W^*)$ *are orthogonal complements,*

(ii) *if* M *in* $\mathcal{O}$ *is nonzero, then* $H^1_f(K,W_M)$ *and* $H^1_f(K,W^*_M)$ *are orthogonal complements.*

Proof. The definition of the local pairings in terms of cup products shows that the diagram

$$\begin{array}{ccccc} H^1(K,V) & \times & H^1(K,V^*) & \longrightarrow & \Phi \\ {\scriptstyle\phi}\uparrow & & \downarrow{\scriptstyle\phi^*} & & \downarrow \\ H^1(K,T) & \times & H^1(K,W^*) & \longrightarrow & \mathbf{D}. \end{array}$$

"commutes", in the sense that if $c \in H^1(K,T)$ and $d \in H^1(K,V^*)$, then

$$\langle \phi(c), d\rangle_K \;=\; \langle c, \phi^*(d)\rangle_K \;\in\; \mathbf{D}.$$

By Proposition 1.4.2, $H^1_f(K,V)$ and $H^1_f(K,V^*)$ are orthogonal complements of each other in all cases. Thus if we write $\cdot^\perp$ to denote the orthogonal complement, then since $H^1_f(K,W^*) = \phi^*(H^1_f(K,V^*))$ we see that

$$H^1_f(K,W^*)^\perp \;=\; \phi^{-1}(H^1_f(K,V^*)^\perp) \;=\; \phi^{-1}(H^1_f(K,V)) \;=\; H^1_f(K,T).$$

This proves (i). The proof of (ii) is similar, using (i), Lemma 1.3.8(i), and the diagram

$$\begin{array}{ccccc} H^1(K,T) & \times & H^1(K,W^*) & \longrightarrow & \mathbf{D} \\ \downarrow & & \uparrow & & \uparrow \\ H^1(K,W_M) & \times & H^1(K,W^*_M) & \longrightarrow & \mathcal{O}/M\mathcal{O}. \end{array}$$

□

Definition 1.4.4. If K is nonarchimedean of residue characteristic different from p, then there is an exact sequence

$$0 \longrightarrow \mathcal{I}' \longrightarrow \mathcal{I} \longrightarrow \mathbf{Z}_p \longrightarrow 0$$

where $\mathcal{I}'$ has trivial pro-p-part (see [**Fr**] §8 Corollary 3). It follows that if M is a power of p then $\mathcal{I}$ has a unique subgroup of index M (the inverse image of $M\mathbf{Z}_p$), and by slight abuse of notation we denote this subgroup by $\mathcal{I}^M$.

There is a natural action of $\mathrm{Gal}(K^{\mathrm{ur}}/K)$ on the cyclic group $\mathcal{I}/\mathcal{I}^M$. The next lemma is essentially Exercice 2, §IV.2 of [**Se3**].

Lemma 1.4.5. *Suppose that* K *is nonarchimedean of residue characteristic different from* p *and that* M *is a power of* p*. Then there is a canonical isomorphism of* $\mathrm{Gal}(K^{\mathrm{ur}}/K)$*-modules*

$$\mathcal{I}/\mathcal{I}^M \xrightarrow{\sim} \boldsymbol{\mu}_M.$$

Proof. We have isomorphisms

$$\mathrm{Hom}(\mathcal{I}/\mathcal{I}^M, \boldsymbol{\mu}_M) = \mathrm{Hom}(\mathcal{I}, \boldsymbol{\mu}_M) \xrightarrow{\sim} (K^{\mathrm{ur}})^\times/((K^{\mathrm{ur}})^\times)^M \xrightarrow{\sim} \mathbf{Z}/M\mathbf{Z},$$

given by Kummer theory and (on the right) by the valuation map (the unit group of the ring of integers of K^{ur} is p-divisible). The inverse image of 1 under this composition is the desired isomorphism.

More concretely, the isomorphism is given by

$$\sigma \mapsto (\lambda^{1/M})^\sigma/(\lambda^{1/M})$$

where λ is any uniformizing parameter of K. □

Definition 1.4.6. If $M \in \mathcal{O}$ is nonzero, we let $\bar{M} \in \mathbf{Z}^+$ denote the smallest power of p which is divisible in $\mathcal{O}$ by M.

Lemma 1.4.7. *Suppose that K is nonarchimedean of residue characteristic different from p, that T is unramified, that $M \in \mathcal{O}$ is nonzero, and that $\boldsymbol{\mu}_{\bar{M}} \subset K$. Fix a generator ζ of $\boldsymbol{\mu}_{\bar{M}}$ and let $\sigma_\zeta \in \mathcal{I}/\mathcal{I}^{\bar{M}}$ be the inverse image of ζ under the isomorphism of Lemma* 1.4.5.

(i) *Evaluating cocycles on* Fr *and σ_ζ induces isomorphisms*

$$H^1_f(K, W_M) \xrightarrow{\sim} W_M/(\mathrm{Fr} - 1)W_M, \quad H^1_s(K, W_M) \xrightarrow{\sim} W_M^{\mathrm{Fr}=1},$$

respectively.

(ii) *With an appropriate choice of sign on the right, the diagram*

$$\begin{array}{ccccc} H^1_f(K, W_M^*) & \times & H^1_s(K, W_M) & \longrightarrow & \mathcal{O}/M\mathcal{O} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pm 1 \otimes \zeta} \\ W_M^*/(\mathrm{Fr} - 1)W_M^* & \times & (W_M)^{\mathrm{Fr}=1} & \longrightarrow & \mathcal{O}(1)/M\mathcal{O}(1) \end{array}$$

commutes, where the first two vertical maps are the isomorphisms of (i)*, the upper pairing is the paring of Theorem* 1.4.1*, and the lower pairing is the natural one.*

Proof. The first isomorphism of (i) is just a restatement of Lemma 1.3.2(i), since by Lemma 1.3.8(ii) we have $H^1_f(K, W_M) = H^1_{\mathrm{ur}}(K, W_M)$. Similarly, Lemma 1.3.2(ii) shows that

$$H^1_s(K, W_M) = H^1(K, W_M)/H^1_{\mathrm{ur}}(K, W_M) \cong H^1(\mathcal{I}, W_M)^{\mathrm{Fr}=1}.$$

Lemma 1.4.5 shows that $\mathcal{I}/\mathcal{I}^{\bar{M}} \cong \boldsymbol{\mu}_{\bar{M}}$, and we have assumed that G_K acts trivially on $\boldsymbol{\mu}_{\bar{M}}$, so we conclude that

$$H^1_s(K, W_M) \cong \mathrm{Hom}(\mathcal{I}/\mathcal{I}^{\bar{M}}, W_M)^{\mathrm{Fr}=1} \cong \mathrm{Hom}(\boldsymbol{\mu}_{\bar{M}}, W_M^{\mathrm{Fr}=1}).$$

Our choice of generator of $\boldsymbol{\mu}_{\bar{M}}$ now completes the proof of (i).

Assertion (ii) can be extracted from Chapter I of [**Mi**], especially Proposition 0.14, Examples 0.8 and 1.6, and Theorem 2.6. □

1.5. Global Cohomology Groups

Suppose for this section that K is a number field, T is a p-adic representation of G_K, and V and W are defined in terms of T as in §1.1. We assume in addition that T is unramified outside a finite set of primes of K. (As usual, we say that T is unramified at a place v if the inertia group of v acts trivially on T.) We write K_v for the completion of K at a place v, and for all primes v dividing p we fix a subspace $H^1_f(K_v, V)$ of $H^1(K_v, V)$.

For every place v of K there is a canonical restriction map

$$H^1(K, \cdot) \longrightarrow H^1(K_v, \cdot),$$

which we will denote either by $c \mapsto \mathrm{res}_v(c)$ or simply $c \mapsto c_v$.

If Σ is a finite set of places of K we write K_Σ for the maximal extension of K unramified outside Σ.

Recall that

$$H^1_s(K_v, W) = H^1(K_v, W)/H^1_f(K_v, W).$$

Definition 1.5.1. Suppose Σ is a finite set of places of K. We define some *Selmer groups* corresponding to Σ as follows. First, define

$$\mathcal{S}_\Sigma(K, W) \subset \mathcal{S}^\Sigma(K, W) \subset H^1(K, W)$$

by

$$\mathcal{S}^\Sigma(K, W) = \ker\Big(H^1(K, W) \longrightarrow \bigoplus_{v \notin \Sigma} H^1_s(K_v, W)\Big),$$
$$\mathcal{S}_\Sigma(K, W) = \ker\Big(\mathcal{S}^\Sigma(K, W) \longrightarrow \bigoplus_{v \in \Sigma} H^1(K_v, W)\Big).$$

(Note that every element of $H^1(K, W)$ restricts to zero in all but finitely many $H^1_s(K_v, W)$ because T is ramified at only finitely many primes.) In other words, $\mathcal{S}^\Sigma(K, W)$ consists of all classes $c \in H^1(K, W)$ satisfying the local conditions

- $c_v \in H^1_f(K_v, W)$ if $v \notin \Sigma$,
- no restriction for $v \in \Sigma$,

and $\mathcal{S}_\Sigma(K, W)$ has the additional restrictions

- $c_v = 0$ if $v \in \Sigma$.

When $\Sigma = \emptyset$ is the empty set we write

$$\mathcal{S}(K, W) = \mathcal{S}^\emptyset(K, W) = \mathcal{S}_\emptyset(K, W).$$

Similarly, we define $\mathcal{S}_\Sigma(K,T) \subset \mathcal{S}^\Sigma(K,T) \subset H^1(K,T)$ by

$$\mathcal{S}^\Sigma(K,T) \;=\; \ker\Bigl(H^1(K,T) \longrightarrow \prod_{v\notin\Sigma} H^1_s(K_v,T)\Bigr),$$

$$\mathcal{S}_\Sigma(K,T) \;=\; \ker\Bigl(\mathcal{S}^\Sigma(K,T) \longrightarrow \bigoplus_{v\in\Sigma} H^1(K_v,T)\Bigr),$$

and likewise for $\mathcal{S}_\Sigma(K,W_M) \subset \mathcal{S}^\Sigma(K,W_M) \subset H^1(K,W_M)$ for every non-zero M in $\mathcal{O}$.

Remark 1.5.2. If Σ contains all primes above p, then the Selmer groups $\mathcal{S}^\Sigma$ and $\mathcal{S}_\Sigma$ are independent of the choice of subspaces $H^1_f(K_v,V)$ for v dividing p.

Lemma 1.5.3. *Suppose Σ contains all infinite places, all primes above p, and all primes of K where T is ramified. If $A = T$, W, or W_M with $M \in \mathcal{O}$, then*

$$\mathcal{S}^\Sigma(K,A) \;=\; H^1(K_\Sigma/K,A).$$

Proof. For every place $v \notin \Sigma$, Lemmas 1.3.5(iv) and 1.3.8(ii) show that $H^1_f(K_v,A) = H^1_{\mathrm{ur}}(K_v,A)$. Therefore, writing $\mathcal{I}_v$ for an inertia group above v, we have

$$\begin{aligned}\mathcal{S}^\Sigma(K,A) \;&=\; \ker\Bigl(H^1(K,A) \longrightarrow \prod_{v\notin\Sigma} \mathrm{Hom}(\mathcal{I}_v,A)\Bigr)\\ &=\; \ker\Bigl(H^1(K,A) \longrightarrow H^1(K_\Sigma,A)\Bigr) \;=\; H^1(K_\Sigma/K,A).\end{aligned}$$

□

Lemma 1.5.4. *If $M \in \mathcal{O}$ is nonzero and Σ is a finite set of primes of K, then the natural map $\iota_M : H^1(K,W_M) \to H^1(K,W)$ induces a surjection*

$$\mathcal{S}^\Sigma(K,W_M) \twoheadrightarrow \mathcal{S}^\Sigma(K,W)_M.$$

Proof. Lemma 1.2.2(i) shows that $\iota_M(H^1(K,W_M)) = H^1(K,W)_M$, and from the definition of $H^1_f(K_v,W_M)$ it is clear that

$$\iota_M^{-1}(\mathcal{S}^\Sigma(K,W)_M) \;=\; \mathcal{S}^\Sigma(K,W_M).$$

□

Remark 1.5.5. Lemma 1.5.4 need *not* be true if we replace $\mathcal{S}^\Sigma$ by $\mathcal{S}_\Sigma$, because it might not be the case that $\iota_M^{-1}(\mathcal{S}_\Sigma(K,W)_M) \subset \mathcal{S}_\Sigma(K,W_M)$.

Proposition 1.5.6. *If Σ is a finite set of primes of K then*

(i) $\mathcal{S}^\Sigma(K,T) = \varprojlim_M \mathcal{S}^\Sigma(K,W_M)$ *and* $\mathcal{S}_\Sigma(K,T) = \varprojlim_M \mathcal{S}_\Sigma(K,W_M)$,

(ii) $\mathcal{S}^\Sigma(K,W) = \varinjlim_M \mathcal{S}^\Sigma(K,W_M)$ *and* $\mathcal{S}_\Sigma(K,W) = \varinjlim_M \mathcal{S}_\Sigma(K,W_M)$.

Proof. We have $H^1(K,W) = \varinjlim H^1(K,W_M)$, and by Proposition B.2.3, $H^1(K,T) = \varprojlim H^1(K,W_M)$. Corollary 1.3.10 shows that all the local conditions behave well under inverse and direct limits, and the proposition follows. □

Lemma 1.5.7. *Suppose $M \in \mathcal{O}$ is nonzero and Σ is a finite set of primes of K. Then*

(i) *$\mathcal{S}^\Sigma(K,W_M)$ is finite,*
(ii) *$\mathcal{S}^\Sigma(K,T)$ is a finitely generated $\mathcal{O}$-module,*
(iii) *the Pontryagin dual of $\mathcal{S}^\Sigma(K,W)$ is a finitely generated $\mathcal{O}$-module.*

Proof. Without loss of generality we may enlarge Σ if necessary so that Σ contains all infinite places, all primes above p, and all primes where T is ramified. Then by Lemma 1.5.3, if A is W_M, T, or W we have $\mathcal{S}^\Sigma(K,A) = H^1(K_\Sigma/K,A)$. As is well known (see Proposition B.2.7) these groups have the desired properties. □

Remark 1.5.8. Suppose F is a finite extension of K, and Σ is a set of places of K. Write Σ_F for the set of primes of F dividing places in Σ. We will often write $\mathcal{S}^\Sigma(F,\cdot)$ instead of $\mathcal{S}^{\Sigma_F}(F,\cdot)$ and $\mathcal{S}_\Sigma(F,\cdot)$ instead of $\mathcal{S}_{\Sigma_F}(F,\cdot)$ for the Selmer groups over F attached to W, W_M, and T.

1.6. Examples of Selmer Groups

Again for this section K will denote a number field.

1.6.A. Ideal class groups I. Suppose $\mathcal{O} = \mathbf{Z}_p$ and $T = \mathbf{Z}_p$ with trivial G_K-action.

Proposition 1.6.1. *If Σ is a set of places of K containing all primes above p, then*

$$\mathcal{S}_\Sigma(K,\mathbf{Q}_p/\mathbf{Z}_p) = \mathrm{Hom}(A_{K,\Sigma},\mathbf{Q}_p/\mathbf{Z}_p),$$
$$\mathcal{S}^\Sigma(K,\mathbf{Q}_p/\mathbf{Z}_p) = \mathrm{Hom}(\mathrm{Gal}(K_\Sigma/K),\mathbf{Q}_p/\mathbf{Z}_p),$$

where $A_{K,\Sigma}$ is the quotient of the ideal class group of K by the subgroup generated by the classes of primes in Σ.

Proof. Suppose v is a place of K not dividing p. Lemma 1.3.5(iv) (for v finite) and Remark 1.3.7 and Lemma 1.3.2(iii) (for v infinite) show that

$$H^1_f(K_v,\mathbf{Q}_p/\mathbf{Z}_p) = H^1_{\mathrm{ur}}(K_v,\mathbf{Q}_p/\mathbf{Z}_p) = \mathrm{Hom}(\mathrm{Gal}(K_v^{\mathrm{ur}}/K_v),\mathbf{Q}_p/\mathbf{Z}_p).$$

Since $H^1(K,\mathbf{Q}_p/\mathbf{Z}_p) = \mathrm{Hom}(G_K,\mathbf{Q}_p/\mathbf{Z}_p)$, the proposition follows easily by class field theory. □

With an appropriate choice of $H^1_f(K_v, \mathbf{Q}_p)$ for primes v dividing p, Proposition 1.6.2 below will show that Proposition 1.6.1 holds even when Σ is empty, i.e.,

$$\mathcal{S}(K, \mathbf{Q}_p/\mathbf{Z}_p) \;=\; \mathrm{Hom}(A_K, \mathbf{Q}_p/\mathbf{Z}_p) \tag{1.4}$$

where A_K is the ideal class group of K.

1.6.B. Ideal class groups II. More generally, suppose that

$$\chi \,:\, G_K \;\longrightarrow\; \mathcal{O}^\times$$

is a character of finite prime-to-p order, and let $T = \mathcal{O}_\chi$, a free rank-one $\mathcal{O}$-module on which G_K acts via χ. Let L be an abelian extension of K of degree prime to p such that χ factors through $\Delta = \mathrm{Gal}(L/K)$. Write $\mathbf{D}_\chi = \mathbf{D} \otimes \mathcal{O}_\chi$ and $\Phi_\chi = \Phi \otimes \mathcal{O}_\chi$.

Suppose v is a place of K (finite or infinite). If w is a place of L above v let $\mathcal{D}_w$ and $\mathcal{I}_w$ denote a decomposition group and inertia group of w, respectively, in G_K. The restriction map gives isomorphisms (Corollary B.5.3(ii))

$$H^1(K_v, V) \;\cong\; (\oplus_{w|v}\mathrm{Hom}(\mathcal{D}_w, V))^\Delta \;=\; (\oplus_{w|v}\mathrm{Hom}(\mathcal{D}_w, \Phi_\chi))^\Delta, \tag{1.5}$$

and if $v \nmid p$ this identifies

$$H^1_f(K_v, V) \;=\; H^1_{\mathrm{ur}}(K_v, V) \;=\; (\oplus_{w|v}\mathrm{Hom}(\mathcal{D}_w/\mathcal{I}_w, V))^\Delta. \tag{1.6}$$

If $v \mid p$ we take (1.6) as the definition of $H^1_f(K_v, V)$ as well; this agrees with the Bloch-Kato definition of H^1_f in this case.

Let A_L denote the ideal class group of L. When $L = K$ the following proposition reduces to (1.4).

Proposition 1.6.2. $\mathcal{S}(K, W) \cong \mathrm{Hom}(A_L, \mathbf{D}_\chi)^\Delta$.

Proof. Since $[L : K]$ is prime to p, the restriction map

$$H^1(K, W) \longrightarrow H^1(L, W)^\Delta \;=\; \mathrm{Hom}(G_L, \mathbf{D}_\chi)^\Delta$$

is an isomorphism. Exactly as in (1.5) and (1.6), and using Lemma 1.3.5(i), for every place v of K we have

$$\begin{array}{ccc} H^1(K_v, W) & \xrightarrow{\sim} & (\oplus_{w|v}\mathrm{Hom}(\mathcal{D}_w, W))^\Delta \\ \cup & & \cup \\ H^1_f(K_v, W) & \xrightarrow{\sim} & (\oplus_{w|v}(\mathrm{Hom}(\mathcal{D}_w/\mathcal{I}_w, W))^\Delta)_{\mathrm{div}}. \end{array}$$

Since each $\mathcal{D}_w/\mathcal{I}_w$ is torsion-free, $\oplus_{w|v}\mathrm{Hom}(\mathcal{D}_w/\mathcal{I}_w, W)$ is divisible. Since Δ has order prime to p, we see that

$$(\oplus_{w|v}\mathrm{Hom}(\mathcal{D}_w/\mathcal{I}_w, W))^\Delta \;=\; \Big(|\Delta|^{-1}\sum_{\delta\in\Delta}\delta\Big)(\oplus_{w|v}\mathrm{Hom}(\mathcal{D}_w/\mathcal{I}_w, W))$$

is divisible and so $H^1_f(K_v, W) = (\oplus_{w|v}\mathrm{Hom}(\mathcal{D}_w/\mathcal{I}_w, W))^\Delta$. Therefore, if H_L is the Hilbert class field of L, we have

$$\begin{aligned}\mathcal{S}(K, W) &\cong \{\phi \in \mathrm{Hom}(G_L, \mathbf{D}_\chi)^\Delta : \phi(\mathcal{I}_w) = 0 \text{ for every } w\} \\ &= \mathrm{Hom}(\mathrm{Gal}(H_L/L), \mathbf{D}_\chi)^\Delta = \mathrm{Hom}(A_L, \mathbf{D}_\chi)^\Delta. \qquad \square\end{aligned}$$

1.6.C. Global units and ideal class groups. Let χ, $T = \mathcal{O}_\chi$, L, A_L, and $\Delta = \mathrm{Gal}(L/K)$ be as in §1.6.B above. Then $T^* = \mathcal{O}_{\chi^{-1}\varepsilon_{\mathrm{cyc}}}$, i.e., T^* is a free rank-one $\mathcal{O}$-module on which G_K acts via $\chi^{-1}\varepsilon_{\mathrm{cyc}}$, where $\varepsilon_{\mathrm{cyc}}$ denotes the cyclotomic character. In particular G_L acts on T^* by the cyclotomic character.

Definition 1.6.3. Suppose B is an abelian group. We define the *p-adic completion* of B to be the double dual

$$B\hat{} = \mathrm{Hom}(\mathrm{Hom}(B, \mathbf{Q}_p/\mathbf{Z}_p), \mathbf{Q}_p/\mathbf{Z}_p)$$

(with continuous homomorphisms, when B comes with a topology). For example, if B is a $\mathbf{Z}_p$-module then $B\hat{} = B$; if B is a finitely generated abelian group then $B\hat{} = B \otimes_{\mathbf{Z}} \mathbf{Z}_p$. In general $B\hat{}$ is a $\mathbf{Z}_p$-module and there is a canonical map from B to $B\hat{}$.

Now suppose further that B is a $\mathbf{Z}[\Delta]$-module. Define the *χ-component* of B by

$$B^\chi = \{b \in B\hat{} \otimes_{\mathbf{Z}_p} \mathcal{O} : \gamma b = \chi(\gamma) b \text{ for every } \gamma \in \Delta\}.$$

We fix once and for all an $\mathcal{O}$-generator of $\mathcal{O}_{\chi^{-1}}$, and with this choice we get an isomorphism

$$B^\chi \cong (B\hat{} \otimes_{\mathbf{Z}_p} \mathcal{O}_{\chi^{-1}})^\Delta.$$

Since $[L : K]$ is prime to p, taking χ-components is an exact functor and

$$B\hat{} \otimes_{\mathbf{Z}_p} \mathcal{O} = \oplus_\chi B^\chi.$$

Suppose now that v is a prime of K, and let $U_{L,v}$ denote the local units of $L \otimes K_v = \prod_{w|v} L_w$. (That is, $U_{L,v} = \prod_{w|v} \mathcal{O}_w^\times$ where $\mathcal{O}_w$ is the ring of integers of L_w.) The restriction map (Corollary B.5.3(ii)) and Kummer theory (Example 1.2.1) give isomorphisms

$$\begin{aligned}H^1(K_v, V^*) &\cong (\oplus_{w|v} H^1(L_w, V^*))^\Delta \\ &= (\oplus_{w|v} H^1(L_w, \mathbf{Q}_p(1)) \otimes \Phi_{\chi^{-1}})^\Delta \cong ((L \otimes K_v)^\times)^\chi \otimes \Phi.\end{aligned}$$

If $v \nmid p$ then with this identification one can check that

$$\begin{array}{ccc} H^1(K_v, V^*) & \xrightarrow{\sim} & ((L \otimes K_v)^\times)^\chi \otimes \Phi \\ \cup & & \cup \\ H^1_f(K_v, V^*) & \xrightarrow{\sim} & U^\chi_{L,v} \otimes \Phi. \end{array} \tag{1.7}$$

If $v \mid p$ we take the bottom row of (1.7) as the definition of $H^1_f(K_v, V^*)$. This agrees with the Bloch-Kato definition of H^1_f in this case. Combining (1.5) and (1.6) with the identifications

$$\oplus_{w|v}\mathcal{D}_w \cong (L \otimes K_v)^\times, \quad \oplus_{w|v}\mathcal{I}_w \cong U_{L,v}$$

of local class field theory gives a similar diagram

$$\begin{array}{ccc} H^1(K_v, V) & \xrightarrow{\sim} & \mathrm{Hom}(((L \otimes K_v)^\times)^\chi, \Phi) \\ \cup & & \cup \\ H^1_f(K_v, V) & \xrightarrow{\sim} & \mathrm{Hom}(((L \otimes K_v)^\times)^\chi / U^\chi_{L,v}, \Phi). \end{array} \tag{1.8}$$

The local pairing $\langle\ ,\ \rangle_v$ is the natural one induced by the identifications of (1.7) and (1.8), and so $H^1_f(K_v, V^*)$ and $H^1_f(K_v, V)$ are orthogonal complements.

Let $\mathcal{O}_L$ denote the ring of integers of L.

Proposition 1.6.4. (i) *There is a natural isomorphism*

$$H^1(K, W^*) \xrightarrow{\sim} (L^\times \otimes \mathbf{Q}_p/\mathbf{Z}_p)^\chi.$$

(ii) *There is an exact sequence*

$$0 \longrightarrow (\mathcal{O}_L^\times \otimes \mathbf{Q}_p/\mathbf{Z}_p)^\chi \longrightarrow \mathcal{S}(K, W^*) \longrightarrow A_L^\chi \longrightarrow 0.$$

Proof. Since $[L : K]$ is prime to p, the restriction map

$$\begin{aligned} H^1(K, W^*) \xrightarrow{\mathrm{res}_{L/K}} H^1(L, W^*)^\Delta &= (H^1(L, \boldsymbol{\mu}_{p^\infty}) \otimes \mathcal{O}_{\chi^{-1}})^\Delta \\ &\cong H^1(L, \boldsymbol{\mu}_{p^\infty})^\chi \cong (L^\times \otimes \mathbf{Q}_p/\mathbf{Z}_p)^\chi \end{aligned}$$

is an isomorphism, which gives (i). It follows easily from (1.7) that for every v there is an isomorphism, compatible with (i),

$$H^1_f(K_v, W^*) \xrightarrow{\sim} U^\chi_{L,v} \otimes \mathbf{Q}_p/\mathbf{Z}_p.$$

Therefore if we define X_L to be

$$\{y \otimes p^{-n} \in L^\times \otimes \mathbf{Q}_p/\mathbf{Z}_p : \mathrm{ord}_w(y) \equiv 0 \pmod{p^n} \text{ for every prime } w \text{ of } L\},$$

then

$$\mathrm{res}_{L/K}(\mathcal{S}(K, W^*)) \cong X_L^\chi.$$

Suppose $x \in X_L$ is represented by $y \otimes p^{-n}$ with $y \in L^\times$. Then the principal fractional ideal $y\mathcal{O}_L$ is of the form $\mathfrak{a}^{p^n}$ for some fractional ideal $\mathfrak{a}$. This map $x \mapsto \mathfrak{a}$ induces a well-defined surjection from X_L to the p-part $A_L^{(p)}$ of the ideal class group of L. Thus there is an exact sequence

$$0 \longrightarrow \mathcal{O}_L^\times \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow X_L \longrightarrow A_L^{(p)} \longrightarrow 0,$$

and taking χ-components gives the exact sequence of the proposition. $\square$

Let Σ_p denote the set of primes of K above p.

Corollary 1.6.5. *If Leopoldt's conjecture is true for L, then $\mathcal{S}_{\Sigma_p}(K, W^*)$ is finite.*

Proof. Leopoldt's conjecture for L is the assertion that the p-adic completion of $\mathcal{O}_L^\times$ injects into $(L \otimes \mathbf{Q}_p)^\times$. This implies that the map

$$(\mathcal{O}_L^\times \otimes \mathbf{Q}_p/\mathbf{Z}_p)^\chi \longrightarrow ((L \otimes \mathbf{Q}_p)^\times \otimes \mathbf{Q}_p/\mathbf{Z}_p)^\chi \cong \oplus_{v|p} H^1(K_v, W^*)$$

has finite kernel, so the corollary follows from Proposition 1.6.4(ii) and the finiteness of the ideal class group. □

Corollary 1.6.6. *With notation as above, suppose that $K = \mathbf{Q}$. If χ is odd (i.e., χ sends complex conjugation to -1) then $\mathcal{S}(\mathbf{Q}, W^*) \cong A_L^\chi$.*

Proof. Since χ is odd, $(\mathcal{O}_L^\times)^\chi$ is finite and so $(\mathcal{O}_L^\times \otimes \mathbf{Q}_p/\mathbf{Z}_p)^\chi = 0$. Thus the corollary follows immediately from Proposition 1.6.4(ii). □

1.6.D. Abelian varieties. Let A be an abelian variety defined over K and let $T = T_p(A)$ be the p-adic Tate module of A as in Example 1.1.5. (See for example [**Si**] for the basic facts in the case of elliptic curves.) Then

$$V = V_p(A) = T_p(A) \otimes \mathbf{Q}_p, \quad W = V_p(A)/T_p(A) = A_{p^\infty},$$

where A_{p^∞} is the p-power torsion in $A(\bar{K})$.

For every place v of K there is a natural injective Kummer map

$$A(K_v)\hat{} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \hookrightarrow H^1(K_v, V_p(A)) \tag{1.9}$$

where $A(K_v)\hat{}$ denotes the p-adic completion of $A(K_v)$. If v is a prime of K above p we define $H^1_f(K_v, V_p(A))$ to be the image of this map. This definition agrees with the Bloch-Kato definition of H^1_f.

Remark 1.6.7. Let A^* denote the dual abelian variety of A. The Weil pairing shows that $V_p(A)^* = V_p(A^*)$, and if we define $H^1_f(K_v, V_p(A^*))$ in the same way as for A, then $H^1_f(K_v, V_p(A))$ and $H^1_f(K_v, V_p(A^*))$ are orthogonal complements under the local pairing $\langle\ ,\ \rangle_{K_v}$.

Note that if we fix a polarization of A, then the Weil pairing gives an isomorphism $V_p(A^*) \cong V_p(A)$. This isomorphism identifies $H^1_f(K_v, V_p(A))$ and $H^1_f(K_v, V_p(A^*))$.

Proposition 1.6.8. *The Selmer group $\mathcal{S}(K, A_{p^\infty})$ is the usual p-power Selmer group attached to the abelian variety A, and there is an exact sequence*

$$0 \longrightarrow A(K) \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow \mathcal{S}(K, A_{p^\infty}) \longrightarrow Ш(A_{/K})_{p^\infty} \longrightarrow 0$$

where $Ш(A_{/K})_{p^\infty}$ denotes the p-part of the Tate-Shafarevich group of A over K.

Proof. Suppose $v \nmid p$. If ℓ is the rational prime below v, then $A(K_v)$ has a subgroup of finite index which is a pro-ℓ group, so the p-adic completion $A(K_v)^\wedge$ is finite. Also in this case $H^1_f(K_v, V_p(A)) = 0$ by Corollary 1.3.3(i) and Remark 1.3.7. Therefore for all v (including those above p), $H^1_f(K_v, V_p(A))$ is the image of the map (1.9). It follows that for every v the subgroup $H^1_f(K_v, A_{p^\infty})$ is the image of $A(K_v)^\wedge \otimes \mathbf{Q}_p/\mathbf{Z}_p$ under the corresponding Kummer map, and so the definition of $\mathcal{S}(K, A_{p^\infty})$ coincides with the classical definition of the Selmer group of A. □

1.7. Global Duality

As in §1.5 we suppose that K is a number field and T is a p-adic representation of G_K ramified at only finitely many primes of K. For all primes v dividing p we also fix special subspaces $H^1_f(K_v, V) \subset H^1(K_v, V)$ and $H^1_f(K_v, V^*) \subset H^1(K_v, V^*)$ which are orthogonal complements under the pairing $\langle\ ,\ \rangle_{K_v}$ of Theorem 1.4.1. We will also denote this pairing by $\langle\ ,\ \rangle_v$.

Remark 1.7.1. If v is a place dividing p, and the representation V is potentially semistable (see [**FPR**] §I.2) at v, then the subspaces $H^1_f(K_v, V)$ and $H^1_f(K_v, V^*)$ defined by Bloch and Kato are orthogonal complements (see [**FPR**] Proposition I.3.3.9(iii) or [**BK**] Proposition 3.8).

Definition 1.7.2. If $\Sigma_0 \subset \Sigma$ are finite sets of places of K we will write

$$\begin{aligned}
\mathrm{loc}_\Sigma :\quad & H^1(K, W_M) \longrightarrow \bigoplus_{v\in\Sigma} H^1(K_v, W_M) \\
\mathrm{loc}^s_{\Sigma,\Sigma_0} :\quad & \mathcal{S}^\Sigma(K, W_M) \longrightarrow \bigoplus_{v\in\Sigma-\Sigma_0} H^1_s(K_v, W_M) \\
\mathrm{loc}^f_{\Sigma,\Sigma_0} :\quad & \mathcal{S}_{\Sigma_0}(K, W_M) \longrightarrow \bigoplus_{v\in\Sigma-\Sigma_0} H^1_f(K_v, W_M)
\end{aligned}$$

for the respective localization maps.

Theorem 1.7.3 (Poitou-Tate duality)**.** *Suppose $M \in \mathcal{O}$ is nonzero, and $\Sigma_0 \subset \Sigma$ are finite sets of places of K.*

(i) *There are exact sequences*

$$0 \longrightarrow \mathcal{S}^{\Sigma_0}(K, W_M) \longrightarrow \mathcal{S}^\Sigma(K, W_M) \xrightarrow{\mathrm{loc}^s_{\Sigma,\Sigma_0}} \bigoplus_{v\in\Sigma-\Sigma_0} H^1_s(K_v, W_M),$$

$$0 \longrightarrow \mathcal{S}_\Sigma(K, W^*_M) \longrightarrow \mathcal{S}_{\Sigma_0}(K, W^*_M) \xrightarrow{\mathrm{loc}^f_{\Sigma,\Sigma_0}} \bigoplus_{v\in\Sigma-\Sigma_0} H^1_f(K_v, W^*_M).$$

(ii) *The images* $\mathrm{loc}^s_{\Sigma,\Sigma_0}(\mathcal{S}^\Sigma(K,W_M))$ *and* $\mathrm{loc}^f_{\Sigma,\Sigma_0}(\mathcal{S}_{\Sigma_0}(K,W_M^*))$ *are orthogonal complements with respect to the pairing* $\sum_{v\in\Sigma-\Sigma_0}\langle\ ,\ \rangle_v$.

(iii) *There is an isomorphism*

$$\mathcal{S}_{\Sigma_0}(K,W_M^*)/\mathcal{S}_\Sigma(K,W_M^*) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\mathrm{coker}(\mathrm{loc}^s_{\Sigma,\Sigma_0}),\mathcal{O}/M\mathcal{O}).$$

Proof. Assertion (i) is immediate from the definitions of the Selmer groups involved.

For (ii), recall that by Theorem 1.4.1 and Proposition 1.4.3(ii), the pairing $\langle\ ,\ \rangle_v$ induces a nondegenerate pairing on $H^1_s(K_v,W_M)\times H^1_f(K_v,W_M^*)$. Suppose first that Σ contains all infinite places, all primes above p, and all primes where T is ramified, so that $\mathcal{S}^\Sigma(K,W_M)=H^1(K_\Sigma/K,W_M)$ and $\mathcal{S}^\Sigma(K,W_M^*)=H^1(K_\Sigma/K,W_M^*)$ by Lemma 1.5.3. Under these conditions, a part of the Poitou-Tate duality exact sequence ([**Mi**] Theorem I.4.10 or [**T1**] Theorem 3.1) is

$$\mathcal{S}^\Sigma(K,W_M) \xrightarrow{\mathrm{loc}_\Sigma} \bigoplus_{v\in\Sigma} H^1(K_v,W_M) \xrightarrow{\mathrm{loc}_\Sigma^\vee} \mathcal{S}^\Sigma(K,W_M^*)^\vee \tag{1.10}$$

where $\mathcal{S}^\Sigma(K,W_M^*)^\vee=\mathrm{Hom}(\mathcal{S}^\Sigma(K,W_M^*),\mathcal{O}/M\mathcal{O})$ and the maps are induced by localization and the local pairings between $H^1(K_v,W_M)$ and $H^1(K_v,W_M^*)$. Using Proposition 1.4.3(ii), we can combine (1.10) and (i) to produce a new exact sequence

$$\begin{aligned}0\longrightarrow \mathcal{S}^{\Sigma_0}(K,W_M)\longrightarrow \mathcal{S}^\Sigma(K,W_M) &\xrightarrow{\mathrm{loc}^s_{\Sigma,\Sigma_0}} \bigoplus_{v\in\Sigma-\Sigma_0} H^1_s(K_v,W_M)\\ &\xrightarrow{\mathrm{loc}^f_{\Sigma,\Sigma_0}{}^\vee} \mathcal{S}_{\Sigma_0}(K,W_M^*)^\vee\longrightarrow \mathcal{S}_\Sigma(K,W_M^*)^\vee\longrightarrow 0.\end{aligned} \tag{1.11}$$

The exactness in the center proves (ii) in this case. (To see the exactness in the center, note that the dual of the tautological exact sequence

$$\begin{aligned}0\longrightarrow \mathcal{S}_{\Sigma_0}(K,W_M^*)\longrightarrow \mathcal{S}^\Sigma(K,W_M^*)&\\ \xrightarrow{\mathrm{loc}_{\Sigma_0}\oplus\mathrm{loc}^s_{\Sigma-\Sigma_0}} \bigoplus_{v\in\Sigma_0} H^1(K_v,W_M^*) &\bigoplus_{v\in\Sigma-\Sigma_0} H^1_s(K_v,W_M^*)\end{aligned}$$

is

$$\begin{aligned}\bigoplus_{v\in\Sigma_0} H^1(K_v,W_M) \bigoplus_{v\in\Sigma-\Sigma_0} H^1_f(K_v,W_M)&\\ \xrightarrow{(\mathrm{loc}_{\Sigma_0}\oplus\mathrm{loc}^s_{\Sigma-\Sigma_0})^\vee} \mathcal{S}^\Sigma(K,W_M^*)^\vee\longrightarrow \mathcal{S}_{\Sigma_0}(K,W_M^*)^\vee&\longrightarrow 0.\end{aligned}$$

Splicing this together with (1.10) and

$$0 \longrightarrow \bigoplus_{v\in\Sigma_0} H^1(K_v, W_M) \bigoplus_{v\in\Sigma-\Sigma_0} H^1_f(K_v, W_M) \longrightarrow \bigoplus_{v\in\Sigma} H^1(K_v, W_M) \longrightarrow \bigoplus_{v\in\Sigma-\Sigma_0} H^1_s(K_v, W_M) \longrightarrow 0$$

gives (1.11).)

Now suppose Σ is arbitrary, and let Σ' be a finite set of places containing Σ, all infinite places, all primes above p, and all primes where T is ramified. Then we have an exact sequence (1.11) for each of the pairs $\Sigma \subset \Sigma'$ and $\Sigma_0 \subset \Sigma'$, so we obtain a diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{S}^{\Sigma'}(K,W_M)/\mathcal{S}^{\Sigma_0}(K,W_M) & \twoheadrightarrow & \mathcal{S}^{\Sigma'}(K,W_M)/\mathcal{S}^{\Sigma}(K,W_M) \\
{\scriptstyle \mathrm{loc}^s_{\Sigma',\Sigma_0}}\downarrow & & {\scriptstyle \mathrm{loc}^s_{\Sigma',\Sigma}}\downarrow \\
\bigoplus_{v\in\Sigma'-\Sigma_0} H^1_s(K_v,W_M) & \twoheadrightarrow & \bigoplus_{v\in\Sigma'-\Sigma} H^1_s(K_v,W_M) \\
{\scriptstyle (\mathrm{loc}^f_{\Sigma',\Sigma_0})^\vee}\downarrow & & {\scriptstyle (\mathrm{loc}^f_{\Sigma',\Sigma})^\vee}\downarrow \\
(\mathcal{S}_{\Sigma_0}(K,W_M^*)/\mathcal{S}_{\Sigma'}(K,W_M^*))^\vee & \twoheadrightarrow & (\mathcal{S}_{\Sigma}(K,W_M^*)/\mathcal{S}_{\Sigma'}(K,W_M^*))^\vee \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

with surjective horizontal maps. The snake lemma gives an exact sequence of kernels of the horizontal maps

$$0 \longrightarrow \mathcal{S}^{\Sigma}(K,W_M)/\mathcal{S}^{\Sigma_0}(K,W_M) \xrightarrow{\mathrm{loc}^s_{\Sigma,\Sigma_0}} \oplus_{v\in\Sigma-\Sigma_0} H^1_s(K_v,W_M) \xrightarrow{(\mathrm{loc}^f_{\Sigma,\Sigma})^\vee} (\mathcal{S}_{\Sigma_0}(K,W_M^*)/\mathcal{S}_{\Sigma}(K,W_M^*))^\vee \longrightarrow 0$$

and the exactness in the center proves (ii) for $\Sigma_0 \subset \Sigma$. Assertion (iii) is just a restatement of (ii). □

Remark 1.7.4. Theorem 1.7.3 will be applied with Σ_0 equal to the empty set or the set of primes dividing p, and with Σ large enough so that $\mathcal{S}_\Sigma(K, W_M^*) = 0$. In that situation, it follows from Theorem 1.7.3(iii) that

$$|\mathcal{S}_{\Sigma_0}(K, W_M^*)| \;=\; |\mathrm{coker}(\mathrm{loc}^s_{\Sigma,\Sigma_0})|.$$

Thus if one can produce "enough" cohomology classes in $\mathcal{S}^\Sigma(K, W_M)$, one obtains a good bound on the size of $\mathcal{S}_{\Sigma_0}(K, W_M^*)$. The purpose of an Euler system is to construct these classes.

Recall that Σ_p denotes the set of primes of K above p.

Corollary 1.7.5. *There is an isomorphism*

$$\mathcal{S}(K,W^*)/\mathcal{S}_{\Sigma_p}(K,W^*) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\mathrm{coker}(\mathrm{loc}^s_{\Sigma_p}),\mathbf{D})$$

where $\mathrm{loc}^s_{\Sigma_p}$ *is the localization map* $\mathcal{S}^{\Sigma_p}(K,T) \to \prod_{v|p} H^1_s(K_v,T)$.

Proof. We apply Theorem 1.7.3(iii) with $\Sigma = \Sigma_p$ and with Σ_0 equal to the empty set, and take the direct limit over M to obtain

$$\varinjlim_M \left(\mathcal{S}(K,W^*_M)/\mathcal{S}_{\Sigma_p}(K,W^*_M)\right) \cong \varinjlim_M \mathrm{Hom}_{\mathcal{O}}(\mathrm{coker}(\mathrm{loc}^s_{\Sigma_p,M}),\mathcal{O}/M\mathcal{O})$$

where $\mathrm{loc}^s_{\Sigma_p,M}$ is the localization map $\mathcal{S}^{\Sigma_p}(K,W_M) \to \oplus_{v|p} H^1_s(K_v,W_M)$. By Proposition 1.5.6(ii),

$$\varinjlim_M \left(\mathcal{S}(K,W^*_M)/\mathcal{S}_{\Sigma_p}(K,W^*_M)\right) = \mathcal{S}(K,W^*)/\mathcal{S}_{\Sigma_p}(K,W^*).$$

By Proposition 1.5.6(i),

$$\varprojlim_M \mathcal{S}^{\Sigma_p}(K,W_M) = \mathcal{S}^{\Sigma_p}(K,T),$$

and by Corollary 1.3.10,

$$\varprojlim_M \oplus_{v|p} H^1_s(K_v,W_M) = \oplus_{v|p} H^1_s(K_v,T).$$

Since all the groups $\mathcal{S}^{\Sigma_p}(K,W_M)$ and $H^1_s(K_v,W_M)$ are finite (Proposition B.2.7(ii) and Lemma 1.5.7), it follows (Proposition B.1.1(ii)) that

$$\varinjlim_M \mathrm{Hom}_{\mathcal{O}}(\mathrm{coker}(\mathrm{loc}^s_{\Sigma_p,M}),\mathcal{O}/M\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}(\varprojlim_M \mathrm{coker}(\mathrm{loc}^s_{\Sigma_p,M}),\mathbf{D})$$

and that

$$\varprojlim_M \mathrm{coker}(\mathrm{loc}^s_{\Sigma_p,M}) = \mathrm{coker}(\mathrm{loc}^s_{\Sigma_p}).$$

This completes the proof. □

CHAPTER 2

Euler Systems: Definition and Main Results

In this chapter we state our main results. The definition of an Euler system is given in §2.1, and the theorems applying Euler systems to study Selmer groups over number fields and over $\mathbf{Z}_p^d$-extensions of number fields are given in §2.2 and §2.3, respectively. Examples and applications are given in Chapter 3; the reader might benefit from following along in those examples while reading this chapter. The proofs, using tools to be developed in Chapter 4, will be given in Chapters 5 and 7. In Chapter 9 we discuss some variants and extensions of the definition of Euler system given below.

For similar results see the papers of Kato [**Ka2**] and Perrin-Riou [**PR5**].

For a first reading, one might want to restrict below to the case $K = \mathbf{Q}$ (so that the group of global units $\mathcal{O}_K^\times$ is finite) and $\mathcal{O} = \mathbf{Z}_p$. This simplifies the notation, while all the main ideas still appear.

2.1. Euler Systems

Fix a number field K, and let $\mathcal{O}_K$ denote the ring of integers of K. Fix also a rational prime p and a p-adic representation T of G_K as in §1.1, with coefficients in the ring of integers $\mathcal{O}$ of some finite extension Φ of $\mathbf{Q}_p$. We assume in addition, as in §1.5, that T is unramified outside a finite set of primes of K.

Suppose $\mathfrak{q}$ is a prime of K not dividing p, and T is unramified at $\mathfrak{q}$. Let $K(\mathfrak{q})$ denote the maximal p-extension of K inside the ray class field of K modulo $\mathfrak{q}$, let $\mathrm{Fr}_\mathfrak{q}$ denote a Frobenius of $\mathfrak{q}$ in G_K, and define

$$P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;x) \;=\; \det(1-\mathrm{Fr}_\mathfrak{q}^{-1}x|T^*) \in \mathcal{O}[x]$$

(the determinant is well-defined because T^* is unramified at $\mathfrak{q}$).

We will write

$$K \subset_{\mathrm{f}} F$$

to indicate that the field F is a *finite* extension of K.

Definition 2.1.1. Suppose $\mathcal{K}$ is an (infinite) abelian extension of K and $\mathcal{N}$ is an ideal of K divisible by p and by all primes where T is ramified, such that

(i) $\mathcal{K}$ contains $K(\mathfrak{q})$ for every prime $\mathfrak{q}$ of K not dividing $\mathcal{N}$,
(ii) $\mathcal{K}$ contains an extension K_∞ of K such that
- $\mathrm{Gal}(K_\infty/K) \cong \mathbf{Z}_p^d$ for some $d \geq 1$,
- no (finite) prime of K splits completely in K_∞/K.

A collection of cohomology classes

$$\mathbf{c} = \{\mathbf{c}_F \in H^1(F,T) : K \subset_{\mathrm{f}} F \subset \mathcal{K}\}$$

is an *Euler system* for $(T,\mathcal{K},\mathcal{N})$ if, whenever $K \subset_{\mathrm{f}} F \subset_{\mathrm{f}} F' \subset \mathcal{K}$, then

$$\mathrm{Cor}_{F'/F}(\mathbf{c}_{F'}) = \Big(\prod_{\mathfrak{q}\in\Sigma(F'/F)} P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;\mathrm{Fr}_\mathfrak{q}^{-1})\Big)\mathbf{c}_F$$

where $\Sigma(F'/F)$ is the set of (finite) primes of K, not dividing $\mathcal{N}$, which ramify in F' but not in F.

We say a collection $\mathbf{c} = \{\mathbf{c}_F \in H^1(F,T)\}$ is an Euler system for T if $\mathbf{c}$ is an Euler system for $(T,\mathcal{K},\mathcal{N})$ for some choice of $\mathcal{N}$ and $\mathcal{K}$ as above.

If K_∞ is a $\mathbf{Z}_p^d$-extension of K in which no finite prime of K splits completely, we say a collection $\mathbf{c} = \{\mathbf{c}_F \in H^1(F,T)\}$ is an Euler system for (T,K_∞) if $\mathbf{c}$ is an Euler system for $(T,\mathcal{K},\mathcal{N})$ for some choice of $\mathcal{N}$ and $\mathcal{K}$ containing K_∞ as above.

Remark 2.1.2. The condition that no finite prime splits completely in K_∞/K is satisfied, for example, if K_∞ contains the cyclotomic $\mathbf{Z}_p$-extension of K.

In general, since $\mathbf{Z}_p^d$ has no proper finite subgroups, to say that a prime does not split completely in K_∞/K is equivalent to saying that its decomposition group is infinite. See §9.2 for additional remarks about this assumption.

Note that since we require $\mathcal{N}$ to be divisible by p, no Euler factors at primes dividing p enter our picture. It follows from our definition that the Euler system cohomology classes are "universal norms" in the K_∞/K direction, i.e., if $K \subset_{\mathrm{f}} F \subset_{\mathrm{f}} F' \subset F'K_\infty$, then $\Sigma(F'/F)$ is empty so

$$\mathrm{Cor}_{F'/F}(\mathbf{c}_{F'}) = \mathbf{c}_F.$$

On the other hand, one might want to include Euler factors for primes where T is ramified. One could easily modify the definition above to take such Euler factors into account. Alternatively, one can choose an ideal $\mathcal{N}'$ prime to p, replace $\mathcal{K}$ by the maximal extension $\mathcal{K}'$ of K in $\mathcal{K}$ which is unramified at all primes dividing $\mathcal{N}'$, and replace $\mathcal{N}$ by $\mathcal{N}\mathcal{N}'$. Then the

Euler factors at primes dividing $\mathcal{N}'$ become irrelevant, and no information has been lost when we apply the theorems below (since the conclusions are independent of $\mathcal{K}$ and $\mathcal{N}$).

Remark 2.1.3. If $\mathfrak{m}$ is a generalized ideal of K (i.e., $\mathfrak{m}$ can be divisible by archimedean places as well as prime ideals), let $K[\mathfrak{m}]$ denote the ray class field of K modulo $\mathfrak{m}$. Given $\mathcal{K}$ and $\mathcal{N}$ as in the definition above, an Euler system for $(T, \mathcal{K}, \mathcal{N})$ is equivalent to a collection

$$\{\tilde{\mathbf{c}}_{\mathfrak{m}} \in H^1(K[\mathfrak{m}] \cap \mathcal{K}, T) \ : \ \text{every generalized ideal } \mathfrak{m}\}$$

satisfying

$$\mathrm{Cor}_{K[\mathfrak{m}\mathfrak{q}]\cap\mathcal{K}/K[\mathfrak{m}]\cap\mathcal{K}}(\tilde{\mathbf{c}}_{\mathfrak{m}\mathfrak{q}}) \ = \ \begin{cases} P(\mathrm{Fr}_{\mathfrak{q}}^{-1}|T^*;\mathrm{Fr}_{\mathfrak{q}}^{-1})\tilde{\mathbf{c}}_{\mathfrak{m}} & \text{if } \mathfrak{q} \nmid \mathfrak{m}\mathcal{N}, \\ \tilde{\mathbf{c}}_{\mathfrak{m}} & \text{if } \mathfrak{q} \mid \mathfrak{m}\mathcal{N}. \end{cases}$$

For, given such a collection, if F is a subfield of $\mathcal{K}$, then we can define

$$\mathbf{c}_F \ = \ \mathrm{Cor}_{K[\mathfrak{m}]\cap\mathcal{K}/F}(\tilde{\mathbf{c}}_{\mathfrak{m}})$$

where $\mathfrak{m}$ is the conductor of F/K. One checks easily that the collection $\{\mathbf{c}_F\}$ is an Euler system. Conversely, given an Euler system $\{\mathbf{c}_F\}$ we can define

$$\tilde{\mathbf{c}}_{\mathfrak{m}} \ = \ \prod P(\mathrm{Fr}_{\mathfrak{q}}^{-1}|T^*;\mathrm{Fr}_{\mathfrak{q}}^{-1})\, \mathbf{c}_{K[\mathfrak{m}]\cap\mathcal{K}}$$

where the product is over primes $\mathfrak{q}$ which divide $\mathfrak{m}$ but do not divide $\mathcal{N}$, and which are unramified in $(K[\mathfrak{m}] \cap \mathcal{K})/K$.

Remark 2.1.4. Suppose now that we are given $\mathcal{N}$ and K_∞/K as in Definition 2.1.1. If $\mathfrak{r} = \mathfrak{q}_1 \cdots \mathfrak{q}_k$ is a product of distinct primes not dividing $\mathcal{N}$, then we define $K(\mathfrak{r})$ to be the compositum

$$K(\mathfrak{r}) \ = \ K(\mathfrak{q}_1) \cdots K(\mathfrak{q}_k).$$

We will write $\mathbf{1}$ for the trivial ideal of $\mathcal{O}_K$, and $K(\mathbf{1})$ will denote the maximal p-extension of K inside the Hilbert class field of K. If $K \subset_{\mathrm{f}} F \subset K_\infty$ we let $F(\mathfrak{r}) = FK(\mathfrak{r})$. Let $\mathcal{K}_{\min}$ be the compositum of K_∞ and all $K(\mathfrak{q})$ for primes $\mathfrak{q}$ not dividing $\mathcal{N}$. Thus $\mathcal{K}_{\min}$ is the smallest extension of K satisfying the conditions of Definition 2.1.1 for $\mathcal{N}$ and K_∞/K. Every finite extension of K in $\mathcal{K}_{\min}$ is contained in $F(\mathfrak{r})$ for some squarefree ideal $\mathfrak{r}$ prime to $\mathcal{N}$ and some $K \subset_{\mathrm{f}} F \subset K_\infty$. It follows easily that an Euler system for $(T, \mathcal{K}_{\min}, \mathcal{N})$ is completely determined by the classes $\mathbf{c}_{F(\mathfrak{r})}$ with $K \subset_{\mathrm{f}} F \subset K_\infty$ and $\mathfrak{r}$ squarefree and prime to $\mathcal{N}$.

Conversely, suppose we are given a collection

$$\{\mathbf{c}_{F(\mathfrak{r})} \in H^1(F(\mathfrak{r}), T) : K \subset_{\mathrm{f}} F \subset K_\infty,\ \mathfrak{r} \text{ squarefree prime to } \mathcal{N}\}$$

satisfying

(a) if $K \subset_f F \subset_f F' \subset K_\infty$ and $\mathfrak{r}$ is a squarefree ideal of K prime to $\mathcal{N}$, then

$$\mathrm{Cor}_{F'(\mathfrak{r})/F(\mathfrak{r})}(\mathbf{c}_{F'(\mathfrak{r})}) = \mathbf{c}_{F(\mathfrak{r})},$$

(b) if $K \subset_f F \subset K_\infty$, and $\mathfrak{q}\mathfrak{r}$ is a squarefree ideal prime to $\mathcal{N}$ with $\mathfrak{q}$ prime such that $K(\mathfrak{q}) \neq K(\mathbf{1})$, then

$$\mathrm{Cor}_{F(\mathfrak{r}\mathfrak{q})/F(\mathfrak{r})}(\mathbf{c}_{F(\mathfrak{r}\mathfrak{q})}) = P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*; \mathrm{Fr}_\mathfrak{q}^{-1})\mathbf{c}_{F(\mathfrak{r})}.$$

(Note that if $K(\mathfrak{q}) = K(\mathbf{1})$ if and only if $F(\mathfrak{r}\mathfrak{q}) = F(\mathfrak{r})$.) Then this collection determines an Euler system: if $K \subset_f L \subset \mathcal{K}_{\min}$ then we can set

$$\mathbf{c}_L = \mathrm{Cor}_{F(\mathfrak{r})/L}(\mathbf{c}_{F(\mathfrak{r})})$$

where $\mathfrak{r}$ and F are minimal such that $L \subset F(\mathfrak{r})$. Thus we may view an Euler system for $(T, \mathcal{K}_{\min}, \mathcal{N})$ as such a collection $\{\mathbf{c}_{F(\mathfrak{r})} \in H^1(F(\mathfrak{r}), T)\}$. This will be the most convenient way to think of an Euler system for the proofs in Chapter 4.

Remark 2.1.5. Kolyvagin's original method ([**Ko2**] or [**Ru3**]) required the Euler system to satisfy an additional "congruence" condition. By expanding on an idea from [**Ru6**], using our assumption that $\mathcal{K}$ contains K_∞ (i.e., that our Euler system extends "in the p-direction"), we will be able to bypass the need for the congruence condition. In fact, the congruence condition follows easily from our techniques in Chapter 4, and although we do not need it, we will state and prove it in §4.8 (Corollary 4.8.1).

On the other hand, if we assume that our Euler system classes satisfy appropriate congruence conditions then we can remove from Definition 2.1.1(ii) the assumption that $\mathcal{K}$ contains K_∞. See Chapter 9 for a discussion of this and other possible variations on the definition of an Euler system.

2.2. Results over K

We now come to the fundamental application of Euler systems. We will use the "derivative" classes associated to an Euler system (see §4.4) and the duality theorems from Galois cohomology stated in §1.7 to bound the order of a Selmer group (Theorems 2.2.2, 2.2.3, and 2.2.10). Theorems 2.2.2 and 2.2.3 will be proved in Chapter 5.

Let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}$ and let $\Bbbk = \mathcal{O}/\mathfrak{p}$ be the residue field. Recall that $K(\mathbf{1})$ is the maximal p-extension of K inside the Hilbert class field of K. We will make use of two different sets of hypotheses on the Galois representation T. Hypotheses $\mathrm{Hyp}(K, T)$ are stronger than $\mathrm{Hyp}(K, V)$, and will allow us to prove a stronger conclusion.

Hypotheses $\mathrm{Hyp}(K,T)$. (i) *There is a* $\tau \in G_K$ *such that*
- τ *acts trivially on* $\boldsymbol{\mu}_{p^\infty}$, *on* $(\mathcal{O}_K^\times)^{1/p^\infty}$, *and on* $K(\mathbf{1})$,
- $T/(\tau-1)T$ *is free of rank one over* $\mathcal{O}$.

(ii) $T \otimes \Bbbk$ *is an irreducible* $\Bbbk[G_K]$*-module.*

Hypotheses $\mathrm{Hyp}(K,V)$. (i) *There is a* $\tau \in G_K$ *such that*
- τ *acts trivially on* $\boldsymbol{\mu}_{p^\infty}$, *on* $(\mathcal{O}_K^\times)^{1/p^\infty}$, *and on* $K(\mathbf{1})$,
- $\dim_\Phi(V/(\tau-1)V) = 1$.

(ii) V *is an irreducible* $\Phi[G_K]$*-module.*

Definition 2.2.1. If $\mathbf{c}$ is an Euler system, we define the *index of divisibility* of $\mathbf{c}$ to be

$$\mathrm{ind}_{\mathcal{O}}(\mathbf{c}) \;=\; \sup\{n : \mathbf{c}_K \in \mathfrak{p}^n H^1(K,T) + H^1(K,T)_{\mathrm{tors}}\} \;\leq\; \infty,$$

i.e., $\mathfrak{p}^{\mathrm{ind}_{\mathcal{O}}(\mathbf{c})}$ is the largest power of the maximal ideal $\mathfrak{p}$ by which $\mathbf{c}_K$ can be divided in $H^1(K,T)/H^1(K,T)_{\mathrm{tors}}$.

Write $\ell_{\mathcal{O}}(B)$ for the length of an $\mathcal{O}$-module B, so that $|B| = |\Bbbk|^{\ell_{\mathcal{O}}(B)}$. We allow $\ell_{\mathcal{O}}(B) = \infty$.

Define $\Omega = K(\mathbf{1})K(W)K(\boldsymbol{\mu}_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$, where $K(W)$ denotes the smallest extension of K such that $G_{K(W)}$ acts trivially on W.

Let Σ_p denote the set of primes of K above p.

Theorem 2.2.2. *Suppose that* $p > 2$ *and that* T *satisfies* $\mathrm{Hyp}(K,T)$. *If* $\mathbf{c}$ *is an Euler system for* T *then*

$$\ell_{\mathcal{O}}(\mathcal{S}_{\Sigma_p}(K,W^*)) \;\leq\; \mathrm{ind}_{\mathcal{O}}(\mathbf{c}) + \mathfrak{n}_W + \mathfrak{n}_W^*$$

where

$$\begin{aligned}\mathfrak{n}_W &= \ell_{\mathcal{O}}(H^1(\Omega/K,W) \cap \mathcal{S}^{\Sigma_p}(K,W)),\\ \mathfrak{n}_W^* &= \ell_{\mathcal{O}}(H^1(\Omega/K,W^*) \cap \mathcal{S}_{\Sigma_p}(K,W^*)).\end{aligned}$$

Theorem 2.2.3. *Suppose that* V *satisfies* $\mathrm{Hyp}(K,V)$ *and* T *is not the one-dimensional trivial representation. If* $\mathbf{c}$ *is an Euler system for* T *and* $\mathbf{c}_K \notin H^1(K,T)_{\mathrm{tors}}$, *then* $\mathcal{S}_{\Sigma_p}(K,W^*)$ *is finite.*

Note that Theorem 2.2.3 holds even if $p = 2$.

Remark 2.2.4. Hypotheses $\mathrm{Hyp}(K,T)$ are satisfied if the image of the Galois representation on T is "sufficiently large". They often hold in practice; see the discussion of the examples in the next chapter. If $\mathrm{rank}_{\mathcal{O}}(T) = 1$, then (i) holds with $\tau = 1$, and (ii) holds trivially.

Remark 2.2.5. Corollary C.2.2 shows that if V is irreducible as a $\Phi[G_K]$-module, then both $H^1(\Omega/K,W)$ and $H^1(\Omega/K,W^*)$ are finite unless either

$T = \mathcal{O}$ with trivial action or $T = \mathcal{O}(1)$. Frequently the "error terms" $\mathfrak{n}_W$ and $\mathfrak{n}_W^*$ in Theorem 2.2.2 are zero; see the examples in Chapter 3.

Remark 2.2.6. Hypothesis Hyp(K,T)(i) will be used to guarantee the existence of sufficiently many primes $\mathfrak{q}$ of K such that $H^1_f(K_\mathfrak{q}, W_M)$ and $H^1_s(K_\mathfrak{q}, W_M^*)$ are free of rank one over $\mathcal{O}/M\mathcal{O}$. This in turn will make it possible to use Theorem 1.7, along with the cohomology classes we will construct from the Euler system in Chapter 4, to bound the Selmer group as in Theorem 2.2.2.

Remark 2.2.7. In the exceptional case of Theorem 2.2.3, when $T = \mathcal{O}$, the Selmer group $\mathcal{S}_{\Sigma_p}(K, W^*)$ is finite if and only if Leopoldt's conjecture holds for K. See Corollary 1.6.5.

Remark 2.2.8. There is always a trivial Euler system, namely the one defined by $\mathbf{c}_F = 0$ for every F. But in that case $\mathrm{ind}_\mathcal{O}(\mathbf{c}) = \infty$ so Theorems 2.2.2 and 2.2.3 say nothing.

Remark 2.2.9. Theorem 2.2.2 gives a bound for the size of the "strict" Selmer group $\mathcal{S}_{\Sigma_p}(K, W^*)$, not the true Selmer group $\mathcal{S}(K, W^*)$. Since we have put no local conditions at p on either our representation T or our Euler system $\mathbf{c}$, that restricted Selmer group is all that the Euler system can "see". Combining the global duality results from §1.7 with Theorems 2.2.2 and 2.2.3 gives Theorem 2.2.10 below concerning $\mathcal{S}(K, W^*)$.

Suppose that, as in §1.7, for every prime v of K dividing p we have subspaces $H^1_f(K_v, V) \subset H^1(K_v, V)$ and $H^1_f(K_v, V^*) \subset H^1(K_v, V^*)$ which are orthogonal complements under the pairing $\langle\ ,\ \rangle_{K_v}$. We write

$$H^1(K_p,\ \cdot\ ) = \oplus_{v|p} H^1(K_v,\ \cdot\ )$$

and similarly for H^1_f and $H^1_s = H^1/H^1_f$, and let

$$\mathrm{loc}^s_{\Sigma_p} : \mathcal{S}^{\Sigma_p}(K,T) \longrightarrow H^1_s(K_p, T)$$

be the localization map as in Corollary 1.7.5.

By Corollary B.3.5 (see also Proposition 4.6.1) and Lemma 1.3.5(ii), if $\mathbf{c}$ is an Euler system then $\mathbf{c}_K \in \mathcal{S}^{\Sigma_p}(K,T)$.

Theorem 2.2.10. *Suppose* $\mathbf{c}$ *is an Euler system for* T *and* $\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_K) \neq 0$.

(i) *Suppose that* T *is not the one-dimensional trivial representation, that* V *satisfies* Hyp(K,V), *and that* $[H^1_s(K_p,T) : \mathcal{O}\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_K)]$ *is finite. Then* $\mathcal{S}(K,W^*)$ *is finite.*

(ii) *Suppose that* $p > 2$ *and* T *satisfies* Hyp(K,T). *Let* $\mathfrak{n}_W$ *and* $\mathfrak{n}_W^*$ *be as in Theorem* 2.2.2. *Then*

$$\ell_\mathcal{O}(\mathcal{S}(K,W^*)) \leq \ell_\mathcal{O}(H^1_s(K_p,T)/\mathcal{O}\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_K)) + \mathfrak{n}_W + \mathfrak{n}_W^*.$$

Proof. We will use Theorems 2.2.2 and 2.2.3 to bound $\mathcal{S}_{\Sigma_p}(K,W^*)$, and Corollary 1.7.5 to bound $[\mathcal{S}(K,W^*):\mathcal{S}_{\Sigma_p}(K,W^*)]$.

For every v, the $\mathcal{O}$-module $H^1_s(K_v,T)$ is torsion-free since by definition it injects into the vector space $H^1_s(K_v,V)$. Hence if $\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_K)$ is not zero then $\mathbf{c}_K \notin H^1(K,T)_{\mathrm{tors}}$. Now Theorem 2.2.3 shows that $\mathcal{S}_{\Sigma_p}(K,W^*)$ is finite, and Corollary 1.7.5 shows that

$$\begin{aligned}[\mathcal{S}(K,W^*):\mathcal{S}_{\Sigma_p}(K,W^*)] &= [H^1_s(K_p,T):\mathcal{O}\mathrm{loc}^s_{\Sigma_p}(\mathcal{S}^{\Sigma_p}(K,T))] \qquad (2.1)\\ &\le [H^1_s(K_p,T):\mathcal{O}\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_K)].\end{aligned}$$

This proves (i).

The definition of $\mathcal{S}^{\Sigma_p}(K,T)$ gives an injective map

$$H^1(K,T)/\mathcal{S}^{\Sigma_p}(K,T) \hookrightarrow \oplus_{v\nmid p} H^1_s(K_v,T),$$

so $H^1(K,T)/\mathcal{S}^{\Sigma_p}(K,T)$ is torsion-free. Since $\mathbf{c}_K \in \mathcal{S}^{\Sigma_p}(K,T)$, It follows that for every $n \ge 0$,

$$\begin{aligned}\mathbf{c}_K \in \mathfrak{p}^n H^1(K,T) + H^1(K,T)_{\mathrm{tors}} &\Rightarrow \mathbf{c}_K \in \mathfrak{p}^n \mathcal{S}^{\Sigma_p}(K,T) + H^1(K,T)_{\mathrm{tors}}\\ &\Rightarrow \mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_K) \in \mathfrak{p}^n \mathrm{loc}^s_{\Sigma_p}(\mathcal{S}^{\Sigma_p}(K,T)).\end{aligned}$$

Therefore if $\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_K) \neq 0$ then

$$\mathrm{ind}_{\mathcal{O}}(\mathbf{c}) \le \ell_{\mathcal{O}}(\mathrm{loc}^s_{\Sigma_p}(\mathcal{S}^{\Sigma_p}(K,T))/\mathcal{O}\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_K)),$$

and so Theorem 2.2.2 shows that

$$\ell_{\mathcal{O}}(\mathcal{S}_{\Sigma_p}(K,W^*)) \le \ell_{\mathcal{O}}(\mathrm{loc}^s_{\Sigma_p}(\mathcal{S}^{\Sigma_p}(K,T))/\mathcal{O}\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_K)) + \mathfrak{n}_W + \mathfrak{n}^*_W.$$

Together with equality (2.1) of Corollary 1.7.5, this proves (ii). □

Remark 2.2.11. Note that, although a full Euler system is required to prove Theorems 2.2.2, 2.2.3, and 2.2.10, only the class $\mathbf{c}_K$ appears in the statements of those theorems.

Remark 2.2.12. The choice of subspace $H^1_f(K_p,V)$ intervenes on both sides of the inequality of Theorem 2.2.10(ii).

Remark 2.2.13. One would like a bound for the order of $\mathcal{S}(K,W^*)$ which involves a value of an appropriate L-function. However, Theorems 2.2.2 and 2.2.10 are purely algebraic and never "see" special values of L-functions. One hopes that (as in the examples of Chapter 3) these L-values will arise as $\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_K)$ for some Euler system $\mathbf{c}$, and thereby come into the bound for the order of $\mathcal{S}(K,W^*)$ via Theorem 2.2.10. See Chapter 8 for a discussion of a general framework in which one expects Euler systems which are related to L-values to exist.

2.3. Results over K_∞

Fix for this section an (infinite) abelian extension K_∞ of K such that $\mathrm{Gal}(K_\infty/K) \cong \mathbf{Z}_p^d$ for some $d \geq 1$, and such that no finite prime of K splits completely in K_∞.

Essentially by proving analogues of Theorem 2.2.2 for each finite extension F of K in K_∞, we can pass to the limit and prove an Iwasawa-theoretic version of Theorem 2.2.2. See [**Lan**] Chapter 5 or [**Wa**] Chapter 13 for basic background on Iwasawa theory, or [**Se1**] for the more general situation of $\mathbf{Z}_p^d$-extensions with $d > 1$.

Theorems 2.3.2, 2.3.3, and 2.3.4 below will be proved in Chapter 7.

Notation. If $K \subset_{\mathrm{f}} F \subset K_\infty$, we will write $\Lambda_F = \mathcal{O}[\mathrm{Gal}(F/K)]$. Let $\Gamma = \mathrm{Gal}(K_\infty/K)$ and let Λ denote the Iwasawa algebra

$$\Lambda = \mathcal{O}[[\Gamma]] = \varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} \Lambda_F,$$

so Λ is (noncanonically) isomorphic to a power series ring over $\mathcal{O}$ in d variables.

We say that a Λ-module B is *pseudo-null* if B is annihilated by an ideal of Λ of height at least two. A *pseudo-isomorphism* is a Λ-module homomorphism with pseudo-null kernel and cokernel, and two Λ-modules are pseudo-isomorphic if there is a pseudo-isomorphism between them. If B is a finitely generated torsion Λ-module then there is an injective pseudo-isomorphism

$$\bigoplus_i \Lambda/f_i\Lambda \hookrightarrow B$$

with $f_i \in \Lambda$, and we define the characteristic ideal of B

$$\mathrm{char}(B) = \prod_i f_i\Lambda.$$

The characteristic ideal is well-defined, although the individual f_i are not. The individual ideals (elementary divisors) $f_i\Lambda$ are uniquely determined if we add the extra requirement that $f_{i+1} \mid f_i$ for every i. If B is a finitely generated Λ-module which is not torsion, we define $\mathrm{char}(B) = 0$. If

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

is an exact sequence of finitely generated Λ-modules, then

$$\mathrm{char}(B) = \mathrm{char}(B')\mathrm{char}(B'').$$

We will need the following weak assumption to rule out some very special bad cases. In particular it is satisfied if $K = \mathbf{Q}$.

Hypothesis $\mathrm{Hyp}(K_\infty/K)$. *If* $\mathrm{rank}_{\mathbf{Z}_p}(\Gamma) = 1$ *and* G_{K_∞} *acts either trivially or by the cyclotomic character on* V*, then either* K *is a totally real field and Leopoldt's conjecture holds for* K *(i.e., the* p*-adic completion of* $\mathcal{O}_K^\times$ *injects into* $(\mathcal{O}_K \otimes \mathbf{Z}_p)^\times$*), or* K *is an imaginary quadratic field.*

We will also write $\mathrm{Hyp}(K_\infty, T)$ (resp. $\mathrm{Hyp}(K_\infty, V)$) for hypotheses $\mathrm{Hyp}(K, T)$ (resp. $\mathrm{Hyp}(K, V)$) with G_K replaced by G_{K_∞}, i.e.,

Hypotheses $\mathrm{Hyp}(K_\infty, T)$. (i) *There is a* $\tau \in G_{K_\infty}$ *such that*
- τ *acts trivially on* $\boldsymbol{\mu}_{p^\infty}$*, on* $(\mathcal{O}_K^\times)^{1/p^\infty}$*, and on* $K(\mathbf{1})$*,*
- $T/(\tau - 1)T$ *is free of rank one over* $\mathcal{O}$*.*

(ii) $T \otimes \Bbbk$ *is an irreducible* $\Bbbk[G_{K_\infty}]$*-module.*

Hypotheses $\mathrm{Hyp}(K_\infty, V)$. (i) *There is a* $\tau \in G_{K_\infty}$ *such that*
- τ *acts trivially on* $\boldsymbol{\mu}_{p^\infty}$*, on* $(\mathcal{O}_K^\times)^{1/p^\infty}$*, and on* $K(\mathbf{1})$*,*
- $\dim_\Phi(V/(\tau - 1)V) = 1$.

(ii) V *is an irreducible* $\Phi[G_{K_\infty}]$*-module.*

There are simple implications

$$\begin{array}{ccc} \mathrm{Hyp}(K_\infty, T) & \Rightarrow & \mathrm{Hyp}(K_\infty, V) \\ \Downarrow & & \Downarrow \\ \mathrm{Hyp}(K, T) & \Rightarrow & \mathrm{Hyp}(K, V). \end{array}$$

Definition 2.3.1. Recall that $\mathbf{D} = \Phi/\mathcal{O}$. Define Λ-modules

$$H^1_\infty(K, T) = \varprojlim_{K \subset_f F \subset K_\infty} H^1(F, T)$$

$$\mathcal{S}_{\Sigma_p}(K_\infty, W^*) = \varinjlim_{K \subset_f F \subset K_\infty} \mathcal{S}_{\Sigma_p}(F, W^*)$$

$$X_\infty = \mathrm{Hom}_\mathcal{O}(\mathcal{S}_{\Sigma_p}(K_\infty, W^*), \mathbf{D}),$$

limits with respect to corestriction and restriction maps, respectively. If $\mathbf{c}$ is an Euler system let $\mathbf{c}_{K,\infty} = \{\mathbf{c}_F\}_{K \subset_f F \subset K_\infty}$ denote the corresponding element of $H^1_\infty(K, T)$ and define an ideal of Λ by

$$\mathrm{ind}_\Lambda(\mathbf{c}) = \{\phi(\mathbf{c}_{K,\infty}) : \phi \in \mathrm{Hom}_\Lambda(H^1_\infty(K, T), \Lambda)\} \subset \Lambda.$$

The ideal $\mathrm{ind}_\Lambda(\mathbf{c})$ measures the Λ-divisibility of $\mathbf{c}_{K,\infty}$, just as $\mathrm{ind}_\mathcal{O}(\mathbf{c})$ of Definition 2.2.1 measures the $\mathcal{O}$-divisibility of $\mathbf{c}_K$.

Recall that $\mathbf{c}$ is an Euler system for (T, K_∞) if it is an Euler system for $(T, \mathcal{K}, \mathcal{N})$ with $K_\infty \subset \mathcal{K}$.

Theorem 2.3.2. *Suppose* $\mathbf{c}$ *is an Euler system for* (T, K_∞)*, and* V *satisfies* $\mathrm{Hyp}(K_\infty, V)$*. If* $\mathbf{c}_{K,\infty}$ *does not belong to the* Λ*-torsion submodule of* $H^1_\infty(K, T)$*, then* X_∞ *is a torsion* Λ*-module.*

Theorem 2.3.3. *Suppose* $\mathbf{c}$ *is an Euler system for* (T, K_∞)*, and* T *satisfies hypotheses* $\mathrm{Hyp}(K_\infty, T)$ *and* $\mathrm{Hyp}(K_\infty/K)$*. Then*

$$\mathrm{char}(X_\infty) \ \textit{divides} \ \mathrm{ind}_\Lambda(\mathbf{c}).$$

Theorem 2.3.4. *Suppose* $\mathbf{c}$ *is an Euler system for* (T, K_∞)*, and* V *satisfies hypotheses* $\mathrm{Hyp}(K_\infty, V)$ *and* $\mathrm{Hyp}(K_\infty/K)$*. Then there is a nonnegative integer* t *such that*

$$\mathrm{char}(X_\infty) \ \textit{divides} \ p^t \mathrm{ind}_\Lambda(\mathbf{c}).$$

Remark 2.3.5. The assertion that X_∞ is a torsion Λ-module is called the weak Leopoldt conjecture for T. See [**Gr2**] or [**PR4**] (§1.3 and Appendice B).

Remark 2.3.6. As with Theorem 2.2.2, these three theorems all give bounds for the size of $\mathcal{S}_{\Sigma_p}(K_\infty, W^*)$ rather than the true Selmer group $\varinjlim \mathcal{S}(F, W^*)$. Combining these results with the global duality results from §1.7 gives Theorem 2.3.8 below concerning the true Selmer group.

Suppose that for every $K \subset_{\mathrm{f}} F \subset K_\infty$ and every prime w dividing p we have subspaces $H^1_f(F_w, V) \subset H^1(F_w, V)$ and $H^1_f(F_w, V^*) \subset H^1(F_w, V^*)$ which are orthogonal complements under the pairing $\langle \ , \ \rangle_{F_w}$, as in §1.7. We suppose further that if $F \subset F'$ and $w' \mid w$ then

$$\begin{aligned} \mathrm{Cor}_{F'_{w'}/F_w} H^1_f(F'_{w'}, V) &\subset H^1_f(F_w, V), \\ \mathrm{Res}_{F'_{w'}/F_w} H^1_f(F_w, V^*) &\subset H^1_f(F'_{w'}, V^*). \end{aligned}$$

(In fact, the local pairing and our assumptions about orthogonality show that these two inclusions are equivalent.) These conditions ensure that, if $K \subset_{\mathrm{f}} F \subset_{\mathrm{f}} F' \subset \mathcal{K}$, the natural restriction and corestriction maps induce maps

$$\mathcal{S}(F, W^*) \longrightarrow \mathcal{S}(F', W^*), \quad H^1_s(F'_p, T) \longrightarrow H^1_s(F_p, T)$$

where we write

$$H^1(F_p, \ \cdot \ ) \ = \ \oplus_{w|p} H^1(F_w, \ \cdot \ ),$$

and similarly for H^1_f and $H^1_s = H^1/H^1_f$. Define

$$\begin{aligned} \mathcal{S}(K_\infty, W^*) \ &= \ \varinjlim_{K \subset_{\mathrm{f}} F \subset K_\infty} \mathcal{S}(F, W^*), \\ H^1_{\infty,s}(K_p, T) \ &= \ \varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^1_s(F_p, T). \end{aligned}$$

Proposition 2.3.7. *There is an exact sequence*

$$\begin{aligned} 0 \longrightarrow H^1_{\infty,s}(K_p, T)/\mathrm{loc}^s_{\Sigma_p}(H^1_\infty(K, T)) \\ \longrightarrow \mathrm{Hom}_{\mathcal{O}}(\mathcal{S}(K_\infty, W^*), \mathbf{D}) \longrightarrow X_\infty \longrightarrow 0 \end{aligned}$$

where $\mathrm{loc}^s_{\Sigma_p} : H^1_\infty(K,T) \to H^1_{\infty,s}(K_p,T)$ *is the localization map.*

Proof. By Corollary B.3.5,

$$H^1_\infty(K,T) = \varprojlim_{K \subset_f F \subset K_\infty} \mathcal{S}^{\Sigma_p}(F,T).$$

Thus the proposition follows from Corollary 1.7.5 by passing to the (direct) limit and applying $\mathrm{Hom}_{\mathcal{O}}(\cdot, \mathbf{D})$. □

Theorem 2.3.8. *Suppose that* $\mathbf{c}$ *is an Euler system for* (T, K_∞)*, and* V *satisfies hypotheses* $\mathrm{Hyp}(K_\infty, V)$ *and* $\mathrm{Hyp}(K_\infty/K)$*. Suppose further that* $\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_{K,\infty}) \notin H^1_{\infty,s}(K_p,T)_{\Lambda\text{-tors}}$ *and* $H^1_{\infty,s}(K_p,T)/\Lambda\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_{K,\infty})$ *is a torsion* Λ*-module. Then* $\mathrm{Hom}_{\mathcal{O}}(\mathcal{S}(K_\infty, W^*), \mathbf{D})$ *is a torsion* Λ*-module and*

(i) *there is a nonnegative integer* t *such that*

$$\mathrm{char}(\mathrm{Hom}_{\mathcal{O}}(\mathcal{S}(K_\infty, W^*), \mathbf{D})) \text{ divides } p^t \mathrm{char}(H^1_{\infty,s}(K_p,T)/\Lambda\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_{K,\infty})),$$

(ii) *if* T *satisfies* $\mathrm{Hyp}(K_\infty, T)$ *then*

$$\mathrm{char}(\mathrm{Hom}_{\mathcal{O}}(\mathcal{S}(K_\infty, W^*), \mathbf{D})) \text{ divides } \mathrm{char}(H^1_{\infty,s}(K_p,T)/\Lambda\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_{K,\infty})).$$

Proof. Since $\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_{K,\infty}) \notin H^1_{\infty,s}(K_p,T)_{\Lambda\text{-tors}}$, we see that $\mathbf{c}_{K,\infty}$ cannot belong to $H^1_\infty(K,T)_{\Lambda\text{-tors}}$. Therefore Theorem 2.3.2 and Proposition 2.3.7 show that $\mathrm{Hom}_{\mathcal{O}}(\mathcal{S}(K_\infty, W^*), \mathbf{D})$ is a torsion Λ-module and that

$$\begin{aligned}\mathrm{char}&(\mathrm{Hom}_{\mathcal{O}}(\mathcal{S}(K_\infty, W^*), \mathbf{D})) \\ &= \mathrm{char}(X_\infty)\mathrm{char}(H^1_{\infty,s}(K_p,T)/\mathrm{loc}^s_{\Sigma_p}(H^1_\infty(K,T))).\end{aligned}$$

Our assumptions ensure that $\mathrm{loc}^s_{\Sigma_p}(H^1_\infty(K,T))$ is a rank-one Λ-module, so there is a map $\psi : \mathrm{loc}^s_{\Sigma_p}(H^1_\infty(K,T)) \to \Lambda$ with pseudo-null cokernel. Then

$$\begin{aligned}\psi(\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_{K,\infty}))\Lambda &= \mathrm{char}(\psi(\mathrm{loc}^s_{\Sigma_p}(H^1_\infty(K,T)))/\psi(\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_{K,\infty}))\Lambda) \\ &\supset \mathrm{char}(\mathrm{loc}^s_{\Sigma_p}(H^1_\infty(K,T))/\Lambda\mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_{K,\infty})),\end{aligned}$$

and by definition $\mathrm{ind}_\Lambda(\mathbf{c})$ divides $\psi \circ \mathrm{loc}^s_{\Sigma_p}(\mathbf{c}_{K,\infty})$. The theorem follows easily from these divisibilities and the divisibilities of Theorems 2.3.4 and 2.3.3. □

2.4. Twisting by Characters of Finite Order

Suppose $\mathbf{c}$ is an Euler system for $(T, \mathcal{K}, \mathcal{N})$ as defined in Definition 2.1.1. The consequences of the existence of such an Euler system described in §2.2 and §2.3 do not depend on $\mathcal{K}$ (except that, in the case of §2.3, the field $\mathcal{K}$ must contain K_∞). We could always take $\mathcal{K}$ to be the "minimal" field $\mathcal{K}_{\min}$ described in Remark 2.1.4, and ignore the Euler system classes $\mathbf{c}_F$ for $F \not\subset \mathcal{K}_{\min}$, and still obtain the results stated above.

However, there is a way to make use of the additional information contained in an Euler system for a non-minimal $\mathcal{K}$. Namely, in this section we show how to take an Euler system for $(T, \mathcal{K}, \mathcal{N})$ and obtain from it an Euler system for twists $T \otimes \chi$ of T by characters χ of finite order of $\mathrm{Gal}(\mathcal{K}/K)$ (see below). For example, if $\mathcal{K}$ is the maximal abelian extension of K, then we will obtain Euler systems for *all* twists of T by characters of finite order, and the results of this chapter then give (possibly trivial, possibly not) bounds for *all* the corresponding Selmer groups.

Suppose $\chi : G_K \to \mathcal{O}^\times$ is a character of finite order. As in Example 1.1.2 we will denote by $\mathcal{O}_\chi$ a free rank-one $\mathcal{O}$-module on which G_K acts via χ, and we fix a generator ξ_χ of $\mathcal{O}_\chi$ We will write $T \otimes \chi$ for the representation $T \otimes_{\mathcal{O}} \mathcal{O}_\chi$.

Definition 2.4.1. Suppose $\mathbf{c}$ is an Euler system for $(T, \mathcal{K}, \mathcal{N})$ and χ is a character of finite order of $\mathrm{Gal}(\mathcal{K}/K)$ with values in $\mathcal{O}^\times$. Let $L = \mathcal{K}^{\ker(\chi)}$ be the field cut out by χ. If $K \subset_{\mathrm{f}} F \subset \mathcal{K}$, define $\mathbf{c}_F^\chi \in H^1(F, T \otimes \chi)$ to be the image of $\mathbf{c}_{FL}$ under the composition

$$H^1(FL, T) \xrightarrow{\otimes \xi_\chi} H^1(FL, T) \otimes \mathcal{O}_\chi \cong H^1(FL, T \otimes \chi) \xrightarrow{\mathrm{Cor}} H^1(F, T \otimes \chi)$$

(we get the center isomorphism since G_{FL} is in the kernel of χ).

Proposition 2.4.2. *Suppose* $\mathbf{c}$ *is an Euler system for* $(T, \mathcal{K}, \mathcal{N})$ *and*

$$\chi : \mathrm{Gal}(\mathcal{K}/K) \longrightarrow \mathcal{O}^\times$$

is a character of finite order. If $\mathfrak{f}$ *is the conductor of* χ *then the collection*

$$\{\mathbf{c}_F^\chi : K \subset_{\mathrm{f}} F \subset \mathcal{K}\}$$

defined above is an Euler system for $(T \otimes \chi, \mathcal{K}, \mathfrak{f}\mathcal{N})$.

Proof. If $K \subset_{\mathrm{f}} F \subset_{\mathrm{f}} F' \subset \mathcal{K}$ then using Definition 2.1.1 we have

$$\begin{aligned}
\mathrm{Cor}_{F'/F}(\mathbf{c}_{F'}^\chi) &= \mathrm{Cor}_{F'L/F}(\mathbf{c}_{F'L} \otimes \xi_\chi) \\
&= \mathrm{Cor}_{FL/F}\left((\mathrm{Cor}_{F'L/FL}\mathbf{c}_{F'L}) \otimes \xi_\chi\right) \\
&= \mathrm{Cor}_{FL/F}\left(\Big(\prod_{\mathfrak{q} \in \Sigma(F'L/FL)} P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*; \mathrm{Fr}_\mathfrak{q}^{-1})\, \mathbf{c}_{FL}\Big) \otimes \xi_\chi\right) \\
&= \mathrm{Cor}_{FL/F}\Big(\prod_{\mathfrak{q} \in \Sigma(F'L/FL)} P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*; \chi(\mathrm{Fr}_\mathfrak{q})\mathrm{Fr}_\mathfrak{q}^{-1})(\mathbf{c}_{FL} \otimes \xi_\chi)\Big) \\
&= \prod_{\mathfrak{q} \in \Sigma(F'L/FL)} P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*; \chi(\mathrm{Fr}_\mathfrak{q})\mathrm{Fr}_\mathfrak{q}^{-1})\, \mathrm{Cor}_{FL/F}(\mathbf{c}_{FL} \otimes \xi_\chi) \\
&= \prod_{\mathfrak{q} \in \Sigma(F'L/FL)} P(\mathrm{Fr}_\mathfrak{q}^{-1}|(T \otimes \chi)^*; \mathrm{Fr}_\mathfrak{q}^{-1})\, \mathbf{c}_F^\chi
\end{aligned}$$

where as usual $P(\mathrm{Fr}_{\mathfrak{q}}^{-1}|(T\otimes\chi)^*;x) = \det(1-\mathrm{Fr}_{\mathfrak{q}}^{-1}x|(T\otimes\chi)^*)$, and

$$\begin{aligned}\Sigma(F'L/FL) &= \{\text{primes } \mathfrak{q} : \mathfrak{q}\nmid\mathcal{N},\ \mathfrak{q} \text{ ramifies in } F'L \text{ but not in } FL\}\\ &= \{\text{primes } \mathfrak{q} : \mathfrak{q}\nmid\mathfrak{f}\mathcal{N},\ \mathfrak{q} \text{ ramifies in } F' \text{ but not in } F\}.\end{aligned}$$

This proves the proposition. □

Lemma 2.4.3. *With notation as in Definition* 2.4.1*, suppose* $K\subset_{\mathfrak{f}} F\subset K_\infty$ *and* $L\subset_{\mathfrak{f}} L'\subset\mathcal{K}$*. If every prime which ramifies in* L'/K *is already ramified in* L/K*, then the image of* $\mathbf{c}_F^\chi$ *under the composition*

$$H^1(F,T\otimes\chi)\xrightarrow{\mathrm{Res}} H^1(FL',T\otimes\chi)\xrightarrow{\otimes\xi_\chi^{-1}} H^1(FL',T)$$

is

$$\sum_{\delta\in\mathrm{Gal}(FL'/F)}\chi(\delta)\delta\mathbf{c}_{FL'}.$$

Proof. Since $\mathbf{c}$ is an Euler system, and every prime which ramifies in L'/K ramifies in L/K, we have $\mathrm{Cor}_{FL'/FL}(\mathbf{c}_{FL'})=\mathbf{c}_{FL}$. Thus the image of $\mathbf{c}_F^\chi$ under the composition above is

$$\begin{aligned}&\big(\mathrm{Res}_{FL'/F}\mathrm{Cor}_{FL/F}(\mathbf{c}_{FL}\otimes\xi_\chi)\big)\otimes\xi_\chi^{-1}\\ &\qquad=\big(\mathrm{Res}_{FL'/F}\mathrm{Cor}_{FL'/F}(\mathbf{c}_{FL'}\otimes\xi_\chi)\big)\otimes\xi_\chi^{-1}\\ &\qquad=\Big(\sum_{\delta\in\mathrm{Gal}(FL'/F)}\delta(\mathbf{c}_{FL'}\otimes\xi_\chi)\Big)\otimes\xi_\chi^{-1}\\ &\qquad=\sum_{\delta\in\mathrm{Gal}(FL'/F)}\chi(\delta)\delta\mathbf{c}_{FL'}.\end{aligned}$$

□

CHAPTER 3

Examples and Applications

In this chapter we give the basic examples of Euler systems and their applications, using the results of Chapter 2.

3.1. Preliminaries

Suppose χ is a character of G_K into $\mathcal{O}^\times$. As in Example 1.1.2 we will denote by $\mathcal{O}_\chi$ a free rank-one $\mathcal{O}$-module on which G_K acts via χ. Recall that $\mathbf{D} = \Phi/\mathcal{O} = \mathcal{O} \otimes (\mathbf{Q}_p/\mathbf{Z}_p)$. We will also write

$$\mathbf{D}_\chi \;=\; \mathbf{D} \otimes_{\mathcal{O}} \mathcal{O}_\chi \;=\; \mathcal{O}_\chi \otimes (\mathbf{Q}_p/\mathbf{Z}_p).$$

For the first three examples (§§3.2, 3.3, and 3.4) we will assume that χ has finite prime-to-p order. As in §1.6.B and §1.6.C we take $T = \mathcal{O}_\chi$, and we then have $W = \mathbf{D}_\chi$ and $T^* = \mathcal{O}(1) \otimes \mathcal{O}_{\chi^{-1}} = \mathcal{O}_{\chi^{-1}\varepsilon_{\rm cyc}}$, where $\varepsilon_{\rm cyc}$ is the cyclotomic character.

Let $L = \bar{K}^{\ker\chi}$ be the abelian extension of K corresponding to χ, and write $\Delta = \mathrm{Gal}(L/K)$. Thus Δ is a cyclic group of order prime to p. As in Definition 1.6.3, if B is a $\mathbf{Z}[\Delta]$-module we write $B^\wedge$ for the p-adic completion of B and B^χ for the χ-component of $B^\wedge \otimes_{\mathbf{Z}_p} \mathcal{O}$. We also fix a generator of $\mathcal{O}_{\chi^{-1}}$, and this choice determines an isomorphism $B^\chi \cong (B^\wedge \otimes_{\mathbf{Z}_p} \mathcal{O}_{\chi^{-1}})^\Delta$.

Lemma 3.1.1. (i) *If $\chi \neq 1$ then $H^1(L(\boldsymbol{\mu}_{p^\infty})/K, W) = 0$.*

(ii) *If χ is not the character giving the action of G_K on $\boldsymbol{\mu}_p$, then $H^1(L(\boldsymbol{\mu}_{p^\infty})/K, W^*) = 0$.*

Proof. Write $\Omega = L(\boldsymbol{\mu}_{p^\infty})$ as in §2.2. Suppose $\rho : G_K \to \mathcal{O}^\times$ is a character. Recall that $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}$ and $\Bbbk = \mathcal{O}/\mathfrak{p}$ is the residue field. Write $\Bbbk_\rho = \Bbbk \otimes \mathcal{O}_\rho$.

Since $|\Delta|$ is prime to p, the inflation-restriction sequence shows that

$$H^1(\Omega/K, \Bbbk_\rho) \;=\; \mathrm{Hom}(\mathrm{Gal}(\Omega/L), \Bbbk_\rho)^\Delta \;=\; \mathrm{Hom}(\mathrm{Gal}(\Omega/L), \Bbbk_\rho^\Delta)$$

(note that Δ acts trivially on $\mathrm{Gal}(\Omega/L)$ because Ω/K is abelian). Further, if π is a generator of $\mathfrak{p}$, it follows from the exact sequence

$$0 \longrightarrow \Bbbk_\rho \longrightarrow \mathbf{D}_\rho \xrightarrow{\pi} \mathbf{D}_\rho \longrightarrow 0$$

that

$$H^1(\Omega/K,\mathbb{k}_\rho)=0 \;\Rightarrow\; H^1(\Omega/K,\mathbf{D}_\rho)_{\mathfrak{p}}=0 \;\Rightarrow\; H^1(\Omega/K,\mathbf{D}_\rho)=0.$$

If ρ is not congruent to 1 modulo $\mathfrak{p}$, then $\mathbb{k}_\rho^{\Delta}=0$ and so $H^1(\Omega/K,\mathbf{D}_\rho)=0$. Applying this with $\rho=\chi$ proves (i), and with $\rho=\chi^{-1}\varepsilon_{\mathrm{cyc}}$ proves (ii). □

3.2. Cyclotomic Units

The Euler system of cyclotomic units is studied in detail in [**Ko2**] and [**Ru3**].

An Euler system for $\mathbf{Z}_p(1)$. Take $K=\mathbf{Q}$. For every extension F of $\mathbf{Q}$, Kummer theory shows as in Example 1.2.1 that

$$H^1(F,\mathbf{Z}_p(1)) \;=\; \varprojlim_n H^1(F,\boldsymbol{\mu}_{p^n}) \;=\; \varprojlim_n F^\times/(F^\times)^{p^n} \;=\; (F^\times)\hat{} \qquad (3.1)$$

where $(F^\times)\hat{}$ is the p-adic completion of $F^\times$.

Fix a collection $\{\zeta_m : m\in\mathbf{Z}^+\}$ such that ζ_m is a primitive m-th root of unity and $\zeta_{mn}^n=\zeta_m$ for every m and n. (For example, we could fix an embedding of $\bar{\mathbf{Q}}$ into $\mathbf{C}$ and choose $\zeta_m=e^{2\pi i/m}$.) For every $m\geq 1$ and every prime ℓ we have the relation

$$\mathbf{N}_{\mathbf{Q}(\boldsymbol{\mu}_{m\ell})/\mathbf{Q}(\boldsymbol{\mu}_m)}(\zeta_{m\ell}-1) \;=\; \begin{cases} (\zeta_m-1) & \text{if } \ell\mid m \\ (\zeta_m-1)^{1-\mathrm{Fr}_\ell^{-1}} & \text{if } \ell\nmid m \text{ and } m>1 \\ (-1)^{\ell-1}\ell & \text{if } m=1 \end{cases} \qquad (3.2)$$

where Fr_ℓ is the Frobenius of ℓ in $\mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_m)/\mathbf{Q})$ (see for example [**Lan**] Theorem 6.3.1). For every $m\geq 1$ we define

$$\tilde{\mathbf{c}}_{m\infty} \;=\; \mathbf{N}_{\mathbf{Q}(\boldsymbol{\mu}_{mp})/\mathbf{Q}(\boldsymbol{\mu}_m)}(\zeta_{mp}-1)\in\mathbf{Q}(\boldsymbol{\mu}_m)^\times \;\subset\; H^1(\mathbf{Q}(\boldsymbol{\mu}_m),\mathbf{Z}_p(1))$$

and $\tilde{\mathbf{c}}_m=\mathbf{N}_{\mathbf{Q}(\boldsymbol{\mu}_m)/\mathbf{Q}(\boldsymbol{\mu}_m)^+}(\tilde{\mathbf{c}}_{m\infty})$ where $\mathbf{Q}(\boldsymbol{\mu}_m)^+$ is the maximal real subfield of $\mathbf{Q}(\boldsymbol{\mu}_m)$. The distribution relation (3.2) shows that the collection

$$\{\tilde{\mathbf{c}}_{m\infty},\tilde{\mathbf{c}}_m : m\in\mathbf{Z}^+\}$$

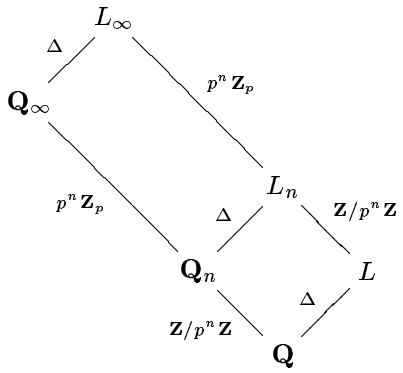
is an Euler system for $(\mathbf{Z}_p(1),\mathbf{Q}^{ab},p)$ (see Definition 2.1.1 and Remark 2.1.3), since for every prime $\ell\neq p$ we have

$$\det(1-\mathrm{Fr}_\ell^{-1}x|\mathbf{Z}_p(1)^*) \;=\; \det(1-\mathrm{Fr}_\ell^{-1}x|\mathbf{Z}_p) \;=\; 1-x.$$

Remark 3.2.1. If $p\mid m$ then (3.2) shows that $\tilde{\mathbf{c}}_{m\infty}=\zeta_m-1$. But if $p\nmid m$, our definition takes into account that our Euler system must satisfy $\mathbf{N}_{\mathbf{Q}(\boldsymbol{\mu}_{mp})/\mathbf{Q}(\boldsymbol{\mu}_m)}(\tilde{\mathbf{c}}_{mp})=\tilde{\mathbf{c}}_m$. This causes us to lose some information, and leads to the unwanted hypothesis $\chi(p)\neq 1$ in Theorem 3.2.3 below. We can remove this hypothesis either by using Theorem 3.2.10 below (see Remark 3.2.5) or by modifying the definition of Euler system as in Example 9.1.1.

The setting. Let $K = \mathbf{Q}$ and $K_\infty = \mathbf{Q}_\infty$, the cyclotomic (and only) $\mathbf{Z}_p$-extension of $\mathbf{Q}$. As in §3.1 we fix a character $\chi : G_{\mathbf{Q}} \to \mathcal{O}^\times$ of finite prime-to-p order, and we assume for the rest of this section that χ is even and nontrivial.

Let f denote the conductor of χ, and recall that L is the field cut out by χ. We will view χ as a Dirichlet character modulo f in the usual way, so that $\chi(\ell) = \chi(\mathrm{Fr}_\ell)$ if the prime ℓ does not divide f, and $\chi(\ell) = 0$ if $\ell \mid f$. For every $n \geq 0$ let $\mathbf{Q}_n$ be the unique subfield of $\mathbf{Q}_\infty$ with degree $[\mathbf{Q}_n : \mathbf{Q}] = p^n$, so $\mathbf{Q}_n \subset \mathbf{Q}(\boldsymbol{\mu}_{p^{n+1}})$ and $[\mathbf{Q}(\boldsymbol{\mu}_{p^{n+1}}) : \mathbf{Q}_n] = p - 1$. Let $L_n = L\mathbf{Q}_n$, and let $L_\infty = L\mathbf{Q}_\infty$. Since $[L : \mathbf{Q}]$ is prime to p, we have $L \cap \mathbf{Q}_n = \mathbf{Q}$ for every n so we can identify $\Delta = \mathrm{Gal}(L/\mathbf{Q})$ with $\mathrm{Gal}(L_n/\mathbf{Q}_n)$ for every n.



Let $T = \mathcal{O}_\chi$ as in §3.1, so that $T^* = \mathbf{Z}_p(1) \otimes \chi^{-1}$. The restriction map gives an isomorphism (using (3.1))

$$H^1(\mathbf{Q}_n, T^*) \cong H^1(L_n, T^*)^\Delta \cong ((L_n^\times)\hat{} \otimes \mathcal{O}_{\chi^{-1}})^\Delta \cong (L_n^\times)^\chi \subset (L_n^\times)\hat{} \otimes \mathcal{O}. \quad (3.3)$$

The Euler system $\tilde{\mathbf{c}}$ for $\mathbf{Z}_p(1)$ constructed above gives rise (by Proposition 2.4.2) to an Euler system $\mathbf{c} = \tilde{\mathbf{c}}^{\chi^{-1}}$ for $(T^*, \mathbf{Q}^{\mathrm{ab}}, pf)$. By Lemma 2.4.3, the image of $\mathbf{c}_{\mathbf{Q}}$ in $L^\times \hat{\otimes} \mathcal{O}$ under (3.3) is

$$\prod_{\delta \in \mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_f)^+/\mathbf{Q})} (\delta \tilde{\mathbf{c}}_f)^{\chi^{-1}(\delta)} = \prod_{\delta \in \mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_{fp})/\mathbf{Q})} (\zeta_{fp}^\delta - 1)^{\chi^{-1}(\delta)}. \quad (3.4)$$

The Selmer group. We have $W = \mathbf{D}_\chi$. Let $\mathbf{Q}_{n,p}$ denote the completion of $\mathbf{Q}_n$ at the unique prime above p, and as in the example of §1.6.B take $H^1_f(\mathbf{Q}_{n,p}, V) = H^1_{\mathrm{ur}}(\mathbf{Q}_{n,p}, V)$.

For every n let A_n be the ideal class group of L_n. We will also write $A_L = A_0$, the ideal class group of L. By Proposition 1.6.2 we have isomorphisms

$$\mathcal{S}(\mathbf{Q}, W) \cong \mathrm{Hom}(A_L, \mathbf{D}_\chi)^\Delta, \quad \mathcal{S}(\mathbf{Q}_\infty, W) \cong \mathrm{Hom}(\varprojlim A_n, \mathbf{D}_\chi)^\Delta. \quad (3.5)$$

The ideal class group of L.

Definition 3.2.2. If $n \geq 0$ we let $\mathcal{E}_n$ denote the group of global units of L_n. We define the group of χ-cyclotomic units $\mathcal{C}_{n,\chi}$ to be the subgroup of $\mathcal{E}_n^\chi$ generated over $\mathcal{O}[\mathrm{Gal}(L_n/\mathbf{Q})]$ by

$$\xi_{n,\chi} = \begin{cases} \displaystyle\prod_{\delta \in \mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_f)/\mathbf{Q})} (\zeta_f^\delta - 1)^{\chi^{-1}(\delta)} & \text{if } n = 0, \\ \displaystyle\prod_{\delta \in \mathrm{Gal}(\mathbf{Q}_n(\boldsymbol{\mu}_{fp^{n+1}})/\mathbf{Q}_n)} (\zeta_{fp^{n+1}}^\delta - 1)^{\chi^{-1}(\delta)} & \text{if } n > 0. \end{cases}$$

We will also write $\mathcal{E}_L = \mathcal{E}_0$, $\mathcal{C}_{L,\chi} = \mathcal{C}_{0,\chi}$ and $\xi_{L,\chi} = \xi_{0,\chi}$.

The following theorem and its Corollary (3.2.4 below) were first proved by Mazur and Wiles [**MW**]; the proof given here is due to Kolyvagin [**Ko2**]. See the additional remarks following the proof.

Theorem 3.2.3. *Suppose that χ is an even character of order prime to p. If $p > 2$ and $\chi(p) \neq 1$, then*

$$|A_L^\chi| \text{ divides } [\mathcal{E}_L^\chi : \mathcal{C}_{L,\chi}].$$

Proof. We will apply Theorem 2.2.2 with the Euler system $\mathbf{c}$ constructed from cyclotomic units above. Since $\mathrm{rank}_\mathcal{O} T^* = 1$, we see that $\mathrm{Hyp}(\mathbf{Q}, T^*)$ is satisfied with $\tau = 1$. Further, in this case $\Omega = L(\boldsymbol{\mu}_{p^\infty})$, and since χ is nontrivial and even, Lemma 3.1.1 shows that the error terms $\mathfrak{n}_{W^*}$ and $\mathfrak{n}_{W^*}^*$ in Theorem 2.2.2 are both zero.

By (3.3) we have maps

$$\mathcal{E}_L^\chi \hookrightarrow (L^\times)^\chi \xrightarrow{\sim} H^1(\mathbf{Q}, T^*).$$

Identifying $\xi_{L,\chi}$ with its image in $H^1(\mathbf{Q}, T^*)$, it follows from (3.2) and (3.4) that

$$\mathbf{c}_\mathbf{Q} = \xi_{L,\chi}^{1-\chi^{-1}(p)} \quad (3.6)$$

where $\chi(p) = 0$ if $p \mid f$. Since $\chi(p) \neq 1$ and χ has order prime to p, we have $1 - \chi^{-1}(p) \in \mathcal{O}^\times$ so $\mathbf{c}_\mathbf{Q}$ generates $\mathcal{C}_{L,\chi}$.

Recall that $\mathrm{ind}_\mathcal{O}(\mathbf{c})$ is the index of divisibility defined in Definition 2.2.1. Since $L^\times/\mathcal{E}_L$ is torsion-free, it follows that $\mathrm{ind}_\mathcal{O}(\mathbf{c})$ is the largest power of p by which a generator of $\mathcal{C}_{L,\chi}$ can be divided in $\mathcal{E}_L^\chi$. Since we have assumed that $p > 2$, that χ is even, and that $\chi \neq 1$, the Dirichlet unit

theorem (see for example [**T5**] §I.4) shows that $\mathcal{E}_L^\chi$ is free of rank one over $\mathcal{O}$, and we conclude that

$$\mathrm{ind}_{\mathcal{O}}(\mathbf{c}) \;=\; \ell_{\mathcal{O}}(\mathcal{E}_L^\chi/\mathcal{C}_{L,\chi}).$$

Putting all of this together, Theorem 2.2.2 in this case yields

$$|\mathcal{S}_{\Sigma_p}(\mathbf{Q},W)| \;\text{ divides }\; [\mathcal{E}_L^\chi : \mathcal{C}_{L,\chi}].$$

Let $\mathcal{I}$ denote an inertia group above p and $\mathrm{Fr}_p \in G_{\mathbf{Q}}$ a Frobenius element. By Lemma 1.3.2(i),

$$H^1_{\mathrm{ur}}(\mathbf{Q}_p,V) \;=\; V^{\mathcal{I}}/(\mathrm{Fr}_p-1)V^{\mathcal{I}} \;=\; V^{\mathcal{I}}/(\chi(p)-1)V^{\mathcal{I}} \;=\; 0$$

since $\chi(p)\neq 1$. Therefore $H^1_f(\mathbf{Q}_p,W)=0$ and

$$\mathcal{S}_{\Sigma_p}(\mathbf{Q},W) \;=\; \mathcal{S}(\mathbf{Q},W) \;=\; \mathrm{Hom}_{\mathcal{O}}(A_L^\chi,\mathbf{D}),$$

the final equality coming from (3.5). This completes the proof. □

A well-known argument using the analytic class number formula takes Theorem 3.2.3 and gives the following strengthening.

Corollary 3.2.4 (Mazur & Wiles [**MW**] Theorem 1.10.1)**.** *Suppose χ is an even character of order prime to p. If $p>2$ and $\chi(p)\neq 1$, then*

$$|A_L^\chi| \;=\; [\mathcal{E}_L^\chi : \mathcal{C}_{L,\chi}].$$

Proof. See for example Theorem 4.2 of [**Ru3**]. □

Remark 3.2.5. When p divides the order of χ, Theorem 2.2.2 still applies to give a bound for $\mathcal{S}(\mathbf{Q},W)$, but (see Proposition 1.6.2) this Selmer group is no longer exactly the ideal class group.

If $\chi(p)=1$, then (3.6) shows that $\mathbf{c}_{\mathbf{Q}}=0$, so Theorem 2.2.2 is of no use. However, in this case Greenberg ([**Gr1**] §5) has shown how to deduce the equality of Corollary 3.2.4 from Theorem 3.2.10 below (Iwasawa's "main conjecture") which we will prove using Theorem 2.3.3. See also §9.1.

The inverse limit of the ideal class groups. Recall that Λ is the Iwasawa algebra $\mathcal{O}[[\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]]$. For every n, let $L_{n,p}=L_n\otimes\mathbf{Q}_p$ and denote by U_n the local units of $L_{n,p}$. Define

$$A_\infty \;=\; \varprojlim_n (A_n)^\wedge,\;\; \mathcal{E}_\infty \;=\; \varprojlim_n (\mathcal{E}_n)^\wedge,\;\; \mathcal{C}_{\infty,\chi} \;=\; \varprojlim_n (\mathcal{C}_{n,\chi})^\wedge,$$

$$U_\infty \;=\; \varprojlim_n (U_n)^\wedge,\;\; Y_\infty \;=\; \varprojlim_n (L_{n,p})^\wedge,$$

inverse limits with respect to norm maps, where $(\cdot\,)^\wedge$ denotes p-adic completion (Definition 1.6.3). Let $\mathcal{E}'_n$ be the group of p-units of L_n (elements which are units at all primes not dividing p) and $\mathcal{E}'_\infty=\varprojlim(\mathcal{E}'_n)^\wedge$. Recall that $H^1_\infty(\mathbf{Q},T^*)=\varprojlim H^1(\mathbf{Q}_n,T^*)$, and set $H^1_\infty(\mathbf{Q}_p,T^*)=\varprojlim H^1(\mathbf{Q}_{n,p},T^*)$

and $H^1_{\infty,s}(\mathbf{Q}_p, T^*) = \varprojlim H^1(\mathbf{Q}_{n,p}, T^*)/H^1_f(\mathbf{Q}_{n,p}, T^*)$, where $H^1_f(\mathbf{Q}_{n,p}, V^*)$ is defined as in the example of §1.6.C.

Proposition 3.2.6. (i) *With the natural horizontal inclusions and surjections, there are vertical isomorphisms making the following diagram commute.*

$$\begin{array}{ccccccc}
\Lambda\{\mathbf{c}_{\mathbf{Q}_n}\} & \hookrightarrow & H^1_\infty(\mathbf{Q}, T^*) & \hookrightarrow & H^1_\infty(\mathbf{Q}_p, T^*) & \twoheadrightarrow & H^1_{\infty,s}(\mathbf{Q}_p, T^*) \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\mathcal{C}_{\infty,\chi} & \hookrightarrow & (\mathcal{E}'_\infty)^\chi & \hookrightarrow & Y^\chi_\infty & \twoheadrightarrow & Y^\chi_\infty/U^\chi_\infty.
\end{array}$$

(ii) *There is a Λ-module isomorphism*

$$Y^\chi_\infty/U^\chi_\infty \;\cong\; \begin{cases} 0 & \text{if } \chi(p) \neq 1, \\ \mathcal{O} & \text{if } \chi(p) = 1. \end{cases}$$

(iii) *There is a Λ-module injection $(\mathcal{E}'_\infty)^\chi/\mathcal{E}^\chi_\infty \hookrightarrow \mathcal{O}$.*

Proof. Just as for (3.4), Lemma 2.4.3 shows that the image of $\mathbf{c}_{\mathbf{Q}_n}$ in $(L^\times_n)^\chi$ under (3.3) is $\xi_{n,\chi}$, so the left-hand vertical isomorphism is clear. As in the example of §1.6.C, the restriction isomorphism (3.3) identifies

$$\mathcal{S}^{\{p\}}(\mathbf{Q}_n, T^*) \;\cong\; (\mathcal{E}'_n)^\chi,$$

and by Corollary B.3.5

$$\varprojlim_n H^1(\mathbf{Q}_n, T^*) \;=\; \varprojlim_n \mathcal{S}^{\{p\}}(\mathbf{Q}_n, T^*),$$

so we obtain the second vertical isomorphism. With H^1_f as defined in the example of §1.6.C, we see as in (1.7) that there are restriction isomorphisms (the top row is the local analogue of (3.3))

$$\begin{array}{ccc}
H^1(\mathbf{Q}_{n,p}, T^*) & \xrightarrow{\sim} & (L^\times_{n,p})^\chi \\
\cup & & \cup \\
H^1_f(\mathbf{Q}_{n,p}, T^*) & \xrightarrow{\sim} & U^\chi_n
\end{array}$$

and the rest of (i) follows. (Note that once we have the vertical isomorphisms, the injectivity of the upper center horizontal map follows from that of the lower center horizontal map; the latter injectivity follows from Leopoldt's conjecture, which is known in this setting.)

Let Δ_p denote the decomposition group of p in Δ. For every $m > n$ there is a commutative diagram in which all maps are isomorphisms

$$\begin{array}{ccccc} L_{m,p}^\times/U_m & \xrightarrow[\sim]{\oplus_{w|p}\mathrm{ord}_w} & \bigoplus_{w|p}\mathbf{Z}w & \xrightarrow{\sim} & \mathbf{Z}[\Delta/\Delta_p] \\ \Big\downarrow{\scriptstyle \mathbf{N}_{L_m/L_n}} & & \Big\downarrow{\scriptstyle w\mapsto w|_{L_n}} & & \| \\ L_{n,p}^\times/U_n & \xrightarrow[\sim]{\oplus_{v|p}\mathrm{ord}_v} & \bigoplus_{v|p}\mathbf{Z}v & \xrightarrow{\sim} & \mathbf{Z}[\Delta/\Delta_p], \end{array}$$

and so $Y_\infty^\chi/U_\infty^\chi \cong \mathbf{Z}_p[\Delta/\Delta_p]^\chi$. Clearly $\mathbf{Z}_p[\Delta/\Delta_p]^\chi = 0$ if χ is nontrivial on Δ_p, i.e., if $\chi(p) \neq 1$, and otherwise $\mathbf{Z}_p[\Delta/\Delta_p]^\chi \cong \mathcal{O}_\chi$. This proves (ii), and (iii) follows from (ii) since $\mathcal{E}_\infty^\chi$ is the kernel of the natural map $(\mathcal{E}'_\infty)^\chi \to Y_\infty^\chi/U_\infty^\chi$. □

Theorem 3.2.7. *If χ is even and nontrivial then*

$$\mathrm{char}(A_\infty^\chi) \ \textit{divides}\ \mathrm{char}(\mathcal{E}_\infty^\chi/\mathcal{C}_{\infty,\chi}).$$

Proof. Hypotheses $\mathrm{Hyp}(\mathbf{Q}_\infty, T^*)$ are satisfied with $\tau = 1$, so we can apply Theorem 2.3.3 and Proposition 2.3.7 to conclude that

$$\mathrm{char}(\mathrm{Hom}_\mathcal{O}(\mathcal{S}(\mathbf{Q}_\infty, W), \mathbf{D}))\ \text{divides}\ \mathrm{ind}_\Lambda(\mathbf{c})\,\mathrm{char}(H^1_{\infty,s}(\mathbf{Q}_p, T^*))$$

with $\mathrm{ind}_\Lambda(\mathbf{c})$ as defined in Definition 2.3.1.

By [**Iw3**] Theorem 25, the Λ-module Y_∞^χ is torsion-free, finitely generated, and rank-one. Since $(\mathcal{E}'_\infty)^\chi$ is a nonzero Λ-submodule of Y_∞^χ, it follows that $(\mathcal{E}'_\infty)^\chi$ is also torsion-free, finitely generated, and rank-one. Combined with the diagram of Proposition 3.2.6(i), it follows easily that $\mathrm{ind}_\Lambda(\mathbf{c}) = \mathrm{char}((\mathcal{E}'_\infty)^\chi/\mathcal{C}_{\infty,\chi})$, and so using Proposition 3.2.6(iii) we see that $\mathrm{ind}_\Lambda(\mathbf{c})$ divides $\mathcal{J}\mathrm{char}(\mathcal{E}_\infty^\chi/\mathcal{C}_{\infty,\chi})$, where $\mathcal{J} = \mathrm{char}(\mathcal{O})$ is the augmentation ideal of Λ. By (3.5) we have $\mathrm{Hom}_\mathcal{O}(\mathcal{S}(\mathbf{Q}_\infty, W), \mathbf{D}) \cong A_\infty^\chi$, and Proposition 3.2.6 shows that $\mathrm{char}(H^1_{\infty,s}(\mathbf{Q}_p, T^*))$ divides $\mathcal{J}$. Therefore

$$\mathrm{char}(A_\infty^\chi)\ \text{divides}\ \mathcal{J}^2\mathrm{char}(\mathcal{E}_\infty^\chi/\mathcal{C}_{\infty,\chi}).$$

Thus to prove the theorem it suffices to show that $\mathrm{char}(A_\infty^\chi)$ is not divisible by $\mathcal{J}$.

We only sketch the proof. A standard elementary Iwasawa theory argument (see for example [**Iw3**] §3.1) shows that $A_\infty^\chi/\mathcal{J}A_\infty^\chi$ is a finitely generated $\mathbf{Z}_p$-module, that

$$\mathcal{J} \mid \mathrm{char}(A_\infty^\chi) \iff A_\infty^\chi/\mathcal{J}A_\infty^\chi \text{ is infinite},$$

and that $A_\infty^\chi/\mathcal{J}A_\infty^\chi = \mathrm{Gal}(M_\infty/L_\infty)$ where M_∞ is an extension of L_∞ which is abelian over L. Since χ is even, we know that L is a real abelian field and that Leopoldt's conjecture holds for L. Therefore class field theory

shows that L has no $\mathbf{Z}_p^2$-extensions, so $\mathrm{Gal}(M_\infty/L)$ has $\mathbf{Z}_p$-rank one and $[M_\infty : L_\infty]$ must be finite. This completes the proof. □

Corollary 3.2.8. *Suppose that χ is even and nontrivial, and that $p > 2$. Then*

$$\mathrm{char}(A_\infty^\chi) = \mathrm{char}(\mathcal{E}_\infty^\chi/\mathcal{C}_{\infty,\chi}).$$

Proof. As was the case for Corollary 3.2.4, this follows from Theorem 3.2.7 and the analytic class number formula. See for example [**MW**] §1.6, or [**Ru3**] p. 414. □

The p-adic L-function. Let $\omega : G_\mathbf{Q} \to (\mathbf{Z}_p^\times)_{\mathrm{tors}}$ denote the Teichmüller character giving the action of $G_\mathbf{Q}$ on $\boldsymbol{\mu}_p$ (if p is odd) or on $\boldsymbol{\mu}_4$ (if $p = 2$). Thus $\omega^{-1}\varepsilon_{\mathrm{cyc}}$ is a character of $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$. Fix an embedding $\mathcal{O} \hookrightarrow \overline{\mathbf{Q}_p} \hookrightarrow \mathbf{C}$ so that we can identify complex and p-adic characters of finite order of $G_\mathbf{Q}$. With this identification, a character ρ of $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ of finite order extends naturally to an $\mathcal{O}$-algebra homomorphism $\rho : \Lambda \to \overline{\mathbf{Q}_p}$.

Let $L(s, \rho)$ denote the Dirichlet L-function attached to a character ρ.

Theorem 3.2.9. *Suppose that χ is even and nontrivial, and that $p > 2$.*

(i) *There is an element $\mathcal{L}_\chi \in \Lambda$ (the p-adic L-function attached to χ) such that for every $k \geq 1$ and every character ρ of finite order of* $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$,

$$(\omega^{-1}\varepsilon_{\mathrm{cyc}})^k\rho(\mathcal{L}_\chi) = (1 - \omega^{-k}\rho\chi(p)p^{k-1})L(1-k, \omega^{-k}\rho\chi).$$

(ii) $\mathrm{char}(U_\infty^\chi/\mathcal{C}_{\infty,\chi}) = \mathcal{L}_\chi\Lambda$.

Proof. See for example [**Iw2**] §6 or [**Wa**] Theorem 7.10 for (i), and [**Iw1**], [**Wa**] Theorem 13.56, [**Lan**] Theorem 7.5.2, or (for the general case) [**Gi**] Théorème 1 for (ii). (See also §D.2 where we carry out the main computation needed to prove (ii).) □

Theorem 3.2.10. *Suppose that χ is even and nontrivial, and that $p > 2$. Let $Z_{\infty,\chi} = \mathrm{Gal}(M_\infty/L_\infty)^\chi$, where M_∞ is the maximal abelian p-extension of L_∞ unramified outside primes above p. Then $Z_{\infty,\chi}$ is a* $\mathrm{Gal}(L/\mathbf{Q})$*-module and a finitely generated Λ-module, and*

$$\mathrm{char}(Z_{\infty,\chi}) = \mathcal{L}_\chi\Lambda$$

where $\mathcal{L}_\chi$ is the p-adic L-function defined in Theorem 3.2.9.

Proof. Class field theory gives an exact sequence (see for example §III.1.7 of [**dS**])

$$0 \longrightarrow \mathcal{E}_\infty^\chi/\mathcal{C}_{\infty,\chi} \longrightarrow U_\infty^\chi/\mathcal{C}_{\infty,\chi} \longrightarrow Z_{\infty,\chi} \longrightarrow A_\infty^\chi \longrightarrow 0.$$

Applying Corollary 3.2.8 and Theorem 3.2.9(ii) proves the corollary. □

Remark 3.2.11. The case $p = 2$ was excluded from Corollary 3.2.8 and Theorems 3.2.9 and 3.2.10 because we did not use the best group of cyclotomic units to define our Euler system and to define $\mathcal{C}_{\infty,\chi}$. By making use of all cyclotomic units one can prove analogues of Corollary 3.2.8 and Theorems 3.2.9 and 3.2.10 for $p = 2$ as well.

3.3. Elliptic Units

Let K be an imaginary quadratic field, K_∞ a $\mathbf{Z}_p$- or $\mathbf{Z}_p^2$-extension of K in which no (finite) prime splits completely[1], $\chi : G_K \to \mathcal{O}^\times$ a character of finite order, and $T = \mathcal{O}_\chi$ as above. Using elliptic units in abelian extensions of K, exactly as with cyclotomic units in §3.2, we can define an Euler system $\mathbf{c}_{\text{ell}}$ for $\mathbf{Z}_p(1)$ over K, from which we get an Euler system for T^*. See [**Ru5**] §1 and §2 for details.

Keep the notation of §3.2, except that if F is an abelian extension of K we now let $\mathcal{C}_{F,\chi}$ denote the elliptic units in $(F^\times)^\chi$. Then Theorems 2.2.2 and 2.3.3, respectively, prove the following two theorems (compare with [**Ru5**] Theorems 3.3 and 4.1), exactly as in §3.2.

Theorem 3.3.1. *Suppose that $p > 2$ and that $\chi(\mathfrak{P}) \neq 1$ for all primes $\mathfrak{P}$ of K above p. Then*

$$|A_L^\chi| \text{ divides } [\mathcal{E}_L^\chi : \mathcal{C}_{L,\chi}].$$

Theorem 3.3.2. *If $\chi(\mathfrak{P}) \neq 1$ for all primes $\mathfrak{P}$ of K above p, then*

$$\operatorname{char}(\varprojlim A_F^\chi) \text{ divides } \operatorname{char}(\varprojlim (\mathcal{E}_F^\chi / \mathcal{C}_{F,\chi})),$$

where the inverse limits are over finite extensions F of L in LK_∞.

Remarks 3.3.3. As with cyclotomic units, one can use the analytic class number formula to turn the divisibility of Theorem 3.3.1 into an equality.

One can remove the hypothesis that $\chi(\mathfrak{P}) \neq 1$ from Theorem 3.3.2 by modifying the definition of an Euler system. See §9.1.

3.4. Stickelberger Elements

The Euler system we present in this section is not the same as the Euler system of Gauss sums introduced by Kolyvagin in [**Ko2**] (see also [**Ru4**]), but it has the same applications to ideal class groups. We use Stickelberger's theorem in the construction of our Euler system, so Gauss sums are implicitly being used.

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[1] In fact, this splitting condition is unnecessary; see §9.2.

Definition 3.4.1. For every integer $m \geq 2$, define the Stickelberger element

$$\theta_m = \sum_{a \in (\mathbf{Z}/m\mathbf{Z})^\times} \left(\frac{\langle a \rangle}{m} - \frac{1}{2} \right) \gamma_a^{-1} \in \mathbf{Q}[\mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_m)/\mathbf{Q})]$$

where $\gamma_a \in \mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_m)/\mathbf{Q})$ is the automorphism which sends every m-th root of unity to its a-th power, and $\langle a \rangle \in \mathbf{Z}$ is such that $0 \leq \langle a \rangle < m$ and $\langle a \rangle \equiv a \pmod{m}$. Also define $\theta_1 = 0$. It is well-known (and easy to check; see for example [**Wa**] Lemma 6.9 or [**Lan**] §2.8) that

$$\text{if } b \in \mathbf{Z} \text{ is prime to } 2m, \text{ then } (b - \gamma_b)\theta_m \in \mathbf{Z}[\mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_m)/\mathbf{Q})] \tag{3.7}$$

and if ℓ is prime, then

$$\theta_{m\ell}|_{\mathbf{Q}(\boldsymbol{\mu}_m)} = \begin{cases} (1 - \mathrm{Fr}_\ell^{-1})\theta_m & \text{if } \ell \nmid m, \\ \theta_m & \text{if } \ell \mid m. \end{cases} \tag{3.8}$$

An Euler system for $\mathbf{Z}_p$. Again we take $K = \mathbf{Q}$. For every finite extension F of $\mathbf{Q}$, class field theory shows that

$$\begin{aligned} H^1(F, \mathbf{Z}_p) = \mathrm{Hom}(G_F, \mathbf{Z}_p) &= \mathrm{Hom}(\mathbf{A}_F^\times / F^\times, \mathbf{Z}_p) \\ &= \mathrm{Hom}(\mathbf{A}_F^\times / (F^\times B_F), \mathbf{Z}_p) \end{aligned} \tag{3.9}$$

where $\mathbf{A}_F^\times$ denotes the group of ideles of F and

$$B_F = \prod_{w|\infty} F_w^\times \times \prod_{w|p} \{1\} \times \prod_{w \nmid p\infty} \mathcal{O}_{F,w}^\times \subset \mathbf{A}_F^\times,$$

since every (continuous) homomorphism from $\mathbf{A}_F^\times$ into $\mathbf{Z}_p$ must vanish on B_F. Further, the map which sends an idele to the corresponding ideal class induces an exact sequence

$$0 \longrightarrow U_F / \bar{\mathcal{E}}_F \longrightarrow \mathbf{A}_F^\times / (F^\times B_F) \longrightarrow A_F \longrightarrow 0 \tag{3.10}$$

where U_F denotes the local units of $F \otimes \mathbf{Q}_p$ (i.e., $U_F = \oplus_{v|p} \mathcal{O}_{F,v}^\times$) and $\bar{\mathcal{E}}_F$ denotes the closure of the global units of F in U_F, and A_F is the ideal class group of F. We will write $\mathbf{Z}_p[\boldsymbol{\mu}_m]^\times = U_{\mathbf{Q}(\boldsymbol{\mu}_m)}$.

Definition 3.4.2. Fix an integer b prime to $2p$ (a precise choice will be made later), and for every $m \in \mathbf{Z}^+$ prime to b we use the Stickelberger elements θ_m to define

$$\bar{\theta}_m^{(b)} = \begin{cases} (b - \gamma_b)\theta_m & \text{if } p \mid m \\ (b - \gamma_b)(1 - \mathrm{Fr}_p^{-1})\theta_m & \text{if } p \nmid m \end{cases} \in \mathbf{Z}[\mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_m)/\mathbf{Q})]$$

(the two separate cases are to ensure, using (3.8), that $\bar{\theta}_{mp}^{(b)}|_{\mathbf{Q}(\boldsymbol{\mu}_m)} = \bar{\theta}_m^{(b)}$ for every m). Stickelberger's theorem (see for example [**Wa**] Theorem 6.10

or [**Lan**] Theorem 1.2.3) shows that $\bar{\theta}_m^{(b)} A_{\mathbf{Q}(\boldsymbol{\mu}_m)} = 0$. Thus, using (3.10) we can view multiplication by $\bar{\theta}_m^{(b)}$ as a map

$$\mathbf{A}_{\mathbf{Q}(\boldsymbol{\mu}_m)}^{\times}/(\mathbf{Q}(\boldsymbol{\mu}_m)^{\times} B_{\mathbf{Q}(\boldsymbol{\mu}_m)}) \longrightarrow \mathbf{Z}_p[\boldsymbol{\mu}_m]^{\times}/\bar{\mathcal{E}}_{\mathbf{Q}(\boldsymbol{\mu}_m)},$$

and we define $\phi_m = \phi_m^{(b)} \in \mathrm{Hom}(\mathbf{A}_{\mathbf{Q}(\boldsymbol{\mu}_m)}^{\times}/(\mathbf{Q}(\boldsymbol{\mu}_m)^{\times} B_{\mathbf{Q}(\boldsymbol{\mu}_m)}), \mathbf{Z}_p)$ to be the composition

$$\begin{aligned}\mathbf{A}_{\mathbf{Q}(\boldsymbol{\mu}_m)}^{\times}/(\mathbf{Q}(\boldsymbol{\mu}_m)^{\times} B_{\mathbf{Q}(\boldsymbol{\mu}_m)}) &\xrightarrow{\bar{\theta}_m^{(b)}} \mathbf{Z}_p[\boldsymbol{\mu}_m]^{\times}/\bar{\mathcal{E}}_{\mathbf{Q}(\boldsymbol{\mu}_m)} \\ &\xrightarrow{1-c} \mathbf{Z}_p[\boldsymbol{\mu}_m]^{\times}/(\mathbf{Z}_p[\boldsymbol{\mu}_m]^{\times})_{\mathrm{tors}} \xrightarrow{\lambda_m} \mathbf{Z}_p\end{aligned}$$

where c denotes complex conjugation in $\mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_m)/\mathbf{Q})$, so $(1-c)\bar{\mathcal{E}}_{\mathbf{Q}(\boldsymbol{\mu}_m)}$ is finite, and λ_m is the map defined in Appendix D, Definition D.1.2. Finally, we define $\tilde{\mathbf{c}}'_m \in H^1(\mathbf{Q}(\boldsymbol{\mu}_m), \mathbf{Z}_p)$ to be the element corresponding to ϕ_m under (3.9).

Proposition 3.4.3. *Suppose m is prime to b and ℓ is a prime not dividing b. Then*

$$\mathrm{Cor}_{\mathbf{Q}(\boldsymbol{\mu}_{m\ell})/\mathbf{Q}(\boldsymbol{\mu}_m)}(\tilde{\mathbf{c}}'_{m\ell}) = \begin{cases} (1 - \mathrm{Fr}_\ell^{-1})\tilde{\mathbf{c}}'_m & \text{if } \ell \nmid mp, \\ \tilde{\mathbf{c}}'_m & \text{if } \ell \mid mp. \end{cases}$$

Proof. It follows from a standard result of class field theory (see for example [**T2**] §11(13)) that, with the identification (3.9), the map $\mathrm{Cor}_{\mathbf{Q}(\boldsymbol{\mu}_{m\ell})/\mathbf{Q}(\boldsymbol{\mu}_m)}$ is induced by the inclusion $\mathbf{A}_{\mathbf{Q}(\boldsymbol{\mu}_m)}^{\times} \hookrightarrow \mathbf{A}_{\mathbf{Q}(\boldsymbol{\mu}_{m\ell})}^{\times}$.

Suppose first that $\ell \nmid mp$. By Lemma D.1.4 the restriction of $\lambda_{m\ell}$ to $\mathbf{Z}_p[\boldsymbol{\mu}_m]^{\times}$ is $\lambda_m \circ (-\mathrm{Fr}_\ell)$, and by (3.8) we have $\bar{\theta}_{m\ell}^{(b)}|_{\mathbf{Q}(\boldsymbol{\mu}_m)} = (1 - \mathrm{Fr}_\ell^{-1})\bar{\theta}_m^{(b)}$. Therefore

$$\phi_{m\ell}|_{\mathbf{A}_{\mathbf{Q}(\boldsymbol{\mu}_m)}^{\times}} = \phi_m \circ (-\mathrm{Fr}_\ell)(1 - \mathrm{Fr}_\ell^{-1}) = \phi_m \circ (1 - \mathrm{Fr}_\ell) = (1 - \mathrm{Fr}_\ell^{-1})\phi_m$$

and hence $\mathrm{Cor}_{\mathbf{Q}(\boldsymbol{\mu}_{m\ell})/\mathbf{Q}(\boldsymbol{\mu}_m)}(\tilde{\mathbf{c}}'_{m\ell}) = (1-\mathrm{Fr}_\ell^{-1})\tilde{\mathbf{c}}'_m$. Similarly (but more simply), if ℓ divides mp then Lemma D.1.4 and (3.8) show that $\phi_{m\ell}|_{\mathbf{A}_{\mathbf{Q}(\boldsymbol{\mu}_m)}^{\times}} = \phi_m$ and then $\mathrm{Cor}_{\mathbf{Q}(\boldsymbol{\mu}_{m\ell})/\mathbf{Q}(\boldsymbol{\mu}_m)}(\tilde{\mathbf{c}}'_{m\ell}) = \tilde{\mathbf{c}}'_m$. □

Remark 3.4.4. Technically we should write $\tilde{\mathbf{c}}'_{m\infty}$ instead of $\tilde{\mathbf{c}}'_m$, since the ray class field of $\mathbf{Q}$ modulo m is the real subfield $\mathbf{Q}(\boldsymbol{\mu}_m)^+$. But

$$\mathrm{Cor}_{\mathbf{Q}(\boldsymbol{\mu}_m)/\mathbf{Q}(\boldsymbol{\mu}_m)^+}(\tilde{\mathbf{c}}'_m) = 0 \in H^1(\mathbf{Q}(\boldsymbol{\mu}_m)^+, \mathbf{Z}_p)$$

(because we annihilated all even components in our definition), so we will never need to deal with those classes and there should be no confusion.

For every prime $\ell \neq p$ we have

$$\det(1 - \mathrm{Fr}_\ell^{-1} x | \mathbf{Z}_p(1)) = 1 - \ell^{-1} x.$$

But Proposition 3.4.3 shows that the collection $\{\tilde{\mathbf{c}}'_m \in H^1(\mathbf{Q}(\boldsymbol{\mu}_m), \mathbf{Z}_p)\}$ satisfies a distribution relation with Frobenius polynomials $1 - \mathrm{Fr}_\ell^{-1}$, not $1 - \ell^{-1}\mathrm{Fr}_\ell^{-1}$, so this collection is *not* an Euler system for the trivial representation $\mathbf{Z}_p$. However, since

$$1 - \ell^{-1}x \equiv 1 - x \pmod{(\ell - 1)\mathbf{Z}_p[x]}$$

we can modify the classes $\tilde{\mathbf{c}}'_m$ (see Lemma 9.6.1 and Example 9.6.2) to produce a new collection

$$\{\tilde{\mathbf{c}}_m \in H^1(\mathbf{Q}(\boldsymbol{\mu}_m), \mathbf{Z}_p) : m \in \mathbf{Z}^+ \text{ and } (m, b) = 1\}$$

which *is* an Euler system for $(\mathbf{Z}_p, \mathbf{Q}^{\mathrm{ab},b}, bp)$, where $\mathbf{Q}^{\mathrm{ab},b}$ denotes the maximal abelian extension of $\mathbf{Q}$ unramified outside b. Further (Lemma 9.6.1(ii)), we have $\tilde{\mathbf{c}}_{p^n} = \tilde{\mathbf{c}}'_{p^n}$ for every n.

Note that this Euler system depends on the choice of b.

The setting. As in §3.2 let $K = \mathbf{Q}$, let $T = \mathcal{O}_\chi$ for a character χ of G_K of finite prime-to-p order, and keep the rest of the notation of the beginning of §3.2 as well.

For the rest of this section we assume that χ is odd, and we let b be a nonzero integer prime to $2p$ and to the conductor f of χ. (A precise choice of b will be made later.) Note that these hypotheses imply that $p > 2$, because there are no odd characters of odd order.

Let $\Delta = \mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_f)/\mathbf{Q})$. Since χ is nontrivial and of order prime to p, we have $H^i(\Delta, \mathcal{O}_\chi) = 0$ for every $i \geq 0$. Therefore the restriction map gives an isomorphism (compare with (3.9))

$$\begin{aligned} H^1(\mathbf{Q}_n, T) &= H^1(\mathbf{Q}_n(\boldsymbol{\mu}_f), \mathcal{O}_\chi)^\Delta \\ &\cong \mathrm{Hom}(\mathbf{A}^\times_{\mathbf{Q}_n(\boldsymbol{\mu}_f)}/\mathbf{Q}_n(\boldsymbol{\mu}_f)^\times, \mathcal{O}_\chi)^\Delta \subset \mathrm{Hom}(\mathbf{A}^\times_{\mathbf{Q}_n(\boldsymbol{\mu}_f)}, \mathcal{O}), \end{aligned} \tag{3.11}$$

the inclusion using our fixed generator of $\mathcal{O}_\chi$. The Euler system $\tilde{\mathbf{c}}$ for $\mathbf{Z}_p$ constructed above gives rise (by Proposition 2.4.2) to an Euler system $\mathbf{c} = \tilde{\mathbf{c}}^\chi$ for $(T, \mathbf{Q}^{\mathrm{ab},b}, bfp)$. By Lemmas 2.4.3 and 9.6.1(iii), the image under (3.11) of $\mathbf{c}_\mathbf{Q}$ in $\mathrm{Hom}(\mathbf{A}^\times_{\mathbf{Q}(\boldsymbol{\mu}_f)}, \mathcal{O})$ is

$$\sum_{\delta \in \Delta} \chi(\delta)\delta\tilde{\mathbf{c}}_f = \sum_{\delta \in \Delta} \chi(\delta)\delta\tilde{\mathbf{c}}'_f = \sum_{\delta \in \Delta} \chi(\delta)\phi_f^\delta. \tag{3.12}$$

The Selmer group. We have $W^* = \mathbf{D}_{\chi^{-1}\varepsilon_{\mathrm{cyc}}}$. As in §3.2, let L be the fixed field of the kernel of χ, let $L_n = L\mathbf{Q}_n$, let $\mathbf{Q}_{n,p}$ be the completion of $\mathbf{Q}_n$ above p, let A_n be the ideal class group of L_n, and let $A_L = A_0$, the ideal class group of L.

We take $H^1_f(\mathbf{Q}_{n,p}, V)$ and $H^1_f(\mathbf{Q}_{n,p}, V^*)$ to be as defined in the examples of §1.6.B and §1.6.C, respectively.

Proposition 3.4.5. (i) $\mathcal{S}(\mathbf{Q}, W^*) \cong A_L^\chi$,
(ii) $\mathcal{S}(\mathbf{Q}_\infty, W^*) \cong \varinjlim_n A_n^\chi$.

Proof. Let $\mathcal{E}_n$ denote the group of global units of L_n. Since χ is odd, $\mathcal{E}_n^\chi$ is finite, so $(\mathcal{E}_n \otimes \mathbf{Q}_p/\mathbf{Z}_p)^\chi = 0$. Now the proposition follows from Proposition 1.6.4(ii). □

The minus part of the ideal class group of L. The following theorem (or more precisely, its Corollary 3.4.7) was first proved by Mazur and Wiles in [**MW**]. A proof using Euler systems, but somewhat different from the one here, was given by Kolyvagin in [**Ko2**], see also [**Ru4**].

Define the generalized Bernouilli number

$$\mathbf{B}_{1,\chi^{-1}} = \frac{1}{f}\sum_{a=1}^{f} \chi^{-1}(a)a = \chi(\theta_f).$$

Recall that $\omega : G_\mathbf{Q} \to (\mathbf{Z}_p^\times)_{\text{tors}}$ is the Teichmüller character giving the action of $G_\mathbf{Q}$ on $\boldsymbol{\mu}_p$ (recall also that $p \neq 2$, since we have assumed that χ is an odd character of order prime to p).

Theorem 3.4.6. *Suppose that χ is an odd character of order prime to p, that $\chi(p) \neq 1$, and that $\chi^{-1}\omega(p) \neq 1$. Then*

$$|A_L^\chi| \leq |\mathcal{O}/\mathbf{B}_{1,\chi^{-1}}\mathcal{O}|.$$

Proof. Since $\chi \neq \omega$, we can choose b prime to $2pf$ so that $b - \chi(b) \in \mathcal{O}^\times$. Let $\mathbf{c}$ be the Euler system for T constructed above from Stickelberger elements, with this choice of b.

Since T has rank one over $\mathcal{O}$, $\text{Hyp}(\mathbf{Q}, T)$ is satisfied with $\tau = 1$, so we can apply Theorem 2.2.10 with this Euler system.

As in the proof of Theorem 3.2.3, since χ is odd and different from ω, Lemma 3.1.1 shows that $\mathfrak{n}_W = \mathfrak{n}_W^* = 0$ in Theorem 2.2.2.

Using the definition of H_f^1 in §1.6.B and local class field theory, we have identifications (the top row is the local analogue of (3.11))

$$\begin{array}{ccccc}
H^1(\mathbf{Q}_p, T) & \xrightarrow{\sim} & \text{Hom}(\oplus_{w|p} G_{\mathbf{Q}(\boldsymbol{\mu}_f)_w}, \mathcal{O}_\chi)^\Delta & \xrightarrow{\sim} & \text{Hom}(\mathbf{Q}_p(\boldsymbol{\mu}_f)^\times, \mathcal{O}_\chi)^\Delta \\
\downarrow & & \downarrow & & \downarrow \\
H_s^1(\mathbf{Q}_p, T) & \xrightarrow{\sim} & \text{Hom}(\oplus_{w|p} \mathcal{I}_w, \mathcal{O}_\chi)^\Delta & \xrightarrow{\sim} & \text{Hom}(\mathbf{Z}_p[\boldsymbol{\mu}_f]^\times, \mathcal{O}_\chi)^\Delta
\end{array}$$

where $\mathbf{Q}_p(\boldsymbol{\mu}_f) = \mathbf{Q}(\boldsymbol{\mu}_f) \otimes \mathbf{Q}_p$ and $\mathcal{I}_w$ is the inertia group in G_{L_w}. Thus

$$H_s^1(\mathbf{Q}_p, T) \cong \text{Hom}(\mathbf{Z}_p[\boldsymbol{\mu}_f]^\times, \mathcal{O}_\chi)^\Delta \cong \text{Hom}(\mathbf{Z}_p[\boldsymbol{\mu}_f]^\times, \mathcal{O})^{\chi^{-1}}. \quad (3.13)$$

With this identification, using (3.12) and Definition 3.4.2 of $\bar{\theta}_f^{(b)}$, and writing c for complex conjugation,

$$\begin{aligned}\mathrm{loc}^s_{\{p\},T}(\mathbf{c}_\mathbf{Q}) &= \sum_{\delta\in\Delta}\chi(\delta)(\lambda_f\circ(1-c)\bar{\theta}_f^{(b)})^\delta \\ &= \sum_{\delta\in\Delta}\chi(\delta)(\lambda_f\circ\delta^{-1}(1-c)\bar{\theta}_f^{(b)}) \\ &= \lambda_f\circ\sum_{\delta\in\Delta}(\chi(\delta)\delta^{-1})(1-c)\bar{\theta}_f^{(b)} \\ &= 2(b-\chi(b))(1-\chi^{-1}(p))\mathbf{B}_{1,\chi^{-1}}\sum_{\delta\in\Delta}\chi(\delta)\lambda_f^\delta.\end{aligned}$$

Since $\chi^{-1}\omega(p)\neq 1$, Lemma D.1.5 shows that $\sum_{\delta\in\Delta}\chi(\delta)\lambda_f^\delta$ generates the (free rank-one) $\mathcal{O}$-module $\mathrm{Hom}(\mathbf{Z}_p[\boldsymbol{\mu}_f]^\times,\mathcal{O})^{\chi^{-1}}$. We chose b so that $b-\chi(b)\in\mathcal{O}^\times$, and we assumed that $\chi(p)\neq 1$ and χ has order prime to p, so $1-\chi(p)\in\mathcal{O}^\times$. Thus (3.13) shows that

$$\mathcal{O}\mathrm{loc}^s_{\{p\},T}(\mathbf{c}_\mathbf{Q}) = \mathbf{B}_{1,\chi^{-1}}H^1_s(\mathbf{Q}_p,T).$$

Now Theorem 2.2.10 yields

$$|\mathcal{S}(\mathbf{Q},W^*)| \le [H^1_s(\mathbf{Q}_p,T):\mathbf{B}_{1,\chi^{-1}}H^1_s(\mathbf{Q}_p,T)] = |\mathcal{O}/\mathbf{B}_{1,\chi^{-1}}\mathcal{O}|. \qquad \square$$

Corollary 3.4.7 (Mazur & Wiles [**MW**] Theorem 1.10.2)**.** *With hypotheses as in Theorem* 3.4.6*,*

$$|A_L^\chi| = |\mathcal{O}/\mathbf{B}_{1,\chi^{-1}}\mathcal{O}|.$$

Proof. As in Corollary 3.2.4, this follows from Theorem 3.4.6 and the analytic class number formula. See for example [**Ru4**] Theorem 4.3. $\square$

Remarks 3.4.8. If $\chi=\omega$, then $A_L^\chi=0$ and $\mathbf{B}_{1,\chi^{-1}}\mathcal{O}=p^{-1}\mathcal{O}$.

If $\chi(p)=1$, or $\chi^{-1}\omega(p)=1$ but $\chi\neq\omega$, the equality of Corollary 3.4.7 can be deduced from Theorem 3.4.13 below (Iwasawa's "main conjecture"). See [**MW**], §1.10 Theorem 2. See also §9.1.

The p-adic L-function. There is a natural map

$$\chi_\Lambda : \mathcal{O}[[\mathrm{Gal}(\mathbf{Q}_\infty(\boldsymbol{\mu}_f)/\mathbf{Q})]] = \mathcal{O}[\Delta][[\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]] \xrightarrow{\chi} \Lambda$$

given by χ on Δ and the identity on $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$. Let

$$\langle\varepsilon\rangle = \omega^{-1}\varepsilon_{\mathrm{cyc}} : \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \longrightarrow 1+p\mathbf{Z}_p,$$

let $\mathrm{Tw}_{\langle\varepsilon\rangle}:\Lambda\to\Lambda$ be the twisting map induced by

$$\gamma \mapsto \langle\varepsilon\rangle(\gamma)\gamma$$

for $\gamma\in\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$, and let $\eta\mapsto\eta^\bullet$ denote the involution of Λ induced by $\gamma\mapsto\gamma^{-1}$ for $\gamma\in\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$.

Write $\boldsymbol{\theta}_{fp^\infty} = \{\theta_{fp^{n+1}}\}_n$. If b is prime to $2fp$ then by (3.7) and (3.8),

$$(b - \gamma_b)\boldsymbol{\theta}_{fp^\infty} \in \mathbf{Z}_p[[\mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_{fp^\infty})/\mathbf{Q})]],$$

and so by restriction we have $\chi_\Lambda((b - \gamma_b)\boldsymbol{\theta}_{fp^\infty}) \in \Lambda$. If $\chi \neq \omega$ then we can fix b so that $b - \chi(b) \in \mathcal{O}^\times$, and then $\chi_\Lambda(b - \gamma_b) \in \Lambda^\times$. We will write

$$\chi_\Lambda(\boldsymbol{\theta}_{fp^\infty}) = \chi_\Lambda(b - \gamma_b)^{-1}\chi_\Lambda((b - \gamma_b)\boldsymbol{\theta}_{fp^\infty}) \in \Lambda$$

which is independent of b.

Theorem 3.4.9. *If χ is odd and $\chi \neq \omega$, then*

$$\chi_\Lambda(\boldsymbol{\theta}_{fp^\infty})^\bullet = \mathrm{Tw}_{\langle\varepsilon\rangle}(\mathcal{L}_{\chi^{-1}\omega})$$

where $\mathcal{L}_{\chi^{-1}\omega}$ is the p-adic L-function defined in Theorem 3.2.9 for the even character $\chi^{-1}\omega$.

Proof. This was proved by Iwasawa; see [**Iw2**] §6 or [**Wa**] Theorem 7.10. If ρ is a character of finite order of $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$, it follows from the definitions that

$$\begin{aligned}\rho(\chi_\Lambda(\boldsymbol{\theta}_{fp^\infty})^\bullet) &= \rho^{-1}(\chi_\Lambda(\boldsymbol{\theta}_{fp^\infty})) = (1 - \chi^{-1}\rho(p))\mathbf{B}_{1,\chi^{-1}\rho} \\ &= (1 - \chi^{-1}\rho(p))L(0, \chi^{-1}\rho) = \langle\varepsilon\rangle\rho(\mathcal{L}_{\chi^{-1}\omega}) = \rho(\mathrm{Tw}_{\langle\varepsilon\rangle}(\mathcal{L}_{\chi^{-1}\omega})).\end{aligned}$$

Since this is true for every ρ, the equality of the theorem holds. □

Direct limit of the ideal class groups. The main result of this section, Theorem 3.4.13 below, is equivalent to Theorem 3.2.10 by standard methods of Iwasawa theory (see for example [**Ru3**] §8), so we will only sketch the proof.

Let $\mathcal{U}$ denote the direct limit (not the inverse limit) of the local units of $\mathbf{Q}_n(\boldsymbol{\mu}_f) \otimes \mathbf{Q}_p$. Recall that $\Delta = \mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_f)/\mathbf{Q}) \cong \mathrm{Gal}(\mathbf{Q}_n(\boldsymbol{\mu}_f)/\mathbf{Q}_n)$.

Lemma 3.4.10. *There is an isomorphism of Λ-modules*

$$\mathrm{Hom}(\mathcal{U}, \mathcal{O}_\chi)^\Delta \cong \begin{cases} \Lambda & \text{if } \chi(p) \neq 1, \\ \Lambda \oplus \mathcal{O} & \text{if } \chi(p) = 1. \end{cases}$$

Sketch of proof. Let Y_∞ denote the inverse limit of the p-adic completions of the multiplicative groups $(\mathbf{Q}(\boldsymbol{\mu}_{fp^n}) \otimes \mathbf{Q}_p)^\times$. There is a natural Kummer pairing

$$\mathcal{U} \times Y_\infty \longrightarrow \mathbf{Z}_p(1)$$

which leads to a Λ-module isomorphism

$$(Y_\infty \otimes \mathcal{O}_{\chi\omega^{-1}})^{\mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_{fp})/\mathbf{Q})} \cong \mathrm{Hom}(\mathcal{U}, \mathcal{O}_\chi)^\Delta \otimes \mathcal{O}_{\langle\varepsilon\rangle}.$$

The lemma then follows from a result of Iwasawa ([**Iw3**] Theorem 25; see also [**Gi**] Proposition 1). □

Corollary 3.4.11. *Suppose that χ is odd and $\chi \neq \omega$. Then we can choose b so that, if $\mathbf{c}$ is the Euler system defined above, then*

$$\begin{aligned}\operatorname{char}(H^1_{\infty,s}(\mathbf{Q}_p,T)/\Lambda\mathrm{loc}^s_{\{p\}}(\{\mathbf{c}_{\mathbf{Q}_n}\}_n)) \\ = \begin{cases}\mathrm{Tw}_{\langle\varepsilon\rangle}(\mathcal{L}_{\chi^{-1}\omega}) & \text{if } \chi^{-1}\omega(p)\neq 1, \chi(p)\neq 1,\\ \mathrm{Tw}_{\langle\varepsilon\rangle}(\mathcal{J}\mathcal{L}_{\chi^{-1}\omega}) & \text{if } \chi^{-1}\omega(p)=1,\\ \mathcal{J}\,\mathrm{Tw}_{\langle\varepsilon\rangle}(\mathcal{L}_{\chi^{-1}\omega}) & \text{if } \chi(p)=1,\end{cases}\end{aligned}$$

where $\mathcal{J}$ is the augmentation ideal of Λ.

Sketch of proof. For every n, exactly as in (3.13) we have

$$H^1_s(\mathbf{Q}_{n,p},T) \;=\; \mathrm{Hom}(U_n,\mathcal{O}_\chi)^\Delta$$

and so $H^1_{\infty,s}(\mathbf{Q}_p,T) = \mathrm{Hom}(\mathcal{U},\mathcal{O}_\chi)^\Delta$. Let

$$\lambda_{fp^\infty,\chi} \;=\; \lim_{n\to\infty}\sum_{\delta\in\Delta}\chi(\delta)\lambda^\delta_{fp^n} \in \mathrm{Hom}(\mathcal{U},\mathcal{O}_\chi)^\Delta.$$

One computes, using Lemma 3.4.10, that there are Λ-module isomorphisms

$$\mathrm{Hom}(\mathcal{U},\mathcal{O}_\chi)^\Delta/\Lambda\lambda_{fp^\infty,\chi} \;\cong\; \begin{cases}0 & \text{if } \chi^{-1}\omega(p)\neq 1,\chi(p)\neq 1,\\ \mathcal{O}_{\langle\varepsilon\rangle} & \text{if } \chi^{-1}\omega(p)=1,\\ \mathcal{O} & \text{if } \chi(p)=1.\end{cases}$$

(The first case follows from Lemma D.1.5; the others require more work.) Also, by the definition of $\mathbf{c}_{\mathbf{Q}_n}$ and Lemma 2.4.3 we have

$$\mathrm{loc}^s_{\{p\}}(\mathbf{c}_{\mathbf{Q}_n}) \;=\; \sum_{\delta\in\Delta}\chi(\delta)\lambda_{fp^{n+1}}\circ\delta^{-1}(1-c)(b-\gamma_b)\theta_{fp^{n+1}}.$$

Thus

$$\begin{aligned}\Lambda\mathrm{loc}^s_{\{p\}}(\{\mathbf{c}_{\mathbf{Q}_n}\}_n) &= \Lambda\lambda_{fp^\infty,\chi}\circ 2(\chi_\Lambda(b-\gamma_b)\chi_\Lambda(\boldsymbol{\theta}_{fp^\infty}))\\ &= \chi_\Lambda(b-\gamma_b)^\bullet\chi_\Lambda(\boldsymbol{\theta}_{fp^\infty})^\bullet\Lambda\lambda_{fp^\infty,\chi}\\ &= \chi_\Lambda(\boldsymbol{\theta}_{fp^\infty})^\bullet\Lambda\lambda_{fp^\infty,\chi}\end{aligned}$$

since b was chosen so that $\chi_\Lambda(b-\gamma_b)\in\Lambda^\times$. The corollary now follows from Theorem 3.4.9. □

Theorem 3.4.12. *If χ is an odd character of order prime to p and $\chi\neq\omega$, then*

$$\mathrm{char}(\mathrm{Hom}_{\mathcal{O}}(\varinjlim A^\chi_n,\mathbf{D})) \;\; \textit{divides} \;\; \mathrm{Tw}_{\langle\varepsilon\rangle}(\mathcal{L}_{\chi^{-1}\omega}).$$

Sketch of proof. Let $\mathbf{c}$ be the Euler system constructed above, with b chosen to satisfy Corollary 3.4.11. Since T has rank one over $\mathcal{O}$, we see that $\mathrm{Hyp}(\mathbf{Q},T)$ is satisfied with $\tau=1$. Thus we can apply Theorem 2.3.8(ii),

and we conclude (using Proposition 3.4.5(ii) to identify the Selmer group with the direct limit of the ideal class groups) that

$$\mathrm{char}(\mathrm{Hom}_{\mathcal{O}}(\varinjlim A_n^{\chi}, \mathbf{D})) \text{ divides } \mathrm{char}(H^1_{\infty,s}(\mathbf{Q}_p, T)/\Lambda\mathrm{loc}^s_{\{p\}}(\{\mathbf{c}_{\mathbf{Q}_n}\}_n)).$$

If $\chi(p) \neq 1$ and $\chi^{-1}\omega(p) \neq 1$, the theorem now follows immediately from Corollary 3.4.11.

The two exceptional cases remain. First suppose that $\chi^{-1}\omega(p) = 1$. In this case we conclude from Corollary 3.4.11 that $\mathrm{char}(\mathrm{Hom}_{\mathcal{O}}(\varinjlim A_n^{\chi}, \mathbf{D}))$ divides $\mathrm{Tw}_{\langle\varepsilon\rangle}(\mathcal{J}\mathcal{L}_{\chi^{-1}\omega})$, so to complete the proof it will suffice to show that $\mathrm{Tw}_{\langle\varepsilon\rangle}(\mathcal{J})$ cannot divide $\mathrm{char}(\mathrm{Hom}_{\mathcal{O}}(\varinjlim A_n^{\chi}, \mathbf{D}))$.

Briefly, if $\mathrm{Tw}_{\langle\varepsilon\rangle}(\mathcal{J})$ divides $\mathrm{char}(\mathrm{Hom}_{\mathcal{O}}(\varinjlim A_n^{\chi}, \mathbf{D}))$ then class field theory and Kummer theory show (see for example [**Lan**] Chapter 6 or [**Wa**] §13.5) that there is a divisible subgroup of $\mathbf{Q}(\boldsymbol{\mu}_{fp})^{\times} \otimes (\mathbf{Q}_p/\mathbf{Z}_p)$ which generates an unramified extension of $\mathbf{Q}(\boldsymbol{\mu}_{fp^{\infty}})$. But this would contradict Leopoldt's conjecture, which holds for $\mathbf{Q}(\boldsymbol{\mu}_{fp})$.

Now suppose $\chi(p) = 1$. In this case, if χ_0 denotes the trivial character then the definition (Theorem 3.2.9) of $\mathcal{L}_{\chi^{-1}\omega}$ shows that

$$\chi_0(\mathrm{Tw}_{\langle\varepsilon\rangle}(\mathcal{L}_{\chi^{-1}\omega})) = \omega^{-1}\varepsilon_{\mathrm{cyc}}(\mathcal{L}_{\chi^{-1}\omega}) = (1-\chi(p))L(0,\chi) = 0.$$

In other words, $\mathcal{J}$ divides $\mathrm{Tw}_{\langle\varepsilon\rangle}(\mathcal{L}_{\chi^{-1}\omega})$ so we cannot hope to show in this case that $\mathrm{char}(\mathrm{Hom}_{\mathcal{O}}(\varinjlim A_n^{\chi}, \mathbf{D}))$ is not divisible by $\mathcal{J}$. Instead, one must "improve" the Euler system $\mathbf{c}$ to remove this extra zero. We omit the details. □

Theorem 3.4.13 (Mazur & Wiles [**MW**]). *If χ is an odd character of order prime to p and $\chi \neq \omega$ then*

$$\mathrm{char}(\mathrm{Hom}_{\mathcal{O}}(\varinjlim A_n^{\chi}, \mathbf{D})) = \mathrm{Tw}_{\langle\varepsilon\rangle}(\mathcal{L}_{\chi^{-1}\omega}).$$

Proof. This follows from Theorem 3.4.12 by the usual analytic class number argument. See [**MW**] §1.6, where this equality is deduced from divisibilities opposite to those of Theorem 3.4.12. □

3.5. Elliptic Curves

The "Heegner point Euler system" for elliptic curves, used by Kolyvagin in [**Ko2**], does not fit precisely into the framework we have established. We will discuss in §9.4 how to adapt Definition 2.1.1 to include the system of Heegner points. However, Kato ([**Ka3**], [**Scho**]) has constructed an Euler system for the Tate module of a modular[2] elliptic curve, using Beilinson elements in the K-theory of modular curves. In this section we describe applications of Kato's Euler system.

[2] See the footnote on page 3.

The setting. Suppose E is an elliptic curve defined over $\mathbf{Q}$, and take $K = \mathbf{Q}$, $K_\infty = \mathbf{Q}_\infty$, $\mathcal{O} = \mathbf{Z}_p$, and $T = T_p(E)$, the p-adic Tate module of E, as in Example 1.1.5. Then $V = V_p(E) = T_p(E) \otimes \mathbf{Q}_p$ and $W = E_{p^\infty}$. The Weil pairing gives isomorphisms $V \cong V^*$, $T \cong T^*$, and $W \cong W^*$. As in the previous sections, $\mathbf{Q}_n$ is the subfield of $\mathbf{Q}_\infty$ with $[\mathbf{Q}_n : \mathbf{Q}] = p^n$, and $\mathbf{Q}_{n,p}$ is the completion of $\mathbf{Q}_n$ at the unique prime above p.

The p-adic cohomology groups. As in §1.6, for every n we let

$$H^1_f(\mathbf{Q}_{n,p}, V) \;=\; \text{image}\big(E(\mathbf{Q}_{n,p}) \otimes \mathbf{Q}_p \;\hookrightarrow\; H^1(\mathbf{Q}_{n,p}, V)\big).$$

Since $V = V^*$, this also fixes a choice of $H^1_f(\mathbf{Q}_{n,p}, V^*) \subset H^1(\mathbf{Q}_{n,p}, V^*)$, and then $H^1_f(\mathbf{Q}_{n,p}, V)$ and $H^1_f(\mathbf{Q}_{n,p}, V^*)$ are orthogonal complements under the local pairing of §1.4.

For every n let $\text{Tan}(E_{/\mathbf{Q}_{n,p}})$ denote the tangent space of $E_{/\mathbf{Q}_{n,p}}$ at the origin and consider the Lie group exponential map

$$\exp_E \;:\; \text{Tan}(E_{/\mathbf{Q}_{n,p}}) \xrightarrow{\sim} E(\mathbf{Q}_{n,p}) \otimes \mathbf{Q}_p.$$

Fix a minimal Weierstrass model of E and let ω_E denote the corresponding holomorphic differential. Then the cotangent space $\text{Cotan}(E_{/\mathbf{Q}_{n,p}})$ is $\mathbf{Q}_{n,p}\omega_E$, and we let ω_E^* be the corresponding dual basis of $\text{Tan}(E)$. Let $E_1(\mathbf{Q}_{n,p})$ be the kernel of reduction in $E(\mathbf{Q}_p)$, let $\mathfrak{p}_n$ be the maximal ideal of $\mathbf{Q}_{n,p}$, and let $\hat{E}$ be the formal group of E. We have a commutative diagram in which all the maps are isomorphisms

$$\begin{array}{ccccc}
\text{Tan}(E_{/\mathbf{Q}_{n,p}}) & & \xrightarrow{\qquad\exp_E\qquad} & & E(\mathbf{Q}_{n,p}) \otimes \mathbf{Q}_p \\
\uparrow{\scriptstyle \cdot\omega_E^*} & & & & \uparrow \\
\mathbf{Q}_{n,p} & \xleftarrow{\lambda_E} & \hat{E}(\mathfrak{p}_n) \otimes \mathbf{Q}_p & \xrightarrow{\sim} & E_1(\mathbf{Q}_{n,p}) \otimes \mathbf{Q}_p
\end{array}$$

where λ_E is the formal group logarithm, and the bottom right isomorphism is defined in [**T3**] Theorem 4.2. Using these identifications we will also view λ_E as a homomorphism from $E(\mathbf{Q}_{n,p})$ to $\mathbf{Q}_{n,p}$.

Since $V \cong V^*$, the local Tate pairing gives the second isomorphism in

$$\text{Hom}(E(\mathbf{Q}_{n,p}), \mathbf{Q}_p) \;\cong\; \text{Hom}(H^1_f(\mathbf{Q}_{n,p}, V), \mathbf{Q}_p) \;\cong\; H^1_s(\mathbf{Q}_{n,p}, V).$$

Thus there is a dual exponential map (see [**Ka1**] §II.1.2)

$$\exp_E^* \;:\; H^1_s(\mathbf{Q}_{n,p}, V) \;\xrightarrow{\sim}\; \text{Cotan}(E_{/\mathbf{Q}_{n,p}}) \;=\; \mathbf{Q}_{n,p}\omega_E.$$

Write $\exp^*_{\omega_E} : H^1_s(\mathbf{Q}_{n,p}, V) \xrightarrow{\sim} \mathbf{Q}_{n,p}$ for the composition $\omega_E^* \circ \exp_E^*$. Since $H^1_s(\mathbf{Q}_{n,p}, T)$ injects into $H^1_s(\mathbf{Q}_{n,p}, V)$, we see that $\exp^*_{\omega_E}$ is injective on

$H^1_s(\mathbf{Q}_{n,p}, T)$. The local pairing allows us to identify

$$\begin{array}{ccc} H^1_s(\mathbf{Q}_{n,p}, V) & \xrightarrow{\sim} & \mathrm{Hom}(E(\mathbf{Q}_{n,p}), \mathbf{Q}_p) \\ \uparrow & & \uparrow \\ H^1_s(\mathbf{Q}_{n,p}, T) & \xrightarrow{\sim} & \mathrm{Hom}(E(\mathbf{Q}_{n,p}), \mathbf{Z}_p). \end{array} \tag{3.14}$$

Explicitly (see [**Ka1**] Theorem II.1.4.1(iv)), $z \in H^1_s(\mathbf{Q}_{n,p}, V)$ is identified with the map

$$x \mapsto \mathrm{Tr}_{\mathbf{Q}_{n,p}/\mathbf{Q}_p} \lambda_E(x) \exp^*_{\omega_E}(z). \tag{3.15}$$

Proposition 3.5.1. *With notation as above, if $p > 2$ then*

$$\exp^*_{\omega_E}(H^1_s(\mathbf{Q}_p, T)) \;=\; [E(\mathbf{Q}_p) : E_1(\mathbf{Q}_p) + E(\mathbf{Q}_p)_{\mathrm{tors}}] p^{-1}\mathbf{Z}_p.$$

Proof. The diagram (3.14) shows that an element of $H^1_s(\mathbf{Q}_p, V)$ belongs to $H^1_s(\mathbf{Q}_p, T)$ if and only if its image in $\mathrm{Hom}(E(\mathbf{Q}_p), \mathbf{Q}_p)$ takes $E(\mathbf{Q}_p)$ into $\mathbf{Z}_p$. Thus by (3.15), we have

$$\exp^*_{\omega_E}(H^1_s(\mathbf{Q}_p, T)) \;=\; p^a \mathbf{Z}_p$$

where $\lambda_E(E(\mathbf{Q}_p)) \;=\; p^{-a}\mathbf{Z}_p$. If $p > 2$ then $\lambda_E(E_1(\mathbf{Q}_p)) = p\mathbf{Z}_p$ and, since $\mathrm{rank}_{\mathbf{Z}_p} E(\mathbf{Q}_p) = 1$, we see that

$$[\lambda_E(E(\mathbf{Q}_p)) : \lambda_E(E_1(\mathbf{Q}_p)] \;=\; [E(\mathbf{Q}_p) : E_1(\mathbf{Q}_p) + E(\mathbf{Q}_p)_{\mathrm{tors}}]. \qquad \square$$

The L-functions.

Definition 3.5.2. Let

$$L(E, s) \;=\; \sum_{n=1}^{\infty} a_n n^{-s} \;=\; \prod_q \ell_q(q^{-s})^{-1}$$

denote the Hasse-Weil L-function of E, where $\ell_q(q^{-s})$ is the usual Euler factor at q. If $m \in \mathbf{Z}^+$ we will also write

$$L_m(E, s) \;=\; \sum_{(n,m)=1} a_n n^{-s} \;=\; \prod_{q \nmid m} \ell_q(q^{-s})^{-1} \;=\; \Big(\prod_{q \mid m} \ell_q(q^{-s})\Big) L(E, s)$$

for the L-function with the Euler factors dividing m removed. If χ is a character of $G_{\mathbf{Q}}$ of conductor f_χ, let

$$L_m(E, \chi, s) \;=\; \sum_{(n, f_\chi m)=1} \chi(n) a_n n^{-s} \;=\; \prod_{q \nmid f_\chi m} \ell_q(q^{-s}\chi(q))^{-1}.$$

When $m = 1$ we write simply $L(E, \chi, s)$, and then we have

$$L_m(E, \chi, s) \;=\; \Big(\prod_{q \mid m} \ell_q(q^{-s}\chi(q))\Big) L(E, \chi, s). \tag{3.16}$$

If E is modular then these functions all have analytic continuations to $\mathbf{C}$.

The Euler system. Kato has constructed an Euler system in this setting. Let N denote the conductor of E, and let Ω_E be the fundamental real period of E (which corresponds to our choice of differential ω_E).

Theorem 3.5.3 (Kato [**Ka3**]; see also [**Scho**]). *Suppose that E is modular. Then there is a positive integer r_E, independent of p, and an Euler system $\mathbf{c}$ for $T_p(E)$ such that for every $n \geq 0$ and every character χ of* $\mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})$,

$$\sum_{\gamma \in \mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})} \chi(\gamma) \exp^*_{\omega_E}(\mathrm{loc}^s_{\{p\}}(\mathbf{c}^\gamma_{\mathbf{Q}_n})) \;=\; r_E L_{Np}(E,\chi,1)/\Omega_E.$$

In particular we have

$$\exp^*_{\omega_E}(\mathrm{loc}^s_{\{p\}}(\mathbf{c}_{\mathbf{Q}})) \;=\; r_E L_{Np}(E,1)/\Omega_E.$$

See [**Scho**], especially §5, for the construction of the Euler system and the proof of the identities in the case where E has good reduction at p. (See also [**Ru9**] Corollary 7.2 to get from [**Scho**] Theorem 5.2.6 to the statement above.)

Consequences of Kato's Euler system. Following Kato, we will apply the results of Chapter 2 to bound the Selmer group of E. Let $Ш(E)$ be the Tate-Shafarevich group of E.

Theorem 3.5.4 (Kato [**Ka3**]). *Suppose E is modular and E does not have complex multiplication.*

(i) *If $L(E,1) \neq 0$ then $E(\mathbf{Q})$ and $Ш(E)$ are finite.*
(ii) *Suppose L is a finite abelian extension of $\mathbf{Q}$ and χ is a character of* $\mathrm{Gal}(L/\mathbf{Q})$. *If $L(E,\chi,1) \neq 0$ then $E(L)^\chi$ and $Ш(E_{/L})^\chi$ are finite.*

Remarks 3.5.5. We will prove a more precise version of Theorem 3.5.4(i) in Theorem 3.5.11 below. Kato actually constructs an Euler system for $(T_p(E), \mathbf{Q}^{\mathrm{ab},DD'}, NpDD')$ for appropriate auxiliary integers D, D', where $\mathbf{Q}^{\mathrm{ab},DD'}$ is the maximal abelian extension of $\mathbf{Q}$ unramified outside DD'. Thus (for some choice of D and D', depending on χ) Proposition 2.4.2 gives an Euler system for $T_p(E) \otimes \chi$ for every character χ of $G_{\mathbf{Q}}$ of finite order, with properties analogous to those of Theorem 3.5.3. These twisted Euler systems are needed to prove Theorem 3.5.4(ii). For simplicity we will not treat this more general setting here, so we will only prove Theorem 3.5.4(i) below. But the method for (ii) is the same.

Theorem 3.5.4(i) was first proved by Kolyvagin in [**Ko2**], using a system of Heegner points, along with work of Gross and Zagier [**GZ**], Bump, Friedberg, and Hoffstein [**BFH**], and Murty and Murty [**MM**]. The Euler

system proof given here, due to Kato, is self-contained in the sense that it replaces [**GZ**], [**BFH**], and [**MM**] by the calculation of Theorem 3.5.3.

Corollary 3.5.6. *Suppose E is modular and E does not have complex multiplication. Then $E(\mathbf{Q}_\infty)$ is finitely generated.*

Proof. A theorem of Rohrlich [**Ro**] shows that $L(E,\chi,1) \neq 0$ for almost all characters χ of finite order of $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$. A result of Serre ([**Se4**] Théorème 3) shows that $E(\mathbf{Q}_\infty)_{\mathrm{tors}}$ is finite, and the corollary follows without difficulty from Theorem 3.5.4(ii). (See for example [**RW**], pp. 242–243.) □

Remark 3.5.7. When E has complex multiplication, the representation $T_p(E)$ does not satisfy hypothesis $\mathrm{Hyp}(\mathbf{Q},V)$(i) (see Remark 3.5.10 below), so we cannot apply the results of §2.2 and §2.3 with Kato's Euler system. However, Theorem 3.5.4 and Corollary 3.5.6 are known in that case, as Theorem 3.5.4 for CM curves can be proved using the elliptic unit Euler system of §3.3. See [**CW**], [**Ru5**] §11, and [**RW**]. See also the final example of §6.5.

Verification of the hypotheses. Fix a $\mathbf{Z}_p$-basis of T and let

$$\rho_{E,p} : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}(T) \xrightarrow{\sim} \mathrm{GL}_2(\mathbf{Z}_p)$$

be the p-adic representation of $G_{\mathbf{Q}}$ attached to E with respect to this basis.

Proposition 3.5.8. (i) *Suppose that E does not have complex multiplication. Then $H^1(\mathbf{Q}(E_{p^\infty})/\mathbf{Q}, E_{p^\infty})$ is finite and $T_p(E)$ satisfies hypotheses* $\mathrm{Hyp}(\mathbf{Q}_\infty, V)$.

(ii) *Suppose that $\rho_{E,p}$ is surjective. Then $H^1(\mathbf{Q}(E_{p^\infty})/\mathbf{Q}, E_{p^\infty}) = 0$ and $T_p(E)$ satisfies hypotheses* $\mathrm{Hyp}(\mathbf{Q}_\infty, T)$.

Proof. The Weil pairing shows that

$$G_{\mathbf{Q}(\boldsymbol{\mu}_{p^\infty})} = \rho_{E,p}^{-1}(\mathrm{SL}_2(\mathbf{Z}_p)).$$

If E does not have complex multiplication then Serre's theorem ([**Se4**] Théorème 3) says that the image of $\rho_{E,p}$ is open in $\mathrm{GL}_2(\mathbf{Z}_p)$. It follows that $V_p(E)$ is an irreducible $G_{\mathbf{Q}_\infty}$-representation, and if $\rho_{E,p}$ is surjective then E_p is an irreducible $\mathbf{F}_p[G_{\mathbf{Q}_\infty}]$-representation.

It also follows that we can find $\tau \in G_{\mathbf{Q}(\boldsymbol{\mu}_{p^\infty})}$ such that

$$\rho_{E,p}(\tau) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

with $x \neq 0$, and such a τ satisfies hypothesis $\mathrm{Hyp}(\mathbf{Q}_\infty, V)$(i). If $\rho_{E,p}$ is surjective we can take $x = 1$, and then τ satisfies hypothesis $\mathrm{Hyp}(\mathbf{Q}_\infty, T)$(i).

We have

$$H^1(\mathbf{Q}(E_{p^\infty})/\mathbf{Q}, E_{p^\infty}) = H^1(\rho_{E,p}(G_{\mathbf{Q}}), (\mathbf{Q}_p/\mathbf{Z}_p)^2)$$

which is zero if $\rho_{E,p}(G_{\mathbf{Q}}) = \mathrm{GL}_2(\mathbf{Z}_p)$, and finite if $\rho_{E,p}(G_{\mathbf{Q}})$ is open in $\mathrm{GL}_2(\mathbf{Z}_p)$ (see also Corollary C.2.2). This completes the proof of the proposition. □

Remark 3.5.9. Serre's theorem (see [**Se4**] Corollaire 1 of Théorème 3) also shows that if E does not have complex multiplication then $\rho_{E,p}$ is surjective for all but finitely many p.

Remark 3.5.10. The conditions on τ in hypothesis $\mathrm{Hyp}(\mathbf{Q}, V)$(i) force $\rho_{E,p}(\tau)$ to be nontrivial and unipotent. Thus if E has complex multiplication then there is no τ satisfying $\mathrm{Hyp}(\mathbf{Q}, V)$(i).

Bounding $\mathcal{S}(\mathbf{Q}, E_{p^\infty})$. Recall that N is the conductor of E.

Theorem 3.5.11. *Suppose E is modular, E does not have complex multiplication, and $L(E,1) \neq 0$.*

(i) *$E(\mathbf{Q})$ and $Ш(E)_{p^\infty}$ are finite.*

(ii) *Suppose in addition that E has good reduction at p, that p does not divide $2r_E|\tilde{E}(\mathbf{F}_p)|$ (where $\tilde{E}$ is the reduction of E modulo p and r_E is as in Theorem* 3.5.3*), and that $\rho_{E,p}$ is surjective. Then*

$$|Ш(E)_{p^\infty}| \text{ divides } \frac{L_N(E,1)}{\Omega_E}.$$

Proof. Recall that $\ell_q(q^{-s})$ is the Euler factor of $L(E,s)$ at q, and that by Proposition 1.6.8, $\mathcal{S}(\mathbf{Q}, E_{p^\infty})$ is the usual p-power Selmer group of E.

Since $L(E,1) \neq 0$, and $\ell_q(q^{-1})$ is easily seen to be nonzero for every q, Theorem 3.5.3 shows that $\mathrm{loc}^s_{\{p\}}(\mathbf{c}_{\mathbf{Q}}) \neq 0$. By Proposition 3.5.8(i) and (3.14) we can apply Theorem 2.2.10(i) to conclude that $\mathcal{S}(\mathbf{Q}, E_{p^\infty})$ is finite. This proves (i), and it follows (see for example Proposition 1.6.8) that $\mathcal{S}(\mathbf{Q}, E_{p^\infty}) = Ш(E)_{p^\infty}$.

If E has good reduction at p then $p\ell_p(p^{-1}) = |\tilde{E}(\mathbf{F}_p)|$ and

$$[E(\mathbf{Q}_p) : E_1(\mathbf{Q}_p) + E(\mathbf{Q}_p)_{\mathrm{tors}}] \text{ divides } |\tilde{E}(\mathbf{F}_p)|.$$

Therefore if $p \nmid 2r_E|\tilde{E}(\mathbf{F}_p)|$ then

$$\begin{array}{ccc} \exp^*_{\omega_E}(H^1_s(\mathbf{Q}_p, T_p(E))) & = & p^{-1}\mathbf{Z}_p \\ \cup & & \cup \\ \exp^*_{\omega_E}(\mathbf{Z}_p \mathrm{loc}^s_{\{p\}}(\mathbf{c}_{\mathbf{Q}})) & = & p^{-1}(L_N(E,1)/\Omega_E)\mathbf{Z}_p \end{array}$$

by Proposition 3.5.1 and Theorem 3.5.3. By Proposition 3.5.8(ii), if further $\rho_{E,p}$ is surjective then we can apply Theorem 2.2.10(ii) (with $\mathfrak{n}_W = 0$ and $\mathfrak{n}^*_W = 0$) and (ii) follows. □

Remarks 3.5.12. In Corollary 3.5.19 below, we will prove using Iwasawa theory that Theorem 3.5.11(ii) holds for almost all p, even when p divides $|\tilde{E}(\mathbf{F}_p)|$. This is needed to prove Theorem 3.5.4(i), since $|\tilde{E}(\mathbf{F}_p)|$ could be divisible by p for infinitely many p. However, since $|\tilde{E}(\mathbf{F}_p)| < 2p$ for all primes $p > 5$, we see that if $E(\mathbf{Q})_{\text{tors}} \neq 0$ then $|\tilde{E}(\mathbf{F}_p)|$ is prime to p for almost all p. Thus for curves E with nontrivial rational torsion points, Theorem 3.5.4(i) follows directly from Theorem 3.5.11.

The Euler system techniques we are using give an upper bound for the order of the Selmer group, but no lower bound. In this case there is no analogue of the analytic class number formula that enabled us to go from the Euler system divisibility to equality in Corollaries 3.2.4 and 3.4.7.

The p-adic L-function and the Coleman map. Suppose for this section that E has either good ordinary reduction or multiplicative reduction at p. Let $\alpha \in \mathbf{Z}_p^\times$ and $\beta = p/\alpha \in p\mathbf{Z}_p$ be the eigenvalues of Frobenius over $\mathbf{F}_p$ if E has good ordinary reduction at p, and let $(\alpha, \beta) = (1, p)$ (resp. $(-1, -p)$) if E has split (resp. nonsplit) multiplicative reduction.

Write $G_n = \text{Gal}(\mathbf{Q}_n/\mathbf{Q}) = \text{Gal}(\mathbf{Q}_{n,p}/\mathbf{Q}_p)$, and fix a generator $\{\zeta_{p^n}\}_n$ of $\varprojlim \boldsymbol{\mu}_{p^n}$. If χ is a character of $\text{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ of conductor p^n define the Gauss sum

$$\tau(\chi) = \sum_{\gamma \in \text{Gal}(\mathbf{Q}(\boldsymbol{\mu}_{p^n})/\mathbf{Q})} \chi(\gamma)\zeta_{p^n}^\gamma.$$

Fix also an embedding of $\overline{\mathbf{Q}_p}$ into $\mathbf{C}$ so that we can identify complex and p-adic characters of $G_\mathbf{Q}$.

The following theorem is proved in [**MSD**] in the case of good ordinary reduction. See [**MTT**] for the (even more) general statement.

Theorem 3.5.13. *Suppose E is modular and E has either good ordinary reduction or multiplicative reduction at p. Let α be as above. Then there are a nonzero integer c_E independent of p, and a p-adic L-function $\mathcal{L}_E \in c_E^{-1}\Lambda$, such that for every character χ of $\text{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ of finite order,*

$$\chi(\mathcal{L}_E) = \begin{cases} (1-\alpha^{-1})^2 L(E,1)/\Omega_E & \text{if } \chi = 1 \text{ and } E \text{ has good reduction at } p \\ (1-\alpha^{-1}) L(E,1)/\Omega_E & \text{if } \chi = 1 \text{ and } E \text{ is multiplicative at } p \\ \alpha^{-n}\tau(\chi) L(E,\chi^{-1},1)/\Omega_E & \text{if } \chi \text{ has conductor } p^n > 1. \end{cases}$$

If $m \in \mathbf{Z}^+$, define

$$\mathcal{L}_{E,m} = \Big(\prod_{q|m, q\neq p} \ell_q(q^{-1}\text{Fr}_q^{-1}) \Big) \mathcal{L}_E \in c_E^{-1}\Lambda.$$

Using (3.16) and Theorem 3.5.13 one obtains expressions for $\chi(\mathcal{L}_{E,m})$ in terms of $L_m(E, \chi^{-1}, 1)$ similar to those in Theorem 3.5.13.

Proposition 3.5.14. *Suppose that E has either good ordinary reduction or multiplicative reduction at p. Then there is a Λ-module map*

$$\mathrm{Col}_\infty : H^1_{\infty,s}(\mathbf{Q}_p, T) \hookrightarrow \Lambda$$

such that for every $z = \{z_n\} \in H^1_{\infty,s}(\mathbf{Q}_p, T)$ and every nontrivial character χ of G_n, we have

$$\chi(\mathrm{Col}_\infty(z)) = \alpha^{-k}\tau(\chi) \sum_{\gamma \in G_n} \chi^{-1}(\gamma) \exp^*_{\omega_E}(z_n^\gamma)$$

where p^k is the conductor of χ. If χ_0 is the trivial character then

$$\chi_0(\mathrm{Col}_\infty(z)) = (1-\alpha^{-1})(1-\beta^{-1})^{-1} \exp^*_{\omega_E}(z_0).$$

Further, if E has split multiplicative reduction at p then the image of Col_∞ is contained in the augmentation ideal of Λ.

Proof. The proof is based on work of Coleman [**Co**]. See the appendix of [**Ru9**] for an explicit construction of Col_∞ in this case, and see §8.1 for a discussion of a generalization due to Perrin-Riou [**PR2**]. □

Using the Coleman map Col_∞ described above, we can relate Kato's Euler system to the p-adic L-function.

Corollary 3.5.15. *With hypotheses as in Theorem* 3.5.13*, with r_E as in Theorem* 3.5.3*, and with other notation as above, we have*

$$\mathrm{Col}_\infty(\mathrm{loc}^s_{\{p\}}(\{\mathbf{c}_{\mathbf{Q}_n}\})) = r_E \mathcal{L}_{E,N}.$$

Proof. Fix a character χ of $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ of finite order. Theorem 3.5.3 and Proposition 3.5.14 allow us to compute $\chi(\mathrm{Col}_\infty(\mathrm{loc}^s_{\{p\}}(\{\mathbf{c}_{\mathbf{Q}_n}\})))$, the definition (Theorem 3.5.13) of $\mathcal{L}_E$ and (3.16) allow us to compute $\chi(r_E\mathcal{L}_{E,N})$, and these values turn out to be equal. For example, if χ is nontrivial then both are equal to

$$r_E \alpha^{-n} \tau(\chi) L_{Np}(E, \chi^{-1}, 1)/\Omega_E.$$

When $\chi = 1$, we need to use the fact that $\ell_p(p^{-1})$ is $(1-\alpha^{-1})(1-\beta^{-1})$ (resp. $(1-\beta^{-1})$) if E has good (resp. multiplicative) reduction at p. Since $\chi(\mathrm{Col}_\infty(\mathrm{loc}^s_{\{p\}}(\{\mathbf{c}_{\mathbf{Q}_n}\}))) = \chi(r_E\mathcal{L}_{E,N})$ for all χ, the corollary follows. □

Bounding $\mathcal{S}(\mathbf{Q}_\infty, E_{p^\infty})$. Recall that N is the conductor of E, and let

$$Z_\infty = \mathrm{Hom}(\mathcal{S}(\mathbf{Q}_\infty, E_{p^\infty}), \mathbf{Q}_p/\mathbf{Z}_p).$$

Theorem 3.5.16. *Suppose E is modular, E does not have complex multiplication, and E has either good ordinary reduction or nonsplit multiplicative reduction at p. Then Z_∞ is a finitely generated torsion Λ-module and there is an integer t such that*

$$\mathrm{char}(Z_\infty) \text{ divides } p^t \mathcal{L}_{E,N}\Lambda.$$

If $\rho_{E,p}$ is surjective and $p \nmid r_E \prod_{q|N, q\neq p} \ell_q(q^{-1})$, then $\mathrm{char}(Z_\infty)$ divides $\mathcal{L}_E\Lambda$.

If E has split multiplicative reduction at p, then the same results hold with $\mathrm{char}(Z_\infty)$ replaced by $\mathcal{J}\mathrm{char}(Z_\infty)$, where $\mathcal{J}$ is the augmentation ideal of Λ.

Proof. Rohrlich [**Ro**] proved that $\mathcal{L}_E \neq 0$. Thus the theorem follows by combining Propositions 3.5.8 and 3.5.14, Corollary 3.5.15, and Theorem 2.3.8. □

Remark 3.5.17. For a discussion of the "extra zero" (the extra factor of $\mathcal{J}$ in Theorem 3.5.16) in the case of split multiplicative reduction, see [**MTT**].

Corollary 3.5.18. *Let E be as in Theorem* 3.5.16. *If p is a prime where E has good ordinary reduction and*

$$p \nmid \prod_{q|N} |E(\mathbf{Q}_q)_{\mathrm{tors}}|,$$

then Z_∞ has no nonzero finite submodules.

Proof. This corollary is due to Greenberg [**Gr2**], [**Gr3**]; we sketch a proof here. Let Σ be the set of places of $\mathbf{Q}$ dividing $Np\infty$, and let $\mathbf{Q}_\Sigma$ be the maximal extension of $\mathbf{Q}$ unramified outside Σ. By Lemma 1.5.3 there is an exact sequence

$$0 \to \mathcal{S}(\mathbf{Q}_\infty, E_{p^\infty}) \to H^1(\mathbf{Q}_\Sigma/\mathbf{Q}_\infty, E_{p^\infty}) \to \oplus_{q\in\Sigma} \oplus_{v|q} H^1_s(\mathbf{Q}_{\infty,v}, E_{p^\infty}). \tag{3.17}$$

It follows from local duality (Theorem 1.4.1 and Proposition 1.4.3) that for every place v of $\mathbf{Q}_\infty$, we have

$$\mathrm{Hom}(H^1_s(\mathbf{Q}_{\infty,v}, E_{p^\infty}), \mathbf{Q}_p/\mathbf{Z}_p) \cong \varprojlim_n E(\mathbf{Q}_{n,v})\hat{}$$

where as usual $(\ )\hat{}$ denotes p-adic completion.

If $v \mid q$ for some $q \neq p$, and $E(\mathbf{Q}_q)$ has no p-torsion, then it is not hard to show that $E(\mathbf{Q}_{\infty,v})$ has no p-torsion and so $E(\mathbf{Q}_{n,v})\hat{} = 0$ for every

n. Thus for p as in the statement of the corollary, the Pontryagin dual of (3.17) is

$$\varprojlim_n E(\mathbf{Q}_{n,p})\hat{} \longrightarrow \mathrm{Hom}(H^1(\mathbf{Q}_\Sigma/\mathbf{Q}_\infty, E_{p^\infty}), \mathbf{Q}_p/\mathbf{Z}_p) \longrightarrow Z_\infty \longrightarrow 0.$$

Since $\mathbf{Q}_\infty/\mathbf{Q}$ is totally ramified at p, we have

$$\varprojlim_n E(\mathbf{Q}_{n,p})\hat{} \;=\; \varprojlim_n E_1(\mathbf{Q}_{n,p}) \;=\; \varprojlim_n \hat{E}(\mathfrak{p}_n)$$

and this is free of rank one over Λ (see for example [**PR1**] Théorème 3.1 or [**Schn**] Lemma 6, §A.1). It now follows, using the fact that Z_∞ is a torsion Λ-module (Theorem 3.5.16) and using Propositions 3, 4, and 5 of [**Gr2**] that $\mathrm{Hom}(H^1(\mathbf{Q}_\Sigma/\mathbf{Q}_\infty, E_{p^\infty}), \mathbf{Q}_p/\mathbf{Z}_p)$ has no nonzero finite submodules. By the Lemma on p. 123 of [**Gr2**] the same is true of Z_∞. □

Corollary 3.5.19. *Suppose that E is modular, that E does not have complex multiplication, that E has good reduction at p, that p does not divide $2r_E \prod_{q|N} \ell_q(q^{-1})|E(\mathbf{Q}_q)_{\mathrm{tors}}|$ (where r_E is as in Theorem 3.5.3), and that $\rho_{E,p}$ is surjective. Then*

$$|Ш(E)_{p^\infty}| \;\; \textit{divides} \;\; \frac{L(E,1)}{\Omega_E}.$$

Proof. First, if E has good supersingular reduction at p then $|\tilde{E}(\mathbf{F}_p)|$ is prime to p, so the corollary follows from Theorem 3.5.11(ii).

Thus we may assume that E has good ordinary reduction at p. In this case the corollary is a well-known consequence of Theorem 3.5.16 and Corollary 3.5.18; see for example [**PR1**] §6 or [**Schn**] §2 for details. The idea is that if Z_∞ has no nonzero finite submodules and $\mathrm{char}(Z_\infty)$ divides $\mathcal{L}_E\Lambda$, then

$$|\mathcal{S}(\mathbf{Q}_\infty, E_{p^\infty})^{\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})}| \;\text{ divides }\; \chi_0(\mathcal{L}_{E,N}),$$

where χ_0 denotes the trivial character, and

$$\chi_0(\mathcal{L}_{E,N}) \;=\; (1-\alpha^{-1})^2 \prod_{q|N} \ell_q(q^{-1})(L(E,1)/\Omega_E).$$

On the other hand, one can show that the restriction map

$$\mathcal{S}(\mathbf{Q}, E_{p^\infty}) \;\longrightarrow\; \mathcal{S}(\mathbf{Q}_\infty, E_{p^\infty})^{\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})}$$

is injective with cokernel of order divisible by $(1-\alpha^{-1})^2$, and the corollary follows. □

Remark 3.5.20. The Birch and Swinnerton-Dyer conjecture predicts that the divisibility of Corollary 3.5.19 holds for almost all, but not all, primes p.

Proof of Theorem 3.5.4(i). Suppose E is modular, E does not have complex multiplication, and $L(E,1) \neq 0$. By Theorem 3.5.11, $E(\mathbf{Q})$ is finite and $Ш(E)_{p^\infty}$ is finite for every p. By Corollary 3.5.19 (and using Serre's theorem, see Remark 3.5.9), $Ш(E)_{p^\infty} = 0$ for almost all p. This proves Theorem 3.5.4(i). □

Remark 3.5.21. We can also now prove part of Theorem 3.5.4(ii), in the special case where L is contained in $\mathbf{Q}_\infty$ and E has either good ordinary or multiplicative reduction at p. For in that case, Theorem 3.5.16 shows that $\chi(\mathrm{char}(\mathrm{Hom}(\mathcal{S}(\mathbf{Q}_\infty, E_{p^\infty}), \mathbf{Q}_p/\mathbf{Z}_p)))$ divides a nonzero multiple of $L(E,\chi,1)/\Omega_E$. If $L(E,\chi,1) \neq 0$ it follows that $\mathcal{S}(\mathbf{Q}_\infty, E_{p^\infty})^\chi$ is finite. The kernel of the restriction map $\mathcal{S}(L, E_{p^\infty}) \to \mathcal{S}(\mathbf{Q}_\infty, E_{p^\infty})$ is contained in the finite group $H^1(\mathbf{Q}_\infty/L, E_{p^\infty}^{G_{\mathbf{Q}_\infty}})$, and so we conclude that both $E(L)^\chi$ and $Ш(E_{/L})^\chi_{p^\infty}$ are finite.

3.6. Symmetric Square of an Elliptic Curve

Let E be an elliptic curve over $\mathbf{Q}$ and let $T_p(E)$ be the p-adic Tate module of E. Let T be the symmetric square of $T_p(E)$, i.e., the three-dimensional $\mathbf{Z}_p$-representation of $G_\mathbf{Q}$ defined by

$$T = \big(T_p(E) \otimes T_p(E)\big)/\{t \otimes t' - t' \otimes t : t, t' \in T_p(E)\}.$$

Suppose τ has eigenvalues α and α^{-1} on $T_p(E)$ with $\alpha^2 \not\equiv 1 \pmod{p}$. Then $\tau \in G_{\mathbf{Q}(\boldsymbol{\mu}_{p^\infty})}$ (as in Proposition 3.5.8), and τ has eigenvalues α^2, 1, and α^{-2} on T, so τ satisfies hypothesis $\mathrm{Hyp}(\mathbf{Q}, T)$(i). If the p-adic representation attached to E is surjective and $p > 3$ then we can always find such a τ, and further T/pT is an irreducible $G_\mathbf{Q}$-module and

$$H^1(\Omega/\mathbf{Q}, W) = H^1(\Omega/\mathbf{Q}, W^*) = 0.$$

Thus *if* we had an Euler system for T, then we could apply Theorem 2.2.10 to study the Selmer group $\mathcal{S}(\mathbf{Q}, W^*)$. See [**Fl**] for important progress in this direction.

CHAPTER 4

Derived Cohomology Classes

The proofs of the main theorems stated in Chapter 2 consist of two steps. First we use an Euler system to construct auxiliary cohomology classes which Kolyvagin calls "derivative" classes, and second we use these derived classes along with the duality theorems of §1.7 to bound Selmer groups.

In this chapter we carry out the first of these steps. In §4.2 and §4.3 we define and study the "universal Euler system" associated to T and K_∞/K. In §4.4 we construct the Kolyvagin derivative classes, and in §4.5 we state the local properties of these derivative classes, which will be crucial in all the applications. The remainder of this chapter is devoted to the proofs of these properties.

4.1. Setup

Keep the notation of §2.1. We have a fixed number field K, a p-adic representation T of G_K with coefficients in the ring of integers $\mathcal{O}$ of some finite extension Φ of $\mathbf{Q}_p$, and we assume that T is unramified outside a finite set of primes of K.

The letter $\mathfrak{q}$ will always denote a prime of K. For every prime $\mathfrak{q}$ of K not dividing p, we let $K(\mathfrak{q})$ denote the maximal p-extension of K inside the ray class field of K modulo $\mathfrak{q}$. Similarly, let $K(\mathbf{1})$ denote the maximal p-extension of K inside the Hilbert class field of K. Class field theory shows that $K(\mathfrak{q})/K(\mathbf{1})$ is unramified outside $\mathfrak{q}$, totally ramified above $\mathfrak{q}$, and cyclic with Galois group canonically isomorphic to the maximal p-quotient of $(\mathcal{O}_K/\mathfrak{q})^\times/(\mathcal{O}_K^\times \pmod{\mathfrak{q}})$. Let $\Gamma_\mathfrak{q} = \mathrm{Gal}(K(\mathfrak{q})/K(\mathbf{1}))$.

Fix an ideal $\mathcal{N}$ of K divisible by p and by all primes where T is ramified, as in Definition 2.1.1. Define

$$\mathcal{R} = \mathcal{R}(\mathcal{N}) = \{\text{squarefree products of primes } \mathfrak{q} \text{ of } K \text{ such that } \mathfrak{q} \nmid \mathcal{N}\}.$$

If $\mathfrak{r} \in \mathcal{R}$, say $\mathfrak{r} = \mathfrak{q}_1 \cdots \mathfrak{q}_k$, then we define $K(\mathfrak{r})$ to be the compositum

$$K(\mathfrak{r}) = K(\mathfrak{q}_1) \cdots K(\mathfrak{q}_k).$$

Note that $K(\mathfrak{r})$ is contained in, but not in general equal to, the maximal p-extension of K inside the ray class field of K modulo $\mathfrak{r}$. We define

$$\Gamma_{\mathfrak{r}} = \mathrm{Gal}(K(\mathfrak{r})/K(\mathbf{1})).$$

Ramification considerations show that the fields $K(\mathfrak{q})$ are linearly disjoint over $K(\mathbf{1})$, so there is a natural isomorphism

$$\Gamma_{\mathfrak{r}} \cong \prod_{\text{primes } \mathfrak{q}|\mathfrak{r}} \Gamma_{\mathfrak{q}}$$

where $\Gamma_{\mathfrak{q}}$ is identified with the inertia group of $\mathfrak{q}$ in $\Gamma_{\mathfrak{r}}$. If $\mathfrak{s} \mid \mathfrak{r}$ this allows us to view $\Gamma_{\mathfrak{s}}$ as a subgroup of $\Gamma_{\mathfrak{r}}$, as well as a quotient.

Fix a $\mathbf{Z}_p^d$-extension K_∞/K in which no finite prime splits completely, as in Definition 2.1.1. If $K \subset F \subset K_\infty$, let $F(\mathfrak{r}) = FK(\mathfrak{r})$. As in Chapter 2, we will write $K \subset_{\mathrm{f}} F$ to indicate that F is a finite extension of K, and if $K \subset_{\mathrm{f}} F \subset K_\infty$ we let

$$\Gamma_{F(\mathfrak{r})} = \mathrm{Gal}(F(\mathfrak{r})/K(\mathbf{1})).$$

Again, we will often identify $\Gamma_{\mathfrak{r}}$ with the subgroup of $\Gamma_{F(\mathfrak{r})}$ generated by the inertia groups of primes dividing $\mathfrak{r}$, and $\Gamma_{F(\mathbf{1})}$ with the subgroup generated by the inertia groups of primes dividing p, and then (since K_∞/K is unramified outside p)

$$\Gamma_{F(\mathfrak{r})} \cong \Gamma_{F(\mathbf{1})} \times \Gamma_{\mathfrak{r}}.$$

As above, if $\mathfrak{s} \mid \mathfrak{r}$ we can also identify $\Gamma_{F(\mathfrak{s})}$ with a subgroup of $\Gamma_{F(\mathfrak{r})}$.

Figure 1 illustrates these fields and Galois groups.

For $\mathfrak{r} \in \mathcal{R}$ define

$$N_{\mathfrak{r}} = \sum_{\sigma \in \Gamma_{\mathfrak{r}}} \sigma \in \mathbf{Z}[\Gamma_{\mathfrak{r}}] \subset \mathbf{Z}[\mathrm{Gal}(K(\mathfrak{r})/K)].$$

If $\mathfrak{s} \mid \mathfrak{r}$ and $K \subset_{\mathrm{f}} F \subset K_\infty$ we can view $N_{\mathfrak{s}} \in \mathbf{Z}[\Gamma_{\mathfrak{r}}] \subset \mathbf{Z}[\mathrm{Gal}(F(\mathfrak{r})/K)]$ as above, and then $N_{\mathfrak{r}} = N_{\mathfrak{s}} N_{\mathfrak{r}/\mathfrak{s}}$.

As in Chapter 2, let $\mathrm{Fr}_{\mathfrak{q}}$ denote a Frobenius of $\mathfrak{q}$ in G_K, and

$$P(\mathrm{Fr}_{\mathfrak{q}}^{-1}|T^*; x) = \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1} x | T^*) \in \mathcal{O}[x].$$

Definition 4.1.1. Suppose $K \subset_{\mathrm{f}} F \subset K_\infty$ and $M \in \mathcal{O}$ is nonzero. We define $\mathcal{R}_{F,M} \subset \mathcal{R}$ to be the set of all $\mathfrak{r} \in \mathcal{R}$ such that for every prime $\mathfrak{q}$ dividing $\mathfrak{r}$,

- $M \mid [K(\mathfrak{q}) : K(\mathbf{1})]$,
- $M \mid P(\mathrm{Fr}_{\mathfrak{q}}^{-1}|T^*; 1)$,
- $\mathfrak{q}$ splits completely in $F(\mathbf{1})/K$.

As in Definition 1.4.6, if $M \in \mathcal{O}$ is nonzero we let $\bar{M} \in \mathbf{Z}^+$ denote the smallest power of p which is divisible in $\mathcal{O}$ by M.

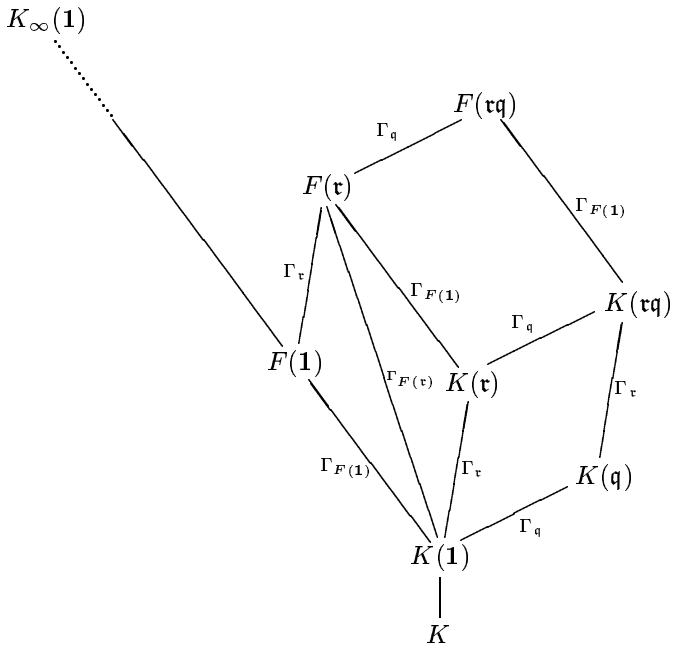


FIGURE 1

Lemma 4.1.2. *Suppose $\mathfrak{q} \in \mathcal{R}$ is a prime of K and $M \in \mathcal{O}$ is nonzero.*

(i) $M \mid [K(\mathfrak{q}) : K(\mathbf{1})] \iff \mathfrak{q}$ *splits completely in* $K(\boldsymbol{\mu}_{\bar{M}}, (\mathcal{O}_K^\times)^{1/\bar{M}})$.

(ii) $P(\mathrm{Fr}_\mathfrak{q}^{-1} | T^*; \mathbf{N}(\mathfrak{q})\mathrm{Fr}_\mathfrak{q}^{-1})$ *annihilates* T.

(iii) *If* $M \mid [K(\mathfrak{q}) : K(\mathbf{1})]$ *then*

$$P(\mathrm{Fr}_\mathfrak{q}^{-1} | T^*; x) \equiv \det(1 - \mathrm{Fr}_\mathfrak{q} x | W_M) \pmod{M}.$$

(iv) *If* $M \mid [K(\mathfrak{q}) : K(\mathbf{1})]$ *then* $P(\mathrm{Fr}_\mathfrak{q}^{-1} | T^*; \mathrm{Fr}_\mathfrak{q}^{-1})$ *annihilates* W_M.

Proof. Class field theory identifies $\mathrm{Gal}(K(\mathfrak{q})/K)$ with the maximal p-quotient of $(\mathcal{O}_K/\mathfrak{q})^\times/(\mathcal{O}_K^\times \pmod{\mathfrak{q}})$. Thus if $\mathfrak{q} \nmid p$, then $[K(\mathfrak{q}) : K(\mathbf{1})]$ divides $|(\mathcal{O}_K/\mathfrak{q})^\times| = (\mathbf{N}(\mathfrak{q}) - 1)$ and

$$\mathrm{Fr}_\mathfrak{q} \text{ fixes } \boldsymbol{\mu}_{\bar{M}} \iff \bar{M} \text{ divides } |(\mathcal{O}_K/\mathfrak{q})^\times| \iff M \text{ divides } |(\mathcal{O}_K/\mathfrak{q})^\times|.$$

If $\mathrm{Fr}_\mathfrak{q}$ fixes $\boldsymbol{\mu}_{\bar{M}}$ we have further

$$\mathrm{Fr}_\mathfrak{q} \text{ fixes } (\mathcal{O}_K^\times)^{1/\bar{M}} \iff (\mathcal{O}_K^\times \pmod{\mathfrak{q}}) \subset ((\mathcal{O}_K/\mathfrak{q})^\times)^{\bar{M}}.$$

This proves (i). One checks easily that

$$P(\mathrm{Fr}_\mathfrak{q}^{-1} | T^*; x) = \det(1 - \mathrm{Fr}_\mathfrak{q}^{-1} x | T^*) = \det(1 - \mathbf{N}(\mathfrak{q})^{-1} \mathrm{Fr}_\mathfrak{q} x | T).$$

This and the Cayley-Hamilton theorem prove (ii), (iii), and (iv). □

The following lemma, together with the Tchebotarev density theorem, will give a large supply of primes in $\mathcal{R}_{F,M}$. By $F(W_M)$ we mean the smallest extension of F whose absolute Galois group acts trivially on W_M (or equivalently, the fixed field of the kernel of the action of G_F on W_M).

Lemma 4.1.3. *Suppose $K \subset_f F \subset K_\infty$ and $M \in \mathcal{O}$ is nonzero. Suppose further that $\tau \in G_K$ acts trivially on $K_\infty(\mathbf{1})K(\boldsymbol{\mu}_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$ and that $T^{\tau=1} \neq 0$. If $\mathfrak{q}$ is a prime of K not dividing $\mathcal{N}$ such that the Frobenius $\mathrm{Fr}_\mathfrak{q}$ of $\mathfrak{q}$ is a conjugate of τ on $F(\mathbf{1})K(W_M, \boldsymbol{\mu}_{\bar{M}}, (\mathcal{O}_K^\times)^{1/\bar{M}})$, then $\mathfrak{q} \in \mathcal{R}_{F,M}$.*

Proof. Note that $\mathfrak{q}$ is unramified in $F(\mathbf{1})K(W_M, \boldsymbol{\mu}_{\bar{M}}, (\mathcal{O}_K^\times)^{1/\bar{M}})/K$. Since $\mathrm{Fr}_\mathfrak{q}$ fixes $K(\boldsymbol{\mu}_{\bar{M}}, (\mathcal{O}_K^\times)^{1/\bar{M}})$, it follows from Lemma 4.1.2(i) that M divides $[K(\mathfrak{q}) : K(\mathbf{1})]$. Since $\mathrm{Fr}_\mathfrak{q}$ fixes $F(\mathbf{1})$, we see that $\mathfrak{q}$ splits completely in $F(\mathbf{1})/K$. By Lemma 4.1.2(iii) we have

$$\begin{aligned} P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;1) \equiv \det(1-\mathrm{Fr}_\mathfrak{q}|W_M) &= \det(1-\tau|W_M) \\ &\equiv \det(1-\tau|T) = 0 \pmod{M}, \end{aligned}$$

the first equality because $\mathrm{Fr}_\mathfrak{q}$ is a conjugate of τ on W_M, the second because $T^{\tau=1} \neq 0$. Thus $\mathfrak{q} \in \mathcal{R}_{F,M}$. □

4.2. The Universal Euler System

We now define the "universal Euler system". This might also be called a universal distribution for the Euler system distribution relation of Definition 2.1.1. In the special case $K = \mathbf{Q}$ and $T = \mathbf{Z}_p(1)$, it is closely related to the universal ordinary distribution studied by Kubert in [**Ku**] (see also [**Lan**] §2.9 or [**Wa**] §12.3).

Definition 4.2.1. Suppose that $\mathfrak{r} \in \mathcal{R}$ and $K \subset_f F \subset K_\infty$. We define an $\mathcal{O}[\mathrm{Gal}(F(\mathfrak{r})/K)]$-module $\mathbf{X}_{F(\mathfrak{r})}$ as follows.

First suppose that every prime $\mathfrak{q}$ dividing $\mathfrak{r}$ satisfies $K(\mathfrak{q}) \neq K(\mathbf{1})$. For every $\mathfrak{s}$ dividing $\mathfrak{r}$ let $x_{F(\mathfrak{s})}$ be an indeterminate. Set $\mathbf{X}_{F(\mathfrak{r})} = Y_{F(\mathfrak{r})}/Z_{F(\mathfrak{r})}$ where:

$Y_{F(\mathfrak{r})}$ is the free $\mathcal{O}[\mathrm{Gal}(F(\mathfrak{r})/K)]$-module on generators $\{x_{F(\mathfrak{s})} : \mathfrak{s} \mid \mathfrak{r}\}$,

$Z_{F(\mathfrak{r})}$ is the submodule of $Y_{F(\mathfrak{r})}$ generated by the relations

$$\begin{aligned} &\sigma x_{F(\mathfrak{s})} = x_{F(\mathfrak{s})} && \text{if } \sigma \in \mathrm{Gal}(F(\mathfrak{r})/F(\mathfrak{s})) = \Gamma_{\mathfrak{r}/\mathfrak{s}}, \\ &N_\mathfrak{q} x_{F(\mathfrak{q}\mathfrak{s})} = P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;\mathrm{Fr}_\mathfrak{q}^{-1})x_{F(\mathfrak{s})} && \text{if } \mathfrak{q}\mathfrak{s} \mid \mathfrak{r}. \end{aligned}$$

For general $\mathfrak{r}$, let $\mathfrak{r}'$ be the product of all primes $\mathfrak{q}$ dividing $\mathfrak{r}$ such that $K(\mathfrak{q}) \neq K(\mathbf{1})$. Then by definition $F(\mathfrak{r}) = F(\mathfrak{r}')$, and further we have $\{F(\mathfrak{s}) : \mathfrak{s} \mid \mathfrak{r}\} = \{F(\mathfrak{s}) : \mathfrak{s} \mid \mathfrak{r}'\}$. We simply let $\mathbf{X}_{F(\mathfrak{r})}$ be $\mathbf{X}_{F(\mathfrak{r}')}$ as defined above.

If $\mathfrak{s} \mid \mathfrak{r}$ and $K \subset_{\mathrm{f}} F \subset_{\mathrm{f}} F' \subset K_\infty$ then there are natural $\mathcal{O}[\mathrm{Gal}(F'(\mathfrak{r})/K)]$-module maps

$$\mathbf{X}_{F'(\mathfrak{r})} \longrightarrow \mathbf{X}_{F(\mathfrak{r})} \text{ induced by } x_{F'(\mathfrak{t})} \mapsto x_{F(\mathfrak{t})} \text{ for } \mathfrak{t} \mid \mathfrak{r}, \tag{4.1}$$

$$\mathbf{X}_{F(\mathfrak{s})} \longrightarrow \mathbf{X}_{F'(\mathfrak{r})} \text{ induced by } x_{F(\mathfrak{t})} \mapsto \mathbf{N}_{F'(\mathfrak{r})/F(\mathfrak{r})} x_{F'(\mathfrak{t})} \text{ for } \mathfrak{t} \mid \mathfrak{s}. \tag{4.2}$$

The map (4.1) is clearly surjective, and Proposition 4.3.1(v) below will show that the map (4.2) is injective.

Definition 4.2.2. The *universal Euler system* (for $(T, \mathcal{N}, K_\infty/K)$) is

$$\mathcal{X} = \mathcal{X}(T, \mathcal{N}, K_\infty/K) = \varinjlim_{F,\mathfrak{r}} \mathbf{X}_{F(\mathfrak{r})}.$$

direct limit with respect to the maps (4.2). Using the maps (4.1), (4.2) we also define

$$\mathbf{X}_{\infty,\mathfrak{r}} = \varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} \mathbf{X}_{F(\mathfrak{r})} \text{ and } \mathbf{X}_{\infty,\mathcal{R}} = \varinjlim_{\mathfrak{r} \in \mathcal{R}} \mathbf{X}_{\infty,\mathfrak{r}}.$$

For every $\mathfrak{r} \in \mathcal{R}$ define

$$H^1_\infty(K(\mathfrak{r}), T) = \varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^1(F(\mathfrak{r}), T).$$

Lemma 4.2.3. *If* $\mathbf{c}$ *is an Euler system for* $(T, \mathcal{K}, \mathcal{N})$ *with* $K_\infty \subset \mathcal{K}$*, then sending* $x_{F(\mathfrak{r})}$ *to* $\mathbf{c}_{F(\mathfrak{r})}$ *induces* G_K*-equivariant maps*

$$\mathbf{X}_{F(\mathfrak{r})} \longrightarrow H^1(F(\mathfrak{r}), T), \qquad \mathbf{X}_{\infty,\mathfrak{r}} \longrightarrow H^1_\infty(K(\mathfrak{r}), T),$$

$$\mathcal{X} \longrightarrow \varinjlim_{F,\mathfrak{r}} H^1(F(\mathfrak{r}), T), \qquad \mathbf{X}_{\infty,\mathcal{R}} \longrightarrow \varinjlim_{\mathfrak{r} \in \mathcal{R}} H^1_\infty(K(\mathfrak{r}), T),$$

direct limits with respect to restriction maps.

Proof. This is immediate, since the Euler system classes $\{\mathbf{c}_{F(\mathfrak{r})}\}$ satisfy all the relations that the $\{x_{F(\mathfrak{r})}\}$ do (see Definition 2.1.1 and Remark 2.1.4). □

Remark 4.2.4. Conversely, although we will not make use of it, it follows from the following lemma that every map

$$\mathbf{X}_{\infty,\mathcal{R}} \longrightarrow \varinjlim_{\mathfrak{r} \in \mathcal{R}} H^1_\infty(K(\mathfrak{r}), T)$$

induces an Euler system for $(T, \mathcal{K}_{\min}, \mathcal{N})$, where $\mathcal{K}_{\min}$ is as in Remark 2.1.4.

Lemma 4.2.5. (i) *If* $K \subset_{\mathrm{f}} F \subset K_\infty$ *and* $\mathfrak{r} \in \mathcal{R}$*, then*

$$T^{G_{F(\mathfrak{r})}} = T^{G_{F(\mathbf{1})}} \text{ and } W^{G_{F(\mathfrak{r})}} = W^{G_{F(\mathbf{1})}}.$$

(ii) *If* $\mathfrak{rs} \in \mathcal{R}$ *then the restriction map induces an isomorphism*

$$H^1_\infty(K(\mathfrak{r}), T) \cong H^1_\infty(K(\mathfrak{rs}), T)^{\Gamma_{\mathfrak{s}}}.$$

Proof. Since $\mathrm{Gal}(F(\mathfrak{r})/F(\mathbf{1})) = \Gamma_{\mathfrak{r}}$ is generated by inertia groups of primes dividing $\mathfrak{r}$, and T is unramified at those primes, $\mathrm{Gal}(F(\mathfrak{r})/F(\mathbf{1}))$ acts trivially on $T^{G_{F(\mathfrak{r})}}$ and $W^{G_{F(\mathfrak{r})}}$. This proves (i).

It is enough to prove (ii) when $\mathfrak{s}$ is prime, and then the general case follows by induction.

Let S be a finite set of places of K containing all places dividing $\mathcal{N}\mathfrak{r}\mathfrak{s}\infty$, and let K_S be the maximal extension of K unramified outside S. (Recall that $\mathcal{N}$ is divisible by p and all primes where T is ramified, so in particular $K_\infty(\mathfrak{r}\mathfrak{s}) \subset K_S$ and T is a $\mathrm{Gal}(K_S/K)$-module). By Propositions B.2.5(ii) and B.2.7(i), we have an inflation-restriction exact sequence

$$\begin{aligned} H^1(F(\mathfrak{r}\mathfrak{s})/F(\mathfrak{r}), T^{G_{F(\mathfrak{r}\mathfrak{s})}}) &\longrightarrow H^1(K_S/F(\mathfrak{r}), T) \\ &\longrightarrow H^1(K_S/F(\mathfrak{r}\mathfrak{s}), T)^{\Gamma_{\mathfrak{s}}} \longrightarrow H^2(F(\mathfrak{r}\mathfrak{s})/F(\mathfrak{r}), T^{G_{F(\mathfrak{r}\mathfrak{s})}}). \quad (4.3) \end{aligned}$$

By (i) (and using our identification $\mathrm{Gal}(F(\mathfrak{r}\mathfrak{s})/F(\mathfrak{r})) \cong \Gamma_{\mathfrak{s}}$),

$$\begin{aligned} H^1(F(\mathfrak{r}\mathfrak{s})/F(\mathfrak{r}), T^{G_{F(\mathfrak{r}\mathfrak{s})}}) &= H^1(F(\mathfrak{r}\mathfrak{s})/F(\mathfrak{r}), T^{G_{F(\mathbf{1})}}) \\ &= \mathrm{Hom}(\Gamma_{\mathfrak{s}}, T^{G_{F(\mathbf{1})}}) = 0 \end{aligned}$$

and similarly (since $\mathfrak{s}$ is prime, so $\Gamma_{\mathfrak{s}}$ is cyclic)

$$H^2(F(\mathfrak{r}\mathfrak{s})/F(\mathfrak{r}), T^{G_{F(\mathfrak{r}\mathfrak{s})}}) = T^{G_{F(\mathbf{1})}}/|\Gamma_{\mathfrak{s}}|T^{G_{F(\mathbf{1})}}.$$

Now pass to the inverse limit over F in (4.3). Using Corollary B.3.6 and our assumption that the decomposition group in $\mathrm{Gal}(K_\infty/K)$ of every finite prime is infinite, we obtain an exact sequence

$$0 \longrightarrow H^1_\infty(K(\mathfrak{r}), T) \longrightarrow H^1_\infty(K(\mathfrak{r}\mathfrak{s}), T)^{\Gamma_{\mathfrak{s}}} \longrightarrow \varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} T^{G_{F(\mathbf{1})}}/|\Gamma_{\mathfrak{s}}|T^{G_{F(\mathbf{1})}}.$$

By Lemma B.3.2, this inverse limit is zero, so this proves (ii). □

4.3. Properties of the Universal Euler System

Recall that Φ is the field of fractions of $\mathcal{O}$.

Proposition 4.3.1. *Suppose $\mathfrak{r} \in \mathcal{R}$ and $K \subset_{\mathrm{f}} F \subset K_\infty$. Then*

(i) $\mathbf{X}_{F(\mathfrak{r})}$ *is a finitely generated free $\mathcal{O}$-module,*
(ii) $\mathbf{X}_{F(\mathfrak{r})} \otimes \Phi$ *is a free rank-one module over $\Phi[\mathrm{Gal}(F(\mathfrak{r})/K)]$,*
(iii) $\mathbf{X}_{F(\mathfrak{r})}$ *is a free $\mathcal{O}[\mathrm{Gal}(F(\mathfrak{r})/K(\mathfrak{r}))]$-module of rank $[K(\mathfrak{r}) : K]$,*
(iv) *for every $F \subset_{\mathrm{f}} F' \subset K_\infty$, the map (4.1) induces an isomorphism $\mathbf{X}_{F'(\mathfrak{r})} \otimes_{\mathcal{O}[\mathrm{Gal}(F'(\mathfrak{r})/K)]} \mathcal{O}[\mathrm{Gal}(F(\mathfrak{r})/K)] \xrightarrow{\sim} \mathbf{X}_{F(\mathfrak{r})}$,*
(v) *for every $\mathfrak{s} \mid \mathfrak{r}$ and $F \subset_{\mathrm{f}} F' \subset K_\infty$, the map (4.2) induces an isomorphism $\mathbf{X}_{F(\mathfrak{s})} \xrightarrow{\sim} \mathbf{X}_{F'(\mathfrak{r})}^{\mathrm{Gal}(F'(\mathfrak{r})/F(\mathfrak{s}))}$.*

Proof. Let $\mathfrak{r}'$ be the product of all primes $\mathfrak{q}$ dividing $\mathfrak{r}$ such that $\Gamma_\mathfrak{q} \neq \{1\}$. Then $\mathbf{X}_{F(\mathfrak{r}')} = \mathbf{X}_{F(\mathfrak{r})}$, $F(\mathfrak{r}') = F(\mathfrak{r})$, and $K(\mathfrak{r}') = K(\mathfrak{r})$, so the proposition for $\mathfrak{r}$ is equivalent to the proposition for $\mathfrak{r}'$. Thus without loss of generality we may replace $\mathfrak{r}$ by $\mathfrak{r}'$, i.e., we may simplify the proof by assuming that $\Gamma_\mathfrak{q} \neq \{1\}$ for every $\mathfrak{q}$ dividing $\mathfrak{r}$.

We will prove the proposition by constructing a specific $\mathcal{O}$-basis of $\mathbf{X}_{F(\mathfrak{r})}$. Fix a set of representatives $A_1 \subset \mathrm{Gal}(F(\mathfrak{r})/K)$ of $\mathrm{Gal}(K(\mathbf{1})/K)$, and for every prime $\mathfrak{q}$ dividing $\mathfrak{r}$ let $A_\mathfrak{q} = \Gamma_\mathfrak{q} - \{1\} \subset \mathrm{Gal}(F(\mathfrak{r})/K)$. For every ideal $\mathfrak{s}$ dividing $\mathfrak{r}$, define a subset $A_{F,\mathfrak{s}} \subset \mathrm{Gal}(F(\mathfrak{r})/K)$ by

$$\begin{aligned} A_{F,\mathfrak{s}} &= \mathrm{Gal}(F(\mathfrak{r})/K(\mathfrak{r}))\, A_1 \prod_{\text{primes } \mathfrak{q}|\mathfrak{s}} A_\mathfrak{q} \\ &= \Big\{ g_F g_1 \prod_{\mathfrak{q}|\mathfrak{s}} g_\mathfrak{q} : g_F \in \mathrm{Gal}(F(\mathfrak{r})/K(\mathfrak{r})), g_1 \in A_1, 1 \neq g_\mathfrak{q} \in \Gamma_\mathfrak{q} \Big\} \end{aligned}$$

and then define a finite subset $B_{F(\mathfrak{r})}$ of $\mathbf{X}_{F(\mathfrak{r})}$ by

$$B_{F(\mathfrak{r})} = \bigcup_{\mathfrak{s}|\mathfrak{r}} A_{F,\mathfrak{s}} x_{F(\mathfrak{s})} \subset \mathbf{X}_{F(\mathfrak{r})}.$$

We will show that $B_{F(\mathfrak{r})}$ is an $\mathcal{O}$-basis of $\mathbf{X}_{F(\mathfrak{r})}$.

Clearly $A_\mathfrak{q} \cup \{N_\mathfrak{q}\}$ generates $\mathcal{O}[\Gamma_\mathfrak{q}]$ over $\mathcal{O}$, so

$$\mathrm{Gal}(F(\mathfrak{r})/K(\mathfrak{r}))\, A_1 \prod_{\mathfrak{q}|\mathfrak{s}} (A_\mathfrak{q} \cup \{N_\mathfrak{q}\})$$

generates $\mathcal{O}[\Gamma_{F(\mathfrak{s})}]$. For every $\mathfrak{q}$ dividing $\mathfrak{s}$ we have a relation

$$N_\mathfrak{q} x_{F(\mathfrak{s})} = P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*; \mathrm{Fr}_\mathfrak{q}^{-1}) x_{F(\mathfrak{s}/\mathfrak{q})} \in \mathcal{O}[\mathrm{Gal}(F(\mathfrak{s}/\mathfrak{q})/K)] x_{F(\mathfrak{s}/\mathfrak{q})}.$$

It follows easily, by induction on the number of primes dividing $\mathfrak{r}$, that $B_{F(\mathfrak{r})}$ generates $\mathbf{X}_{F(\mathfrak{r})}$ over $\mathcal{O}$. Further,

$$\begin{aligned} |B_{F(\mathfrak{r})}| \leq \sum_{\mathfrak{s}|\mathfrak{r}} |A_{F,\mathfrak{s}}| &= [F(\mathbf{1}) : K] \prod_{\mathfrak{q}|\mathfrak{r}} (|A_\mathfrak{q}| + 1) \\ &= [F(\mathbf{1}) : K] \prod_{\mathfrak{q}|\mathfrak{r}} |\Gamma_\mathfrak{q}| = [F(\mathfrak{r}) : K]. \end{aligned}$$

On the other hand, we claim that $\mathrm{rank}_\mathcal{O}(\mathbf{X}_{F(\mathfrak{r})}) \geq [F(\mathfrak{r}) : K]$. To see this, let $Y_{F(\mathfrak{r})}$ and $Z_{F(\mathfrak{r})}$ be as in Definition 4.2.1 of $\mathbf{X}_{F(\mathfrak{r})}$. One can check directly that the assignment

$$x_{F(\mathfrak{s})} \mapsto \prod_{\mathfrak{q}|(\mathfrak{r}/\mathfrak{s})} N_\mathfrak{q} \prod_{\mathfrak{q}|\mathfrak{s}} \Big(|\Gamma_\mathfrak{q}| + (P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*; \mathrm{Fr}_\mathfrak{q}^{-1}) - |\Gamma_\mathfrak{q}|) \frac{N_\mathfrak{q}}{|\Gamma_\mathfrak{q}|} \Big)$$

induces a well-defined homomorphism from $Y_{F(\mathfrak{r})}$ to $\mathcal{O}[\mathrm{Gal}(F(\mathfrak{r})/K)]$ which is zero on $Z_{F(\mathfrak{r})}$. Thus we obtain a map

$$\varphi \,:\, \mathbf{X}_{F(\mathfrak{r})} \otimes \Phi \longrightarrow \Phi[\mathrm{Gal}(F(\mathfrak{r})/K)].$$

If χ is a character of $\mathrm{Gal}(F(\mathfrak{r})/K)$ into an algebraic closure of Φ, say χ has conductor exactly $\mathfrak{s}$, then

$$\chi(\varphi(x_{F(\mathfrak{s})})) \,=\, \prod_{\mathfrak{q}|\mathfrak{r}} |\Gamma_{\mathfrak{q}}| \,\neq\, 0.$$

It follows that φ is surjective, and in particular

$$\mathrm{rank}_{\mathcal{O}}(\mathbf{X}_{F(\mathfrak{r})}) \,=\, \dim_{\Phi}(\mathbf{X}_{F(\mathfrak{r})} \otimes \Phi) \,\geq\, [F(\mathfrak{r}) : K] \,\geq\, |B_{F(\mathfrak{r})}|.$$

Since $B_{F(\mathfrak{r})}$ generates $\mathbf{X}_{F(\mathfrak{r})}$ over $\mathcal{O}$, we conclude that equality holds, that $B_{F(\mathfrak{r})}$ is an $\mathcal{O}$-basis of $\mathbf{X}_{F(\mathfrak{r})}$, that $\mathbf{X}_{F(\mathfrak{r})}$ is torsion-free, and that φ is an isomorphism. This proves (i) and (ii). Further, since $\mathrm{Gal}(F(\mathfrak{r})/K(\mathfrak{r}))$ permutes the elements of the basis $B_{F(\mathfrak{r})}$, (iii) follows as well.

The map (4.1) defined by $x_{F'(\mathfrak{s})} \mapsto x_{F(\mathfrak{s})}$ induces a surjective map

$$\mathbf{X}_{F'(\mathfrak{r})} \otimes_{\mathcal{O}[\mathrm{Gal}(F'(\mathfrak{r})/K)]} \mathcal{O}[\mathrm{Gal}(F(\mathfrak{r})/K)] \twoheadrightarrow \mathbf{X}_{F(\mathfrak{r})}.$$

By (iii) applied to F and F', this map must be injective as well, which proves (iv).

It is immediate from the definitions that the map (4.2) induces a surjection

$$\mathbf{X}_{F(\mathfrak{r})} \twoheadrightarrow \mathbf{X}_{F'(\mathfrak{r})}^{\mathrm{Gal}(F'(\mathfrak{r})/F(\mathfrak{r}))},$$

which then must be injective by (iii). Also we see that $B_{F(\mathfrak{s})} \subset B_{F(\mathfrak{r})}$, so the map $\mathbf{X}_{F(\mathfrak{s})} \to \mathbf{X}_{F(\mathfrak{r})}$ is injective and its cokernel is torsion free. By (ii), $\mathbf{X}_{F(\mathfrak{r})}^{\mathrm{Gal}(F(\mathfrak{r})/F(\mathfrak{s}))}/\mathbf{X}_{F(\mathfrak{s})}$ is finite, so it must be zero. Now (v) follows. □

If G is a profinite abelian group, we write $\mathcal{O}[[G]] = \varprojlim_{U \subset G} \mathcal{O}[G/U]$.

Corollary 4.3.2. *The $\mathcal{O}[[\mathrm{Gal}(K_\infty(\mathfrak{r})/K(\mathfrak{r}))]]$-module $\mathbf{X}_{\infty,\mathfrak{r}}$ is free of rank $[K(\mathfrak{r}) : K]$, and for every $K \subset_{\mathrm{f}} F \subset K_\infty$ we have*

$$\mathbf{X}_{\infty,\mathfrak{r}} \otimes_{\mathcal{O}[[\mathrm{Gal}(K_\infty(\mathfrak{r})/K)]]} \mathcal{O}[\mathrm{Gal}(F(\mathfrak{r})/K)] \,\cong\, \mathbf{X}_{F(\mathfrak{r})}.$$

Proof. This is immediate from Proposition 4.3.1(iii) and (iv). □

Lemma 4.3.3. *Suppose R is a ring, G is a profinite abelian group, and H is an open subgroup of G. Suppose B is an $R[[G]]$-module.*

(i) $\mathrm{Hom}_{R[[G]]}(B, R[[G]]) \cong \mathrm{Hom}_{R[[H]]}(B, R[[H]])$ *as $R[[H]]$-modules.*
(ii) *If B is free as an $R[[H]]$-module then* $\mathrm{Ext}^1_{R[[G]]}(B, R[[G]]) = 0$.

Proof. Write $S = R[[H]]$ and $S' = R[[G]]$. Fix a (finite) set $C \subset G$, containing 1, of coset representatives of G/H. Then C is an S-basis of S', and we let $\pi : S' \to S$ be the S-module map

$$\sum_{\eta \in C} a_\eta \eta \mapsto a_1.$$

Define a homomorphism $\mathrm{Hom}_{S'}(B, S') \to \mathrm{Hom}_S(B, S)$ by composition with π. One can check directly that this map is both injective and surjective, which proves (i).

It follows from (i) that $\mathrm{Ext}^1_{S'}(B, S') = \mathrm{Ext}^1_S(B, S)$, and if B is free over S this is zero. □

Proposition 4.3.4. *Suppose that $\mathfrak{r} \in \mathcal{R}$, that $k \geq 0$, and that $M \in \mathcal{O}$ is nonzero.*

(i) *If $K \subset_{\mathrm{f}} F \subset K_\infty$ and $G = \mathrm{Gal}(F(\mathfrak{r})/K)$, then*

$$\mathrm{Ext}^1_{(\mathcal{O}/M\mathcal{O})[G]}(\mathbf{X}_{F(\mathfrak{r})}/M\mathbf{X}_{F(\mathfrak{r})}, (\mathcal{O}/M\mathcal{O})[G]^k) = 0.$$

(ii) *If $G = \mathrm{Gal}(K_\infty(\mathfrak{r})/K)$, then*

$$\mathrm{Ext}^1_{(\mathcal{O}/M\mathcal{O})[[G]]}(\mathbf{X}_{\infty,\mathfrak{r}}/M\mathbf{X}_{\infty,\mathfrak{r}}, (\mathcal{O}/M\mathcal{O})[[G]]^k) = 0.$$

Proof. Apply Lemma 4.3.3(ii) with $R = \mathcal{O}/M\mathcal{O}$ and with

$$G = \mathrm{Gal}(F(\mathfrak{r})/K), \quad H = \{1\}, \quad B = \mathbf{X}_{F(\mathfrak{r})}/M\mathbf{X}_{F(\mathfrak{r})}$$

to prove (i), and with

$$G = \mathrm{Gal}(K_\infty(\mathfrak{r})/K), \quad H = \mathrm{Gal}(K_\infty(\mathfrak{r})/K(\mathfrak{r})), \quad B = \mathbf{X}_{\infty,\mathfrak{r}}/M\mathbf{X}_{\infty,\mathfrak{r}}$$

to prove (ii). That B is free over $R[[H]]$ is Proposition 4.3.1(i) in the first case, and Corollary 4.3.2 in the second. □

Remark 4.3.5. Alternatively, Proposition 4.3.4(i) can be proved by observing that $(\mathcal{O}/M\mathcal{O})[G]$ is injective (as a module over itself) when G is finite. However, this is not true for $(\mathcal{O}/M\mathcal{O})[[\mathrm{Gal}(K_\infty(\mathfrak{r})/K)]]$.

4.4. Kolyvagin's Derivative Construction

Following Kolyvagin [**Ko2**], we will associate to an Euler system a collection of "derivative" classes

$$\kappa_{[F,\mathfrak{r},M]} \in H^1(F, W_M)$$

for every $K \subset_{\mathrm{f}} F \subset K_\infty$, every nonzero $M \in \mathcal{O}$, and every $\mathfrak{r} \in \mathcal{R}_{F,M}$ (where $\mathcal{R}_{F,M}$ is the subset of $\mathcal{R}$ given by Definition 4.1.1).

Definition 4.4.1. Fix a generator ξ of $\varprojlim \boldsymbol{\mu}_{p^n}$, and for every prime $\mathfrak{q}$ of K not dividing p fix a prime $\mathfrak{Q}$ of $\bar{K}$ above $\mathfrak{q}$. We will fix a generator $\sigma_\mathfrak{q}$ of $\Gamma_\mathfrak{q}$ as follows.

Let $M = |\Gamma_\mathfrak{q}| = [K(\mathfrak{q}) : K(\mathbf{1})]$ and let $\mathcal{I}_\mathfrak{Q}$ denote the inertia group of $\mathfrak{Q}$ in G_K. Since M is a power of p and $\mathfrak{q}$ is prime to p, Lemma 1.4.5 shows that $\mathcal{I}_\mathfrak{Q}$ has a unique cyclic quotient of order M, and this quotient is canonically isomorphic to $\boldsymbol{\mu}_M$. Since $\Gamma_\mathfrak{q}$ itself is a cyclic quotient of $\mathcal{I}_\mathfrak{Q}$, this allows us to identify $\Gamma_\mathfrak{q}$ with $\boldsymbol{\mu}_M$. The chosen generator ξ gives us a generator ζ of $\boldsymbol{\mu}_M$; we define $\sigma_\mathfrak{q} \in \Gamma_\mathfrak{q}$ to be the corresponding generator of $\Gamma_\mathfrak{q}$. (This definition depends on the choices of $\mathfrak{Q}$ and ξ, but we will suppress this dependence from the notation.)

Now define, for every prime $\mathfrak{q}$ not dividing p,

$$D_\mathfrak{q} = \sum_{i=0}^{|\Gamma_\mathfrak{q}|-1} i\sigma_\mathfrak{q}^i \in \mathbf{Z}[\Gamma_\mathfrak{q}].$$

If $\mathfrak{r} \in \mathcal{R}$ and $\mathfrak{q} \mid \mathfrak{r}$ we view $D_\mathfrak{q} \in \mathbf{Z}[\Gamma_\mathfrak{r}]$ and define

$$D_\mathfrak{r} = \prod_{\text{primes } \mathfrak{q}|\mathfrak{r}} D_\mathfrak{q} \in \mathbf{Z}[\Gamma_\mathfrak{r}].$$

If $K \subset_f F \subset K_\infty$, fix an element $N_{F(\mathbf{1})/F} \in \mathbf{Z}[G_F]$ whose image under restriction to $F(\mathbf{1})$ is the norm element $\sum_{\gamma \in \mathrm{Gal}(F(\mathbf{1})/F)} \gamma \in \mathbf{Z}[\mathrm{Gal}(F(\mathbf{1})/F)]$, and define

$$D_{\mathfrak{r},F} = N_{F(\mathbf{1})/F} D_\mathfrak{r}.$$

We have the easy "telescoping" identity

$$(\sigma_\mathfrak{q} - 1)D_\mathfrak{q} = |\Gamma_\mathfrak{q}| - N_\mathfrak{q}. \tag{4.4}$$

This is the key step in the following lemma, which in turn is crucial for the construction of the derivative classes.

Lemma 4.4.2. *Suppose $M \in \mathcal{O}$ is nonzero, $K \subset_f F \subset K_\infty$, and $\mathfrak{r} \in \mathcal{R}_{F,M}$. Let $\bar{x}_{F(\mathfrak{r})}$ denote the image of $x_{F(\mathfrak{r})}$ in $\mathbf{X}_{F(\mathfrak{r})}/M\mathbf{X}_{F(\mathfrak{r})}$. Then*

(i) $D_{\mathfrak{r},F}\bar{x}_{F(\mathfrak{r})} \in (\mathbf{X}_{F(\mathfrak{r})}/M\mathbf{X}_{F(\mathfrak{r})})^{\mathrm{Gal}(F(\mathfrak{r})/F)}$,

(ii) *$D_{\mathfrak{r},F}\bar{x}_{F(\mathfrak{r})}$ is independent of the choice of $N_{F(\mathbf{1})/F}$.*

Proof. We will show that

$$(\sigma - 1)D_\mathfrak{r} x_{F(\mathfrak{r})} \in M\mathbf{X}_{F(\mathfrak{r})} \text{ for every } \sigma \in \mathrm{Gal}(F(\mathfrak{r})/F(\mathbf{1})),$$

and then both assertions of the lemma follow.

The proof is by induction on the number of primes dividing $\mathfrak{r}$. If $\mathfrak{r} = \mathbf{1}$ there is nothing to prove. In general, suppose $\mathfrak{q}$ is a prime dividing $\mathfrak{r}$, say

$\mathfrak{r} = \mathfrak{q}\mathfrak{s}$. Since $\mathfrak{q} \in \mathcal{R}_{F,M}$, we have that $\mathrm{Fr}_\mathfrak{q} \in \mathrm{Gal}(F(\mathfrak{r})/F(\mathbf{1}))$ and that M divides both $|\Gamma_\mathfrak{q}|$ and $P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;1)$. Since $N_\mathfrak{q} x_{F(\mathfrak{r})} = P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;\mathrm{Fr}_\mathfrak{q}^{-1})x_{F(\mathfrak{s})}$, using (4.4) and the induction hypothesis we obtain

$$\begin{aligned}(\sigma_\mathfrak{q} - 1)D_\mathfrak{r} x_{F(\mathfrak{r})} &= (\sigma_\mathfrak{q} - 1)D_\mathfrak{q} D_\mathfrak{s} x_{F(\mathfrak{r})} \\ &= |\Gamma_\mathfrak{q}| D_\mathfrak{s} x_{F(\mathfrak{r})} - P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;\mathrm{Fr}_\mathfrak{q}^{-1}) D_\mathfrak{s} x_{F(\mathfrak{s})} \\ &\equiv |\Gamma_\mathfrak{q}| D_\mathfrak{s} x_{F(\mathfrak{r})} - P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;1) D_\mathfrak{s} x_{F(\mathfrak{s})} \pmod{(\mathrm{Fr}_\mathfrak{q} - 1)D_\mathfrak{s} x_{F(\mathfrak{s})}} \\ &\in M\mathbf{X}_{F(\mathfrak{r})}.\end{aligned}$$

Since $\mathrm{Gal}(F(\mathfrak{r})/F(\mathbf{1}))$ is generated by the $\sigma_\mathfrak{q}$, this proves the lemma. □

Remark 4.4.3. The idea of the construction of the derivative class $\kappa_{[F,\mathfrak{r},M]}$ is as follows.

By Lemmas 4.4.2 and 4.2.3, the image of $D_{\mathfrak{r},F}\mathbf{c}_{F(\mathfrak{r})}$ in $H^1(F(\mathfrak{r}), W_M)$ is fixed by $\mathrm{Gal}(F(\mathfrak{r})/F)$. If $W^{G_{F(\mathfrak{r})}} = 0$ then the restriction map

$$H^1(F, W_M) \longrightarrow H^1(F(\mathfrak{r}), W_M)^{\mathrm{Gal}(F(\mathfrak{r})/F)} \tag{4.5}$$

is an isomorphism, and we define $\kappa_{[F,\mathfrak{r},M]} \in H^1(F, W_M)$ to be the inverse image of $D_{\mathfrak{r},F}\mathbf{c}_{F(\mathfrak{r})}$.

When $W^{G_{F(\mathfrak{r})}} \neq 0$, the map (4.5) need not be an isomorphism. The rest of this section will be devoted to showing, using Proposition 4.3.4 and the universal Euler system, that the image of $D_{\mathfrak{r},F}\mathbf{c}_{F(\mathfrak{r})}$ always has a canonical inverse image under (4.5). That inverse image will be our class $\kappa_{[F,\mathfrak{r},M]}$ (see Definition 4.4.10). Our construction will also be quite explicit, so that we can use it to prove the local properties of the derivative classes which we state in §4.5 below.

Fix, for the rest of this section, a nonzero $M \in \mathcal{O}$.

Definition 4.4.4. Let $\mathbb{W}_M = \mathrm{Ind}(W_M)$ denote the induced module defined (and called $\mathrm{Ind}_{\{1\}}^{G_K}(W_M)$) in §B.4:

$$\mathbb{W}_M = \mathrm{Maps}(G_K, W_M),$$

i.e., the $\mathcal{O}$-module of continuous maps (not necessarily homomorphisms) from G_K to W_M, with G_K acting via

$$(\gamma f)(g) = f(g\gamma) \text{ for all } \gamma, g \in G_K.$$

There is a natural G_K-module inclusion $W_M \hookrightarrow \mathbb{W}_M$ which sends t to the map $g \mapsto gt$, and we will identify W_M with a submodule of $\mathbb{W}_M$ using this inclusion.

Proposition 4.4.5. *For every $\mathfrak{r} \in \mathcal{R}$ and every finite extension L of K in $K_\infty(\mathfrak{r})$, there is a canonical map*

$$\delta_L : (\mathbb{W}_M/W_M)^{G_L} \longrightarrow H^1(L, W_M)$$

such that

(i) *there is an exact sequence*

$$0 \longrightarrow W_M^{G_L} \longrightarrow \mathbb{W}_M^{G_L} \longrightarrow (\mathbb{W}_M/W_M)^{G_L} \xrightarrow{\delta_L} H^1(L, W_M) \longrightarrow 0,$$

(ii) *if* $f \in (\mathbb{W}_M/W_M)^{G_L}$ *and* $\hat{f} \in \mathbb{W}_M$ *lifts* f*, then* $\delta_L(f)$ *is represented by the cocycle*

$$\gamma \mapsto (\gamma - 1)\hat{f} \in W_M \quad \text{for } \gamma \in G_L,$$

(iii) *if* $K \subset_{\mathrm{f}} L \subset_{\mathrm{f}} L' \subset K_\infty(\mathfrak{r})$ *then the following diagram commutes:*

$$\begin{array}{ccccc}
(\mathbb{W}_M/W_M)^{G_L} & \hookrightarrow & (\mathbb{W}_M/W_M)^{G_{L'}} & \xrightarrow{\mathbf{N}_{L'/L}} & (\mathbb{W}_M/W_M)^{G_L} \\
\downarrow{\scriptstyle \delta_L} & & \downarrow{\scriptstyle \delta_{L'}} & & \downarrow{\scriptstyle \delta_L} \\
H^1(L, W_M) & \xrightarrow{\mathrm{Res}_{L'}} & H^1(L', W_M) & \xrightarrow{\mathrm{Cor}_{L'/L}} & H^1(L, W_M).
\end{array}$$

Proof. By Proposition B.4.5, taking G_L-cohomology of the exact sequence

$$0 \longrightarrow W_M \longrightarrow \mathbb{W}_M \longrightarrow \mathbb{W}_M/W_M \longrightarrow 0$$

gives the exact sequence of (i) and the commutativity of (iii). Assertion (ii) is just the standard calculation of the connecting map in Galois cohomology. □

Lemma 4.4.6. *Let* $d = \mathrm{rank}_{\mathcal{O}}(T)$*, and suppose* $\mathfrak{r} \in \mathcal{R}$*.*

(i) *If* $K \subset_{\mathrm{f}} F \subset K_\infty$ *then* $\mathbb{W}_M^{G_{F(\mathfrak{r})}}$ *is a free* $(\mathcal{O}/M\mathcal{O})[\mathrm{Gal}(F(\mathfrak{r})/K)]$*-module of rank* d*.*

(ii) *Let* $\Lambda_{\mathfrak{r}} = \mathcal{O}[[\mathrm{Gal}(K_\infty(\mathfrak{r})/K)]]$*. Then* $\varprojlim \mathbb{W}_M^{G_{F(\mathfrak{r})}}$ *(inverse limit over* $K \subset_{\mathrm{f}} F \subset K_\infty$ *with respect to the norm maps) is a free* $\Lambda_{\mathfrak{r}}/M\Lambda_{\mathfrak{r}}$*-module of rank* d*, and if* $K \subset_{\mathrm{f}} F' \subset K_\infty$ *then*

$$\varprojlim_F \mathbb{W}_M^{G_{F(\mathfrak{r})}} \otimes_{\Lambda_{\mathfrak{r}}} \mathcal{O}[\mathrm{Gal}(F'(\mathfrak{r})/K)] \cong \mathbb{W}_M^{G_{F'(\mathfrak{r})}}.$$

Proof. Let W_M^0 denote the $\mathcal{O}$-module W_M with trivial G_K-action, and abbreviate $H = \mathrm{Gal}(F(\mathfrak{r})/K)$. Then there are Galois-equivariant isomorphisms

$$\mathbb{W}_M^{G_{F(\mathfrak{r})}} = \mathrm{Maps}(H, W_M) = \mathrm{Maps}(H, \mathcal{O}/M\mathcal{O}) \otimes_{\mathcal{O}} W_M^0.$$

Since W_M is free of rank d over $\mathcal{O}/M\mathcal{O}$, and $\mathrm{Maps}(H, \mathcal{O}/M\mathcal{O})$ is free of rank one over $(\mathcal{O}/M\mathcal{O})[H]$, this proves (i). The second assertion follows by taking inverse limits. □

If $\mathfrak{r} \in \mathcal{R}$ we write $H^1_\infty(K(\mathfrak{r}), W_M) = \varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^1(F(\mathfrak{r}), W_M)$.

Proposition 4.4.7. *Suppose $\mathfrak{r} \in \mathcal{R}$. Then the maps $\delta_{F(\mathfrak{r})}$ of Proposition* 4.4.5 *induce an exact sequence*

$$0 \longrightarrow \varprojlim_F \mathbb{W}_M^{G_{F(\mathfrak{r})}} \longrightarrow \varprojlim_F (\mathbb{W}_M/W_M)^{G_{F(\mathfrak{r})}} \xrightarrow{\boldsymbol{\delta}_\mathfrak{r}} H^1_\infty(K(\mathfrak{r}), W_M) \longrightarrow 0.$$

Proof. By Lemma 4.4.6(i), $\mathbb{W}_M^{G_{F(\mathfrak{r})}}$ is finite whenever $K \subset_{\mathrm{f}} F \subset K_\infty$. Therefore taking inverse limits over F of the exact sequence of Proposition 4.4.5(i) (with respect to norm maps for the first three terms and corestriction for the fourth; see Proposition 4.4.5(iii)) yields a new exact sequence (see Proposition B.1.1(i))

$$0 \longrightarrow \varprojlim_F W_M^{G_{F(\mathfrak{r})}} \longrightarrow \varprojlim_F \mathbb{W}_M^{G_{F(\mathfrak{r})}} \longrightarrow \varprojlim_F (\mathbb{W}_M/W_M)^{G_{F(\mathfrak{r})}} \xrightarrow{\boldsymbol{\delta}_\mathfrak{r}} H^1_\infty(K(\mathfrak{r}), W_M) \longrightarrow 0.$$

By Lemma B.3.2, $\varprojlim W_M^{G_{F(\mathfrak{r})}} = 0$, and the proposition follows. □

Proposition 4.4.8. *Suppose* $\mathbf{c}$ *is an Euler system and $\mathfrak{r} \in \mathcal{R}$. There is a family of $\mathcal{O}[G_K]$-module maps*

$$\{\mathbf{d}_F : \mathbf{X}_{F(\mathfrak{r})} \to (\mathbb{W}_M/W_M)^{G_{F(\mathfrak{r})}} : K \subset_{\mathrm{f}} F \subset K_\infty\}$$

lifting $\mathbf{c}$*, i.e., such that the following diagrams commute*

$$\begin{array}{ccc} & & (\mathbb{W}_M/W_M)^{G_{F(\mathfrak{r})}} \\ & \overset{\mathbf{d}_F}{\nearrow} & \big\downarrow{\scriptstyle \delta_{F(\mathfrak{r})}} \\ \mathbf{X}_{F(\mathfrak{r})} & \xrightarrow[\mathbf{c}]{} & H^1(F(\mathfrak{r}), W_M) \end{array} \qquad \begin{array}{ccc} \mathbf{X}_{F'(\mathfrak{r})} & \xrightarrow{\mathbf{d}_{F'}} & (\mathbb{W}_M/W_M)^{G_{F'(\mathfrak{r})}} \\ \big\downarrow{\scriptstyle \mathbf{N}_{F'(\mathfrak{r})/F(\mathfrak{r})}} & & \big\downarrow{\scriptstyle \mathbf{N}_{F'(\mathfrak{r})/F(\mathfrak{r})}} \\ \mathbf{X}_{F(\mathfrak{r})} & \xrightarrow{\mathbf{d}_F} & (\mathbb{W}_M/W_M)^{G_{F(\mathfrak{r})}} \end{array}$$

where the bottom map on the left sends $x_{F(\mathfrak{s})} \mapsto \mathbf{c}_{F(\mathfrak{s})}$ for all $\mathfrak{s}$ dividing $\mathfrak{r}$ as in Lemma 4.2.3*, and on the right $K \subset_{\mathrm{f}} F \subset_{\mathrm{f}} F' \subset K_\infty$. These conditions determine each $\mathbf{d}_F$ uniquely up to an element of* $\mathrm{Hom}_{\mathcal{O}[G_K]}(\mathbf{X}_{F(\mathfrak{r})}, \mathbb{W}_M)$.

Proof. We first illustrate the proof in a simplified setting. If $W_M^{G_{F(\mathfrak{r})}} = 0$, then Proposition 4.4.5(i) becomes a short exact sequence which (abbreviating $R = (\mathcal{O}/M\mathcal{O})[\mathrm{Gal}(F(\mathfrak{r})/K)]$ and $\mathbf{X}_{F(\mathfrak{r})}/M = \mathbf{X}_{F(\mathfrak{r})}/M\mathbf{X}_{F(\mathfrak{r})}$) induces an exact sequence

$$0 \to \mathrm{Hom}_R(\mathbf{X}_{F(\mathfrak{r})}/M, \mathbb{W}_M^{G_{F(\mathfrak{r})}}) \to \mathrm{Hom}_R(\mathbf{X}_{F(\mathfrak{r})}/M, (\mathbb{W}_M/W_M)^{G_{F(\mathfrak{r})}}) \xrightarrow{\delta_{F(\mathfrak{r})}} \mathrm{Hom}_R(\mathbf{X}_{F(\mathfrak{r})}/M, H^1(F(\mathfrak{r}), W_M)) \to \mathrm{Ext}^1_R(\mathbf{X}_{F(\mathfrak{r})}/M, \mathbb{W}_M^{G_{F(\mathfrak{r})}}).$$

By Lemma 4.4.6(i) and Proposition 4.3.4(i), $\mathrm{Ext}^1_R(\mathbf{X}_{F(\mathfrak{r})}/M, \mathbb{W}_M^{G_{F(\mathfrak{r})}}) = 0$, so we can choose a map $\mathbf{d}_F$ lifting $\mathbf{c}$ in this case.

In general, since $W_M^{G_{F(\mathfrak{1})}}$ may be nonzero, we pass to the limit and use the short exact sequence of Proposition 4.4.7 instead of Proposition 4.4.5(i). Arguing as above, using Lemma 4.4.6(ii) and Propositions 4.4.7 and 4.3.4(ii), and writing $\Lambda_{\mathfrak{r}} = \mathcal{O}[[\mathrm{Gal}(K_\infty(\mathfrak{r})/K)]]$, we obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\Lambda_{\mathfrak{r}}}(\mathbf{X}_{\infty,\mathfrak{r}}/M\mathbf{X}_{\infty,\mathfrak{r}}, \varprojlim_F \mathbb{W}_M^{G_{F(\mathfrak{r})}}) \\ \longrightarrow \mathrm{Hom}_{\Lambda_{\mathfrak{r}}}(\mathbf{X}_{\infty,\mathfrak{r}}/M\mathbf{X}_{\infty,\mathfrak{r}}, \varprojlim_F (\mathbb{W}_M/W_M)^{G_{F(\mathfrak{r})}}) \\ \xrightarrow{\boldsymbol{\delta}_{\mathfrak{r}}} \mathrm{Hom}_{\Lambda_{\mathfrak{r}}}(\mathbf{X}_{\infty,\mathfrak{r}}/M\mathbf{X}_{\infty,\mathfrak{r}}, H^1_\infty(K(\mathfrak{r}), W_M)) \longrightarrow 0. \quad (4.6) \end{aligned}$$

Therefore there is a map $\mathbf{d}_\infty : \mathbf{X}_{\infty,\mathfrak{r}} \to \varprojlim_F (\mathbb{W}_M/W_M)^{G_{F(\mathfrak{r})}}$ such that

$$\boldsymbol{\delta}_{\mathfrak{r}} \circ \mathbf{d}_\infty(\{x_{F(\mathfrak{s})}\}_F) = \{\mathbf{c}_{F(\mathfrak{s})}\}_F$$

for every $\mathfrak{s}$ dividing $\mathfrak{r}$. We define $\mathbf{d}_F$ to be the composition

$$\begin{aligned} \mathbf{X}_{F(\mathfrak{r})} \xrightarrow{\sim} \mathbf{X}_{\infty,\mathfrak{r}} \otimes_{\Lambda_{\mathfrak{r}}} \mathcal{O}[\mathrm{Gal}(F(\mathfrak{r})/K)] \\ \xrightarrow{\mathbf{d}_\infty \otimes 1} \varprojlim_{F'} (\mathbb{W}_M/W_M)^{G_{F'(\mathfrak{r})}} \otimes_{\Lambda_{\mathfrak{r}}} \mathcal{O}[\mathrm{Gal}(F(\mathfrak{r})/K)] \\ \longrightarrow (\mathbb{W}_M/W_M)^{G_{F(\mathfrak{r})}} \end{aligned}$$

where the left-hand isomorphism comes from Corollary 4.3.2 and the right-hand map is the natural projection. (Explicitly, $\mathbf{d}_F(x_{F(\mathfrak{s})})$ is the projection of $\mathbf{d}_\infty(\{x_{F'(\mathfrak{s})}\})$ to $(\mathbb{W}_M/W_M)^{G_{F(\mathfrak{r})}}$.) It is straightforward to check that these maps have the desired properties. By (4.6), $\mathbf{d}_\infty$ is unique up to an element of $\mathrm{Hom}_{G_K}(\mathbf{X}_{\infty,\mathfrak{r}}, \varprojlim_F \mathbb{W}_M^{G_{F(\mathfrak{r})}})$, and it follows that $\mathbf{d}_F$ is well-defined up to an element of $\mathrm{Hom}_{\mathcal{O}[G_K]}(\mathbf{X}_{F(\mathfrak{r})}, \mathbb{W}_M)$. □

Remark 4.4.9. We will only need to use the existence of the maps $\mathbf{d}_F$ of Proposition 4.4.8 for individual F. The compatibility as F varies (the right-hand diagram of the proposition) is needed in order to get the uniqueness portion of the proposition, i.e., to make the map $\mathbf{d}_F$ well-defined up to an element of $\mathrm{Hom}_{\mathcal{O}[G_K]}(\mathbf{X}_{F(\mathfrak{r})}, \mathbb{W}_M)$.

Definition 4.4.10. Suppose $\mathbf{c}$ is an Euler system, $M \in \mathcal{O}$ is nonzero, $K \subset_{\mathrm{f}} F \subset K_\infty$, and $\mathfrak{r} \in \mathcal{R}_{F,M}$. Fix a map

$$\mathbf{d} = \mathbf{d}_F : \mathbf{X}_{F(\mathfrak{r})} \longrightarrow \mathbb{W}_M/W_M$$

in a family lifting $\mathbf{c}$ as in Proposition 4.4.8.

Lemma 4.4.2 shows that

$$\mathbf{d}(D_{\mathfrak{r},F} x_{F(\mathfrak{r})}) \in (\mathbb{W}_M/W_M)^{G_F},$$

where $D_{\mathfrak{r},F} = N_{F(\mathbf{1})/F} D_{\mathfrak{r}}$ is as in Definition 4.4.1, and we define

$$\kappa_{[F,\mathfrak{r},M]} = \delta_F(\mathbf{d}(D_{\mathfrak{r},F} x_{F(\mathfrak{r})})) \in H^1(F, W_M).$$

We can describe this definition with the following diagram

$$\begin{array}{ccccccc}
\mathbf{d}(D_{\mathfrak{r},F} x_{F(\mathfrak{r})}) & \in & (\mathbb{W}_M/W_M)^{G_{F(\mathfrak{r})}} & \xrightarrow{\delta_{F(\mathfrak{r})}} & H^1(F(\mathfrak{r}), W_M) & \ni & D_{\mathfrak{r},F}\mathbf{c}_{F(\mathfrak{r})} \\
\uparrow & & \uparrow & & \uparrow \mathrm{res} & & \\
\mathbf{d}(D_{\mathfrak{r},F} x_{F(\mathfrak{r})}) & \in & (\mathbb{W}_M/W_M)^{G_F} & \xrightarrow{\delta_F} & H^1(F, W_M) & \ni & \kappa_{[F,\mathfrak{r},M]}
\end{array}$$

where the commutativity of the inner square is part of Proposition 4.4.5(iii).

Remark 4.4.11. The class $\kappa_{[F,\mathfrak{r},M]}$ is independent of the choice of $N_{F(\mathbf{1})/F}$ used to define $D_{\mathfrak{r},F}$, since by Lemma 4.4.2, $D_{\mathfrak{r},F} x_{F(\mathfrak{r})} \in \mathbf{X}_{F(\mathfrak{r})}/M\mathbf{X}_{F(\mathfrak{r})}$ is independent of this choice. The definition of $\kappa_{[F,\mathfrak{r},M]}$ is also independent the choice of $\mathbf{d}$ in Proposition 4.4.8. For if $\mathbf{d}'$ is another choice, then $\mathbf{d} - \mathbf{d}' \in \mathrm{Hom}_{G_K}(\mathbf{X}_{F(\mathfrak{r})}, \mathbb{W}_M)$, so by Lemma 4.4.2 and Proposition 4.4.5(i),

$$\mathbf{d}(D_{\mathfrak{r},F} x_{F(\mathfrak{r})}) - \mathbf{d}'(D_{\mathfrak{r},F} x_{F(\mathfrak{r})}) \in \mathrm{image}((\mathbb{W}_M)^{G_F}) = \ker(\delta_F).$$

Also, note that $\kappa_{[F,\mathfrak{r},M]}$ depends only on the images of the Euler system classes $\mathbf{c}_L$ in $H^1(L, W_M)$, not in $H^1(L, T)$. However, the extra information in $H^1(L, T)$ will be used to prove Theorem 4.5.1 below. See §9.3 for a further discussion in this direction.

The class $\kappa_{[F,\mathfrak{r},M]}$ does depend (because $D_{\mathfrak{r}}$ does) on the choice of generators $\sigma_{\mathfrak{q}}$ of the groups $\Gamma_{\mathfrak{q}}$. Making another set of choices will multiply $\kappa_{[F,\mathfrak{r},M]}$ by a unit in $(\mathcal{O}/M\mathcal{O})^\times$.

For the next two lemmas, suppose $\mathbf{c}$ is an Euler system, $M \in \mathcal{O}$ is nonzero, $K \subset_{\mathrm{f}} F \subset K_\infty$, and $\mathfrak{r} \in \mathcal{R}_{F,M}$ as in Definition 4.4.10.

Lemma 4.4.12. *Suppose $\mathbf{d} : \mathbf{X}_{F(\mathfrak{r})} \to \mathbb{W}_M/W_M$ is a lifting of the Euler system $\mathbf{c}$ as in Proposition 4.4.8. Let $f \in \mathbb{W}_M$ be any lifting of $\mathbf{d}(D_{\mathfrak{r},F} x_{F(\mathfrak{r})})$. Then $\kappa_{[F,\mathfrak{r},M]}$ is represented by the cocycle*

$$\gamma \mapsto (\gamma - 1)f \in W_M \quad \text{for } \gamma \in G_F.$$

Proof. This is a combination of the definition of $\kappa_{[F,\mathfrak{r},M]}$ above with the explicit description of the connecting map δ_F (Proposition 4.4.5(ii)). □

Lemma 4.4.13. (i) *The class $\kappa_{[F,\mathbf{1},M]}$ is the image of $\mathbf{c}_F$ under the map $H^1(F, T) \to H^1(F, W_M)$.*

(ii) *More generally, the restriction of $\kappa_{[F,\mathfrak{r},M]}$ to $H^1(F(\mathfrak{r}), W_M)$ is equal to the image of $D_{\mathfrak{r},F}\mathbf{c}_{F(\mathfrak{r})}$ in $H^1(F(\mathfrak{r}), W_M)$.*

(iii) *If $M \mid M'$ and $\mathfrak{r} \in \mathcal{R}_{F,M'}$ then under the natural maps we have*

$$H^1(F, W_{M'}) \longrightarrow H^1(F, W_M), \qquad H^1(F, W_M) \longrightarrow H^1(F, W_{M'}).$$

$$\kappa_{[F,\mathfrak{r},M']} \longmapsto \kappa_{[F,\mathfrak{r},M]} \qquad \kappa_{[F,\mathfrak{r},M]} \longmapsto (M'/M)\kappa_{[F,\mathfrak{r},M']}$$

Proof. All three assertions follow from Definition 4.4.10. For the first we take $\mathfrak{r} = \mathbf{1}$, so $D_{\mathfrak{r},F} = N_{F(\mathbf{1})/F}$, and use Proposition 4.4.5(iii) and the Euler system relation $\mathrm{Cor}_{F(\mathbf{1})/F}(\mathbf{c}_{F(\mathbf{1})}) = \mathbf{c}_F$. □

4.5. Local Properties of the Derivative Classes

Fix an Euler system $\mathbf{c}$ for T. In this section we will state the main results describing the local behavior of the derivative classes $\kappa_{[F,\mathfrak{r},M]}$ of §4.4. We will see (Theorem 4.5.1) that $\kappa_{[F,\mathfrak{r},M]}$ belongs to the Selmer group $\mathcal{S}^{\Sigma}(F, W_M)$ where Σ is the set of primes of K dividing $p\mathfrak{r}$ (see Remark 1.5.8 for this slight abuse of notation). For our applications it will be crucial to understand (Theorem 4.5.4) the ramification of $\kappa_{[F,\mathfrak{r},M]}$ at primes dividing $\mathfrak{r}$.

The proofs will be given in the remaining sections of this chapter.

Theorem 4.5.1. *Suppose that $M \in \mathcal{O}$ is nonzero, that $K \subset_{\mathrm{f}} F \subset K_\infty$, and that $\mathfrak{r} \in \mathcal{R}_{F,M}$. For every place w of F not dividing $p\mathfrak{r}$,*

$$(\kappa_{[F,\mathfrak{r},M]})_w \in H^1_f(F_w, W_M).$$

In other words,

$$\kappa_{[F,\mathfrak{r},M]} \in \mathcal{S}^{\Sigma_{p\mathfrak{r}}}(F, W_M)$$

where $\Sigma_{p\mathfrak{r}}$ is the set of primes of K dividing $p\mathfrak{r}$.

Theorem 4.5.1 will be proved in §4.6.

Lemma 4.5.2. *Suppose $M \in \mathcal{O}$ is nonzero and $\mathfrak{q} \in \mathcal{R}_{K,M}$ is prime. Then there is a unique $Q_{\mathfrak{q}}(x) \in (\mathcal{O}/M\mathcal{O})[x]$ such that*

$$P(\mathrm{Fr}_{\mathfrak{q}}^{-1} | T^*; x) \equiv (x-1)Q_{\mathfrak{q}}(x) \pmod{M}.$$

Proof. Take

$$Q_{\mathfrak{q}}(x) = \frac{P(\mathrm{Fr}_{\mathfrak{q}}^{-1} | T^*; x) - P(\mathrm{Fr}_{\mathfrak{q}}^{-1} | T^*; 1)}{x-1}.$$

Since $\mathfrak{q} \in \mathcal{R}_{K,M}$, we know that M divides $P(\mathrm{Fr}_{\mathfrak{q}}^{-1} | T^*; 1)$ so this polynomial has the desired property. The uniqueness comes from the fact that $x - 1$ is not a zero divisor in $(\mathcal{O}/M\mathcal{O})[x]$. □

Definition 4.5.3. Suppose $M \in \mathcal{O}$ is nonzero and $\mathfrak{q} \in \mathcal{R}_{K,M}$ is prime. The choices of $\sigma_\mathfrak{q} \in \Gamma_\mathfrak{q}$ (Definition 4.4.1) and $\mathrm{Fr}_\mathfrak{q}$ depend on the choice of a prime $\mathfrak{Q}$ of $\bar{K}$ above $\mathfrak{q}$. We use the same choice for both, and we further fix $\bar{\sigma}_\mathfrak{q}$ in the inertia group of $\mathfrak{Q}$ extending $\sigma_\mathfrak{q}$.

By Lemma 1.4.7(i) (which applies thanks to Lemma 4.1.2(i)) there are well-defined isomorphisms

$$\alpha_\mathfrak{q} : H^1_s(K_\mathfrak{q}, W_M) \xrightarrow{\sim} W_M^{\mathrm{Fr}_\mathfrak{q}=1}$$
$$\beta_\mathfrak{q} : H^1_f(K_\mathfrak{q}, W_M) \xrightarrow{\sim} W_M/(\mathrm{Fr}_\mathfrak{q} - 1)W_M$$

given on cocycles by

$$\alpha_\mathfrak{q}(c) = c(\bar{\sigma}_\mathfrak{q}), \quad \beta_\mathfrak{q}(c) = c(\mathrm{Fr}_\mathfrak{q}).$$

If $\mathfrak{q} \in \mathcal{R}_{K,M}$, then $P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*; \mathrm{Fr}_\mathfrak{q}^{-1})$ annihilates W_M by Lemma 4.1.2(iv). Thus the polynomial $Q_\mathfrak{q}$ of Lemma 4.5.2 induces a map

$$Q_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1}) : W_M/(\mathrm{Fr}_\mathfrak{q} - 1)W_M \longrightarrow W_M^{\mathrm{Fr}_\mathfrak{q}=1}.$$

We define the "finite-singular comparison" map

$$\phi^{fs}_\mathfrak{q} : H^1_f(K_\mathfrak{q}, W_M) \longrightarrow H^1_s(K_\mathfrak{q}, W_M)$$

to be the composition

$$H^1_f(K_\mathfrak{q}, W_M) \xrightarrow{\beta_\mathfrak{q}} W_M/(\mathrm{Fr}_\mathfrak{q} - 1)W_M \xrightarrow{Q_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})} W_M^{\mathrm{Fr}_\mathfrak{q}=1} \xrightarrow{\alpha_\mathfrak{q}^{-1}} H^1_s(K_\mathfrak{q}, W_M).$$

If $K \subset_\mathrm{f} F \subset K_\infty$ and $\mathfrak{q} \in \mathcal{R}_{F,M}$, then $F_\mathfrak{Q} = K_\mathfrak{q}$, and we can view $\phi^{fs}_\mathfrak{q}$ as a map from $H^1_f(F_\mathfrak{Q}, W_M)$ to $H^1_s(F_\mathfrak{Q}, W_M)$. We will still write $\phi^{fs}_\mathfrak{q}$ in this case, and suppress the dependence on $\mathfrak{Q}$.

Theorem 4.5.4. *Suppose $M \in \mathcal{O}$ is nonzero, $K \subset_\mathrm{f} F \subset K_\infty$, $\mathfrak{q}$ is prime, and $\mathfrak{r}\mathfrak{q} \in \mathcal{R}_{F,M}$. If $\phi^{fs}_\mathfrak{q}$ is the map defined above, and $(\kappa_{[F,\mathfrak{r}\mathfrak{q},M]})^s_\mathfrak{q}$ denotes the image of $\kappa_{[F,\mathfrak{r}\mathfrak{q},M]}$ in $H^1_s(F_\mathfrak{Q}, W_M)$, then*

$$(\kappa_{[F,\mathfrak{r}\mathfrak{q},M]})^s_\mathfrak{q} = \phi^{fs}_\mathfrak{q}(\kappa_{[F,\mathfrak{r},M]}).$$

In other words, the singular part of $\kappa_{[F,\mathfrak{r}\mathfrak{q},M]}$ at $\mathfrak{q}$ is controlled by the (finite) localization of $\kappa_{[F,\mathfrak{r},M]}$ at $\mathfrak{q}$. Theorem 4.5.4 will be proved in §4.7.

Corollary 4.5.5. *Suppose $M \in \mathcal{O}$ is nonzero, $\mathfrak{q}$ is prime, and $\mathfrak{r}\mathfrak{q} \in \mathcal{R}_{K,M}$. Suppose further that $W_M/(\mathrm{Fr}_\mathfrak{q} - 1)W_M$ is free of rank one over $\mathcal{O}/M\mathcal{O}$. Then the order of $(\kappa_{[K,\mathfrak{r}\mathfrak{q},M]})^s_\mathfrak{q}$ in $H^1_s(K_\mathfrak{q}, W_M)$ is equal to the order of $(\kappa_{[K,\mathfrak{r},M]})_\mathfrak{q}$ in $H^1_f(K_\mathfrak{q}, W_M)$.*

Proof. The maps $\alpha_{\mathfrak{q}}$ and $\beta_{\mathfrak{q}}$ in Definition 4.5.3 are both isomorphisms, and by Lemma 4.1.2(iii) and Corollary A.2.7 (applied with $\tau = \mathrm{Fr}_{\mathfrak{q}}^{-1}$ and $Q(x) = Q_{\mathfrak{q}}(x)$), so is the map $Q_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})$. Thus $\phi_{\mathfrak{q}}^{fs}$ is an isomorphism and the corollary follows from Theorem 4.5.4. □

4.6. Local Behavior at Primes Not Dividing $p\mathfrak{r}$

For this section fix an Euler system $\mathbf{c}$ for T and a nonzero element $M \in \mathcal{O}$. If $K \subset_{\mathrm{f}} F \subset K_\infty$ and $\mathfrak{r} \in \mathcal{R}_{F,M}$, we need to show that $(\kappa_{[F,\mathfrak{r},M]})_w \in H^1_f(F_w, W_M)$ for every place w of F not dividing $p\mathfrak{r}$. When w is archimedean (Lemma 4.6.3), or when w is nonarchimedean and T is unramified at w (Corollary 4.6.2(ii)), this is not difficult. We treat those cases first, and then go on to the general case.

Proposition 4.6.1. *Suppose $K \subset_{\mathrm{f}} F \subset K_\infty$ and $\mathfrak{r} \in \mathcal{R}$. For every prime $\mathcal{Q}$ of $F(\mathfrak{r})$ not dividing p, and every $\gamma \in G_K$, we have*

$$(\gamma \mathbf{c}_{F(\mathfrak{r})})_{\mathcal{Q}} \in H^1_{\mathrm{ur}}(F(\mathfrak{r})_{\mathcal{Q}}, T), \quad (\gamma \bar{\mathbf{c}}_{F(\mathfrak{r})})_{\mathcal{Q}} \in H^1_f(F(\mathfrak{r})_{\mathcal{Q}}, W_M)$$

where $\bar{\mathbf{c}}_{F(\mathfrak{r})}$ is the image of $\mathbf{c}_{F(\mathfrak{r})}$ under $H^1(F(\mathfrak{r}), T) \to H^1(F(\mathfrak{r}), W_M)$.

Proof. Since $\{\gamma \mathbf{c}_{F(\mathfrak{r})}\}_F \in H^1_\infty(K(\mathfrak{r}), T)$, the first inclusion follows from Corollary B.3.5 and the second from Lemma 1.3.8(i). □

Corollary 4.6.2. *Suppose $K \subset_{\mathrm{f}} F \subset K_\infty$ and $\mathfrak{r} \in \mathcal{R}_{F,M}$. If $\mathcal{Q}$ is a prime of F not dividing $p\mathfrak{r}$, then*

(i) $(\kappa_{[F,\mathfrak{r},M]})_{\mathcal{Q}} \in H^1_{\mathrm{ur}}(F_{\mathcal{Q}}, W_M)$,

(ii) *if T is unramified at $\mathcal{Q}$ then $(\kappa_{[F,\mathfrak{r},M]})_{\mathcal{Q}} \in H^1_f(F_{\mathcal{Q}}, W_M)$.*

Proof. Let $D_{\mathfrak{r},F}$ be as in Definition 4.4.10 and write $\mathcal{I}$ for an inertia group of $\mathcal{Q}$ in G_F. Since $F(\mathfrak{r})/F$ is unramified at $\mathcal{Q}$ we have $\mathcal{I} \subset G_{F(\mathfrak{r})}$, so by Lemma 4.4.13(ii) the restriction of $\kappa_{[F,\mathfrak{r},M]}$ to $\mathcal{I}$ is equal to the image of $D_{\mathfrak{r},F}\mathbf{c}_{F(\mathfrak{r})}$ in $H^1(\mathcal{I}, W_M)$. By Proposition 4.6.1, the latter is zero. This shows that $(\kappa_{[F,\mathfrak{r},M]})_{\mathcal{Q}} \in H^1_{\mathrm{ur}}(F_{\mathcal{Q}}, W_M)$, and if T is unramified at $\mathcal{Q}$ then Lemma 1.3.8(ii) shows that $H^1_f(F_{\mathcal{Q}}, W_M) = H^1_{\mathrm{ur}}(F_{\mathcal{Q}}, W_M)$. □

Lemma 4.6.3. *Suppose $K \subset_{\mathrm{f}} F \subset K_\infty$ and $\mathfrak{r} \in \mathcal{R}_{F,M}$. If w is an infinite place of F, then $(\kappa_{[F,\mathfrak{r},M]})_w \in H^1_f(F_w, W_M)$.*

Proof. Let $\tilde{w}$ be a place of $F(\mathfrak{r})$ above w. Since $F(\mathfrak{r})/F$ ramifies only at primes dividing $\mathfrak{r}$, the place w splits completely in $F(\mathfrak{r})/F$. Thus Lemma 4.4.13(ii) shows that $(\kappa_{[F,\mathfrak{r},M]})_w$ is the image of $(D_{\mathfrak{r},F}\mathbf{c}_{F(\mathfrak{r})})_{\tilde{w}}$ under

$$H^1(F(\mathfrak{r})_{\tilde{w}}, T) = H^1(F_w, T) \longrightarrow H^1(F_w, W_M).$$

By Remark 1.3.7 we have $H^1_f(F_w, T) = H^1(F_w, T)$, so the lemma follows from Lemma 1.3.8(i). □

Remark 4.6.4. In the nonarchimedean case, if w is a prime of K not dividing $p\mathfrak{r}$, then Corollary 4.6.2(i) shows that $(\kappa_{[F,\mathfrak{r},M]})_w \in H^1_{\mathrm{ur}}(F_w, W_M)$. Unfortunately, for primes w where T is ramified it may not be true that $H^1_f(F_w, W_M) = H^1_{\mathrm{ur}}(F_w, W_M)$. However, we do get immediately the following corollary, with only a slightly stronger assumption on $\mathfrak{r}$.

Corollary 4.6.5. *There is a nonzero $m \in \mathcal{O}$, independent of M, such that for every $K \subset_{\mathrm{f}} F \subset K_\infty$, every $\mathfrak{r} \in \mathcal{R}_{F,Mm}$, and every prime $\mathcal{Q}$ of F not dividing $p\mathfrak{r}$, we have $(\kappa_{[F,\mathfrak{r},M]})_{\mathcal{Q}} \in H^1_f(F_{\mathcal{Q}}, W_M)$.*

Proof. Choose $m \in \mathcal{O}$ such that for every prime $\mathfrak{q}$ of K not dividing p, m annihilates $W^{\mathcal{I}_\mathfrak{q}}/(W^{\mathcal{I}_\mathfrak{q}})_{\mathrm{div}}$, where $\mathcal{I}_\mathfrak{q}$ is an inertia group for $\mathfrak{q}$ in G_K and $(W^{\mathcal{I}_\mathfrak{q}})_{\mathrm{div}}$ is the maximal divisible submodule of $W^{\mathcal{I}_\mathfrak{q}}$. Clearly we can take m to be nonzero, since $W^{\mathcal{I}_\mathfrak{q}}/(W^{\mathcal{I}_\mathfrak{q}})_{\mathrm{div}}$ is always finite and is zero whenever W is unramified at $\mathfrak{q}$.

Suppose $K \subset_{\mathrm{f}} F \subset K_\infty$. If $\mathcal{Q}$ is a prime of F not dividing p, and $\mathfrak{q}$ is the prime of K below $\mathcal{Q}$, then $\mathcal{I}_\mathfrak{q}$ is also an inertia group of $\mathcal{Q}$ in G_F. Therefore Lemma 1.3.5(iii) shows that m annihilates $H^1_{\mathrm{ur}}(F_{\mathcal{Q}}, W_{Mm})/H^1_f(F_{\mathcal{Q}}, W_{Mm})$, so by Corollary 4.6.2 we have $(m\kappa_{[F,\mathfrak{r},Mm]})_{\mathcal{Q}} \in H^1_f(F_{\mathcal{Q}}, W_{Mm})$. Lemma 4.4.13(iii) shows that $m\kappa_{[F,\mathfrak{r},Mm]}$ is the image of $\kappa_{[F,\mathfrak{r},M]}$ in $H^1(F, W_{Mm})$, and the corollary follows. □

Corollary 4.6.5 is already strong enough to use in place of Theorem 4.5.1 in proving the theorems of Chapter 2. Thus one could skip the rest of this section if one were so inclined.

To prove Theorem 4.5.1 for primes $\mathcal{Q}$ where T may be ramified is much more delicate. We will mimic the construction of $\kappa_{[F,\mathfrak{r},M]}$ locally, and use Proposition 4.6.1 to show that $(\kappa_{[F,\mathfrak{r},M]})_{\mathcal{Q}}$ can be constructed inside $H^1(F_{\mathcal{Q}}, T^{\mathcal{I}_\mathcal{Q}}/MT^{\mathcal{I}_\mathcal{Q}})$. The theorem will follow directly from this.

Fix for the rest of this section an ideal $\mathfrak{r} \in \mathcal{R}$ and a prime $\mathfrak{q}$ of K not dividing $p\mathfrak{r}$ (but not necessarily in $\mathcal{R}$).

Definition 4.6.6. Fix inertia and decomposition groups $\mathcal{I} \subset \mathcal{D} \subset G_K$ of $\mathfrak{q}$. If L is a finite extension of K, unramified at $\mathfrak{q}$, let S_L denote the set of primes of L above $\mathfrak{q}$ and abbreviate

$$H^i(L_\mathfrak{q}, W_M) = \bigoplus_{\mathcal{Q} \in S_L} H^i(L_\mathcal{Q}, W_M),$$
$$H^i(L_\mathfrak{q}, T^{\mathcal{I}}/MT^{\mathcal{I}}) = \bigoplus_{\mathcal{Q} \in S_L} H^i(L_\mathcal{Q}, T^{\mathcal{I}_\mathcal{Q}}/MT^{\mathcal{I}_\mathcal{Q}})$$

where for each $\mathcal{Q} \in S_L$, we write $\mathcal{I}_\mathcal{Q}$ for the inertia subgroup of $G_{L_\mathcal{Q}}$. (Since L/K is unramified at $\mathfrak{q}$, each $\mathcal{I}_\mathcal{Q}$ is conjugate to $\mathcal{I}$.) Write $(\ \cdot\ )_\mathfrak{q}$

or $\mathrm{res}_{\mathfrak{q}} : H^i(L, W_M) \to H^i(L_{\mathfrak{q}}, W_M)$ for the sum of the restriction maps. Note that $H^i(L_{\mathfrak{q}}, W_M)$ and $H^i(L_{\mathfrak{q}}, T^{\mathcal{I}}/MT^{\mathcal{I}})$ are $\mathrm{Gal}(L/K)$-modules: this can be seen directly (every $\sigma \in \mathrm{Gal}(L/K)$ induces an isomorphism

$$H^i(L_{\mathcal{Q}}, T^{\mathcal{I}_{\mathcal{Q}}}/MT^{\mathcal{I}_{\mathcal{Q}}}) \xrightarrow{\sim} H^i(L_{\sigma\mathcal{Q}}, \sigma(T^{\mathcal{I}_{\mathcal{Q}}}/MT^{\mathcal{I}_{\mathcal{Q}}})) = H^i(L_{\sigma\mathcal{Q}}, T^{\mathcal{I}_{\sigma\mathcal{Q}}}/MT^{\mathcal{I}_{\sigma\mathcal{Q}}})$$

for every $\mathcal{Q}$, and summing these maps over $\mathcal{Q} \in S_L$ gives an automorphism of $H^i(L_{\mathfrak{q}}, T^{\mathcal{I}}/MT^{\mathcal{I}})$ and similarly for $H^i(L_{\mathfrak{q}}, W_M)$), or see Corollary B.5.2.

Write

$$W_M^f = T^{\mathcal{I}}/MT^{\mathcal{I}} \cong ((W^{\mathcal{I}})_{\mathrm{div}})_M \subset (W_M)^{\mathcal{I}} \subset W_M$$

and define a G_K-submodule $\mathbb{W}_M^f \subset \mathbb{W}_M$ by

$$\mathbb{W}_M^f = \mathrm{Ind}(W_M^f) = \mathrm{Maps}(G_K, W_M^f) \subset \mathbb{W}_M.$$

As in §B.4, let $\mathrm{Ind}_{\mathcal{D}}(W_M) \subset \mathbb{W}_M$ denote the G_K-submodule of maps satisfying $f(hg) = hf(g)$ for every $h \in \mathcal{D}$, and similarly for $\mathrm{Ind}_{\mathcal{D}}(W_M^f) \subset \mathbb{W}_M^f$.

Lemma 4.6.7. *If L is a finite extension of K, unramified at $\mathfrak{q}$, then with notation as above we have a natural commutative diagram with exact columns*

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
W_M^{G_L} & \hookrightarrow & H^0(L_{\mathfrak{q}}, W_M) & \hookleftarrow & H^0(L_{\mathfrak{q}}, W_M^f) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{W}_M^{G_L} & \xrightarrow{\sim} & \mathbb{W}_M^{G_L} & \leftarrow & (\mathbb{W}_M^f)^{G_L} \\
\downarrow & & \downarrow & & \downarrow \\
(\mathbb{W}_M/W_M)^{G_L} & \to & (\mathbb{W}_M/\mathrm{Ind}_{\mathcal{D}}(W_M))^{G_L} & \leftarrow & (\mathbb{W}_M^f/\mathrm{Ind}_{\mathcal{D}}(W_M^f))^{G_L} \\
\downarrow{\scriptstyle \delta_L} & & \downarrow{\scriptstyle \delta_{L_{\mathfrak{q}}}} & & \downarrow{\scriptstyle \delta_{L_{\mathfrak{q}}, W_M^f}} \\
H^1(L, W_M) & \xrightarrow{\mathrm{res}_{\mathfrak{q}}} & H^1(L_{\mathfrak{q}}, W_M) & \leftarrow & H^1(L_{\mathfrak{q}}, W_M^f) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}$$

Proof. The three columns come from G_L-cohomology of the short exact sequences

$$\begin{aligned} 0 &\longrightarrow W_M \longrightarrow \mathbb{W}_M \longrightarrow \mathbb{W}_M/W_M \longrightarrow 0 \\ 0 &\longrightarrow \mathrm{Ind}_{\mathcal{D}}(W_M) \longrightarrow \mathbb{W}_M \longrightarrow \mathbb{W}_M/\mathrm{Ind}_{\mathcal{D}}(W_M) \longrightarrow 0 \\ 0 &\longrightarrow \mathrm{Ind}_{\mathcal{D}}(W_M^f) \longrightarrow \mathbb{W}_M^f \longrightarrow \mathbb{W}_M^f/\mathrm{Ind}_{\mathcal{D}}(W_M^f) \longrightarrow 0 \end{aligned}$$

respectively (the left-hand column is Proposition 4.4.5(i)), using Corollaries B.4.4 and B.5.2. The horizontal arrows are the natural ones, and the commutativity follows from the functoriality of all the maps involved. □

We now need a local analogue of Proposition 4.4.8. If $K \subset_{\mathrm{f}} F \subset K_\infty$, $\mathcal{Q} \in S_{F(\mathfrak{r})}$, and $\mathfrak{s} \mid \mathfrak{r}$, then by Proposition 4.6.1,

$$(\mathbf{c}_{F(\mathfrak{s})})_{\mathcal{Q}} \in H^1_{\mathrm{ur}}(F(\mathfrak{s})_{\mathcal{Q}}, T) = H^1(F(\mathfrak{s})^{\mathrm{ur}}_{\mathcal{Q}}/F(\mathfrak{s})_{\mathcal{Q}}, T^{\mathcal{I}_{\mathcal{Q}}}) \subset H^1(F(\mathfrak{s})_{\mathcal{Q}}, T^{\mathcal{I}_{\mathcal{Q}}}) \tag{4.7}$$

so $(\mathbf{c}_{F(\mathfrak{s})})_{\mathfrak{q}}$ maps naturally to $H^1(F(\mathfrak{r})_{\mathfrak{q}}, W_M^f)$.

Proposition 4.6.8. *Suppose* $\mathbf{c}$ *is an Euler system and* $\mathfrak{r} \in \mathcal{R}$. *There are two families of* $\mathcal{O}[G_K]$*-module maps*

$$\{\mathbf{d}_{F,\mathfrak{q}} : \mathbf{X}_{F(\mathfrak{r})} \to (\mathbb{W}_M/\mathrm{Ind}_{\mathcal{D}}(W_M))^{G_{F(\mathfrak{r})}} : K \subset_{\mathrm{f}} F \subset K_\infty\}$$
$$\{\mathbf{d}^f_{F,\mathfrak{q}} : \mathbf{X}_{F(\mathfrak{r})} \to (\mathbb{W}^f_M/\mathrm{Ind}_{\mathcal{D}}(W^f_M))^{G_{F(\mathfrak{r})}} : K \subset_{\mathrm{f}} F \subset K_\infty\}$$

lifting $\mathbf{c}$, *i.e., such that if* $K \subset_{\mathrm{f}} F \subset_{\mathrm{f}} F' \subset K_\infty$ *then*

(i) *the maps* $\mathbf{d}_{F,\mathfrak{q}}$ *(resp* $\mathbf{d}^f_{F,\mathfrak{q}}$*) are compatible with respect to the norm maps*

$$\mathbf{X}_{F'(\mathfrak{r})} \to \mathbf{X}_{F(\mathfrak{r})}, \quad (\mathbb{W}_M/\mathrm{Ind}_{\mathcal{D}}(W_M))^{G_{F'(\mathfrak{r})}} \to (\mathbb{W}_M/\mathrm{Ind}_{\mathcal{D}}(W_M))^{G_{F(\mathfrak{r})}},$$
$$(\mathbb{W}^f_M/\mathrm{Ind}_{\mathcal{D}}(W^f_M))^{G_{F'(\mathfrak{r})}} \to (\mathbb{W}^f_M/\mathrm{Ind}_{\mathcal{D}}(W^f_M))^{G_{F(\mathfrak{r})}},$$

(ii) *for every* $K \subset_{\mathrm{f}} F \subset K_\infty$ *and every* $\mathfrak{s}$ *dividing* $\mathfrak{r}$, *the compositions*

$$\mathbf{X}_{F(\mathfrak{r})} \xrightarrow{\mathbf{d}_{F,\mathfrak{q}}} (\mathbb{W}_M/\mathrm{Ind}_{\mathcal{D}}(W_M))^{G_{F(\mathfrak{r})}} \xrightarrow{\delta_{F(\mathfrak{r})_{\mathfrak{q}}}} H^1(F(\mathfrak{r})_{\mathfrak{q}}, W_M)$$
$$\mathbf{X}_{F(\mathfrak{r})} \xrightarrow{\mathbf{d}^f_{F,\mathfrak{q}}} (\mathbb{W}^f_M/\mathrm{Ind}_{\mathcal{D}}(W^f_M))^{G_{F(\mathfrak{r})}} \xrightarrow{\delta_{F(\mathfrak{r})_{\mathfrak{q}}, W^f_M}} H^1(F(\mathfrak{r})_{\mathfrak{q}}, W^f_M)$$

both (using (4.7) *for the latter) send* $x_{F(\mathfrak{s})}$ *to* $(\mathbf{c}_{F(\mathfrak{s})})_{\mathfrak{q}}$.

These two conditions determine every $\mathbf{d}_{F,\mathfrak{q}}$ *(resp.* $\mathbf{d}^f_{F,\mathfrak{q}}$*) uniquely up to an element of* $\mathrm{Hom}_{\mathcal{O}[G_K]}(\mathbf{X}_{F(\mathfrak{r})}, \mathbb{W}_M)$ *(resp.* $\mathrm{Hom}_{\mathcal{O}[G_K]}(\mathbf{X}_{F(\mathfrak{r})}, \mathbb{W}^f_M)$*).*

Proof. For each $K \subset_{\mathrm{f}} F \subset K_\infty$ we have maps (see Lemma 4.2.3)

$$\begin{array}{ccccc} x_{F(\mathfrak{s})} & \mapsto & \mathbf{c}_{F(\mathfrak{s})} & \mapsto & (\mathbf{c}_{F(\mathfrak{s})})_{\mathfrak{q}} \\ \mathbf{X}_{F(\mathfrak{r})} & \longrightarrow & H^1(F(\mathfrak{r}),T) & \longrightarrow & H^1(F(\mathfrak{r})_{\mathfrak{q}},T) \\ & & \downarrow & & \downarrow \\ & & H^1(F(\mathfrak{r}),W_M) & \longrightarrow & H^1(F(\mathfrak{r})_{\mathfrak{q}},W_M). \end{array}$$

and by (4.7) the composition $\mathbf{X}_{F(\mathfrak{r})} \to H^1(F(\mathfrak{r})_{\mathfrak{q}},W_M)$ factors through a G_K-equivariant map

$$\mathbf{X}_{F(\mathfrak{r})} \longrightarrow H^1(F(\mathfrak{r})_{\mathfrak{q}},W_M^f). \tag{4.8}$$

To prove the proposition we need to lift these to maps

$$\mathbf{X}_{F(\mathfrak{r})} \longrightarrow \mathbb{W}_M/\mathrm{Ind}_{\mathcal{D}}(W_M), \quad \mathbf{X}_{F(\mathfrak{r})} \longrightarrow \mathbb{W}_M^f/\mathrm{Ind}_{\mathcal{D}}(W_M^f)$$

in the center and right-hand columns, respectively, of the diagram of Lemma 4.6.7 with $L = F(\mathfrak{r})$. We will do this by mimicking the proof of Proposition 4.4.8. We describe the proof only for the right-hand column; the other proof is exactly the same (and see Remark 4.6.9 below).

Since we have assumed that the decomposition group of $\mathfrak{q}$ in K_∞/K is infinite, we can find a $\mathbf{Z}_p$-extension K'_∞ of K in K_∞ such that K'_∞ has only finitely many primes above $\mathfrak{q}$. Then for each finite extension L of K we see that $\varinjlim_{K \subset_{\mathrm{f}} F \subset K'_\infty L} H^0(F(\mathfrak{r})_{\mathfrak{q}},W_M^f)$ is finite, so by Lemma B.3.2,

$$\varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^0(F(\mathfrak{r})_{\mathfrak{q}},W_M^f) = \varprojlim_{K \subset_{\mathrm{f}} L \subset K_\infty} \varprojlim_{KL \subset_{\mathrm{f}} F \subset K'_\infty L} H^0(F(\mathfrak{r})_{\mathfrak{q}},W_M^f) = 0$$

(inverse limits with respect to the norm maps). Proposition B.2.7(ii) shows that each $H^1(F(\mathfrak{r})_{\mathfrak{q}},W_M^f)$ is finite, so exactly as in Proposition 4.4.7 the inverse limit over $K \subset_{\mathrm{f}} F \subset K_\infty$ of the right-hand column of the diagram of Lemma 4.6.7 is a short exact sequence

$$0 \longrightarrow \varprojlim_F (\mathbb{W}_M^f)^{G_{F(\mathfrak{r})}} \longrightarrow \varprojlim_F (\mathbb{W}_M^f/\mathrm{Ind}_{\mathcal{D}}(W_M^f))^{G_{F(\mathfrak{r})}} \longrightarrow \varprojlim_F H^1(F(\mathfrak{r})_{\mathfrak{q}},W_M^f) \longrightarrow 0.$$

Taking the inverse limit over F of (4.8) yields a map

$$\mathbf{X}_{\infty,\mathfrak{r}} \longrightarrow \varprojlim_F H^1(F(\mathfrak{r})_{\mathfrak{q}},W_M^f), \tag{4.9}$$

and exactly as in Lemma 4.4.6 we see that the $(\mathcal{O}/M\mathcal{O})[[\mathrm{Gal}(K_\infty(\mathfrak{r})/K)]]$-module $\varprojlim (\mathbb{W}_M^f)^{G_{F(\mathfrak{r})}}$ is free of finite rank. Just as in Proposition 4.4.8,

Proposition 4.3.4 now shows that (4.9) lifts to a map

$$\mathbf{X}_{\infty,\mathfrak{r}} \longrightarrow \varprojlim_{F} (\mathbb{W}_M^f/\mathrm{Ind}_{\mathcal{D}}(W_M^f))^{G_{F(\mathfrak{r})}}.$$

Also as in Proposition 4.4.8, Corollary 4.3.2 shows that this in turn induces maps

$$\mathbf{d}_{F,\mathfrak{q}}^f : \mathbf{X}_{F(\mathfrak{r})} \longrightarrow (\mathbb{W}_M^f/\mathrm{Ind}_{\mathcal{D}}(W_M^f))^{G_{F(\mathfrak{r})}}$$

having the desired properties. The uniqueness is clear from the diagram of Lemma 4.6.7. □

Remark 4.6.9. To construct the maps $\mathbf{d}_{F,\mathfrak{q}}$ in Proposition 4.6.8 it suffices to construct *either* the global maps $\mathbf{d}_F$ of Proposition 4.4.8 or the "unramified" maps $\mathbf{d}_{F,\mathfrak{q}}^f$ of Proposition 4.6.8 and then compose them with the appropriate map to $(\mathbb{W}_M/\mathrm{Ind}_{\mathcal{D}}(W_M))^{G_{F(\mathfrak{r})}}$ in the diagram of Lemma 4.6.7.

In fact, that is how our proof of Theorem 4.5.1 will proceed. We construct the maps $\mathbf{d}_F$ and $\mathbf{d}_{F,\mathfrak{q}}^f$ lifting our Euler system $\mathbf{c}$. This gives us two different constructions of $\mathbf{d}_{F,\mathfrak{q}}$ and we compare them using the uniqueness assertion of Proposition 4.6.8.

Proof of Theorem 4.5.1. Keep the notation from the beginning of this section, so $M \in \mathcal{O}$ is nonzero and we now suppose that $\mathfrak{r} \in \mathcal{R}_{F,M}$. Fix a prime $\mathfrak{q}$ of K, not necessarily in $\mathcal{R}$. Fix a lifting $\mathbf{d} : \mathbf{X}_{F(\mathfrak{r})} \to \mathbb{W}_M/W_M$ (resp. $\mathbf{d}_{\mathfrak{q}}^f : \mathbf{X}_{F(\mathfrak{r})} \to \mathbb{W}_M^f/\mathrm{Ind}_{\mathcal{D}}(W_M^f)$) of $\mathbf{c}$ as in Proposition 4.4.8 (resp. Proposition 4.6.8). Write $\mathbf{d}_{\mathfrak{q}}$ (resp $\mathbf{d}'_{\mathfrak{q}}$) for the image of $\mathbf{d}$ (resp $\mathbf{d}_{\mathfrak{q}}^f$) in $\mathrm{Hom}(\mathbf{X}_{F(\mathfrak{r})}, \mathbb{W}_M/\mathrm{Ind}_{\mathcal{D}}(W_M))$ in the diagram of Lemma 4.6.7. From the uniqueness portion of Proposition 4.6.8 it follows that

$$\mathbf{d}_{\mathfrak{q}} - \mathbf{d}'_{\mathfrak{q}} \in \mathrm{image}(\mathrm{Hom}(\mathbf{X}_{F(\mathfrak{r})}, \mathbb{W}_M^{G_{F(\mathfrak{r})}})).$$

In particular, Lemma 4.4.2 shows that, in the center column of the diagram of Lemma 4.6.7 with $L = F$, we have

$$\mathbf{d}_{\mathfrak{q}}(D_{\mathfrak{r},F}x_{F(\mathfrak{r})}) - \mathbf{d}'_{\mathfrak{q}}(D_{\mathfrak{r},F}x_{F(\mathfrak{r})}) \in \mathrm{image}(\mathbb{W}_M^{G_F}) = \ker(\delta_{F_{\mathfrak{q}}}).$$

By definition $\kappa_{[F,\mathfrak{r},M]} = \delta_F(\mathbf{d}(D_{\mathfrak{r},F}x_{F(\mathfrak{r})}))$, so the diagram of Lemma 4.6.7 shows that $(\kappa_{[F,\mathfrak{r},M]})_{\mathfrak{q}}$ is equal to the image of $\kappa_{[F,\mathfrak{r},M]}^{(\mathfrak{q},f)}$ in $H^1(F_{\mathfrak{q}}, W_M)$, where

$$\kappa_{[F,\mathfrak{r},M]}^{(\mathfrak{q},f)} = \delta_{F_{\mathfrak{q}},W_M^f}(\mathbf{d}_{\mathfrak{q}}^f(D_{\mathfrak{r},F}x_{F(\mathfrak{r})})) \in H^1(F_{\mathfrak{q}}, W_M^f).$$

As in Lemma 4.4.13, the restriction of $\kappa_{[F,\mathfrak{r},M]}^{(\mathfrak{q},f)}$ to $H^1(F(\mathfrak{r})_{\mathfrak{q}}, W_M^f)$ is the image of $(D_{\mathfrak{r},F}\mathbf{c}_{F(\mathfrak{r})})_{\mathfrak{q}}$ under the map

$$H^1(F(\mathfrak{r})_{\mathfrak{q}}, T^{\mathcal{I}}) \longrightarrow H^1(F(\mathfrak{r})_{\mathfrak{q}}, W_M^f)$$

(using (4.7) to view $(D_{\mathfrak{r},F}\mathbf{c}_{F(\mathfrak{r})})_{\mathfrak{q}} \in H^1(F(\mathfrak{r})_{\mathfrak{q}}, T^{\mathcal{I}})$). Suppose $\mathcal{Q}$ is a prime of F above $\mathfrak{q}$, and $\mathcal{I}_{\mathcal{Q}}$ is an inertia group of $\mathcal{Q}$ in G_F. Then $\mathcal{I}_{\mathcal{Q}} \subset G_{F(\mathfrak{r})}$, and as in Proposition 4.6.1 the restriction of $(D_{\mathfrak{r},F}\mathbf{c}_{F(\mathfrak{r})})_{\mathfrak{q}}$ to $\mathcal{I}_{\mathcal{Q}}$ is zero. We conclude that the restriction of $\kappa^{(\mathfrak{q},f)}_{[F,\mathfrak{r},M]}$ to $\mathcal{I}_{\mathcal{Q}}$ is zero and hence

$$(\kappa^{(\mathfrak{q},f)}_{[F,\mathfrak{r},M]})_{\mathcal{Q}} \in H^1(F^{\mathrm{ur}}_{\mathcal{Q}}/F_{\mathcal{Q}}, T^{\mathcal{I}_{\mathcal{Q}}}/MT^{\mathcal{I}_{\mathcal{Q}}}).$$

Since $\mathrm{Gal}(F^{\mathrm{ur}}_{\mathcal{Q}}/F_{\mathcal{Q}})$ has cohomological dimension one, taking $\mathrm{Gal}(F^{\mathrm{ur}}_{\mathcal{Q}}/F_{\mathcal{Q}})$-cohomology of the short exact sequence

$$0 \longrightarrow T^{\mathcal{I}_{\mathcal{Q}}} \xrightarrow{M} T^{\mathcal{I}_{\mathcal{Q}}} \longrightarrow T^{\mathcal{I}_{\mathcal{Q}}}/MT^{\mathcal{I}_{\mathcal{Q}}} \longrightarrow 0$$

gives a surjective map

$$H^1_{\mathrm{ur}}(F_{\mathcal{Q}}, T) = H^1(F^{\mathrm{ur}}_{\mathcal{Q}}/F_{\mathcal{Q}}, T^{\mathcal{I}_{\mathcal{Q}}}) \twoheadrightarrow H^1(F^{\mathrm{ur}}_{\mathcal{Q}}/F_{\mathcal{Q}}, T^{\mathcal{I}_{\mathcal{Q}}}/MT^{\mathcal{I}_{\mathcal{Q}}}).$$

Thus we conclude finally that $(\kappa_{[F,\mathfrak{r},M]})_{\mathcal{Q}}$ lies in the image of $H^1_{\mathrm{ur}}(F_{\mathcal{Q}}, T)$, so by Lemmas 1.3.5(ii) and 1.3.8(i),

$$(\kappa_{[F,\mathfrak{r},M]})_{\mathcal{Q}} \in H^1_f(F_{\mathcal{Q}}, W_M). \qquad \square$$

4.7. Local Behavior at Primes Dividing $\mathfrak{r}$

Fix for this section an Euler system $\mathbf{c}$ for T, a nonzero $M \in \mathcal{O}$, an ideal $\mathfrak{r} \in \mathcal{R}$, a prime $\mathfrak{q} \in \mathcal{R}$ (which may or may not divide $\mathfrak{r}$), and a finite extension F of K in K_∞.

Fix a prime $\mathfrak{Q}$ of $\bar{K}$ above $\mathfrak{q}$ and let $\mathcal{I} \subset \mathcal{D}$ be the inertia and decomposition groups, respectively, of $\mathfrak{Q}$ in G_K. Since $K(\mathfrak{q})/K(\mathbf{1})$ is totally ramified at $\mathfrak{q}$, the natural map $\mathcal{I} \to \Gamma_{\mathfrak{q}}$ is surjective, so we can choose a lift of $\sigma_{\mathfrak{q}}$ to $\mathcal{I}$ which we will also denote by $\sigma_{\mathfrak{q}}$. With this choice we will view

$$N_{\mathfrak{q}} = \sum_{i=0}^{[K(\mathfrak{q}):K(\mathbf{1})]-1} \sigma_{\mathfrak{q}}^i, \quad D_{\mathfrak{q}} = \sum_{i=0}^{[K(\mathfrak{q}):K(\mathbf{1})]-1} i\sigma_{\mathfrak{q}}^i \quad \in \mathbf{Z}[\mathcal{I}].$$

However, writing $m = [K(\mathfrak{q}) : K(\mathbf{1})]$, we no longer have $\sigma_{\mathfrak{q}}^m = 1$ in $\mathcal{I}$, so instead of the identity (4.4) we have

$$(\sigma_{\mathfrak{q}} - 1)D_{\mathfrak{q}} = m\sigma_{\mathfrak{q}}^m - N_{\mathfrak{q}} \tag{4.10}$$

in $\mathbf{Z}[\mathcal{I}]$. Fix also some choice $\mathrm{Fr}_{\mathfrak{q}} \in \mathcal{D}$ of Frobenius for $\mathfrak{Q}$, and fix a lift of the element $N_{F(\mathbf{1})/F}$ of Definition 4.4.10 to $\mathbf{Z}[G_F]$, so that we can view $D_{\mathfrak{r},F} \in \mathbf{Z}[G_F]$.

Lemma 4.7.1. *Suppose* $\mathbf{d} : \mathbf{X}_{F(\mathfrak{r})} \to \mathbb{W}_M/W_M$ *is a lifting of* $\mathbf{c}$ *as in Proposition* 4.4.8, *and* $\hat{\mathbf{d}}(x_{F(\mathfrak{r})}) \in \mathbb{W}_M$ *is a lifting of* $\mathbf{d}(x_{F(\mathfrak{r})})$. *Then for every* $\gamma \in G_K$ *and* $\rho, \rho' \in \mathcal{D}$,

$$\rho\rho'\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{r})}) = \rho'\rho\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{r})}).$$

Proof. Let $\mathcal{I}_{F(\mathfrak{r})} = \mathcal{I} \cap G_{F(\mathfrak{r})}$. Since T is unramified at $\mathfrak{q}$, Proposition 4.6.1 shows that

$$\mathrm{res}_{\mathcal{I}_{F(\mathfrak{r})}}(\gamma \mathbf{c}_{F(\mathfrak{r})}) = 0 \text{ in } H^1(\mathcal{I}_{F(\mathfrak{r})}, T) = \mathrm{Hom}(\mathcal{I}_{F(\mathfrak{r})}, T).$$

Thus every cocycle representing $\gamma \mathbf{c}_{F(\mathfrak{r})}$ vanishes on $\mathcal{I}_{F(\mathfrak{r})}$. In particular by Proposition 4.4.5(ii),

$$(\sigma - 1)\gamma \hat{\mathbf{d}}(x_{F(\mathfrak{r})}) = 0 \quad \text{in } \mathbb{W}_M, \text{ for every } \sigma \in \mathcal{I}_{F(\mathfrak{r})}. \tag{4.11}$$

Since $\mathcal{D}/\mathcal{I}$ and $\mathrm{Gal}(F(\mathfrak{r})/K)$ are abelian, the commutator subgroup of $\mathcal{D}$ is contained in both $\mathcal{I}$ and $G_{F(\mathfrak{r})}$. In particular if we apply (4.11) with $\sigma = \rho^{-1}\rho'^{-1}\rho\rho' \in \mathcal{I}_{F(\mathfrak{r})}$, the lemma follows. □

Remark 4.7.2. Suppose that ρ and ρ' belong to G_K, but not necessarily to $\mathcal{D}$. Then $\rho\rho'\mathbf{d}(x_{F(\mathfrak{r})}) = \rho'\rho\mathbf{d}(x_{F(\mathfrak{r})})$, because $\mathbf{d}$ is G_K-equivariant and the action of G_K on $x_{F(\mathfrak{r})}$ factors through an abelian extension of K. However, the action of G_K on $\hat{\mathbf{d}}(x_{F(\mathfrak{r})})$ will not in general factor through an abelian extension of K so it is not in general true that $\rho\rho'\hat{\mathbf{d}}(x_{F(\mathfrak{r})}) = \rho'\rho\hat{\mathbf{d}}(x_{F(\mathfrak{r})})$. However, Lemma 4.7.1 shows that this does hold if $\rho, \rho' \in \mathcal{D}$. We will use this repeatedly below.

Note that Lemma 4.7.1 is true whether or not $\mathfrak{q}$ divides $\mathfrak{r}$.

Theorem 4.5.4 which will follow easily from the following lemma.

Lemma 4.7.3. *Suppose that $\mathfrak{q} \in \mathcal{R}_{K,M}$ and that $\mathfrak{q}$ does not divide $\mathfrak{r}$. Fix a lifting $\mathbf{d} : \mathbf{X}_{F(\mathfrak{rq})} \to \mathbb{W}_M/W_M$ of $\mathbf{c}$ as in Proposition 4.4.8, and fix liftings $\hat{\mathbf{d}}(x_{F(\mathfrak{rq})}), \hat{\mathbf{d}}(x_{F(\mathfrak{r})}) \in \mathbb{W}_M$ of $\mathbf{d}(x_{F(\mathfrak{rq})})$ and $\mathbf{d}(x_{F(\mathfrak{r})})$, respectively. Then for every $\gamma \in G_K$,*

$$N_{\mathfrak{q}}\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{rq})}) = P(\mathrm{Fr}_{\mathfrak{q}}^{-1}|T^*; \mathrm{Fr}_{\mathfrak{q}}^{-1})\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{r})}).$$

Proof. We will abbreviate $P_{\mathfrak{q}}(x) = P(\mathrm{Fr}_{\mathfrak{q}}^{-1}|T^*; x)$. Note that

$$N_{\mathfrak{q}}\gamma\mathbf{d}(x_{F(\mathfrak{rq})}) = P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\gamma\mathbf{d}(x_{F(\mathfrak{r})})$$

since $\mathbf{d}$ is G_K-equivariant and $N_{\mathfrak{q}}x_{F(\mathfrak{rq})} = P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})x_{F(\mathfrak{r})}$, so

$$N_{\mathfrak{q}}\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{rq})}) - P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{r})}) \in W_M.$$

First we will show that $N_{\mathfrak{q}}\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{rq})}) - P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{r})})$ is independent of the choices of $\mathbf{d}$ and $\hat{\mathbf{d}}$. Suppose we replace $\mathbf{d}$ by some other choice $\mathbf{d}'$. By Proposition 4.4.8, $\mathbf{d}' = \mathbf{d} + \mathbf{d}_0$ with $\mathbf{d}_0 \in \mathrm{Hom}_{G_K}(\mathbf{X}_{F(\mathfrak{rq})}, \mathbb{W}_M)$. Therefore if we choose liftings $\hat{\mathbf{d}}'(x_{F(\mathfrak{rq})})$ and $\hat{\mathbf{d}}'(x_{F(\mathfrak{r})}) \in \mathbb{W}_M$ of $\mathbf{d}'(x_{F(\mathfrak{rq})})$ and $\mathbf{d}'(x_{F(\mathfrak{r})})$, respectively, they must satisfy

$$\begin{aligned}\hat{\mathbf{d}}'(x_{F(\mathfrak{rq})}) &= \hat{\mathbf{d}}(x_{F(\mathfrak{rq})}) + \mathbf{d}_0(x_{F(\mathfrak{rq})}) + t,\\ \hat{\mathbf{d}}'(x_{F(\mathfrak{r})}) &= \hat{\mathbf{d}}(x_{F(\mathfrak{r})}) + \mathbf{d}_0(x_{F(\mathfrak{r})}) + t'\end{aligned}$$

where $t, t' \in W_M$. Thus, since $\mathbf{d}_0$ is G_K-equivariant,

$$\begin{aligned}(N_\mathfrak{q}\gamma\hat{\mathbf{d}}'(x_{F(\mathfrak{r}\mathfrak{q})}) - P_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\gamma\hat{\mathbf{d}}'(x_{F(\mathfrak{r})}))& \\ - (N_\mathfrak{q}\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}) - P_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{r})}))& \\ = \mathbf{d}_0(\gamma(N_\mathfrak{q}x_{F(\mathfrak{r}\mathfrak{q})} - P_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})x_{F(\mathfrak{r})})) + N_\mathfrak{q}\gamma t - P_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\gamma t'& \\ = N_\mathfrak{q}\gamma t - P_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\gamma t'.&\end{aligned}$$

Since $\sigma_\mathfrak{q}$ fixes W_M, and M divides $[K(\mathfrak{q}) : K(\mathbf{1})]$ (because $\mathfrak{q} \in \mathcal{R}_{K,M}$), we see that $N_\mathfrak{q}$ annihilates W_M. By Lemma 4.1.2(iv), $P_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})$ annihilates W_M as well. Therefore $N_\mathfrak{q}\gamma t - P_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\gamma t' = 0$ as desired.

Next we will make useful choices of $\hat{\mathbf{d}}(x_{F(\mathfrak{r})})$ and $\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})})$. Choose $k \in \mathbf{Z}^+$ so that $\mathrm{Fr}_\mathfrak{q}^k$ is the identity on both $F(\mathfrak{r}\mathfrak{q})$ and W_M, and let k_p be the largest power of p dividing k. Since the decomposition group of $\mathfrak{q}$ in $\mathrm{Gal}(K_\infty/K)$ is infinite, we can fix a finite extension F' of F in K_∞ such that the decomposition group of $\mathfrak{q}$ in $F'(\mathfrak{r}\mathfrak{q})/F(\mathfrak{r}\mathfrak{q})$ has order divisible by k_pM. Choose a lifting $\mathbf{d} : \mathbf{X}_{F'(\mathfrak{r}\mathfrak{q})} \to \mathbb{W}_M/W_M$ of $\mathbf{c}$ as in Proposition 4.4.8.

Let $H \subset \mathrm{Gal}(F'(\mathfrak{r}\mathfrak{q})/F(\mathfrak{r}\mathfrak{q}))$ be the subgroup generated by $\mathrm{Fr}_\mathfrak{q}^k$. Fix a set $B \subset G_{F(\mathfrak{r}\mathfrak{q})}$ of coset representatives of $\mathrm{Gal}(F'(\mathfrak{r}\mathfrak{q})/F(\mathfrak{r}\mathfrak{q}))/H$. Write

$$\mathbf{N}' = \sum_{i=0}^{|H|-1} \mathrm{Fr}_\mathfrak{q}^{ki}, \quad \mathbf{N}'' = \sum_{\beta\in B} \beta \quad \in \mathbf{Z}[G_{F(\mathfrak{r}\mathfrak{q})}].$$

The product $\mathbf{N}'\mathbf{N}''$ restricts to the norm from $F'(\mathfrak{r}\mathfrak{q})$ to $F(\mathfrak{r}\mathfrak{q})$, so in particular

$$\mathbf{N}'\mathbf{N}''x_{F'(\mathfrak{r}\mathfrak{q})} = x_{F(\mathfrak{r}\mathfrak{q})} \text{ and } \mathbf{N}'\mathbf{N}''x_{F'(\mathfrak{r})} = x_{F(\mathfrak{r})} \tag{4.12}$$

in $\mathbf{X}_{F'(\mathfrak{r}\mathfrak{q})}$.

Choose liftings $\hat{\mathbf{d}}(x_{F'(\mathfrak{r}\mathfrak{q})}), \hat{\mathbf{d}}(x_{F'(\mathfrak{r})}) \in \mathbb{W}_M$ of $\mathbf{d}(x_{F'(\mathfrak{r}\mathfrak{q})}), \mathbf{d}(x_{F'(\mathfrak{r})}) \in \mathbb{W}_M/W_M$, respectively, and define

$$\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}) = \gamma^{-1}\mathbf{N}'\mathbf{N}''\gamma\hat{\mathbf{d}}(x_{F'(\mathfrak{r}\mathfrak{q})}), \quad \hat{\mathbf{d}}(x_{F(\mathfrak{r})}) = \gamma^{-1}\mathbf{N}'\mathbf{N}''\gamma\hat{\mathbf{d}}(x_{F'(\mathfrak{r})}).$$

It follows from (4.12) that these are liftings of $\mathbf{d}(x_{F(\mathfrak{r}\mathfrak{q})})$ and $\mathbf{d}(x_{F(\mathfrak{r})})$, respectively, to $\mathbb{W}_M$. We will prove the lemma by showing that with these choices, $N_\mathfrak{q}\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{r})}) - P_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}) = 0$.

Note that $\mathbf{N}'$, $P_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})$, and $N_\mathfrak{q}$ all belong to $\mathcal{O}[\mathcal{D}]$ because $\mathrm{Fr}_\mathfrak{q}$ and $\sigma_\mathfrak{q}$ do, so by Lemma 4.7.1 these elements commute in their action on $\mathbf{N}''\gamma\hat{\mathbf{d}}(x_{F'(\mathfrak{r}\mathfrak{q})})$ and $\mathbf{N}''\gamma\hat{\mathbf{d}}(x_{F'(\mathfrak{r})})$. Thus

$$\begin{aligned}N_\mathfrak{q}\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{r})}) - P_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})})& \\ = N_\mathfrak{q}\mathbf{N}'\mathbf{N}''\gamma\hat{\mathbf{d}}(x_{F'(\mathfrak{r}\mathfrak{q})}) - P_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\mathbf{N}'\mathbf{N}''\gamma\hat{\mathbf{d}}(x_{F'(\mathfrak{r})})& \\ = \mathbf{N}'(N_\mathfrak{q}\mathbf{N}''\gamma\hat{\mathbf{d}}(x_{F'(\mathfrak{r}\mathfrak{q})}) - P_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\mathbf{N}''\gamma\hat{\mathbf{d}}(x_{F'(\mathfrak{r})})) \in \mathbf{N}'W_M,&\end{aligned}$$

the last inclusion because $N_\mathfrak{q}\mathbf{N}''\gamma\hat{\mathbf{d}}(x_{F'(\mathfrak{r}\mathfrak{q})}) - P_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\mathbf{N}''\gamma\hat{\mathbf{d}}(x_{F'(\mathfrak{r})}) \in \mathbb{W}_M$ projects to $\mathbf{N}''\gamma\mathbf{d}(N_\mathfrak{q}x_{F'(\mathfrak{r}\mathfrak{q})} - P_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})x_{F'(\mathfrak{r})}) = 0$ in $\mathbb{W}_M/W_M$. Since $\mathrm{Fr}_\mathfrak{q}^k$ fixes W_M, we have

$$\mathbf{N}'W_M \subset |H|\, W_M.$$

Now observe that H has index dividing k_p in the decomposition group of $\mathfrak{q}$ in $F'(\mathfrak{r}\mathfrak{q})/F(\mathfrak{r}\mathfrak{q})$, so in particular M divides $|H|$. This completes the proof. □

Remark 4.7.4. The proof of Lemma 4.7.3 used in an essential way the Euler system classes $\mathbf{c}_{F'(\mathfrak{r}\mathfrak{q})}$ and $\mathbf{c}_{F'(\mathfrak{r})}$ for $F \subset_\mathrm{f} F' \subset K_\infty$, and not just $\mathbf{c}_{F(\mathfrak{r}\mathfrak{q})}$ and $\mathbf{c}_{F(\mathfrak{r})}$.

Proof of Theorem 4.5.4. Keep the notation from the beginning of this section, and suppose now that $\mathfrak{r}\mathfrak{q} \in \mathcal{R}_{F,M}$. Choose $Q_\mathfrak{q} \in \mathcal{O}[x]$ as in Lemma 4.5.2, so that $Q_\mathfrak{q}(x)(x-1) \equiv P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;x) \pmod{M}$. To prove the theorem we need to show that, for some (or equivalently, for every) choice of cocycles representing $\kappa_{[F,\mathfrak{r},M]}$ and $\kappa_{[F,\mathfrak{r}\mathfrak{q},M]}$, we have

$$Q_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\kappa_{[F,\mathfrak{r},M]}(\mathrm{Fr}_\mathfrak{q}) = \kappa_{[F,\mathfrak{r}\mathfrak{q},M]}(\sigma_\mathfrak{q}) \in W_M.$$

Fix $\mathbf{d} : \mathbf{X}_{F(\mathfrak{r}\mathfrak{q})} \to \mathbb{W}_M/W_M$ lifting $\mathbf{c}$ as in Proposition 4.4.8, and choose liftings $\hat{\mathbf{d}}(x_{F(\mathfrak{r})})$, $\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}) \in \mathbb{W}_M$ of $\mathbf{d}(x_{F(\mathfrak{r})})$, $\mathbf{d}(x_{F(\mathfrak{r}\mathfrak{q})}) \in \mathbb{W}_M/W_M$, respectively. Lemma 4.4.12 shows that

$$\begin{aligned}\kappa_{[F,\mathfrak{r},M]}(\mathrm{Fr}_\mathfrak{q}) &= (\mathrm{Fr}_\mathfrak{q} - 1)D_{\mathfrak{r},F}\hat{\mathbf{d}}(x_{F(\mathfrak{r})}) \in W_M,\\ \kappa_{[F,\mathfrak{r}\mathfrak{q},M]}(\sigma_\mathfrak{q}) &= (\sigma_\mathfrak{q} - 1)D_\mathfrak{q}D_{\mathfrak{r},F}\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}) \in W_M.\end{aligned}$$

Also

$$Q_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})(\mathrm{Fr}_\mathfrak{q}^{-1} - 1)\kappa_{[F,\mathfrak{r},M]}(\mathrm{Fr}_\mathfrak{q}) = P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;\mathrm{Fr}_\mathfrak{q}^{-1})\kappa_{[F,\mathfrak{r},M]}(\mathrm{Fr}_\mathfrak{q}) = 0$$

by Lemma 4.1.2(iv). Thus, using Lemma 4.7.1 repeatedly to commute elements of $\mathcal{O}[\mathcal{D}]$, and using (4.10), we see

$$\begin{aligned}&Q_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\kappa_{[F,\mathfrak{r},M]}(\mathrm{Fr}_\mathfrak{q}) - \kappa_{[F,\mathfrak{r}\mathfrak{q},M]}(\sigma_\mathfrak{q})\\ &\quad= Q_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\mathrm{Fr}_\mathfrak{q}^{-1}\kappa_{[F,\mathfrak{r},M]}(\mathrm{Fr}_\mathfrak{q}) - \kappa_{[F,\mathfrak{r}\mathfrak{q},M]}(\sigma_\mathfrak{q})\\ &\quad= Q_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\mathrm{Fr}_\mathfrak{q}^{-1}(\mathrm{Fr}_\mathfrak{q} - 1)D_{\mathfrak{r},F}\hat{\mathbf{d}}(x_{F(\mathfrak{r})}) - (\sigma_\mathfrak{q} - 1)D_\mathfrak{q}D_{\mathfrak{r},F}\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})})\\ &\quad= -P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;\mathrm{Fr}_\mathfrak{q}^{-1})D_{\mathfrak{r},F}\hat{\mathbf{d}}(x_{F(\mathfrak{r})}) + N_\mathfrak{q}D_{\mathfrak{r},F}\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})})\\ &\qquad\qquad - [K(\mathfrak{q}) : K(\mathbf{1})]\sigma_\mathfrak{q}^{[K(\mathfrak{q}):K(\mathbf{1})]}D_{\mathfrak{r},F}\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}).\end{aligned}$$

Since $\mathfrak{q} \in \mathcal{R}_{F,M}$ we have $M \mid [K(\mathfrak{q}) : K(\mathbf{1})]$. Thus by Lemma 4.7.3 we conclude that $Q_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\kappa_{[F,\mathfrak{r},M]}(\mathrm{Fr}_\mathfrak{q}) - \kappa_{[F,\mathfrak{r}\mathfrak{q},M]}(\sigma_\mathfrak{q}) = 0$ in W_M, as desired. □

4.8. The Congruence

Although we will not need it, we can now prove the following corollary (the "congruence condition" for an Euler system) which was promised in Remark 2.1.5. We again write $P_{\mathfrak{q}}(x) = P(\mathrm{Fr}_{\mathfrak{q}}^{-1}|T^*;x)$.

Corollary 4.8.1. *Suppose that* $\mathbf{c}$ *is an Euler system for* T *and* $\mathfrak{q} \in \mathcal{R}$ *is prime. If* $K \subset_{\mathrm{f}} F \subset K_\infty$ *and* $\mathfrak{r}\mathfrak{q} \in \mathcal{R}$*, then for every prime* $\mathcal{Q}$ *of* $F(\mathfrak{r}\mathfrak{q})$ *dividing* $\mathfrak{q}$*,*

$$(\mathbf{c}_{F(\mathfrak{r}\mathfrak{q})})_{\mathcal{Q}} = \frac{P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1}) - P_{\mathfrak{q}}(\mathbf{N}(\mathfrak{q})\mathrm{Fr}_{\mathfrak{q}}^{-1})}{[K(\mathfrak{q}):K(\mathbf{1})]}(\mathbf{c}_{F(\mathfrak{r})})_{\mathcal{Q}} \in H^1(F(\mathfrak{r}\mathfrak{q})_{\mathcal{Q}}, T).$$

Proof. Write

$$R(x) = \frac{P_{\mathfrak{q}}(x) - P_{\mathfrak{q}}(\mathbf{N}(\mathfrak{q})x)}{[K(\mathfrak{q}):K(\mathbf{1})]}.$$

Since $[K(\mathfrak{q}):K(\mathbf{1})]$ divides $(\mathbf{N}(\mathfrak{q})-1)$, we see that $R(x) \in \mathcal{O}[x]$.

Keep the notation and setting from the beginning of the previous section, and let

$$c = \mathbf{c}_{F(\mathfrak{r}\mathfrak{q})} - R(\mathrm{Fr}_{\mathfrak{q}}^{-1})\mathbf{c}_{F(\mathfrak{r})} \in H^1(F(\mathfrak{r}\mathfrak{q}), T).$$

For every nonzero $M \in \mathcal{O}$ let $(c)_{\mathcal{Q},M}$ be the image of c in $H^1(F(\mathfrak{r}\mathfrak{q})_{\mathcal{Q}}, W_M)$. Proposition B.2.3 shows that $H^1(F(\mathfrak{r}\mathfrak{q})_{\mathcal{Q}}, T) = \varprojlim H^1(F(\mathfrak{r}\mathfrak{q})_{\mathcal{Q}}, W_M)$, so to prove the corollary it will suffice to show that $(c)_{\mathcal{Q},M} = 0$ for every M.

Fix an M divisible by $[K(\mathfrak{q}):K(\mathbf{1})]$, and a lifting

$$\mathbf{d} : \mathbf{X}_{F(\mathfrak{r}\mathfrak{q})} \to \mathbb{W}_M/W_M$$

of $\mathbf{c}$ as in Proposition 4.4.8. Choose $\hat{\mathbf{d}}(x_{F(\mathfrak{r})}), \hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}) \in \mathbb{W}_M$ lifting $\mathbf{d}(x_{F(\mathfrak{r})}), \mathbf{d}(x_{F(\mathfrak{r}\mathfrak{q})}) \in \mathbb{W}_M/W_M$, respectively. Fix a Frobenius element $\mathrm{Fr}_{\mathfrak{q}}$ of $\mathfrak{q}$ corresponding to a prime of $\bar{K}$ above $\mathcal{Q}$. Then a Frobenius element for $\mathcal{Q}$ in $G_{F(\mathfrak{r}\mathfrak{q})}$ is given by $\varphi = \mathrm{Fr}_{\mathfrak{q}}^k$ for some k. By Proposition 4.6.1, $(c)_{\mathcal{Q},M} \in H^1_{\mathrm{ur}}(F(\mathfrak{r}\mathfrak{q})_{\mathcal{Q}}, W_M)$, and by Lemma 1.3.2(i) there is an isomorphism

$$\begin{array}{ccc} H^1_{\mathrm{ur}}(F(\mathfrak{r}\mathfrak{q})_{\mathcal{Q}}, W_M) & \xrightarrow{\sim} & W_M/(\varphi-1)W_M \\ (c)_{\mathcal{Q},M} & \mapsto & c(\varphi). \end{array}$$

Proposition 4.4.5(ii) shows that

$$\gamma \mapsto (\gamma-1)(\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}) - R(\mathrm{Fr}_{\mathfrak{q}}^{-1})\hat{\mathbf{d}}(x_{F(\mathfrak{r})})) \in W_M$$

is a cocycle representing $(c)_{\mathcal{Q},M}$, so

$$(c)_{\mathcal{Q},M} = 0 \iff (\varphi-1)(\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}) - R(\mathrm{Fr}_{\mathfrak{q}}^{-1})\hat{\mathbf{d}}(x_{F(\mathfrak{r})})) \in (\varphi-1)W_M.$$

Note that

$$(\varphi-1)\hat{\mathbf{d}}(x_{F(\mathfrak{r})}),\ (\varphi-1)\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}) \in W_M$$

and

$$N_{\mathfrak{q}},\ \varphi - 1,\ P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1}) \ \in \mathcal{O}[\mathcal{D}].$$

Therefore

$$\begin{aligned}[K(\mathfrak{q}) : K(\mathbf{1})](\varphi - 1)(\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}) - R(\mathrm{Fr}_{\mathfrak{q}}^{-1})\hat{\mathbf{d}}(x_{F(\mathfrak{r})})) \\ = N_{\mathfrak{q}}(\varphi - 1)\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}) - P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})(\varphi - 1)\hat{\mathbf{d}}(x_{F(\mathfrak{r})}) \\ = (\varphi - 1)(N_{\mathfrak{q}}\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}) - P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\hat{\mathbf{d}}(x_{F(\mathfrak{r})}))\end{aligned}$$

the first equality since $\sigma_{\mathfrak{q}}$ fixes W_M and $P_{\mathfrak{q}}(\mathbf{N}(\mathfrak{q})\mathrm{Fr}_{\mathfrak{q}}^{-1})$ annihilates W_M (Lemma 4.1.2(ii)), and the second by Lemma 4.7.1. Lemma 4.7.3 shows that the image of $N_{\mathfrak{q}}\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}) - P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\hat{\mathbf{d}}(x_{F(\mathfrak{r})})$ under the projection $W_M \twoheadrightarrow W_{[K(\mathfrak{q}):K(\mathbf{1})]}$ is zero, and we conclude that

$$\begin{aligned}[K(\mathfrak{q}) : K(\mathbf{1})](\varphi - 1)(\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}) - R(\mathrm{Fr}_{\mathfrak{q}}^{-1})\hat{\mathbf{d}}(x_{F(\mathfrak{r})})) \\ \in [K(\mathfrak{q}) : K(\mathbf{1})](\varphi - 1)W_M.\end{aligned}$$

It follows that $(c)_{\mathcal{Q},M/[K(\mathfrak{q}):K(\mathbf{1})]} = 0$, and since this holds for every M, the corollary is proved. □

Example 4.8.2. Suppose $T = \mathbf{Z}_p(1)$. Then for every $\mathfrak{r} \in \mathcal{R}$ and every prime $\mathcal{Q}$ of $F(\mathfrak{r})$ not dividing p (see Example 1.2.1)

$$H^1(F(\mathfrak{r}), T) \ = \ (F(\mathfrak{r})^\times)^\wedge, \ \ H^1(F(\mathfrak{r})_{\mathcal{Q}}, T) \ = \ (F(\mathfrak{r})_{\mathcal{Q}}^\times)^\wedge \ \cong \ (\mathbb{k}_{\mathcal{Q}}^\times)^\wedge$$

where $(\ \cdot\ )^\wedge$ denotes the p-adic completion and $\mathbb{k}_{\mathcal{Q}}$ is the residue field of $F(\mathfrak{r})$ modulo $\mathcal{Q}$. In this case $P_{\mathfrak{q}}(x) = \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1}x|\mathbf{Z}_p) = 1 - x$, so

$$\frac{P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1}) - P_{\mathfrak{q}}(\mathbf{N}(\mathfrak{q})\mathrm{Fr}_{\mathfrak{q}}^{-1})}{[K(\mathfrak{q}) : K(\mathbf{1})]} \ = \ \frac{\mathbf{N}(\mathfrak{q}) - 1}{[K(\mathfrak{q}) : K(\mathbf{1})]}\mathrm{Fr}_{\mathfrak{q}}^{-1}.$$

Thus viewing $\mathbf{c}_{F(\mathfrak{r})}, \mathbf{c}_{F(\mathfrak{r}\mathfrak{q})} \in (F(\mathfrak{r})^\times)^\wedge$, Corollary 4.8.1 in this case says

$$\mathbf{c}_{F(\mathfrak{r}\mathfrak{q})} \ = \ \mathbf{c}_{F(\mathfrak{r})}^{\frac{\mathbf{N}(\mathfrak{q})-1}{[K(\mathfrak{q}):K(\mathbf{1})]}\mathrm{Fr}_{\mathfrak{q}}^{-1}} \quad \text{in } (\mathbb{k}_{\mathcal{Q}}^\times)^\wedge. \tag{4.13}$$

For the Euler system of cyclotomic units discussed in §3.2, (4.13) is the "p-part" of the congruence

$$1 - \zeta_{rq} \ \equiv \ 1 - \zeta_r^{\mathrm{Fr}_q^{-1}} \ \text{ modulo every prime of } \bar{\mathbf{Q}} \text{ above } q$$

(which in turn follows from the observation that $\zeta_q \equiv 1$ modulo every prime above q).

CHAPTER 5

Bounding the Selmer Group

In this chapter we will prove Theorems 2.2.2 (in §5.2) and 2.2.3 (in §5.3). For every power M of p we will choose inductively a finite subset Σ of primes in $\mathcal{R}_{K,M}$. As $\mathfrak{r}$ runs through products of primes in Σ, Theorem 4.5.1 shows that the derivative cohomology classes $\kappa_{[K,\mathfrak{r},M]}$ defined in Chapter 4 belong to $\mathcal{S}^{\Sigma\cup\Sigma_p}(K,W_M)$, where Σ_p is the set of primes of K above p, and Theorem 4.5.4 tells us about the singular parts of these classes at primes in Σ. This information and the duality results of §1.7 will allow us to bound the index $[\mathcal{S}_{\Sigma_p}(K,W_M^*) : \mathcal{S}_{\Sigma\cup\Sigma_p}(K,W_M^*)]$. By taking Σ large enough so that $\mathcal{S}_{\Sigma\cup\Sigma_p}(K,W_M^*)=0$, and letting M go to infinity, we will obtain the bound of Theorem 2.2.2.

5.1. Preliminaries

Keep the notation of §2.1 and §2.2. Fix an Euler system $\mathbf{c}$ for $(T,\mathcal{K},\mathcal{N})$ for some $\mathcal{K}$ and $\mathcal{N}$. If M is a power of p we will write $\mathcal{R}_M=\mathcal{R}_{K,M}$ (as defined in Definition 4.1.1), the set of ideals in $\mathcal{R}$ divisible only by primes $\mathfrak{q}$ such that $\mathfrak{q}\nmid\mathcal{N}$, both $[K(\mathfrak{q}):K(\mathbf{1})]$ and $P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;1)$ are divisible by M, and $\mathfrak{q}$ splits completely in $K(\mathbf{1})$. If $\mathfrak{r}\in\mathcal{R}_M$ then $\kappa_{[\mathfrak{r},M]}\in H^1(K,W_M)$ will denote the derivative class $\kappa_{[K,\mathfrak{r},M]}$ defined in §4.4.

Suppose B is an $\mathcal{O}$-module. Recall that $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}$ and $\ell_\mathcal{O}(B)$ denotes the length of B. If $b\in B$, define

$$\mathrm{order}(b,B)\;=\;\inf\{n\ge 0:\mathfrak{p}^n b=0\}\;\le\;\infty,$$

the *exponent* of the smallest power of $\mathfrak{p}$ which annihilates b. Recall that (Definition 2.2.1) $\mathrm{ind}_\mathcal{O}(\mathbf{c})$ is the largest integer n such that $\mathbf{c}_K$ is divisible by $\mathfrak{p}^n$ in $H^1(K,T)/H^1(K,T)_{\mathrm{tors}}$. We will suppose that $\mathrm{ind}_\mathcal{O}(\mathbf{c})$ is finite, or else there is nothing to prove.

If $M\in\mathcal{O}$ is nonzero, we let $\iota_M:H^1(K,W_M)\to H^1(K,W)$ denote the map induced by the inclusion of W_M in W. If L is an extension of K and $\eta\in H^1(K,W_M^*)$, we write $(\eta)_L$ for the restriction of η to L, and similarly with W_M in place of W_M^*.

Lemma 5.1.1. *Suppose M is a power of p and* $\mathrm{ord}_{\mathfrak{p}} M \geq \mathrm{ind}_{\mathcal{O}}(\mathbf{c})$. *Then*

$$\mathrm{order}(\iota_M(\kappa_{[\mathbf{1},M]}), H^1(K,W)) \;=\; \mathrm{ord}_{\mathfrak{p}} M - \mathrm{ind}_{\mathcal{O}}(\mathbf{c}).$$

Proof. Lemma 4.4.13(i) shows that $\iota_M(\kappa_{[\mathbf{1},M]})$ is the image of $\mathbf{c}_K$ under the composition

$$H^1(K,T) \longrightarrow H^1(K,W_M) \longrightarrow H^1(K,W).$$

The kernel of this composition is $MH^1(K,T) + H^1(K,T)_{\mathrm{tors}}$ by Lemma 1.2.2(iii), so

$$\begin{aligned}\mathrm{order}(&\iota_M(\kappa_{[\mathbf{1},M]}), H^1(K,W)) \\ &= \mathrm{order}(\mathbf{c}_K, H^1(K,T)/(MH^1(K,T) + H^1(K,T)_{\mathrm{tors}}).\end{aligned}$$

Since $H^1(K,T)/H^1(K,T)_{\mathrm{tors}}$ is a torsion-free $\mathcal{O}$-module, it follows from the definition (Definition 2.2.1) of $\mathrm{ind}_{\mathcal{O}}(\mathbf{c})$ that

$$\mathrm{order}(\mathbf{c}_K, H^1(K,T)/(MH^1(K,T) + H^1(K,T)_{\mathrm{tors}})) = \mathrm{ord}_{\mathfrak{p}} M - \mathrm{ind}_{\mathcal{O}}(\mathbf{c}).$$

This proves the lemma. □

5.2. Bounding the Order of the Selmer Group

We divide the proof of Theorem 2.2.2 into two main steps. The first step (Lemma 5.2.3) is to produce inductively a sequence of primes of K with useful properties. The second step (Lemma 5.2.5) is to show that the Kolyvagin derivative classes we construct with these primes generate a subgroup which has large image when we project to the singular part of the local cohomology groups. Once this is accomplished, we have only to plug this information into Theorem 1.7.3, the global duality theorem, and we obtain the desired bound.

Suppose throughout this section that $p > 2$ and that T satisfies hypotheses $\mathrm{Hyp}(K,T)$. Fix a $\tau \in G_K$ as in $\mathrm{Hyp}(K,T)$(i), i.e., τ acts trivially on $K(\mathbf{1})K(\boldsymbol{\mu}_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$ and $T/(\tau-1)T$ is free of rank one over $\mathcal{O}$. Then for every power M of p the $\mathcal{O}/M\mathcal{O}$-module $W_M/(\tau-1)W_M$ is free of rank one, and by Corollary A.2.6, so is $W_M^{\tau=1}$ and hence so is

$$W_M^*/(\tau-1)W_M^* \;\cong\; \mathrm{Hom}(W_M^{\tau=1}, \boldsymbol{\mu}_M \otimes \mathcal{O}).$$

Lemma 5.2.1. *Fix a power M of p. Suppose L is a Galois extension of K such that G_L acts trivially on W_M and on W_M^*. If $\kappa \in H^1(K,W_M)$ and $\eta \in H^1(K,W_M^*)$, then there is an element $\gamma \in G_L$ satisfying*

(i) $\mathrm{order}(\kappa(\gamma\tau), W_M/(\tau-1)W_M)) \geq \mathrm{order}((\kappa)_L, H^1(L,W_M))$,
(ii) $\mathrm{order}(\eta(\gamma\tau), W_M^*/(\tau-1)W_M^*) \geq \mathrm{order}((\eta)_L, H^1(L,W_M^*))$.

Proof. For $\gamma \in G_L$, the image of $\kappa(\gamma\tau)$ in $W_M/(\tau-1)W_M$ is well-defined independent of the choice of cocycle representing κ, and

$$\kappa(\gamma\tau) \equiv \kappa(\gamma) + \kappa(\tau) \pmod{(\tau-1)W_M}, \tag{5.1}$$

and similarly for η.

Define subsets of G_L

$$\begin{aligned} B_\kappa &= \{\gamma \in G_L : \\ &\qquad \mathrm{order}(\kappa(\gamma\tau), W_M/(\tau-1)W_M) < \mathrm{order}((\kappa)_L, \mathrm{Hom}(G_L, W_M))\} \\ B_\eta &= \{\gamma \in G_L : \\ &\qquad \mathrm{order}(\eta(\gamma\tau), W_M^*/(\tau-1)W_M^*) < \mathrm{order}((\eta)_L, \mathrm{Hom}(G_L, W_M^*))\}. \end{aligned}$$

Every $\gamma \in G_L - (B_\kappa \cup B_\eta)$ satisfies the conclusions of the lemma, so we need only show that $B_\kappa \cup B_\eta$ is a proper subset of G_L.

Define a subgroup J of G_L by

$$\begin{aligned} J &= \{\gamma \in G_L : \\ &\qquad \mathrm{order}(\kappa(\gamma), W_M/(\tau-1)W_M) < \mathrm{order}((\kappa)_L, \mathrm{Hom}(G_L, W_M))\}. \end{aligned}$$

By (5.1), if $\gamma, \gamma' \in B_\kappa$ then $\gamma^{-1}\gamma' \in J$. Therefore B_κ either is empty or is a coset of J.

Write $d = \mathrm{order}((\kappa)_L, \mathrm{Hom}(G_L, W_M))$, and consider the image $\kappa(G_L)$ of κ on G_L. Since $(\kappa)_L \in \mathrm{Hom}(G_L, W_M)^{\mathrm{Gal}(L/K)}$, we see that

$$\gamma(\kappa(h)) = \kappa(\gamma h \gamma^{-1}) \text{ for every } h \in G_L \text{ and } \gamma \in G_K,$$

so $\kappa(G_L)$ is a G_K-stable subgroup of $W_{\mathfrak{p}^d}$, not contained in $W_{\mathfrak{p}^{d-1}}$. By hypothesis $\mathrm{Hyp}(K,T)$(ii), $W_\mathfrak{p} = T \otimes \Bbbk$ is irreducible so $\mathfrak{p}^{d-1}\kappa(G_L) = W_\mathfrak{p}$ and therefore $\mathcal{O}\kappa(G_L) = W_{\mathfrak{p}^d}$. Since $W_M/(\tau-1)W_M \cong \mathcal{O}/M\mathcal{O}$, we conclude that

$$\mathcal{O}\kappa(J) \subset W_{\mathfrak{p}^{d-1}} + (\tau-1)W_M \subsetneq W_{\mathfrak{p}^d} = \mathcal{O}\kappa(G_L)$$

and so J has index at least p in G_L.

In exactly the same way, B_η either is empty or is a coset of a subgroup of G_L of index at least p. Since $p > 2$, we cannot have $B_\kappa \cup B_\eta = G_L$. This completes the proof. □

Remark 5.2.2. The end of the previous proof is the only place where we use the assumption that $p > 2$ in Theorem 2.2.2.

Let

$$\Omega = K(\mathbf{1})K(W)K(\boldsymbol{\mu}_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$$

where $K(W)$ denotes the smallest extension of K such that $G_{K(W)}$ acts trivially on W. Note that $K(W^*) \subset \Omega$.

Lemma 5.2.3. *Fix a power M of p. Suppose C is a finite subset of $H^1(K, W_M^*)$ and let $k = |C|$.*

Then there is a finite set $\Sigma = \{\mathfrak{q}_1, \dots, \mathfrak{q}_k\}$ of primes of K such that for $1 \le i \le k$,

(i) $\mathfrak{q}_i \in \mathcal{R}_M$,

(ii) $\mathrm{Fr}_{\mathfrak{q}_i}$ *is in the conjugacy class of* τ *in* $\mathrm{Gal}(K(W_M)/K)$,

(iii) *writing* $\mathfrak{r}_j = \prod_{t=1}^{j} \mathfrak{q}_t$ *(so* $\mathfrak{r}_0 = \mathbf{1}$*), we have*

$$\mathrm{order}((\kappa_{[\mathfrak{r}_{i-1},M]})_{\mathfrak{q}_i}, H^1_f(K_{\mathfrak{q}_i}, W_M)) \ge \mathrm{order}((\kappa_{[\mathfrak{r}_{i-1},M]})_\Omega, H^1(\Omega, W_M)),$$

(iv) $\{\eta \in C : (\eta)_{\mathfrak{q}} = 0$ *for every* $\mathfrak{q} \in \Sigma\} \subset H^1(\Omega/K, W_M^*)$.

Proof. Number the elements of C so that $C = \{\eta_1, \eta_2, \dots, \eta_k\}$. We will choose the $\mathfrak{q}_i$ inductively to satisfy (i), (ii), (iii), and

$$(\eta)_{\mathfrak{q}_i} \in H^1_f(K_{\mathfrak{q}_i}, W_M^*) \text{ for every } \eta \in C, \tag{5.2}$$

$$\mathrm{order}((\eta_i)_{\mathfrak{q}_i}, H^1_f(K_{\mathfrak{q}_i}, W_M^*)) \ge \mathrm{order}((\eta_i)_\Omega, H^1(\Omega, W_M^*)). \tag{5.3}$$

Suppose $1 \le i \le k$ and we have chosen $\{\mathfrak{q}_1, \dots, \mathfrak{q}_{i-1}\}$ satisfying (i), (ii), (iii), (5.2), and (5.3). We need to find $\mathfrak{q}_i$ also satisfying these properties. Define $\mathcal{N}'$ to be the (finite) product of $\mathcal{N}$ and all primes $\mathfrak{q}$ of K such that $\{(\eta)_{\mathfrak{q}} : \eta \in C\} \not\subset H^1_f(K_{\mathfrak{q}}, W_M^*)$. (Recall that $\mathcal{N}$ is divisible by p and all primes where W_M is ramified.)

Let $L = K(\mathbf{1})K(W_M, \boldsymbol{\mu}_M, (\mathcal{O}_K^\times)^{1/M})$, so L is a finite extension of K contained in Ω, and G_L acts trivially on both W_M and W_M^*. Apply Lemma 5.2.1 with this L, with $\kappa = \kappa_{[r_{i-1},M]}$, and with $\eta = \eta_i$, to produce an element $\gamma \in G_L$. Let L' denote the (finite) extension of L which is the fixed field of

$$\ker((\kappa_{[r_{i-1},M]})_L) \cap \ker((\eta_i)_L)$$

where we view $(\kappa_{[r_{i-1},M]})_L \in \mathrm{Hom}(G_L, W_M)$ and $(\eta_i)_L \in \mathrm{Hom}(G_L, W_M^*)$. Let $\mathfrak{q}_i$ be a prime of K not dividing $\mathcal{N}'\mathfrak{r}_{i-1}$, whose Frobenius in L'/K, for some choice of prime above $\mathfrak{q}_i$, is $\gamma\tau$. The Tchebotarev theorem guarantees the existence of infinitely many such primes.

Property (i) holds by Lemma 4.1.3, and (ii) and (5.2) are immediate from the definition. By Lemma 1.4.7(i), evaluating cocycles at $\mathrm{Fr}_{\mathfrak{q}_i}$ induces an isomorphism

$$H^1_f(K_{\mathfrak{q}_i}, W_M) \cong W_M/(\mathrm{Fr}_{\mathfrak{q}_i} - 1)W_M = W_M/(\tau - 1)W_M$$

and similarly for W_M^*, so (iii) and (5.3) follow from Lemma 5.2.1(i) and (ii).

It remains only to check (iv). Define $\Sigma = \{\mathfrak{q}_1, \dots, \mathfrak{q}_k\}$, and suppose that for some i, we have $(\eta_i)_{\mathfrak{q}} = 0$ for every $\mathfrak{q} \in \Sigma$. Then in particular

$(\eta_i)_{\mathfrak{q}_i} = 0$, so (5.3) shows that

$$\eta_i \in \ker\bigl(H^1(K, W_M^*) \to H^1(\Omega, W_M^*)\bigr) = H^1(\Omega/K, W_M^*). \qquad \square$$

Definition 5.2.4. Suppose Σ is a finite set of primes in $\mathcal{R}$. For every M we have an exact sequence

$$0 \longrightarrow \mathcal{S}^{\Sigma_p}(K, W_M) \longrightarrow \mathcal{S}^{\Sigma\cup\Sigma_p}(K, W_M) \xrightarrow{\mathrm{loc}^s_{\Sigma, W_M}} \bigoplus_{\mathfrak{q}\in\Sigma} H^1_s(K_\mathfrak{q}, W_M) \tag{5.4}$$

where we recall that

$$H^1_s(K_\mathfrak{q}, W_M) = H^1(K_\mathfrak{q}, W_M)/H^1_f(K_\mathfrak{q}, W_M)$$

and $\mathrm{loc}^s_{\Sigma, W_M}$ is the sum of the localization maps (in Theorem 1.7.3 the map $\mathrm{loc}^s_{\Sigma, W_M}$ was denoted $\mathrm{loc}^s_{\Sigma\cup\Sigma_p, \Sigma_p}$). We define $\mathrm{loc}^s_{\Sigma, W}$ in exactly the same way with W_M replaced by W.

If $c \in H^1(K, W_M)$ and $\mathfrak{q}$ is a prime, we write $(c)^s_\mathfrak{q}$ for the projection of the localization $(c)_\mathfrak{q}$ into $H^1_s(K_\mathfrak{q}, W_M)$.

If $\mathfrak{a}$ is an ideal of K let $\Sigma_\mathfrak{a}$ denote the set of primes dividing $\mathfrak{a}$. Let

$$\mathfrak{n}_W = \ell_\mathcal{O}\left(H^1(\Omega/K, W) \cap \mathcal{S}^{\Sigma_p}(K, W)\right)$$

as in Theorem 2.2.2.

Lemma 5.2.5. *Suppose that $\mathfrak{m} = \mathfrak{p}^n$ is a nonzero ideal of $\mathcal{O}$, that $k \in \mathbf{Z}^+$, and that M is a power of p satisfying*

$$\mathrm{ord}_\mathfrak{p} M \geq n + (k+1)\mathfrak{n}_W + \mathrm{ind}_\mathcal{O}(\mathbf{c}).$$

Suppose further that

$$\Sigma = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_k\} \subset \mathcal{R}_M$$

is a finite set of primes of K such that for $1 \leq i \leq k$,

(a) $\mathrm{Fr}_{\mathfrak{q}_i}$ *is in the conjugacy class of τ in* $\mathrm{Gal}(K(W_M)/K)$,

(b) *writing* $\mathfrak{r}_j = \prod_{t=1}^j \mathfrak{q}_t$ *(so* $\mathfrak{r}_0 = \mathbf{1}$*), we have*

$$\mathrm{order}((\kappa_{[\mathfrak{r}_{i-1}, M]})_{\mathfrak{q}_i}, H^1_f(K_{\mathfrak{q}_i}, W_M)) \geq \mathrm{order}((\kappa_{[\mathfrak{r}_{i-1}, M]})_\Omega, H^1(\Omega, W_M)).$$

Then the map $\mathrm{loc}^s_{\Sigma, W_\mathfrak{m}}$ *of* (5.4) *satisfies*

$$\ell_\mathcal{O}\left(\mathrm{coker}(\mathrm{loc}^s_{\Sigma, W_\mathfrak{m}})\right) \leq \mathrm{ind}_\mathcal{O}(\mathbf{c}) + \mathfrak{n}_W.$$

Remark 5.2.6. Since the proof of Lemma 5.2.5 is a rather technical calculation, we first give a proof under the mild additional hypotheses

$$W^{G_K} = 0 \text{ and } H^1(\Omega/K, W) = 0. \tag{5.5}$$

We will follow this immediately by the general proof; we include the easier special case only because it makes the important ideas clearer.

Proof of Lemma 5.2.5 *under the assumption* (5.5). Note that by assumption (a) of the lemma, $W_M/(\mathrm{Fr}_{\mathfrak{q}_i}-1)W_M$ is free of rank one over $\mathcal{O}/M\mathcal{O}$ for every i. Therefore we can apply Corollary 4.5.5 with $\mathfrak{q}=\mathfrak{q}_i$ and $\mathfrak{r}=\mathfrak{r}_{i-1}$ to relate $\kappa_{[\mathfrak{r}_i,M]}$ and $\kappa_{[\mathfrak{r}_{i-1},M]}$. This will be the key to the proof.

By Lemma 1.2.2(i) and (5.5), all of the maps

$$H^1(K,W_{\mathfrak{m}}) \xrightarrow{\iota_{\mathfrak{m},M}} H^1(K,W_M) \xrightarrow{\iota_M} H^1(K,W) \xrightarrow{(\ )_\Omega} H^1(\Omega,W)$$

are injective. Therefore for $0\le i\le k$ we can define

$$\begin{aligned}\mathfrak{d}_i &= \mathrm{order}(\kappa_{[\mathfrak{r}_i,M]},H^1(K,W_M)) = \mathrm{order}(\iota_M(\kappa_{[\mathfrak{r}_i,M]}),H^1(K,W))\\ &= \mathrm{order}((\iota_M(\kappa_{[\mathfrak{r}_i,M]}))_\Omega,H^1(\Omega,W))\\ &= \mathrm{order}((\kappa_{[\mathfrak{r}_i,M]})_\Omega,H^1(\Omega,W_M)).\end{aligned}$$

By Lemma 5.1.1,

$$\mathfrak{d}_0 = \mathrm{ord}_{\mathfrak{p}}M - \mathrm{ind}_{\mathcal{O}}(\mathbf{c}) \ge n. \tag{5.6}$$

For $i\ge 1$,

$$\begin{aligned}\mathfrak{d}_i &\ge \mathrm{order}((\kappa_{[\mathfrak{r}_i,M]})^s_{\mathfrak{q}_i},H^1_s(K_{\mathfrak{q}_i},W_M))\\ &= \mathrm{order}((\kappa_{[\mathfrak{r}_{i-1},M]})_{\mathfrak{q}_i},H^1_f(K_{\mathfrak{q}_i},W_M)) \ge \mathfrak{d}_{i-1},\end{aligned} \tag{5.7}$$

the equality by Corollary 4.5.5, and the inequality on the right by assumption (b) of the lemma. Combining (5.6) and (5.7) we see that $\mathfrak{d}_i\ge n$ for every i.

It follows from Lemma 1.5.4 and the injectivity of ι_M that the homomorphism $\iota_{\mathfrak{m},M}: H^1(K,W_{\mathfrak{m}})\to H^1(K,W_M)$ sends $\mathcal{S}^{\Sigma_{p\mathfrak{r}_i}}(K,W_{\mathfrak{m}})$ onto $\mathcal{S}^{\Sigma_{p\mathfrak{r}_i}}(K,W_M)_{\mathfrak{m}}$. Theorem 4.5.1 shows that $\kappa_{[\mathfrak{r}_i,M]}\in\mathcal{S}^{\Sigma_{p\mathfrak{r}_i}}(K,W_M)$, so for each $i\ge 1$ we can choose $\bar\kappa_i\in\mathcal{S}^{\Sigma_{p\mathfrak{r}_i}}(K,W_{\mathfrak{m}})$ such that $\mathcal{O}\iota_{\mathfrak{m},M}(\bar\kappa_i)=\mathfrak{p}^{\mathfrak{d}_i-n}\kappa_{[\mathfrak{r}_i,M]}$.

If $1\le i\le k$ let $A^{(i)}$ denote the $\mathcal{O}$-submodule of $H^1(K,W_{\mathfrak{m}})$ generated by $\{\bar\kappa_1,\dots,\bar\kappa_i\}$, and let $A^{(0)}=0$. Then

$$A^{(i)} \subset \mathcal{S}^{\Sigma_{p\mathfrak{r}_i}}(K,W_{\mathfrak{m}}) \subset \mathcal{S}^{\Sigma\cup\Sigma_p}(K,W_{\mathfrak{m}})$$

so for $1\le i\le k$, writing loc^s_Σ for $\mathrm{loc}^s_{\Sigma,W_{\mathfrak{m}}}$, restriction to $\mathfrak{q}_i$ induces a surjective map

$$\mathrm{loc}^s_\Sigma(A^{(i)})/\mathrm{loc}^s_\Sigma(A^{(i-1)}) \twoheadrightarrow \mathcal{O}(\bar\kappa_i)^s_{\mathfrak{q}_i} \subset H^1_s(K_{\mathfrak{q}_i},W_{\mathfrak{m}}).$$

Hence for $1\le i\le k$, (5.7) shows that

$$\begin{aligned}\ell_{\mathcal{O}}(\mathrm{loc}^s_\Sigma(A^{(i)})/\mathrm{loc}^s_\Sigma(A^{(i-1)})) &\ge \mathrm{order}((\bar\kappa_i)^s_{\mathfrak{q}_i},H^1_s(K_{\mathfrak{q}_i},W_{\mathfrak{m}}))\\ &\ge \mathrm{order}((\kappa_{[\mathfrak{r}_i,M]})^s_{\mathfrak{q}_i},H^1_s(K_{\mathfrak{q}_i},W_M)) - (\mathfrak{d}_i-n)\\ &\ge n+\mathfrak{d}_{i-1}-\mathfrak{d}_i.\end{aligned}$$

Using the filtration

$$\mathrm{loc}_\Sigma^s(A^{(k)}) \supset \mathrm{loc}_\Sigma^s(A^{(k-1)}) \supset \cdots \supset \mathrm{loc}_\Sigma^s(A^{(1)}) \supset \mathrm{loc}_\Sigma^s(A^{(0)}) = 0$$

we conclude, using (5.6) and the trivial estimate $\mathfrak{d}_k \leq \mathrm{ord}_\mathfrak{p} M$, that

$$\begin{aligned}\ell_\mathcal{O}(\mathrm{loc}_\Sigma^s(\mathcal{S}^{\Sigma\cup\Sigma_p}(K,W_\mathfrak{m}))) &\geq \ell_\mathcal{O}(\mathrm{loc}_\Sigma^s(A^{(k)}))\\ &\geq \sum_{i=1}^{k}(n+\mathfrak{d}_{i-1}-\mathfrak{d}_i) = kn+\mathfrak{d}_0-\mathfrak{d}_k \geq kn-\mathrm{ind}_\mathcal{O}(\mathbf{c}).\end{aligned}$$

For every prime $\mathfrak{q} \in \mathcal{R}_M$, we have $H_s^1(K_\mathfrak{q}, W_\mathfrak{m}) = W_\mathfrak{m}^{\mathrm{Fr}_\mathfrak{q}=1}$ by Lemma 1.4.7(i), so

$$\ell_\mathcal{O}(\oplus_{\mathfrak{q}\in\Sigma} H_s^1(K_\mathfrak{q}, W_\mathfrak{m})) = k\ell_\mathcal{O}(W_\mathfrak{m}^{\tau=1}) = k\ell_\mathcal{O}(W_\mathfrak{m}/(\tau-1)W_\mathfrak{m}) = kn.$$

Thus

$$\ell_\mathcal{O}(\mathrm{coker}(\mathrm{loc}_\Sigma^s)) \leq \mathrm{ind}_\mathcal{O}(\mathbf{c})$$

as desired. □

Proof of Lemma 5.2.5 *in general.* Recall that ι_M is the natural map from $H^1(K,W_M)$ to $H^1(K,W)$. For $0 \leq i \leq k$ define

$$\begin{aligned}\mathfrak{d}_i' &= \mathrm{order}(\iota_M(\kappa_{[\mathfrak{r}_i,M]}), H^1(K,W)),\\ \mathfrak{d}_i &= \mathrm{order}((\kappa_{[\mathfrak{r}_i,M]})_\Omega, H^1(\Omega,W_M)).\end{aligned}$$

By Lemma 5.1.1,

$$\mathfrak{d}_0' = \mathrm{ord}_\mathfrak{p} M - \mathrm{ind}_\mathcal{O}(\mathbf{c}) \geq n + (k+1)\mathfrak{n}_W. \tag{5.8}$$

Since $\mathfrak{p}^{\mathfrak{d}_i}(\kappa_{[\mathfrak{r}_i,M]})_\Omega = 0$, we see that

$$\mathfrak{p}^{\mathfrak{d}_i}\iota_M(\kappa_{[\mathfrak{r}_i,M]}) \subset H^1(\Omega/K,W).$$

By Theorem 4.5.1, $\mathfrak{p}^{\mathfrak{d}_i}\iota_M(\kappa_{[\mathfrak{r}_i,M]}) \in \mathcal{S}^{\Sigma_p\mathfrak{r}_i}(K,W)$. Lemma 1.3.5(iv) shows that for every $\mathfrak{q} \nmid \mathcal{N}$ we have $H_f^1(K_\mathfrak{q},W) = H_{\mathrm{ur}}^1(K_\mathfrak{q},W)$, and Ω/K is unramified outside primes dividing $\mathcal{N}$, so $H^1(\Omega/K,W) \subset \mathcal{S}^{\Sigma_\mathcal{N}}(K,W)$. Therefore, since $\mathfrak{r}_i$ is prime to $\mathcal{N}$ we conclude that

$$\begin{aligned}\mathfrak{p}^{\mathfrak{d}_i}\iota_M(\kappa_{[\mathfrak{r}_i,M]}) &\in H^1(\Omega/K,W)\cap\mathcal{S}^{\Sigma_p\mathfrak{r}_i}(K,W)\\ &= H^1(\Omega/K,W)\cap\mathcal{S}^{\Sigma_p}(K,W).\end{aligned} \tag{5.9}$$

Therefore for every i, we have $\mathfrak{p}^{\mathfrak{d}_i+\mathfrak{n}_W}\iota_M(\kappa_{[\mathfrak{r}_i,M]}) = 0$, so

$$\mathfrak{n}_W + \mathfrak{d}_i \geq \mathfrak{d}_i'. \tag{5.10}$$

Suppose $i \geq 1$. If $\mathcal{I}_{\mathfrak{q}_i}$ is an inertia group of $\mathfrak{q}_i$, then (using Lemmas 1.3.8(ii), 1.3.5(iv), and 1.3.2(ii)) we have a commutative diagram

$$\begin{array}{ccccc} H^1_s(K_{\mathfrak{q}_i}, W_M) & = & H^1(K_{\mathfrak{q}_i}, W_M)/H^1_{\mathrm{ur}}(K_{\mathfrak{q}_i}, W_M) & \subset & \mathrm{Hom}(\mathcal{I}_{\mathfrak{q}_i}, W_M) \\ \downarrow{\scriptstyle \iota_M} & & \downarrow & & \cap \\ H^1_s(K_{\mathfrak{q}_i}, W) & = & H^1(K_{\mathfrak{q}_i}, W)/H^1_{\mathrm{ur}}(K_{\mathfrak{q}_i}, W) & \subset & \mathrm{Hom}(\mathcal{I}_{\mathfrak{q}_i}, W). \end{array}$$

Therefore the map $\iota_M : H^1_s(K_{\mathfrak{q}_i}, W_M) \to H^1_s(K_{\mathfrak{q}_i}, W)$ is injective. This gives the first equality of

$$\begin{aligned} \mathfrak{d}_i' &\geq \mathrm{order}(\iota_M(\kappa_{[\mathfrak{r}_i,M]})^s_{\mathfrak{q}_i}, H^1_s(K_{\mathfrak{q}_i}, W)) \\ &= \mathrm{order}((\kappa_{[\mathfrak{r}_i,M]})^s_{\mathfrak{q}_i}, H^1_s(K_{\mathfrak{q}_i}, W_M)) \\ &= \mathrm{order}((\kappa_{[\mathfrak{r}_{i-1},M]})_{\mathfrak{q}_i}, H^1_f(K_{\mathfrak{q}_i}, W_M)) \geq \mathfrak{d}_{i-1}, \end{aligned} \tag{5.11}$$

the second equality comes from Corollary 4.5.5 and assumption (a) of the lemma, and the final inequality comes from assumption (b). Combining (5.11) with (5.8) and (5.10), we conclude by induction that

$$\mathfrak{d}_i \geq \mathfrak{d}_0' - (i+1)\mathfrak{n}_W \geq n.$$

For $0 \leq i \leq k$ let $A^{(i)}$ denote the $\mathcal{O}$-submodule of $H^1(K, W)$ generated by

$$\{\mathfrak{p}^{\mathfrak{d}_j - n}\iota_M(\kappa_{[\mathfrak{r}_j,M]}) : 0 \leq j \leq i\},$$

and write $A = A^{(k)}$. By Theorem 4.5.1,

$$A^{(i)} \subset \mathcal{S}^{\Sigma_{p\mathfrak{r}_i}}(K, W), \tag{5.12}$$

so for $1 \leq i \leq k$ restriction to $\mathfrak{q}_i$ induces a surjective map

$$\mathrm{loc}^s_{\Sigma,W}(A^{(i)})/\mathrm{loc}^s_{\Sigma,W}(A^{(i-1)}) \twoheadrightarrow \mathfrak{p}^{\mathfrak{d}_i - n}(\kappa_{[\mathfrak{r}_i,M]})^s_{\mathfrak{q}_i} \subset H^1_s(K_{\mathfrak{q}_i}, W).$$

For $1 \leq i \leq k$, (5.11) shows that

$$\mathrm{order}(\mathfrak{p}^{\mathfrak{d}_i - n}(\kappa_{[\mathfrak{r}_i,M]})^s_{\mathfrak{q}_i}, H^1_s(K_{\mathfrak{q}_i}, W_M)) \geq \mathfrak{d}_{i-1} - \mathfrak{d}_i + n,$$

so using the filtration

$$\begin{aligned} \mathrm{loc}^s_{\Sigma,W}(A) = \mathrm{loc}^s_{\Sigma,W}(A^{(k)}) \supset \mathrm{loc}^s_{\Sigma,W}(A^{(k-1)}) \supset \cdots \\ \cdots \supset \mathrm{loc}^s_{\Sigma,W}(A^{(1)}) \supset \mathrm{loc}^s_{\Sigma,W}(A^{(0)}) \end{aligned}$$

we conclude that

$$\ell_{\mathcal{O}}(\mathrm{loc}^s_{\Sigma,W}(A)) \geq \sum_{i=1}^{k}(\mathfrak{d}_{i-1} - \mathfrak{d}_i + n) = kn + \mathfrak{d}_0 - \mathfrak{d}_k \geq kn + \mathfrak{d}_0 - \mathrm{ord}_{\mathfrak{p}} M. \tag{5.13}$$

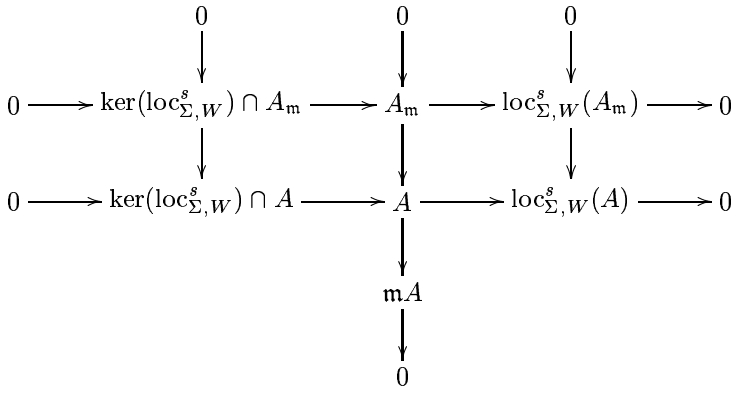
Since $\mathfrak{m} = \mathfrak{p}^n$, (5.9) shows that

$$\mathfrak{m}A \subset H^1(\Omega/K, W) \cap \mathcal{S}^{\Sigma_p}(K, W). \tag{5.14}$$

Let $A_{\mathfrak{m}}$ denote the submodule of A killed by $\mathfrak{m}$. By (5.12) and Lemma 1.5.4,

$$A_{\mathfrak{m}} \subset \mathcal{S}^{\Sigma\cup\Sigma_p}(K,W)_{\mathfrak{m}} = \iota_{\mathfrak{m}}(\mathcal{S}^{\Sigma\cup\Sigma_p}(K,W_{\mathfrak{m}})). \tag{5.15}$$

From the exact diagram



we see that

$$\begin{aligned}\ell_{\mathcal{O}}(\mathrm{loc}^s_{\Sigma,W}(A_{\mathfrak{m}})) &= \ell_{\mathcal{O}}(\mathrm{loc}^s_{\Sigma,W}(A)) + \\ &\quad \ell_{\mathcal{O}}((\ker(\mathrm{loc}^s_{\Sigma,W})\cap A)/(\ker(\mathrm{loc}^s_{\Sigma,W})\cap A_{\mathfrak{m}})) - \ell_{\mathcal{O}}(\mathfrak{m}A).\end{aligned} \tag{5.16}$$

By (5.12) with $i = 0$, we have

$$A^{(0)} = \mathfrak{p}^{\mathfrak{d}_0-n}\iota_M(\kappa_{[\mathbf{1},M]}) \subset \ker(\mathrm{loc}^s_{\Sigma,W}).$$

Since $A^{(0)}$ is a cyclic $\mathcal{O}$-module, we conclude using (5.8) that

$$\begin{aligned}\ell_{\mathcal{O}}((\ker(\mathrm{loc}^s_{\Sigma,W})\cap A)/(\ker(\mathrm{loc}^s_{\Sigma,W})\cap A_{\mathfrak{m}})) &\geq \ell_{\mathcal{O}}(A^{(0)}) - \ell_{\mathcal{O}}(A^{(0)}\cap A_{\mathfrak{m}}) \\ &\geq (\mathfrak{d}_0' - (\mathfrak{d}_0 - n)) - n \\ &= \mathrm{ord}_{\mathfrak{p}}M - \mathrm{ind}_{\mathcal{O}}(\mathbf{c}) - \mathfrak{d}_0.\end{aligned}$$

Combining this with (5.15), (5.16), (5.13), and (5.14) yields

$$\begin{aligned}\ell_{\mathcal{O}}(\mathrm{loc}^s_{\Sigma,W_{\mathfrak{m}}}(\mathcal{S}^{\Sigma\cup\Sigma_p}(K,W_{\mathfrak{m}}))) &\geq \ell_{\mathcal{O}}(\mathrm{loc}^s_{\Sigma,W}(A_{\mathfrak{m}})) \\ &\geq (kn + \mathfrak{d}_0 - \mathrm{ord}_{\mathfrak{p}}M) + (\mathrm{ord}_{\mathfrak{p}}M - \mathrm{ind}_{\mathcal{O}}(\mathbf{c}) - \mathfrak{d}_0) - \mathfrak{n}_W \\ &= kn - \mathrm{ind}_{\mathcal{O}}(\mathbf{c}) - \mathfrak{n}_W.\end{aligned}$$

For every prime $\mathfrak{q} \in \mathcal{R}_M$, we have $H^1_s(K_{\mathfrak{q}},W_{\mathfrak{m}}) = W_{\mathfrak{m}}^{\mathrm{Fr}_{\mathfrak{q}}=1}$ by Lemma 1.4.7(i), so

$$\ell_{\mathcal{O}}(\oplus_{\mathfrak{q}\in\Sigma}H^1_s(K_{\mathfrak{q}},W_{\mathfrak{m}})) = k\ell_{\mathcal{O}}(W_{\mathfrak{m}}^{\tau=1}) = k\ell_{\mathcal{O}}(W_{\mathfrak{m}}/(\tau-1)W_{\mathfrak{m}}) = kn.$$

Thus

$$\begin{aligned}\ell_{\mathcal{O}}(\mathrm{coker}(\mathrm{loc}^s_{\Sigma,W_{\mathfrak{m}}}))&\\ = \ell_{\mathcal{O}}(\oplus_{\mathfrak{q}\in\Sigma}H^1_s(K_{\mathfrak{q}},W_{\mathfrak{m}})) - \ell_{\mathcal{O}}(\mathrm{loc}^s_{\Sigma,W_{\mathfrak{m}}}(\mathcal{S}^{\Sigma\cup\Sigma_p}(K,W_{\mathfrak{m}})))&\\ \leq \mathrm{ind}_{\mathcal{O}}(\mathbf{c}) + \mathfrak{n}_W&\end{aligned}$$

as desired. □

Proof of Theorem 2.2.2. Fix a nonzero ideal $\mathfrak{m} = \mathfrak{p}^n$ of $\mathcal{O}$. Let C be the image of $\mathcal{S}_{\Sigma_p}(K,W^*_{\mathfrak{m}})$ (which is finite by Lemma 1.5.7(i)) in $H^1(K,W^*_M)$ where M is a power of p large enough so that

$$\mathrm{ord}_{\mathfrak{p}}M > n + (|\mathcal{S}_{\Sigma_p}(K,W^*_{\mathfrak{m}})|+1)\mathfrak{n}_W + \mathrm{ind}_{\mathcal{O}}(\mathbf{c})$$

(if $\mathrm{ind}_{\mathcal{O}}(\mathbf{c})$ is infinite then there is nothing to prove). Apply Lemma 5.2.3 with this group C, let Σ be a set of primes of K produced by that lemma, and apply Lemma 5.2.5 with this set Σ.

Combining the inequality of Lemma 5.2.5 with Theorem 1.7.3(iii) shows that

$$\ell_{\mathcal{O}}(\mathcal{S}_{\Sigma_p}(K,W^*_{\mathfrak{m}})/\mathcal{S}_{\Sigma\cup\Sigma_p}(K,W^*_{\mathfrak{m}})) \leq \mathfrak{n}_W + \mathrm{ind}_{\mathcal{O}}(\mathbf{c}).$$

Therefore

$$\ell_{\mathcal{O}}(\iota_{\mathfrak{m}}(\mathcal{S}_{\Sigma_p}(K,W^*_{\mathfrak{m}}))) \leq \ell_{\mathcal{O}}(\iota_{\mathfrak{m}}(\mathcal{S}_{\Sigma\cup\Sigma_p}(K,W^*_{\mathfrak{m}}))) + \mathfrak{n}_W + \mathrm{ind}_{\mathcal{O}}(\mathbf{c}).$$

Lemma 5.2.3(iv) shows that

$$\iota_{\mathfrak{m}}(\mathcal{S}_{\Sigma\cup\Sigma_p}(K,W^*_{\mathfrak{m}})) \subset H^1(\Omega/K,W^*)\cap\mathcal{S}_{\Sigma_p}(K,W^*),$$

and by definition $\mathfrak{n}^*_W = \ell_{\mathcal{O}}(H^1(\Omega/K,W^*)\cap\mathcal{S}_{\Sigma_p}(K,W^*))$, so we see that

$$\ell_{\mathcal{O}}(\iota_{\mathfrak{m}}(\mathcal{S}_{\Sigma_p}(K,W^*_{\mathfrak{m}}))) \leq \mathrm{ind}_{\mathcal{O}}(\mathbf{c}) + \mathfrak{n}_W + \mathfrak{n}^*_W.$$

Since this holds for every $\mathfrak{m}$, and $\mathcal{S}_{\Sigma_p}(K,W^*) = \varinjlim_{\mathfrak{m}}\iota_{\mathfrak{m}}(\mathcal{S}_{\Sigma_p}(K,W^*_{\mathfrak{m}}))$, Theorem 2.2.2 follows. □

5.3. Bounding the Exponent of the Selmer Group

The proof of Theorem 2.2.3 is similar to that of Theorem 2.2.2; it is easier in the sense that one can work with a single prime $\mathfrak{q}$ instead of a finite set of primes, but more difficult in the sense that one must keep track of extra "error terms".

The idea is as follows. Given $\eta\in\mathcal{S}_{\Sigma_p}(K,W^*_M)$, we use Lemma 5.3.1 below to choose a prime $\mathfrak{q}$ such that

- $H^1_f(K_{\mathfrak{q}},W^*_M)$ and $H^1_s(K_{\mathfrak{q}},W_M)$ are "almost" isomorphic to $\mathcal{O}/M\mathcal{O}$,
- $\mathrm{order}((\kappa_{[\mathfrak{q},M]})^s_{\mathfrak{q}},H^1_s(K_{\mathfrak{q}},W_M))$ is approximately $\mathrm{ord}_{\mathfrak{p}}M - \mathrm{ind}_{\mathcal{O}}(\mathbf{c})$,
- $\mathrm{order}((\eta)_{\mathfrak{q}},H^1_f(K_{\mathfrak{q}},W^*_M))$ is approximately $\mathrm{order}(\eta,H^1(K,W^*_M))$.

Since the Kolyvagin derivative class $\kappa_{[\mathfrak{q},M]}$ belongs to $\mathcal{S}^{\Sigma_{p\mathfrak{q}}}(K,W_M)$, the duality Theorem 1.7.3 shows that

$$\mathrm{order}((\kappa_{[\mathfrak{q},M]})^s_{\mathfrak{q}}, H^1_s(K_{\mathfrak{q}},W_M)) + \mathrm{order}((\eta)_{\mathfrak{q}}, H^1_f(K_{\mathfrak{q}},W^*_M))$$

is "approximately" bounded by $\mathrm{ord}_{\mathfrak{p}}M$, and so $\mathrm{order}(\eta, H^1(K,W^*_M))$ is "approximately" bounded by $\mathrm{ind}_{\mathcal{O}}(\mathbf{c})$. Since $\eta \in \mathcal{S}_{\Sigma_p}(K,W^*_M)$ is arbitrary, if we can bound all the error terms independently of M, this will prove Theorem 2.2.3. In the remainder of this section we sketch the details of this argument.

Keep the notation of §5.1 and §5.2. Suppose the Euler system $\mathbf{c}$ satisfies the hypotheses $\mathrm{Hyp}(K,V)$, and fix a $\tau \in G_K$ as in hypothesis $\mathrm{Hyp}(K,V)$(i). We now allow $p = 2$.

Let a be the smallest positive integer such that $\mathfrak{p}^a$ annihilates the maximal G_K-stable $\mathcal{O}$-submodule of $(\tau-1)W$ and of $(\tau-1)W^*$. Hypothesis $\mathrm{Hyp}(K,V)$(ii) ensures that a is finite, since any divisible G_K-stable subgroup of $(\tau-1)W$ would be the image of a G_K-stable subgroup of $(\tau-1)V$, which must be zero.

We have the following variant of Lemma 5.2.1.

Lemma 5.3.1. *Fix a power M of p. Suppose L is a Galois extension of K such that G_L acts trivially on W_M and on W^*_M. If*

$$\kappa \in H^1(K,W_M), \quad \eta \in H^1(K,W^*_M)$$

then there is an element $\gamma \in G_L$ satisfying

(i) $\mathrm{order}(\kappa(\gamma\tau), W_M/(\tau-1)W_M)) \geq \mathrm{order}((\kappa)_L, H^1(L,W_M)) - a - 1$,
(ii) $\mathrm{order}(\eta(\gamma\tau), W^*_M/(\tau-1)W^*_M) \geq \mathrm{order}((\eta)_L, H^1(L,W^*_M)) - a - 1$.

Proof. The proof is similar to that of Lemma 5.2.1, once we note that every G_K-submodule of W_M which projects to zero in $W_M/(\tau-1)W_M$ is killed by $\mathfrak{p}^a$, and similarly for W^*_M. The extra '1' takes care of the case $p = 2$. □

Let $\Omega = K(\mathbf{1})K(W, \boldsymbol{\mu}_{p^\infty}, (\mathcal{O}^\times_K)^{1/p^\infty})$ as in §5.2.

Lemma 5.3.2. *If $T \neq \mathcal{O}$ and $T \neq \mathcal{O}(1)$ then both $H^1(\Omega/K,W)$ and $H^1(\Omega/K,W^*)$ are finite.*

Proof. This is Corollary C.2.2 applied with $F = K$. □

Proof of Theorem 2.2.3. If $T = \mathcal{O}(1)$ then Proposition 1.6.1 shows that

$$\mathcal{S}_{\Sigma_p}(K,W^*) \subset \mathrm{Hom}(A_K, \mathbf{D}),$$

where A_K is the ideal class group of K, so $\mathcal{S}_{\Sigma_p}(K,W^*)$ is finite. The theorem assumes that $T \neq \mathcal{O}$, so by Lemma 5.3.2 we may assume from

now on that $H^1(\Omega/K, W)$ and $H^1(\Omega/K, W^*)$ are finite. Let

$$n = \max\{\ell_{\mathcal{O}}(H^1(\Omega/K, W)), \ell_{\mathcal{O}}(H^1(\Omega/K, W^*))\}.$$

Suppose M is a power of p and $\eta \in \mathcal{S}_{\Sigma_p}(K, W_M^*)$. Apply Lemma 5.3.1 with $L = K(\mathbf{1})K(W_M, \boldsymbol{\mu}_M, (\mathcal{O}_K^\times)^{1/M}) \subset \Omega$, with this η, and with $\kappa = \kappa_{[\mathbf{1},M]} \in H^1(K, W_M)$, and let $\gamma \in G_L$ be an element satisfying the conclusions of that lemma. Then since $H^1(\Omega/K, W)$ is the kernel of the restriction map $H^1(K, W) \to H^1(\Omega, W)$,

$$\begin{aligned}\operatorname{order}(\kappa_{[\mathbf{1},M]}(\gamma\tau), W_M/(\tau-1)W_M)) & \\ &\geq \operatorname{order}(\iota_M(\kappa_{[\mathbf{1},M]})_\Omega, H^1(\Omega, W)) - a - 1 \\ &\geq \operatorname{order}(\iota_M(\kappa_{[\mathbf{1},M]}), H^1(K, W)) - a - 1 - n \\ &= \operatorname{ord}_{\mathfrak{p}} M - \operatorname{ind}_{\mathcal{O}}(\mathbf{c}) - a - 1 - n \qquad (5.17)\end{aligned}$$

by Lemma 5.1.1. Similarly

$$\operatorname{order}(\eta(\gamma\tau), W_M^*/(\tau-1)W_M^*) \geq \operatorname{order}(\eta, H^1(K, W_M^*)) - a - 1 - n. \qquad (5.18)$$

Let L' denote the fixed field of

$$\ker((\kappa_{[\mathbf{1},M]})_L) \cap \ker((\eta)_L)$$

and, using the Tchebotarev theorem, choose a prime $\mathfrak{q}$ of K, not dividing $\mathcal{N}$, whose Frobenius in L'/K, for some choice of prime above $\mathfrak{q}$, is $\gamma\tau$. By Lemma 4.1.3, we have $\mathfrak{q} \in \mathcal{R}_M$.

As in the proof of Lemma 5.2.3, we conclude from (5.17) and (5.18) that

$$\operatorname{order}((\kappa_{[\mathbf{1},M]})_{\mathfrak{q}}, H_f^1(K_{\mathfrak{q}}, W_M)) \geq \operatorname{ord}_{\mathfrak{p}} M - \operatorname{ind}_{\mathcal{O}}(\mathbf{c}) - a - 1 - n$$

and

$$\operatorname{order}((\eta)_{\mathfrak{q}}, H_f^1(K_{\mathfrak{q}}, W_M^*)) \geq \operatorname{order}(\eta, H^1(K, W_M^*)) - a - 1 - n. \qquad (5.19)$$

Let $b = \ell_{\mathcal{O}}(W^{\tau=1}/(W^{\tau=1})_{\mathrm{div}})$, where $(W^{\tau=1})_{\mathrm{div}}$ is the maximal divisible submodule of $W^{\tau=1}$. By Theorem 4.5.4 and Corollary A.2.6,

$$\begin{aligned}\operatorname{order}((\kappa_{[\mathfrak{q},M]})_{\mathfrak{q}}^s, H_s^1(K_{\mathfrak{q}}, W_M)) &\geq \operatorname{order}((\kappa_{[\mathbf{1},M]})_{\mathfrak{q}}, H_f^1(K_{\mathfrak{q}}, W_M)) - 2b \\ &\geq \operatorname{ord}_{\mathfrak{p}} M - \operatorname{ind}_{\mathcal{O}}(\mathbf{c}) - a - 1 - n - 2b.\end{aligned}$$

By Lemma 1.4.7(i),

$$\ell_{\mathcal{O}}(H_s^1(K_{\mathfrak{q}}, W_M)) = \ell_{\mathcal{O}}((W_M)^{\tau=1}) = \ell_{\mathcal{O}}((W^{\tau=1})_M) \leq \operatorname{ord}_{\mathfrak{p}} M + b.$$

Thus, applying Theorem 1.7.3(iii) with

$$\Sigma = \Sigma_{p\mathfrak{q}}, \quad \Sigma_0 = \Sigma_p, \quad \eta \in \mathcal{S}_{\Sigma_p}(K, W_M^*),$$

we conclude that

$$\begin{aligned}\mathrm{order}((\eta)_{\mathfrak{q}}, H^1_f(K_{\mathfrak{q}}, W^*_M)) &\leq \ell_{\mathcal{O}}(\mathrm{coker}(\mathrm{loc}^s_{\Sigma_{p\mathfrak{q}},\Sigma_p})) \\ &\leq \ell_{\mathcal{O}}(H^1_s(K_{\mathfrak{q}}, W_M)) - \mathrm{order}((\kappa_{[\mathfrak{q},M]})^s_{\mathfrak{q}}, H^1_s(K_{\mathfrak{q}}, W_M)) \\ &\leq \mathrm{ind}_{\mathcal{O}}(\mathbf{c}) + a + 1 + n + 3b\end{aligned}$$

since $\kappa_{[\mathfrak{q},M]} \in \mathcal{S}^{\Sigma_{p\mathfrak{q}}}(K, W_M)$. Combining this with (5.19) shows that

$$\mathrm{order}(\eta, H^1(K, W^*_M)) \leq 2 + 2a + 3b + 2n + \mathrm{ind}_{\mathcal{O}}\mathbf{c}.$$

This inequality holds for every M and every $\eta \in \mathcal{S}_{\Sigma_p}(K, W^*_M)$. Since $\mathcal{S}_{\Sigma_p}(K, W^*)$ is the direct limit of the $\mathcal{S}_{\Sigma_p}(K, W^*_M)$, if we set

$$\mathfrak{m} = \mathfrak{p}^{2+2a+3b+2n+\mathrm{ind}_{\mathcal{O}}\mathbf{c}}$$

then we conclude that $\mathfrak{m}\mathcal{S}_{\Sigma_p}(K, W^*) = 0$.

As is well-known, this implies that $\mathcal{S}_{\Sigma_p}(K, W^*)$ is finite: Lemma 1.5.4 shows that

$$\mathcal{S}_{\Sigma_p}(K, W^*) = \mathcal{S}_{\Sigma_p}(K, W^*)_{\mathfrak{m}} \subset \mathcal{S}(K, W^*)_{\mathfrak{m}} = \iota_{\mathfrak{m}}(\mathcal{S}(K, W^*_{\mathfrak{m}}))$$

and the latter is finite by Lemma 1.5.7(i). □

CHAPTER 6

Twisting

In this chapter we extend the methods of §2.4 to twist Euler systems by characters of infinite order. This will be used in Chapter 7 when we prove Theorems 2.3.2, 2.3.3, and 2.3.4. If ρ is a character of $\mathrm{Gal}(K_\infty/K)$, then

- Theorem 6.3.5 says that an Euler system $\mathbf{c}$ for (T, K_∞) gives rise to an Euler system $\mathbf{c}^\rho$ for $(T \otimes \rho, K_\infty)$,
- Theorem 6.4.1 shows that the theorems of §2.3 hold for T and $\mathbf{c}$ if and only if they hold for $T \otimes \rho$ and $\mathbf{c}^\rho$, and
- Lemma 6.1.3 allows us to choose a particular ρ which avoids certain complications.

We keep the setting of Chapter 2, so K is a number field, T is a p-adic representation of G_K ramified at only finitely many primes, and K_∞ is an abelian extension of K satisfying $\mathrm{Gal}(K_\infty/K) \cong \mathbf{Z}_p^d$. Let $\Gamma = \mathrm{Gal}(K_\infty/K)$, and recall that Λ is the Iwasawa algebra

$$\Lambda = \mathcal{O}[[\Gamma]] = \varprojlim_{K \subset_f F \subset K_\infty} \mathcal{O}[\mathrm{Gal}(F/K)],$$

a complete local noetherian unique factorization domain. The characteristic ideal $\mathrm{char}(B)$ of a finitely generated Λ-module B was defined in §2.3.

6.1. Twisting Representations

Definition 6.1.1. Suppose $\rho : G_K \to \mathcal{O}^\times$ is a continuous character, possibly of infinite order. As in Example 1.1.2 we will write $\mathcal{O}_\rho$ for a free rank-one $\mathcal{O}$-module with G_K acting via ρ, and if B is a G_K-module we will abbreviate

$$B \otimes \rho = B \otimes_{\mathcal{O}} \mathcal{O}_\rho.$$

Then $B \otimes \rho$ is isomorphic to B as an $\mathcal{O}$-module but not (in general) as a G_K-module.

If $\rho : \Gamma \to \mathcal{O}^\times$ define

$$\mathrm{Tw}_\rho : \Lambda \xrightarrow{\sim} \Lambda$$

to be the $\mathcal{O}$-linear isomorphism induced by $\gamma \mapsto \rho(\gamma)\gamma$ for $\gamma \in \Gamma$.

Lemma 6.1.2. *Suppose B is a finitely generated torsion Λ-module and $\rho : \Gamma \to \mathcal{O}^\times$ is a character. Then $B \otimes \rho$ is a finitely generated torsion Λ-module and*

(i) $\mathrm{Tw}_\rho(\mathrm{char}(B \otimes \rho)) = \mathrm{char}(B)$,
(ii) *if $f \in \Lambda$ then $f \cdot (B \otimes \rho) = 0 \iff \mathrm{Tw}_\rho(f) \cdot B = 0$.*

Proof. If $f \in \Lambda$ and $\xi_\rho \in \mathcal{O}_\rho$ then

$$f \cdot (b \otimes \xi_\rho) \;=\; (\mathrm{Tw}_\rho(f)b) \otimes \xi_\rho.$$

The lemma follows easily from this, along with (for (i)) the fact that twisting preserves the heights of ideals of Λ. □

Lemma 6.1.3. (i) *Suppose B is G_K-module, free of finite rank over $\mathcal{O}$, and $\gamma_1, \ldots, \gamma_k$ are elements of G_K whose projections to Γ are nontrivial. Then the set*

$$\{\rho \in \mathrm{Hom}(\Gamma, \mathcal{O}^\times) : (B \otimes \rho)^{\gamma_i^{p^n}=1} = 0 \text{ for } 1 \le i \le k \text{ and every } n \ge 0\}$$

contains an open dense subset of $\mathrm{Hom}(\Gamma, \mathcal{O}^\times)$.

(ii) *Suppose B is a finitely generated torsion Λ-module. Then the set*

$$\{\rho \in \mathrm{Hom}(\Gamma, \mathcal{O}^\times) : (B \otimes \rho) \otimes_\Lambda \mathcal{O}[\mathrm{Gal}(F/K)] \text{ is finite for every } K \subset_t F \subset K_\infty\}$$

is dense in $\mathrm{Hom}(\Gamma, \mathcal{O}^\times)$.

Proof. Consider (i) first. Recall that Φ is the field of fractions of $\mathcal{O}$, and let $\bar{\Phi}$ denote an algebraic closure of Φ. For each i define

$$\begin{aligned} R_i &= \{\text{eigenvalues of } \gamma_i \text{ acting on } B \otimes \bar{\Phi}\}, \\ P_i &= \{x \in \mathcal{O}^\times : xR_i \cap \boldsymbol{\mu}_{p^\infty} \neq \emptyset\}, \\ Z_i &= \{\rho \in \mathrm{Hom}(\Gamma, \mathcal{O}^\times) : \rho(\gamma_i) \notin P_i\}. \end{aligned}$$

Each R_i is finite, and $\boldsymbol{\mu}_{p^\infty} \cap \mathcal{O}^\times$ is finite, so each P_i is finite and thus $Z = \cap_i Z_i$ is an open dense subset of $\mathrm{Hom}(\Gamma, \mathcal{O}^\times)$. We will show that Z is contained in the set of (i).

Suppose $\zeta \in \boldsymbol{\mu}_{p^\infty}$. Then

$$\begin{aligned} &\zeta \text{ is an eigenvalue of } \gamma_i \text{ acting on } (B \otimes \rho) \otimes \bar{\Phi} \\ &\qquad \iff \rho^{-1}(\gamma_i)\zeta \text{ is an eigenvalue of } \gamma_i \text{ acting on } B \otimes \bar{\Phi} \\ &\qquad \iff \zeta \in \rho(\gamma_i)R_i \\ &\qquad \implies \rho(\gamma_i) \in P_i. \end{aligned}$$

Therefore if $\rho \in Z_i$ and $n \geq 0$, then 1 is not an eigenvalue of $\gamma_i^{p^n}$ acting on $(B \otimes \rho) \otimes \Phi$. It follows that for $1 \leq i \leq k$, every $n \geq 0$, and every $\rho \in Z$, we have

$$(B \otimes \rho)^{\gamma_i^{p^n}=1} \otimes \Phi \;=\; ((B \otimes \rho) \otimes \Phi)^{\gamma_i^{p^n}=1} \;=\; 0$$

and since B has no p-torsion we conclude that $(B \otimes \rho)^{\gamma_i^{p^n}=1} = 0$.

Let $U \subset \mathrm{Hom}(\Gamma, \mathcal{O}^\times)$ be the set defined in (ii). We will show that U contains a countable intersection of dense open sets, so the Baire category theorem shows that U is dense. Since B is a quotient of a finite direct sum of cyclic torsion Λ-modules, it is enough to prove this when $B = \Lambda/f\Lambda$ with a nonzero $f \in \Lambda$.

Suppose $B = \Lambda/f\Lambda$, so $B \otimes \rho \cong \Lambda/\mathrm{Tw}_{\rho^{-1}}(f)\Lambda$. If $K \subset_{\mathrm{f}} F \subset K_\infty$ then

$$\begin{aligned}(\Lambda/\mathrm{Tw}_{\rho^{-1}}(f)\Lambda) \otimes_\Lambda \mathcal{O}[\mathrm{Gal}(F/K)] \text{ is finite} \iff \\ \rho^{-1}\chi(f) \neq 0 \text{ for every character } \chi : \mathrm{Gal}(F/K) \to \bar{\Phi}^\times. \quad (6.1)\end{aligned}$$

Let Ξ be the set of characters of finite order of Γ into $\bar{\Phi}^\times$, and for $\chi \in \Xi$ let

$$Y_\chi \;=\; \{\rho \in \mathrm{Hom}(\Gamma, \mathcal{O}^\times) : \rho^{-1}\chi(f) \neq 0\}.$$

Since $f \neq 0$, each Y_χ is open and dense in $\mathrm{Hom}(\Gamma, \mathcal{O}^\times)$, and (6.1) shows that $U = \cap_{\chi \in \Xi} Y_\chi$. Since Ξ is countable, this concludes the proof. □

6.2. Twisting Cohomology Groups

For every extension L of K, write

$$H^1_\infty(L, T) \;=\; \varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^1(FL, T).$$

Proposition 6.2.1. *Suppose $K \subset_{\mathrm{f}} L$ and $\rho : \mathrm{Gal}(LK_\infty/K) \to \mathcal{O}^\times$ is a character. The natural map on cocycles induces G_K-isomorphisms*

(i) $H^1_\infty(L, T) \otimes \rho \xrightarrow{\sim} H^1_\infty(L, T \otimes \rho)$

(ii) $\mathcal{S}_{\Sigma_p}(LK_\infty, W) \otimes \rho \xrightarrow{\sim} \mathcal{S}_{\Sigma_p}(LK_\infty, W \otimes \rho)$

where Σ_p is the set of primes of K dividing p.

Proof. Let $L_\infty = LK_\infty$. Choose a sequence fields $L \subset_{\mathrm{f}} L_1 \subset_{\mathrm{f}} L_2 \subset_{\mathrm{f}} \cdots \subset L_\infty$ such that $L_\infty = \cup L_n$ and such that $\mathrm{Gal}(L_\infty/L_n)$ is contained in the kernel of $\mathrm{Gal}(LK_\infty/K) \xrightarrow{\rho} (\mathcal{O}/p^n\mathcal{O})^\times$. Then $\mathcal{O}_\rho/p^n\mathcal{O}_\rho$ is a trivial G_{L_n}-module, so the natural map on cocycles induces G_K-equivariant isomorphisms

$$H^1(L_n, T/p^nT) \otimes \rho \xrightarrow{\sim} H^1(L_n, (T/p^nT) \otimes \rho). \quad (6.2)$$

Combining these isomorphisms with Lemma B.3.1 gives a sequence of isomorphisms

$$H^1_\infty(L,T)\otimes\rho \;=\; \varprojlim_n H^1(L_n,T/p^nT)\otimes\rho$$
$$\xrightarrow{\sim}\; \varprojlim_n H^1(L_n,(T\otimes\rho)/p^n(T\otimes\rho)) \;=\; H^1_\infty(L,T\otimes\rho).$$

This proves (i).

Suppose w is a place of LK_∞. The isomorphisms (6.2), and their analogues for the completions $L_{n,w}$, induce the horizontal isomorphisms in the commutative diagram

$$\begin{array}{ccc} \varinjlim_n H^1(L_n,W)\otimes\rho & \xrightarrow{\sim} & \varinjlim_n H^1(L_n,W\otimes\rho) \\ \Big\downarrow{\scriptstyle \mathrm{res}_w} & & \Big\downarrow{\scriptstyle \mathrm{res}_w} \\ \varinjlim_n H^1(L_{n,w},W)\otimes\rho & \xrightarrow{\sim} & \varinjlim_n H^1(L_{n,w},W\otimes\rho). \end{array} \tag{6.3}$$

By Lemmas B.3.3 and 1.3.5(i), if w does not divide p then

$$\varinjlim_n H^1_f(L_{n,w},W)\otimes\rho = \varinjlim_n H^1_f(L_{n,w},W\otimes\rho) = 0.$$

It follows that the top row of (6.3) induces the isomorphism of (ii). □

Corollary 6.2.2. *Suppose $\rho:\Gamma\to\mathcal{O}^\times$ is a character. Then there is an isomorphism of Λ-modules*

$$X_\infty(T\otimes\rho)\;\cong\; X_\infty(T)\otimes\rho,$$

where $X_\infty(T)=X_\infty=\mathrm{Hom}(\mathcal{S}_{\Sigma_p}(K_\infty,W^*),\mathbf{D})$ and

$$X_\infty(T\otimes\rho)=\mathrm{Hom}(\mathcal{S}_{\Sigma_p}(K_\infty,(W\otimes\rho)^*),\mathbf{D})$$

is the corresponding Λ-module associated to $T\otimes\rho$.

Proof. The corollary follows immediately from Proposition 6.2.1(ii) applied with W replaced by W^* and with $L=K$. □

Remark 6.2.3. Note that Proposition 6.2.1 does not assert the existence of an isomorphism, or even a map, from $H^1(L,T)$ to $H^1(L,T\otimes\rho)$.

6.3. Twisting Euler Systems

Definition 6.3.1. Suppose $\mathbf{c}$ is an Euler system for (T,K_∞), more specifically (in the notation of Definition 2.1.1) for $(T,\mathcal{K},\mathcal{N})$, where $K_\infty\subset\mathcal{K}$ and $\mathcal{N}$ is divisible by p and the primes where T is ramified. If $K\subset_{\mathrm{f}}L\subset\mathcal{K}$ then we write $\mathbf{c}_{L,\infty}=\{\mathbf{c}_{LF}\}_{K\subset_{\mathrm{f}}F\subset K_\infty}$ for the corresponding element of $H^1_\infty(L,T)$.

Suppose $\rho : \mathrm{Gal}(\mathcal{K}/K) \to \mathcal{O}^\times$ is a character which factors through a finite extension of K_∞. (We can always ensure this latter property by taking K_∞ to be the compositum of all $\mathbf{Z}_p$-extensions of K in $\mathcal{K}$.) Let L be a finite extension of K in $\mathcal{K}$ such that, writing $L_\infty = LK_\infty$,

(i) ρ factors through $\mathrm{Gal}(L_\infty/K)$,

(ii) L_∞/K is ramified only at primes dividing $\mathcal{N}$, ∞, and the conductor of ρ.

(For example, L could be a finite extension of K such that LK_∞ is the fixed field of $\ker(\rho) \cap G_{K_\infty}$.) Fix a generator ξ_ρ of $\mathcal{O}_\rho$. We define a collection of cohomology classes

$$\mathbf{c}^\rho \;=\; \{\mathbf{c}^\rho_F \in H^1(F, T \otimes \rho) : K \subset_{\mathrm{f}} F \subset \mathcal{K}\}$$

as follows. If $K \subset_{\mathrm{f}} F \subset \mathcal{K}$ let $\mathbf{c}^\rho_F$ be the image of $\mathbf{c}_{FL,\infty} \otimes \xi_\rho \in H^1_\infty(FL, T) \otimes \rho$ under the composition

$$\begin{aligned} H^1_\infty(FL, T) \otimes \rho \;\xrightarrow{\sim}\; & H^1_\infty(FL, T \otimes \rho) \\ \longrightarrow\; & H^1(FL, T \otimes \rho) \xrightarrow{\mathrm{Cor}_{FL/F}} H^1(F, T \otimes \rho) \end{aligned}$$

where the first map is the isomorphism of Proposition 6.2.1(i) and the second is the natural projection from H^1_∞ to H^1.

Remark 6.3.2. The definition of $\mathbf{c}^\rho_F$ is independent of our choice of L. To see this, suppose L' is another choice satisfying the properties above. Without loss of generality we may suppose that $L \subset L'$. If $K \subset_{\mathrm{f}} F \subset \mathcal{K}$, then FL'/FL is unramified outside $\mathcal{N}$, ∞, and the conductor $\mathfrak{f}$ of ρ. The primes which divide $\mathfrak{f}$ but do not divide p are already ramified in FL/K, so the Euler system distribution relation shows that $\mathrm{Cor}_{FL'/FL}(\mathbf{c}_{FL'}) = \mathbf{c}_{FL}$.

The definition of $\mathbf{c}^\rho_F$ does depend on the choice of ξ_ρ, but only up to a unit in $\mathcal{O}^\times$.

Remark 6.3.3. Let $\xi_{\rho,n}$ denote the image in $\mathcal{O}_\rho/p^n\mathcal{O}_\rho$ of the chosen generator ξ_ρ of $\mathcal{O}_\rho$. An examination of the proof of Proposition 6.2.1(i) shows that for $K \subset_{\mathrm{f}} F \subset K_\infty$, with L_n as in that proof, we have

$$\mathbf{c}_{FL_n} \otimes \xi_{\rho,n} \;\in\; H^1(FL_n, (T \otimes \rho)/p^n(T \otimes \rho))$$

and the definition of $\mathbf{c}^\rho_F$ easily seen to be equivalent to

$$\begin{aligned} \mathbf{c}^\rho_F \;=\; & \lim_{n\to\infty} \mathrm{Cor}_{FL_n/F}(\mathbf{c}_{FL_n} \otimes \xi_{\rho,n}) \\ & \in \varprojlim_n H^1(F, (T \otimes \rho)/p^n(T \otimes \rho)) \;=\; H^1(F, T \otimes \rho). \end{aligned}$$

Remark 6.3.4. When ρ is a character of finite order, this definition of $\mathbf{c}^\rho$ agrees with the one given in Definition 2.4.1. (Just take L to be the fixed field of $\ker(\rho)$.)

Theorem 6.3.5. *Suppose that* $\mathbf{c}$ *is an Euler system for* $(T, \mathcal{K}, \mathcal{N})$ *where* $K_\infty \subset \mathcal{K}$, *and* $\rho : \mathrm{Gal}(\mathcal{K}/K) \to \mathcal{O}^\times$ *is a character which factors through a finite extension of* K_∞. *Then the collection of classes* $\{\mathbf{c}_F^\rho \in H^1(F, T \otimes \rho)\}$ *defined above is an Euler system for* $(T \otimes \rho, \mathcal{K}, \mathfrak{f}\mathcal{N})$ *where* $\mathfrak{f}$ *is the non-archimedean, non-p part of the conductor of* ρ.

Proof. Suppose $K \subset_{\mathrm{f}} F \subset_{\mathrm{f}} F' \subset \mathcal{K}$. We have a commutative diagram

$$\begin{array}{ccccc} H^1_\infty(F'L, T) \otimes \rho & \xrightarrow{\sim} & H^1_\infty(F'L, T \otimes \rho) & \xrightarrow{\mathrm{Cor}} & H^1(F', T \otimes \rho) \\ \downarrow{\scriptstyle \mathrm{Cor} \otimes 1} & & \downarrow{\scriptstyle \mathrm{Cor}} & & \downarrow{\scriptstyle \mathrm{Cor}_{F'/F}} \\ H^1_\infty(FL, T) \otimes \rho & \xrightarrow{\sim} & H^1_\infty(FL, T \otimes \rho) & \xrightarrow{\mathrm{Cor}} & H^1(F, T \otimes \rho). \end{array}$$

Since $\mathbf{c}$ is an Euler system, we have

$$\mathrm{Cor}_{F'LK_\infty/FLK_\infty}(\mathbf{c}_{F'L,\infty}) = \Big(\prod_{\mathfrak{q} \in S} P(\mathrm{Fr}_\mathfrak{q}^{-1} | T^*; \mathrm{Fr}_\mathfrak{q}^{-1})\Big)\mathbf{c}_{FL,\infty}$$

where

$$\begin{aligned} S &= \{\mathfrak{q} \text{ of } K : \mathfrak{q} \text{ ramifies in } F'L/K \text{ but not in } FL/K, \text{ and } \mathfrak{q} \nmid \mathcal{N}\} \\ &= \{\mathfrak{q} \text{ of } K : \mathfrak{q} \text{ ramifies in } F'/K \text{ but not in } F/K, \text{ and } \mathfrak{q} \nmid \mathfrak{f}\mathcal{N}\}, \end{aligned}$$

the last equality because the conductor of L/K is divisible by $\mathfrak{f}$ and divides a power of $\mathfrak{f}\mathcal{N}\infty$. Therefore

$$\begin{aligned} (\mathrm{Cor}_{F'LK_\infty/FLK_\infty}(\mathbf{c}_{F'L,\infty})) \otimes \xi_\rho &= (\prod_{\mathfrak{q} \in S} P(\mathrm{Fr}_\mathfrak{q}^{-1} | T^*; \mathrm{Fr}_\mathfrak{q}^{-1}) \mathbf{c}_{FL,\infty}) \otimes \xi_\rho \\ &= \prod_{\mathfrak{q} \in S} P(\mathrm{Fr}_\mathfrak{q}^{-1} | T^*; \rho(\mathrm{Fr}_\mathfrak{q})\mathrm{Fr}_\mathfrak{q}^{-1})(\mathbf{c}_{FL,\infty} \otimes \xi_\rho) \end{aligned}$$

and so, using the diagram above,

$$\mathrm{Cor}_{F'/F}(\mathbf{c}_{F'}^\rho) = \prod_{\mathfrak{q} \in S} P(\mathrm{Fr}_\mathfrak{q}^{-1} | T^*; \rho(\mathrm{Fr}_\mathfrak{q})\mathrm{Fr}_\mathfrak{q}^{-1}) \mathbf{c}_F^\rho.$$

Since

$$\begin{aligned} \det(1 - \mathrm{Fr}_\mathfrak{q}^{-1} x | (T \otimes \rho)^*) &= \det(1 - \rho(\mathrm{Fr}_\mathfrak{q})\mathrm{Fr}_\mathfrak{q}^{-1} x | T^*) \\ &= P(\mathrm{Fr}_\mathfrak{q}^{-1} | T^*; \rho(\mathrm{Fr}_\mathfrak{q}) x), \end{aligned}$$

this shows that $\mathbf{c}^\rho$ is an Euler system for $(T \otimes \rho, \mathcal{K}, \mathfrak{f}\mathcal{N})$. □

Lemma 6.3.6. *Suppose that* $\mathbf{c}$ *is an Euler system for* $(T, \mathcal{K}, \mathcal{N})$ *where* $K_\infty \subset \mathcal{K}$, *and* $\rho, \rho' : \mathrm{Gal}(\mathcal{K}/K) \to \mathcal{O}^\times$ *are characters which factor through a finite extension of* K_∞. *Let* $\mathfrak{f}_\rho$, $\mathfrak{f}_{\rho'}$, $\mathfrak{f}_{\rho\rho'}$ *be the non-archimedean, non-*p *part of the conductors of* ρ, ρ', *and* $\rho\rho'$, *respectively. Suppose that the generator* $\xi_{\rho\rho'}$ *of* $\mathcal{O}_{\rho\rho'} = \mathcal{O}_\rho \otimes \mathcal{O}_{\rho'}$ *is chosen so that* $\xi_{\rho\rho'} = \xi_\rho \otimes \xi_{\rho'}$.

If every prime divisor of $\mathfrak{f}_\rho \mathfrak{f}_{\rho'}$ *divides* $\mathfrak{f}_{\rho\rho'}\mathcal{N}$, *then* $(\mathbf{c}^\rho)^{\rho'} = \mathbf{c}^{\rho\rho'}$. *In particular, if* $\mathfrak{f}_\rho \mid \mathcal{N}$ *then* $(\mathbf{c}^\rho)^{\rho^{-1}} = \mathbf{c}$.

Proof. Let L_ρ be a finite extension of K satisfying (i) and (ii) of Definition 6.3.1 for ρ, and similarly for $L_{\rho'}$.

The assumption on the conductors of ρ, ρ', and $\rho\rho'$ ensures that the compositum $L_\rho L_{\rho'}$ satisfies Definition 6.3.1(i) and (ii) for $\rho\rho'$. The lemma now follows easily from the definitions of $\mathbf{c}^\rho$, $\mathbf{c}^{\rho'}$, and $\mathbf{c}^{\rho\rho'}$ (and Remark 6.3.2). □

6.4. Twisting Theorems

Recall that $\Gamma = \mathrm{Gal}(K_\infty/K)$.

Theorem 6.4.1. *If* $\rho : \Gamma \to \mathcal{O}^\times$ *is a character then Theorems* 2.3.2, 2.3.3, *and* 2.3.4 *for* T *and* $\mathbf{c}$ *are equivalent to Theorems* 2.3.2, 2.3.3, *and* 2.3.4, *respectively, for* $T \otimes \rho$ *and* $\mathbf{c}^\rho$, *where* $\mathbf{c}^\rho$ *is the Euler system for* $T \otimes \rho$ *given by Theorem* 6.3.5.

Proof. The hypotheses $\mathrm{Hyp}(K_\infty, T)$, $\mathrm{Hyp}(K_\infty, V)$, and $\mathrm{Hyp}(K_\infty/K)$ depend only on the action of G_{K_∞} on T, so they are not affected by twisting by characters of Γ.

Write $X_\infty(T) = X_\infty$ and let $X_\infty(T \otimes \rho)$ be the corresponding Λ-module associated to $T \otimes \rho$, as defined in Corollary 6.2.2. That corollary says that

$$X_\infty(T \otimes \rho) \cong X_\infty(T) \otimes \rho,$$

so by Lemma 6.1.2(i),

$$\mathrm{Tw}_\rho(\mathrm{char}(X_\infty(T \otimes \rho))) = \mathrm{char}(X_\infty(T)).$$

The argument of Lemma 6.1.2, along with Proposition 6.2.1(i), shows that

$$\mathrm{Tw}_\rho(\mathrm{ind}_\Lambda(\mathbf{c}^\rho)) = \mathrm{ind}_\Lambda(\mathbf{c}).$$

The theorem follows from these equalities. □

6.5. Examples and Applications

Recall that $\varepsilon_{\mathrm{cyc}} : G_K \to \mathbf{Z}_p^\times \subset \mathcal{O}^\times$ is the cyclotomic character, and let $\omega : G_K \to (\mathbf{Z}_p^\times)_{\mathrm{tors}}$ be the Teichmüller character giving the action of G_K on $\boldsymbol{\mu}_p$ (if p is odd) or on $\boldsymbol{\mu}_4$ (if $p = 2$).

Tate twists. Suppose $\boldsymbol{\mu}_{p^\infty} \subset \mathcal{K}$, so $\varepsilon_{\mathrm{cyc}}$ factors through $\mathrm{Gal}(\mathcal{K}/K)$. If T is a p-adic representation of G_K, then for every integer n we write $T(n)$ for the Tate twist $T \otimes \varepsilon_{\mathrm{cyc}}^n$. By Theorem 6.3.5, an Euler system $\mathbf{c}$ for $(T, \mathcal{K}, \mathcal{N})$ gives an Euler system $\mathbf{c}^{\varepsilon_{\mathrm{cyc}}^n}$ for $(T(n), \mathcal{K}, \mathcal{N})$, and Lemma 6.3.6 shows that $(\mathbf{c}^{\varepsilon_{\mathrm{cyc}}^n})^{\varepsilon_{\mathrm{cyc}}^m} = \mathbf{c}^{\varepsilon_{\mathrm{cyc}}^{n+m}}$ for every n and m.

Now take K_∞ to be the cyclotomic $\mathbf{Z}_p$-extension of K. Then $\varepsilon_{\mathrm{cyc}}$ does not necessarily factor through $\mathrm{Gal}(K_\infty/K)$, but $\omega^{-1}\varepsilon_{\mathrm{cyc}}$ does. Thus if $\mathbf{c}$ is an Euler system for (T, K_∞), then Theorem 6.4.1 shows that for every n, Theorems 2.3.2, 2.3.3, and 2.3.4 for T and $\mathbf{c}$ are equivalent to those same theorems for $T \otimes \omega^{-n}\varepsilon_{\mathrm{cyc}}^n$ and $\mathbf{c}^{\omega^{-n}\varepsilon_{\mathrm{cyc}}^n}$.

Cyclotomic fields. In §3.2 and §3.4 we used cyclotomic units and Stickelberger elements, respectively, to construct Euler systems $\mathbf{c}_{\mathrm{cyc}}$ for $\mathbf{Z}_p(1)$ and $\mathbf{c}_{\mathrm{St}}$ for $\mathbf{Z}_p$.

Exercise. Both $\mathbf{c}_{\mathrm{St}}^{\varepsilon_{\mathrm{cyc}}}$ and $\mathbf{c}_{\mathrm{cyc}}$ are Euler systems for $\mathbf{Z}_p(1)$. Determine the relation between them.

Elliptic curves with complex multiplication. Let K be an imaginary quadratic field and K_∞ the $\mathbf{Z}_p^2$-extension of K. Suppose E is an elliptic curve defined over K with complex multiplication by the ring of integers $\mathcal{O}_K$ of K. Fix a prime $\mathfrak{p}$ of K above p, and let $\mathcal{O}$ be the completion of $\mathcal{O}_K$ at $\mathfrak{p}$. Let $T_{\mathfrak{p}}(E)$ denote the $\mathfrak{p}$-adic Tate module of E (see Example 1.1.5), which is a free rank-one $\mathcal{O}$-module. Let ψ be the character

$$\psi : G_K \longrightarrow \mathrm{Aut}_{\mathcal{O}_K}(E_{\mathfrak{p}^\infty}) \cong \mathcal{O}^\times.$$

Then $T_{\mathfrak{p}}(E) \cong \mathcal{O}_\psi$.

Let $\mathbf{c}_{\mathrm{ell}}$ denote the Euler system of elliptic units for $\mathcal{O}(1)$ over K of §3.3. The character $\psi\varepsilon_{\mathrm{cyc}}^{-1}$ factors through a finite extension of K_∞, so by Theorem 6.3.5 we obtain an Euler system

$$\mathbf{c}_{E,\mathfrak{p}} = \mathbf{c}_{\mathrm{ell}}^{\psi\varepsilon_{\mathrm{cyc}}^{-1}}$$

for $\mathcal{O}(1) \otimes \psi\varepsilon_{\mathrm{cyc}}^{-1} = \mathcal{O}_\psi \cong T_{\mathfrak{p}}(E)$. In particular we get an element

$$\mathbf{c}_{E,\mathfrak{p},K} \in H^1(K, T_{\mathfrak{p}}(E)).$$

Let $V_{\mathfrak{p}}(E) = T_{\mathfrak{p}}(E) \otimes K_{\mathfrak{p}}$. If v divides p, then as in §1.6.D we define

$$H^1_f(K_v, V_{\mathfrak{p}}(E)) = \mathrm{image}\big(E(K_v) \otimes \mathbf{Q}_p \hookrightarrow H^1(K_v, V_{\mathfrak{p}}(E))\big).$$

Corollary 1.3.3 and Theorem 1.4.1 show that $H^1(K_v, V_{\mathfrak{p}}(E)) = 0$ for all v not dividing p. Write $\mathfrak{p}^*$ for the conjugate of $\mathfrak{p}$ under the nontrivial

automorphism of $K/\mathbf{Q}$. If p splits in K, so $p = \mathfrak{p}\mathfrak{p}^*$, then one can show that

$$H^1_f(K_\mathfrak{p}, V_\mathfrak{p}(E)) = H^1(K_\mathfrak{p}, V_\mathfrak{p}(E)), \quad H^1_f(K_{\mathfrak{p}^*}, V_\mathfrak{p}(E)) = 0.$$

Therefore (whether or not p splits) $H^1(K_v, T_\mathfrak{p}(E)) = H^1_f(K_v, T_\mathfrak{p}(E))$ for all $v \neq \mathfrak{p}^*$, so

$$H^1(K, T_\mathfrak{p}(E)) = \mathcal{S}^{\{\mathfrak{p}^*\}}(K, T_\mathfrak{p}(E)).$$

In particular, we have

$$\mathbf{c}_{E,\mathfrak{p},K} \in \mathcal{S}^{\{\mathfrak{p}^*\}}(K, T_\mathfrak{p}(E)).$$

Now suppose further that the L-function $L(E_{/K}, s)$ of E vanishes at $s = 1$. Then one can show (see [**Ru7**] §4 for the proof in the case that $\mathfrak{p}$ splits in K; the general case is essentially the same) that the image of $\mathbf{c}_{E,\mathfrak{p},K}$ in $H^1_s(K_{\mathfrak{p}^*}, T_\mathfrak{p}(E))$ is zero, so in fact

$$\mathbf{c}_{E,\mathfrak{p},K} \in \mathcal{S}(K, T_\mathfrak{p}(E)).$$

If we assume further that the $\mathfrak{p}$-part of the Tate-Shafarevich group $Ш(E_{/K})$ is finite (and this is known if E is defined over $\mathbf{Q}$ and $\mathrm{ord}_{s=1} L(E, s) = 1$; see [**Ko2**] or [**Ru5**]) then the exact sequence

$$0 \longrightarrow E(K) \otimes \mathbf{Z}_p \longrightarrow \mathcal{S}(K, T_\mathfrak{p}(E)) \longrightarrow \varprojlim Ш(E_{/K})_{\mathfrak{p}^n} \longrightarrow 0$$

shows that $\mathcal{S}(K, T_\mathfrak{p}(E)) \cong E(K) \otimes \mathbf{Z}_p$, so

$$\mathbf{c}_{E,\mathfrak{p},K} \in E(K) \otimes \mathbf{Z}_p.$$

If $\mathfrak{p}$ splits in K, one can compute the p-adic height of $\mathbf{c}_{E,\mathfrak{p},K}$ in terms of the derivative of the $\mathfrak{p}$-adic L-function of E at $s = 1$. In particular one can prove the following theorem.

Theorem 6.5.1. *Suppose E is defined over $\mathbf{Q}$ and p splits into two distinct primes in K. If $\mathrm{ord}_{s=1} L(E_{/\mathbf{Q}}, s) = 1$, then $\mathbf{c}_{E,\mathfrak{p},K}$ has infinite order in $E(K) \otimes \mathbf{Z}_p$.*

See [**Ru7**] §9 for the details.

CHAPTER 7

Iwasawa Theory

In this chapter we use the cohomology classes constructed in Chapter 4, along with the duality results of §1.7, to prove Theorems 2.3.2, 2.3.3, and 2.3.4. The proofs follow generally along the same lines as the proof of Theorem 2.2.2 given in Chapter 5, except that where in Chapter 5 we dealt with $\mathcal{O}$-modules, we must now deal with $\mathcal{O}[\mathrm{Gal}(F/K)]$-modules for $K \subset_f F \subset K_\infty$. This makes the algebra much more complicated.

In §7.1 we give the proofs of Theorems 2.3.3 and 2.3.4, assuming Theorem 2.3.2 and two propositions (Propositions 7.1.7 and 7.1.9), whose proofs will be given in the following sections.

We keep the notation of Chapter 2. In particular $\Gamma = \mathrm{Gal}(K_\infty/K)$ and $\Lambda = \mathcal{O}[[\Gamma]]$. If $K \subset_f F \subset K_\infty$ and $M \in \mathcal{O}$ is nonzero, then we write $\Lambda_F = \mathcal{O}[\mathrm{Gal}(F/K)]$ and

$$\Lambda_{F,M} = \Lambda_F/M\Lambda_F = (\mathcal{O}/M\mathcal{O})[\mathrm{Gal}(F/K)].$$

We assume throughout this chapter that we have a p-adic representation T of G_K and an Euler system $\mathbf{c}$ for (T, K_∞) such that

$$\mathbf{c}_{K,\infty} = \{\mathbf{c}_F\}_F \notin H^1_\infty(K,T)_{\mathrm{tors}}$$

(or else there is nothing to prove). We assume that hypotheses $\mathrm{Hyp}(K_\infty, V)$ are satisfied, and we fix once and for all a $\tau \in G_K$ as in hypothesis $\mathrm{Hyp}(K_\infty, V)$(i), i.e., τ is the identity on $K(\mathbf{1})$, on K_∞, on $\boldsymbol{\mu}_{p^\infty}$, and on $(\mathcal{O}_K^\times)^{1/p^\infty}$, and $\dim_\Phi(V/(\tau-1)V) = 1$.

7.1. Overview

Since τ fixes $\boldsymbol{\mu}_{p^\infty}$, we have

$$\dim_\Phi(V^*/(\tau-1)V^*) = \dim_\Phi(V^{\tau=1}) = 1.$$

Definition 7.1.1. Fix an isomorphism

$$\theta^* : W^*/(\tau-1)W^* \xrightarrow{\sim} \mathbf{D}.$$

Recall that $\Omega = K(\mathbf{1})K(W, \boldsymbol{\mu}_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$. Define $\Omega_\infty = K_\infty\Omega$ and let $\Omega_\infty^{\langle\tau\rangle}$ be the fixed field of τ in Ω_∞.

There is a natural evaluation homomorphism

$$\mathrm{Ev}^* : G_{\Omega_\infty^{\langle\tau\rangle}} \longrightarrow \mathrm{Hom}(\mathcal{S}_{\Sigma_p}(K_\infty, W^*), \mathbf{D}) = X_\infty,$$

defined as follows. For every $\sigma \in G_{\Omega_\infty^{\langle\tau\rangle}}$ and class $c \in \mathcal{S}_{\Sigma_p}(K_\infty, W^*)$, we set

$$\mathrm{Ev}^*(\sigma)(c) = \theta^*(c(\sigma))$$

where $c(\sigma)$ means any cocycle in the class c, evaluated at σ. Then $c(\sigma)$ is well-defined modulo $(\sigma - 1)W^*$, and σ acts on W^* through $\mathrm{Gal}(\Omega_\infty/\Omega_\infty^{\langle\tau\rangle})$ which is (topologically) generated by τ, so

$$(\sigma - 1)W^* \subset (\tau - 1)W^* = \ker(\theta^*).$$

Thus $\mathrm{Ev}^*(\sigma)$ is well-defined, and the cocycle relation shows that Ev^* is a homomorphism.

Definition 7.1.2. Define a positive integer a_τ by

$$a_\tau = [W^{\tau=1} : (W^{\tau=1})_{\mathrm{div}}] \cdot \max\{|Z|, |Z^*|\}$$

where $(W^{\tau=1})_{\mathrm{div}}$ is the maximal divisible submodule of $W^{\tau=1}$, and Z (resp., Z^*) is the maximal G_{K_∞}-stable submodule of $(\tau - 1)W$ (resp., $(\tau - 1)W^*$).

Lemma 7.1.3. (i) *a_τ is finite.*

(ii) *If T and τ satisfy hypotheses* $\mathrm{Hyp}(K_\infty, T)$ *then $a_\tau = 1$.*

Proof. Let Z be as in Definition 7.1.2. The maximal divisible submodule of Z gives rise to a G_{K_∞}-stable subspace $V_0 \subset (\tau - 1)V \subsetneq V$. Hypothesis $\mathrm{Hyp}(K_\infty, V)$(ii) asserts that V is irreducible, so we must have $V_0 = 0$. Hence Z is finite, and similarly Z^* is finite. The index $[W^{\tau=1} : (W^{\tau=1})_{\mathrm{div}}]$ is finite simply because W has finite $\mathbf{Z}_p$-corank. This proves (i).

Now suppose hypotheses $\mathrm{Hyp}(K_\infty, T)$ hold. Then $W_\mathfrak{p}$ is an irreducible G_{K_∞}-module (where $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}$), and $W_\mathfrak{p} \not\subset (\tau - 1)W$ because $W_M/(\tau - 1)W_M$ is free of rank one over $\mathcal{O}/M\mathcal{O}$ for every nonzero M, so it follows that $Z = 0$. Similarly $Z^* = 0$, and Proposition A.2.5 shows that $W^{\tau=1} = (W^{\tau=1})_{\mathrm{div}}$. This proves (ii). □

Recall that $\mathcal{N}$ is the ideal of Definition 2.1.1 corresponding to $\mathbf{c}$. Consider the following two extra assumptions.

Assumption 7.1.4. *For every $K \subset_f F \subset K_\infty$, both $\Lambda_F/\mathrm{char}(X_\infty)\Lambda_F$ and $X_\infty \otimes \Lambda_F$ are finite.*

Assumption 7.1.5. *For every prime λ of K dividing $\mathcal{N}$, the decomposition group of λ in G_K contains an element γ_λ with the property that*

$$T^{\gamma_\lambda^{p^n}=1} = (T^*)^{\gamma_\lambda^{p^n}=1} = 0 \text{ for every } n \geq 0.$$

Remark 7.1.6. Suppose that X_∞ is a torsion Λ-module. If $\rho : \Gamma \to \mathcal{O}^\times$ is a character, and we replace T by its twist $T \otimes \rho$, then Corollary 6.2.2 shows that X_∞ is replaced by $X_\infty \otimes \rho$. Thus Lemma 6.1.3 applied to $T \oplus T^*$ (in part (i)) and to $X_\infty \oplus \Lambda/\mathrm{char}(X_\infty)$ (in part (ii)) shows that there is a ρ such that after twisting by ρ, Assumptions 7.1.4 and 7.1.5 are satisfied. Theorem 6.4.1 shows that Theorems 2.3.2, 2.3.3, and 2.3.4 for T and $\mathbf{c}$ are equivalent to Theorems 2.3.2, 2.3.3, and 2.3.4, respectively, for $T \otimes \rho$ and the twisted Euler system $\mathbf{c}^\rho$ of §6.3.

Suppose now that Theorem 2.3.2 holds (the proof will be given in §7.3), so that X_∞ is a torsion Λ-module. Then the discussion above shows that, without loss of generality, to prove Theorems 2.3.3 and 2.3.4 we may assume that 7.1.4 and 7.1.5 are satisfied. We will assume this for the rest of this section.

As discussed in §2.3, since X_∞ is a torsion Λ-module we can fix an injective pseudo-isomorphism

$$\bigoplus_{i=1}^{r} \Lambda/f_i\Lambda \longrightarrow X_\infty, \tag{7.1}$$

where $f_1, \dots, f_r \in \Lambda$ satisfy $f_{i+1} \mid f_i$ for $1 \le i \le r-1$. The sequence of principal ideals (elementary divisors) $f_1\Lambda, \dots, f_r\Lambda$ is uniquely determined by these conditions, and the characteristic ideal of X_∞ is

$$\mathrm{char}(X_\infty) = \prod_{i=1}^{r} f_i\Lambda. \tag{7.2}$$

Since X_∞ is a torsion Λ-module, all the f_i are nonzero.

Assume for the rest of this section that, in addition to hypotheses $\mathrm{Hyp}(K_\infty, V)$, hypothesis $\mathrm{Hyp}(K_\infty/K)$ is satisfied as well.

Proposition 7.1.7. *With r as above, there are elements $z_1, \dots, z_r \in X_\infty$ and ideals $\mathfrak{g}_1, \dots, \mathfrak{g}_r \subset \Lambda$ such that for $1 \le k \le r$ we have*

(i) $z_k \in \mathrm{Ev}^*(\tau G_{\Omega_\infty})$,

(ii) $a_\tau \mathfrak{g}_k \subset f_k\Lambda$, *and if* $k < r$ *then* $\mathfrak{g}_k \subset \mathfrak{g}_{k+1}$,

(iii) *there is a split exact sequence*

$$0 \longrightarrow \sum_{i=1}^{k-1} \Lambda z_i \longrightarrow \sum_{i=1}^{k} \Lambda z_i \longrightarrow \Lambda/\mathfrak{g}_k \longrightarrow 0,$$

(iv) $a_\tau(X_\infty / \sum_{i=1}^{r} \Lambda z_i)$ *is pseudo-null.*

The proof of Proposition 7.1.7 will be given in §7.6. Using (7.1) it is easy to find $\{z_i\}$, with $\mathfrak{g}_i = f_i\Lambda$, satisfying (ii), (iii), and (iv), but condition (i) will be essential for our purposes.

Definition 7.1.8. Fix a sequence $z_1, \dots, z_r \in X_\infty$ as in Proposition 7.1.7 and define

$$Z_\infty = \sum_{i=1}^{r} \Lambda z_i \subset X_\infty.$$

Let $\mathcal{M}$ denote the maximal ideal of Λ. If $0 \le k \le r$, a *Selmer sequence* $\boldsymbol{\sigma}$ of length k is a k-tuple $(\sigma_1, \dots, \sigma_k)$ of elements of τG_{Ω_∞} satisfying

$$\mathrm{Ev}^*(\sigma_i) - z_i \in \mathcal{M} Z_\infty$$

for $1 \le i \le k$. (When $k = 0$, the empty sequence is a Selmer sequence.) Note that by Proposition 7.1.7(i), Selmer sequences exist, for example with all the above differences equal to zero.

Suppose M is a power of p. Let $\Omega_M = K(\mathbf{1})K(W_M, \boldsymbol{\mu}_M, (\mathcal{O}_K^\times)^{1/M})$, and if $K \subset_f F \subset K_\infty$ let $L_{F,M} \supset F\Omega_M$ be the fixed field of the subgroup

$$\bigcap_{c \in \mathcal{S}_{\Sigma_p}(F, W_M^*)} \ker((c)_{F\Omega_M}) \subset G_{F\Omega_M}.$$

The restriction of $\mathcal{S}_{\Sigma_p}(F, W_M^*)$ to $F\Omega_M$ is a finite (Lemma 1.5.7) subgroup of $\mathrm{Hom}(G_{F\Omega_M}, W_M^*)$, so $L_{F,M}$ is a finite abelian extension of $F\Omega_M$. It is straightforward to check that $L_{F,M}/K$ is Galois and unramified outside primes above p, ∞, and primes where T is ramified.

For $0 \le k \le r$ we call a k-tuple $(\mathfrak{q}_1, \dots, \mathfrak{q}_k)$ of primes of K a *Kolyvagin sequence* (for F and M) if there is a Selmer sequence $\boldsymbol{\sigma}$ of length k such that for $1 \le i \le k$,

- $\mathfrak{q}_i$ does not divide the ideal $\mathcal{N}$ of Definition 2.1.1, and
- $\mathrm{Fr}_{\mathfrak{q}_i}$ is (a conjugate of) σ_i on $L_{F,M}$

(all primes not dividing $\mathcal{N}$ are unramified in $L_{F,M}/K$). If $\boldsymbol{\pi}$ is a Kolyvagin sequence of length k we define

$$\mathfrak{r}(\boldsymbol{\pi}) = \prod_{i=1}^{k} \mathfrak{q}_i.$$

By Lemma 4.1.3, $\mathfrak{r}(\boldsymbol{\pi})$ belongs to the set $\mathcal{R}_{F,M}$ defined in Definition 4.1.1.

Let $\Pi(k, F, M)$ be the set of all Kolyvagin sequences of length k for F and M. When $k = 0$, we make the convention that $\Pi(k, F, M)$ has a single element, the empty sequence (independent of F and M). Define an ideal in $\Lambda_{F,M}$

$$\Psi(k, F, M) = \sum_{\boldsymbol{\pi} \in \Pi(k,F,M)} \sum_{\psi} \psi(\kappa_{[F,\mathfrak{r}(\boldsymbol{\pi}),M]}) \subset \Lambda_{F,M}$$

where $\kappa_{[F,\mathfrak{r}(\boldsymbol{\pi}),M]}$ is the Euler system derivative class constructed in §4.4, $\langle \kappa_{[F,\mathfrak{r}(\boldsymbol{\pi}),M]} \rangle$ is the $\Lambda_{F,M}$-submodule of $H^1(F, W_M)$ generated by $\kappa_{[F,\mathfrak{r}(\boldsymbol{\pi}),M]}$, and the inner sum is over $\psi \in \mathrm{Hom}_\Lambda(\langle \kappa_{[F,\mathfrak{r}(\boldsymbol{\pi}),M]} \rangle, \Lambda_{F,M})$. In other words,

$\Psi(k,F,M)$ is the ideal of $\Lambda_{F,M}$ generated by all homomorphic images of modules $\langle \kappa_{[F,\mathfrak{r}(\boldsymbol{\pi}),M]} \rangle$ as $\boldsymbol{\pi}$ runs through $\Pi(k,F,M)$.

Proposition 7.1.9. *There is an element $h \in \Lambda$ such that*

(i) *h is relatively prime to* $\mathrm{char}(X_\infty)$,

(ii) *for every $K \subset_{\mathrm{f}} F \subset K_\infty$ there is a power N_F of p such that if M is a power of p and $0 \le k < r$, then*

$$a_\tau^5 h \Psi(k,F,MN_F)\Lambda_{F,M} \subset f_{k+1}\Psi(k+1,F,M).$$

Proposition 7.1.9 is the key to the proofs of Theorems 2.3.3 and 2.3.4; it will be proved in §7.7. We now show how to use Proposition 7.1.9 to complete the proof of Theorems 2.3.3 and 2.3.4. Recall that if Σ is a set of places of K, then K_Σ denotes the maximal extension of K in $\bar{K}$ which is unramified outside Σ.

Corollary 7.1.10. *Suppose $K \subset_{\mathrm{f}} F \subset K_\infty$ and $h \in \Lambda$ satisfies Proposition 7.1.9. Let Σ be a set of places of K containing all primes above p, all primes where T is ramified, and all infinite places,*

If $\psi \in \mathrm{Hom}_\Lambda(H^1(K_\Sigma/F,T),\Lambda_F)$, then $a_\tau^{5r} h^r \psi(\mathbf{c}_F) \in \mathrm{char}(X_\infty)\Lambda_F$.

Proof. Note that $\mathbf{c}_F \in H^1(K_\Sigma/F,T)$ by Corollary B.3.6.

Let N_F be as in Proposition 7.1.9(ii). Suppose $0 \le k < r$ and M is a power of p. Proposition 7.1.9(ii) shows that

$$a_\tau^5 h\Psi(k,F,MN_F^{r-k})\Lambda_{F,M} \subset f_{k+1}\Psi(k+1,F,MN_F^{r-k-1})\Lambda_{F,M},$$

so by induction, writing $M' = MN_F^r$ and using (7.2), we conclude that

$$\begin{aligned} a_\tau^{5r} h^r \Psi(0,F,M')\Lambda_{F,M} &\subset \Big(\prod_{i=1}^r f_i\Big)\Psi(r,F,M) \\ &\subset \Big(\prod_{i=1}^r f_i\Big)\Lambda_{F,M} = \mathrm{char}(X_\infty)\Lambda_{F,M}. \end{aligned} \tag{7.3}$$

By Lemma 4.4.13(i), $\kappa_{[F,\mathbf{1},M']}$ is the image of $\mathbf{c}_F$ under the injection

$$H^1(K_\Sigma/F,T)/M'H^1(K_\Sigma/F,T) \hookrightarrow H^1(K_\Sigma/F,W_{M'}) \hookrightarrow H^1(F,W_{M'}).$$

Let $\bar{\psi}$ denote the composition

$$\Lambda_{F,M'}\kappa_{[F,\mathbf{1},M']} \hookrightarrow H^1(K_\Sigma/F,T)/M'H^1(K_\Sigma/F,T) \xrightarrow{\psi} \Lambda_{F,M'} \to \Lambda_{F,M}$$

induced by the inverse of this injection and by ψ. By definition

$$\bar{\psi}(\kappa_{[F,\mathbf{1},M']}) \in \Psi(0,F,M')\Lambda_{F,M},$$

so (7.3) shows

$$a_\tau^{5r} h^r \bar{\psi}(\kappa_{[F,\mathbf{1},M']}) \in \mathrm{char}(X_\infty)\Lambda_{F,M}.$$

Since this holds for every sufficiently large M, and $\bar{\psi}(\kappa_{[F,\mathbf{1},M']})$ is the reduction of $\psi(\mathbf{c}_F)$ modulo M, this completes the proof of the corollary. □

Lemma 7.1.11. *Suppose that G is a finite abelian group, that R is a principal ideal domain, that B is finitely generated $R[G]$-module with no R-torsion, and that $f \in R[G]$ is not a zero-divisor. If $b \in B$ is such that*

$$\{\psi(b) : \psi \in \mathrm{Hom}_{R[G]}(B, R[G])\} \subset fR[G],$$

then $b \in fB$.

Proof. Let $B' = Rb + fB$. Since f is not a zero-divisor, we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{R[G]}(B', fR[G]) & \xleftarrow{f} & \mathrm{Hom}_{R[G]}(B', R[G]) & \xrightarrow{\sim} & \mathrm{Hom}_R(B', R) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}_{R[G]}(fB, fR[G]) & \xleftarrow{f} & \mathrm{Hom}_{R[G]}(fB, R[G]) & \xrightarrow{\sim} & \mathrm{Hom}_R(fB, R)
\end{array}$$

in which the horizontal maps are all isomorphisms (see for example Lemma 4.3.3 for the isomorphisms on the right).

Suppose $\bar{\varphi} \in \mathrm{Hom}_{R[G]}(fB, fR[G])$. Since B has no R-torsion, $\bar{\varphi}$ extends uniquely to a map $\varphi : B \to R[G]$, and by our assumption on b, the restriction of φ belongs to $\mathrm{Hom}_{R[G]}(B', fR[G])$. Thus all the vertical maps in the diagram above are surjective. Since B' and fB are free R-modules, the surjectivity of the right-hand map shows that $B' = fB$, which proves the lemma. □

Let $\mathrm{ind}_\Lambda(\mathbf{c})$ be as in Definition 2.3.1.

Theorem 7.1.12. $\mathrm{char}(X_\infty)$ *divides* $a_\tau^{5r}\mathrm{ind}_\Lambda(\mathbf{c})$.

Proof. Suppose $h \in \Lambda$ is as in Proposition 7.1.9. Let Σ be a finite set of places of K containing all primes above p, all primes where T is ramified, and all infinite places. If $K \subset_{\mathrm{f}} F \subset K_\infty$, Corollary 7.1.10 and Lemma 7.1.11 applied with $B = H^1(K_\Sigma/F, T)/H^1(K_\Sigma/F, T)_{\mathrm{tors}}$ and $b = h^r a_\tau^{5r}\mathbf{c}_F$ show (note that $H^1(K_\Sigma/F, T)$ is finitely generated over $\mathbf{Z}_p$ by Proposition B.2.7) that

$$a_\tau^{5r}h^r\mathbf{c}_F \in \mathrm{char}(X_\infty)(H^1(K_\Sigma/F, T)/H^1(K_\Sigma/F, T)_{\mathrm{tors}}).$$

It follows from Lemma 1.2.2(ii) that if $K \subset_{\mathrm{f}} F \subset K_\infty$, then $H^1(F, T)_{\mathrm{tors}}$ is annihilated by the annihilator in Λ of $W^{G_{K_\infty}}$, so

$$\varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} (H^1(F, T)_{\mathrm{tors}}) \subset H^1_\infty(K, T)_{\mathrm{tors}}$$

(where the latter group is the Λ-torsion submodule). Passing to the inverse limit we deduce that

$$a_\tau^{5r} h^r \mathbf{c}_{K,\infty} \in \mathrm{char}(X_\infty)(H^1_\infty(K,T)/H^1_\infty(K,T)_{\mathrm{tors}}).$$

Therefore if $\phi \in \mathrm{Hom}_\Lambda(H^1_\infty(K,T),\Lambda)$ then

$$a_\tau^{5r} h^r \phi(\mathbf{c}_{K,\infty}) \in \mathrm{char}(X_\infty).$$

Since h is relatively prime to (the principal ideal) $\mathrm{char}(X_\infty)$, it follows that

$$a_\tau^{5r} \phi(\mathbf{c}_{K,\infty}) \in \mathrm{char}(X_\infty).$$

This completes the proof. □

Proof of Theorems 2.3.3 *and* 2.3.4. Lemma 7.1.3(i) shows that a_τ is a (finite) positive integer, so Theorem 2.3.4 is immediate from Theorem 7.1.12. If in addition T and τ satisfy hypotheses $\mathrm{Hyp}(K_\infty,T)$, then $a_\tau = 1$ by Lemma 7.1.3(ii), and Theorem 2.3.3 follows as well. □

7.2. Galois Groups and the Evaluation Map

Keep the notation of the previous section.

Definition 7.2.1. Define $q_\tau(x) = \det(1-\tau^{-1}x|T^*)/(x-1)$. Our assumptions on τ ensure that

$$q_\tau(x) = \det(1-\tau x|T)/(x-1) \in \mathcal{O}[x]$$

and that, by Lemma A.2.4(ii) (applied with $\sigma = \tau^{-1}$),

$$q_\tau(\tau^{-1}) : V/(\tau-1)V \xrightarrow{\sim} V^{\tau=1}$$

is an isomorphism of one-dimensional vector spaces.

The $\mathbf{D}(1)$-dual of the isomorphism θ^* of Definition 7.1.1 is an isomorphism

$$\mathcal{O}(1) \xrightarrow{\sim} T^{\tau=1}.$$

The inverse of this isomorphism, together with the generator ξ of $\mathcal{O}(1)$ chosen in Definition 4.4.1, gives an isomorphism

$$\theta : (W^{\tau=1})_{\mathrm{div}} \xrightarrow{\sim} \mathbf{D}.$$

Define $\bar{\theta}$ to be the (surjective, by Lemma A.2.4) composition

$$W/(\tau-1)W \xrightarrow{q_\tau(\tau^{-1})} (W^{\tau=1})_{\mathrm{div}} \xrightarrow{\theta} \mathbf{D}.$$

We also fix once and for all an extension of θ to $W^{\tau=1}$

$$\theta : W^{\tau=1} \longrightarrow \mathbf{D}.$$

This extension is not in general unique, but the difference between any two choices lies in $\mathrm{Hom}(W^{\tau=1}/(W^{\tau=1})_{\mathrm{div}},\mathbf{D})$ which is killed by a_τ.

Definition 7.2.2. Recall the evaluation homomorphism

$$\mathrm{Ev}^* : G_{\Omega_\infty^{\langle\tau\rangle}} \longrightarrow X_\infty$$

of Definition 7.1.1. Similarly we define

$$\mathrm{Ev} : G_{\Omega_\infty^{\langle\tau\rangle}} \longrightarrow \mathrm{Hom}(H^1(K_\infty, W), \mathbf{D})$$

by

$$\mathrm{Ev}(\sigma)(c) = \bar{\theta}(c(\sigma)) = \theta \circ q_\tau(\tau^{-1})(c(\sigma))$$

for all $\sigma \in G_{\Omega_\infty^{\langle\tau\rangle}}$ and $c \in H^1(K_\infty, W)$, where $c(\sigma)$ means any cocycle representing c, evaluated at σ.

Suppose $M \in \mathcal{O}$ is nonzero and $K \subset_{\mathrm{f}} F \subset K_\infty$. Let $\Omega_M^{\langle\tau\rangle}$ denote the fixed field of τ in $\Omega_M = K(\mathbf{1})K(W_M, \boldsymbol{\mu}_M, (\mathcal{O}_K^\times)^{1/M})/K)$. In exactly the same way, we define evaluation maps

$$\mathrm{Ev}^*_{F,M} : G_{F\Omega_M^{\langle\tau\rangle}} \longrightarrow \mathrm{Hom}(\mathcal{S}_{\Sigma_p}(F, W_M^*), \mathcal{O}/M\mathcal{O})$$
$$\mathrm{Ev}_{F,M} : G_{F\Omega_M^{\langle\tau\rangle}} \longrightarrow \mathrm{Hom}(H^1(F, W_M), \mathcal{O}/M\mathcal{O})$$

by $\mathrm{Ev}^*_{F,M}(\sigma)(c) = \theta^*(c(\sigma))$ and $\mathrm{Ev}_{F,M}(\sigma)(c) = \bar{\theta}(c(\sigma))$, where we identify $\mathcal{O}/M\mathcal{O}$ with $M^{-1}\mathcal{O}/\mathcal{O} \subset \mathbf{D}$.

Definition 7.2.3. If $\eta \in \Lambda$, we will denote by $\eta^\bullet$ the image of η under the involution of Λ induced by $\gamma \mapsto \gamma^{-1}$ for $\gamma \in \Gamma$. Similarly if $\mathcal{A}$ is an ideal of Λ we will write $\mathcal{A}^\bullet$ for the ideal which is the image of $\mathcal{A}$ under this involution.

We will use repeatedly below that if B is a Λ-module and $\mathcal{A}$ is an ideal of Λ which annihilates B, then $\mathcal{A}^\bullet$ annihilates $\mathrm{Hom}(B, \mathbf{D})$.

If B is a Λ-module, $\mathrm{Ann}_\Lambda(B)$ will denote the annihilator in Λ of B.

Lemma 7.2.4. (i) *If* $c \in H^1(K_\infty, W)$ *and* $\mathrm{Ev}(\gamma)(c) = 0$ *for every* $\gamma \in G_{\Omega_\infty}$, *then* $a_\tau \mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W))c = 0$.

(ii) $a_\tau \mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W))^\bullet \mathrm{Hom}(H^1(K_\infty, W), \mathbf{D}) \subset \mathcal{O}\mathrm{Ev}(G_{\Omega_\infty})$.

(iii) $a_\tau \mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W^*))^\bullet X_\infty \subset \mathcal{O}\mathrm{Ev}^*(G_{\Omega_\infty})$.

Proof. Unwinding the definition of Ev, we see that the dual of Ev on G_{Ω_∞} is given by the composition

$$H^1(K_\infty, W) \xrightarrow{\mathrm{res}_{\Omega_\infty}} \mathrm{Hom}(G_{\Omega_\infty}, W)^{G_{K_\infty}}$$
$$\longrightarrow \mathrm{Hom}(G_{\Omega_\infty}, W/(\tau - 1)W) \xrightarrow{\bar{\theta}} \mathrm{Hom}(G_{\Omega_\infty}, \mathbf{D}). \quad (7.4)$$

The kernel of the first map is $H^1(\Omega_\infty/K_\infty, W)$. The kernel of the second map is

$$\mathrm{Hom}(G_{\Omega_\infty}, W)^{G_{K_\infty}} \cap \mathrm{Hom}(G_{\Omega_\infty}, (\tau - 1)W).$$

If ψ belongs to this intersection, then $\psi(G_{\Omega_\infty})$ is a G_{K_∞}-stable submodule of $(\tau-1)W$. The kernel of $\bar{\theta}$ is $W^{q_\tau(\tau^{-1})=0}/(\tau-1)W$, which has the same order as $W^{\tau=1}/W^{\tau=1}_{\mathrm{div}}$ by Proposition A.2.5 (applied with $\sigma=\tau^{-1}$). Thus a_τ annihilates the kernel of the composition of the second and third maps.

If $\mathrm{Ev}(\gamma)(c)=0$ for every $\gamma\in G_{\Omega_\infty}$, then c maps to zero under (7.4), so this proves (i). Applying $\mathrm{Hom}_{\mathcal{O}}(\,\cdot\,,\mathbf{D})$ to (7.4) yields

$$G_{\Omega_\infty}\otimes\mathcal{O}\xrightarrow{\mathrm{Ev}}\mathrm{Hom}(H^1(K_\infty,W),\mathbf{D})$$

and (ii) follows. The proof of (iii) is the same as the proof of (ii) (except that in that case the third map of the analogue of (7.4) is induced by θ^*, which is injective). □

Definition 7.2.5. Suppose $K\subset_{\mathrm{f}}F\subset K_\infty$ and M is a power of p. Define

$$\mathcal{R}_{F,M,\tau}=\{\mathfrak{r}\in\mathcal{R}:\text{for every prime }\mathfrak{q}\text{ dividing }\mathfrak{r},\ \mathrm{Fr}_\mathfrak{q}\text{ belongs to the conjugacy class of }\tau\text{ in }\mathrm{Gal}(F\Omega_M/K)\}$$

where Ω_M is as above (and as in Definition 7.1.8). By Lemma 4.1.3, $\mathcal{R}_{F,M,\tau}\subset\mathcal{R}_{F,M}$ where $\mathcal{R}_{F,M}$ is the set defined in Definition 4.1.1.

Suppose $\mathfrak{q}\in\mathcal{R}_{F,M,\tau}$. Let $\mathfrak{Q}$ be a prime of $\bar{K}$ above $\mathfrak{q}$ such that the corresponding Frobenius $\mathrm{Fr}_\mathfrak{q}$ of $\mathfrak{q}$ is τ on $F\Omega_M$. Recall the generator $\sigma_\mathfrak{q}$ of $\mathrm{Gal}(K(\mathfrak{q})/K)$ given by Definition 4.4.1, and fix a lift of $\sigma_\mathfrak{q}$ to the inertia group $\mathcal{I}_\mathfrak{Q}$ of $\mathfrak{Q}$ in G_K. Then both $\mathrm{Fr}_\mathfrak{q}$ and $\sigma_\mathfrak{q}$ belong to $G_{F\Omega_M^{\langle\tau\rangle}}$. We define the *finite evaluation maps*

$$\mathrm{Ev}^*_{\mathfrak{q},f}=\mathrm{Ev}^*_{F,M}(\mathrm{Fr}_\mathfrak{q}):\mathcal{S}_{\Sigma_p}(F,W^*_M)\longrightarrow\mathcal{O}/M\mathcal{O}$$
$$\mathrm{Ev}_{\mathfrak{q},f}=\mathrm{Ev}_{F,M}(\mathrm{Fr}_\mathfrak{q}):H^1(F,W_M)\longrightarrow\mathcal{O}/M\mathcal{O}.$$

By Lemma 1.4.7(i) (which applies thanks to Lemma 4.1.2(i)), evaluation at $\sigma_\mathfrak{q}$ induces

$$H^1(F,W_M)\longrightarrow H^1(F_\mathfrak{Q},W_M)\longrightarrow H^1_s(F_\mathfrak{Q},W_M)\xrightarrow{\sim}W_M^{\mathrm{Fr}_\mathfrak{q}=1}=W_M^{\tau=1}$$

and we define the *singular evaluation map*

$$\mathrm{Ev}_{\mathfrak{q},s}:H^1(F,W_M)\longrightarrow\mathcal{O}/M\mathcal{O}$$

by $\mathrm{Ev}_{\mathfrak{q},s}(c)=\theta(c(\sigma_\mathfrak{q}))$.

Note that $\mathrm{Ev}^*_{\mathfrak{q},f}$, $\mathrm{Ev}_{\mathfrak{q},f}$, and $\mathrm{Ev}_{\mathfrak{q},s}$ depend on F, M, and the choice of $\mathfrak{Q}$ (as well as the specific choice of $\mathrm{Fr}_\mathfrak{q}$ and $\sigma_\mathfrak{q}$), but we will suppress this from the notation.

Remark 7.2.6. We will usually apply $\mathrm{Ev}_{\mathfrak{q},f}$ and $\mathrm{Ev}^*_{\mathfrak{q},f}$ to cohomology classes which are finite at $\mathfrak{q}$, so we think of these maps (via Lemma 1.4.7(i)) as measuring the finite part at $\mathfrak{q}$ of a cohomology class. Similarly, we view $\mathrm{Ev}_{\mathfrak{q},s}$ as measuring the singular part at $\mathfrak{q}$.

Recall that $\Lambda_F = \mathcal{O}[\mathrm{Gal}(F/K)]$ and $\Lambda_{F,M} = \Lambda_F/M\Lambda_F$.

Lemma 7.2.7. *Suppose that $K \subset_{\mathrm{f}} F \subset K_\infty$, that M is a power of p, and that B is a Λ_F-module.*

(i) *The map*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}}(B, \mathcal{O}/M\mathcal{O}) & \longrightarrow & \mathrm{Hom}_{\Lambda}(B, \Lambda_{F,M}) \\ \psi & \mapsto & \tilde{\psi} \end{array}$$

defined by

$$\tilde{\psi}(b) = \sum_{\eta \in \mathrm{Gal}(F/K)} \psi(\eta b)\eta^{-1}$$

is an $\mathcal{O}$-module isomorphism.

(ii) *If $\psi \in \mathrm{Hom}_{\mathcal{O}}(B, \mathcal{O}/M\mathcal{O})$ and $\sigma \in \mathrm{Gal}(F/K)$ then $\widetilde{\sigma\psi} = \sigma^{-1}\tilde{\psi}$.*

Proof. The map $\mathrm{Hom}_{\Lambda}(B, \Lambda_{F,M}) \to \mathrm{Hom}_{\mathcal{O}}(B, \mathcal{O}/M\mathcal{O})$ induced by composition with $\sum_{\eta \in \mathrm{Gal}(F/K)} a_\eta \eta \mapsto a_1$ is a two-sided inverse of the map in (i), and (ii) is easily checked. (Note that σ acts on $\psi \in \mathrm{Hom}_{\mathcal{O}}(B, \mathcal{O}/M\mathcal{O})$ by $(\sigma\psi)(b) = \psi(\sigma^{-1}b)$ for every $b \in B$, and on $\tilde{\psi} \in \mathrm{Hom}_{\Lambda}(B, \Lambda_{F,M})$ by $(\sigma\tilde{\psi})(b) = \sigma(\tilde{\psi}(b))$.) □

Remark 7.2.8. Note that in Lemma 7.2.7, (ii) says that the bijection of (i) is not in general a $\Lambda_{F,M}$-module homomorphism.

Definition 7.2.9. Suppose $K \subset_{\mathrm{f}} F \subset K_\infty$ and M is a power of p. If $\gamma \in G_{F\Omega_M^{\langle\tau\rangle}}$ and $\mathfrak{q} \in \mathcal{R}_{F,M,\tau}$ we will write

$$\widetilde{\mathrm{Ev}}_{F,M}(\gamma), \widetilde{\mathrm{Ev}}_{\mathfrak{q},f}, \widetilde{\mathrm{Ev}}_{\mathfrak{q},s} \in \mathrm{Hom}_{\Lambda}(H^1(F, W_M), \Lambda_{F,M})$$

for the images of $\mathrm{Ev}_{F,M}(\gamma)$, $\mathrm{Ev}_{\mathfrak{q},f}$, and $\mathrm{Ev}_{\mathfrak{q},s}$, respectively, under the map of Lemma 7.2.7(i). Thus

$$\widetilde{\mathrm{Ev}}_{\mathfrak{q},f}(c) = \sum_{\eta \in \mathrm{Gal}(F/K)} \mathrm{Ev}_{\mathfrak{q},f}(\eta c)\eta^{-1}$$

and similarly for $\widetilde{\mathrm{Ev}}_{F,M}(\gamma)$ and $\widetilde{\mathrm{Ev}}_{\mathfrak{q},s}$.

The next two results, Theorems 7.2.10 and 7.2.11, are crucial for the proof of Theorem 2.3.2 and Proposition 7.1.9. They are restatements of Theorems 4.5.4 and 1.7.3(ii), respectively, in the language of these evaluation maps.

Theorem 7.2.10. *Suppose that $\mathbf{c}$ is an Euler system, that $K \subset_{\mathrm{f}} F \subset K_\infty$, that M is a power of p, that $\mathfrak{r} \in \mathcal{R}_{F,M}$, and that $\mathfrak{q} \in \mathcal{R}_{F,M,\tau}$ is a prime not dividing $\mathfrak{r}$. Let $\kappa_{[F,\mathfrak{r},M]}$ be the derivative class constructed in §4.4. Then*

$$\widetilde{\mathrm{Ev}}_{\mathfrak{q},f}(\kappa_{[F,\mathfrak{r},M]}) = \widetilde{\mathrm{Ev}}_{\mathfrak{q},s}(\kappa_{[F,\mathfrak{r}\mathfrak{q},M]}).$$

Proof. Suppose $\rho \in G_K$. Theorem 4.5.4 applied to the Euler system $\{\rho\mathbf{c}_{F(\mathfrak{r})}\}$ shows that, with $Q_\mathfrak{q}(x)$ as in Lemma 4.5.2,

$$\begin{aligned}\mathrm{Ev}_{\mathfrak{q},f}(\rho\kappa_{[F,\mathfrak{r},M]}) &= \theta \circ q_\tau(\tau^{-1})((\rho\kappa_{[F,\mathfrak{r},M]})(\mathrm{Fr}_\mathfrak{q})) \\ &= \theta \circ Q_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})((\rho\kappa_{[F,\mathfrak{r},M]})(\mathrm{Fr}_\mathfrak{q})) \\ &= \theta((\rho\kappa_{[F,\mathfrak{r}\mathfrak{q},M]})(\sigma_\mathfrak{q})) \\ &= \mathrm{Ev}_{\mathfrak{q},s}(\rho\kappa_{[F,\mathfrak{r}\mathfrak{q},M]}).\end{aligned}$$

(Note that one consequence of Theorem 4.5.4 is that $(\rho\kappa_{[F,\mathfrak{r}\mathfrak{q},M]})(\sigma_\mathfrak{q}) \in W_{\mathrm{div}}^{\tau=1}$, so $\mathrm{Ev}_{\mathfrak{q},s}(\rho\kappa_{[F,\mathfrak{r}\mathfrak{q},M]})$ does not depend on any choice made in extending θ from $W_{\mathrm{div}}^{\tau=1}$ to $W^{\tau=1}$.) The theorem follows immediately. □

Notation. If B is a G_K-module, v is a place of K, and $K \subset_\mathrm{f} F \subset K_\infty$, we will abbreviate

$$\begin{aligned}F_v &= F \otimes_K K_v = \oplus_{w|v} F_w, \\ H^1(F_v, B) &= \oplus_{w|v} H^1(F_w, B), \\ H^1_f(F_v, B) &= \oplus_{w|v} H^1_f(F_w, B), \\ c_v &= \oplus_{w|v} c_w \in H^1(F_v, B) \text{ for every } c \in H^1(F, B).\end{aligned}$$

There is a natural action of $\mathrm{Gal}(F/K)$ on $H^1(F_v, B)$. Concretely, every $\sigma \in \mathrm{Gal}(F/K)$ induces an isomorphism

$$H^1(F_w, B) \xrightarrow{\sim} H^1(F_{\sigma w}, B)$$

for every w, and summing these maps over w lying above v gives an automorphism of $H^1(F_v, B)$; see also Corollary B.5.2. In applying Theorem 1.7.3 over the base field F instead of K, all of the maps are $\mathrm{Gal}(F/K)$-homomorphisms.

Theorem 7.2.11. *Suppose that $K \subset_\mathrm{f} F \subset K_\infty$, that M is a power of p, that $\mathfrak{r}\mathfrak{q} \in \mathcal{R}_{F,M}$ and that $\mathfrak{q}$ is a prime in $\mathcal{R}_{F,M,\tau}$. Let $\Sigma_{p\mathfrak{r}}$ and $\Sigma_{p\mathfrak{r}\mathfrak{q}}$ denote the sets of primes of K dividing $p\mathfrak{r}$ and $p\mathfrak{r}\mathfrak{q}$, respectively. Then*

$$a_\tau \widetilde{\mathrm{Ev}}_{\mathfrak{q},s}(\mathcal{S}^{\Sigma_{p\mathfrak{r}\mathfrak{q}}}(F, W_M))\ \mathrm{Ev}^*_{\mathfrak{q},f}|_{\mathcal{S}_{\Sigma_{p\mathfrak{r}}}(F,W_M^*)} = 0.$$

Proof. Note that $\widetilde{\mathrm{Ev}}_{\mathfrak{q},s}(\mathcal{S}^{\Sigma_{p\mathfrak{r}\mathfrak{q}}}(F, W_M)) \subset \Lambda_{F,M}$ and that $\mathrm{Ev}^*_{\mathfrak{q},f}|_{\mathcal{S}_{\Sigma_{p\mathfrak{r}}}(F,W_M^*)}$ is in the $\Lambda_{F,M}$-module $\mathrm{Hom}(\mathcal{S}_{\Sigma_{p\mathfrak{r}}}(F, W_M^*), \mathbf{D})$.

Suppose $c \in \mathcal{S}_{\Sigma_{p\mathfrak{r}}}(F, W_M^*)$ and $d \in \mathcal{S}^{\Sigma_{p\mathfrak{r}\mathfrak{q}}}(F, W_M)$. Theorem 1.7.3(ii), applied with $\Sigma = \Sigma_{p\mathfrak{r}\mathfrak{q}}$ and $\Sigma_0 = \Sigma_{p\mathfrak{r}}$, shows that $\langle c, d\rangle_\mathfrak{q} = 0$, where $\langle\ ,\ \rangle_\mathfrak{q} = \sum_{\Omega|\mathfrak{q}} \langle\ ,\ \rangle_\mathfrak{q}$ is the sum of the local pairings of Theorem 1.4.1 at primes above $\mathfrak{q}$.

Let $\mathfrak{Q}$ be the prime above $\mathfrak{q}$ corresponding to our choices of $\mathrm{Fr}_\mathfrak{q}$ and $\sigma_\mathfrak{q}$. Consider the diagram

$$\begin{array}{ccccc}
H^1_f(F_\mathfrak{Q}, W_M^*) & \times & H^1_s(F_\mathfrak{Q}, W_M) & \xrightarrow{\langle\ ,\ \rangle_\mathfrak{Q}} & \mathcal{O}/M\mathcal{O} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \pm 1\otimes\xi} \\
W_M^*/(\tau-1)W_M^* & \times & (W_M)^{\tau=1} & \xrightarrow{\langle\ ,\ \rangle_{W_M}} & \mathcal{O}(1)/M\mathcal{O}(1) \\
{\scriptstyle\theta^*}\downarrow & & {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle a_\tau\otimes\xi^{-1}} \\
\mathcal{O}/M\mathcal{O} & \times & \mathcal{O}/M\mathcal{O} & \xrightarrow{a_\tau} & \mathcal{O}/M\mathcal{O}
\end{array}$$

where

- the upper part (including the ambiguity of sign) comes from Lemma 1.4.7 (so the upper left and upper center vertical maps are isomorphisms given by evaluation at $\mathrm{Fr}_\mathfrak{q}$ and $\sigma_\mathfrak{q}$, respectively),
- ξ is the chosen generator of $\mathbf{Z}_p(1)$ from which we defined $\sigma_\mathfrak{q}$,
- $\langle\ ,\ \rangle_{W_M}$ is induced by the natural pairing $W_M^* \times W_M \to \mathcal{O}(1)$, and
- the pairing on the bottom is $(x,y) \mapsto a_\tau xy$.

Since a_τ annihilates $(W^{\tau=1})/(W^{\tau=1})_{\mathrm{div}}$, it follows from Definitions 7.1.1 and 7.2.1 of θ^* and θ, respectively, that the bottom half of the diagram commutes. In other words,

$$a_\tau \langle c, d\rangle_\mathfrak{Q} \;=\; \pm a_\tau \theta(d(\sigma_\mathfrak{q}))\theta^*(c(\mathrm{Fr}_\mathfrak{q})) \;=\; \pm a_\tau \mathrm{Ev}_{\mathfrak{q},s}(d)\mathrm{Ev}^*_{\mathfrak{q},f}(c).$$

Therefore

$$\begin{aligned}
(a_\tau \widetilde{\mathrm{Ev}}_{\mathfrak{q},s}(d)\mathrm{Ev}^*_{\mathfrak{q},f})(c) \;&=\; a_\tau \sum_{\rho\in\mathrm{Gal}(F/K)} \mathrm{Ev}_{\mathfrak{q},s}(\rho d)(\mathrm{Ev}^*_{\mathfrak{q},f})^{\rho^{-1}}(c) \\
&=\; a_\tau \sum_{\rho\in\mathrm{Gal}(F/K)} \mathrm{Ev}_{\mathfrak{q},s}(\rho d)\mathrm{Ev}^*_{\mathfrak{q},f}(\rho c) \\
&=\; \pm a_\tau \sum_{\rho\in\mathrm{Gal}(F/K)} \langle \rho c, \rho d\rangle_\mathfrak{Q} \\
&=\; \pm a_\tau \sum_{\rho\in\mathrm{Gal}(F/K)} \langle c, d\rangle_{\mathfrak{Q}^\rho} \;=\; \pm a_\tau \langle c, d\rangle_\mathfrak{q} \;=\; 0. \qquad \square
\end{aligned}$$

Corollary 7.2.12. *Suppose that $K \subset_f F \subset K_\infty$, that M is a power of p, that $\mathfrak{r} \in \mathcal{R}_{F,M}$, and that $\gamma \in \tau G_{\Omega_\infty}$. Then*

$$a_\tau \widetilde{\mathrm{Ev}}_{F,M}(\gamma)(\kappa_{[F,\mathfrak{r},M]})\ \mathrm{Ev}^*_{F,M}(\gamma)|_{\mathcal{S}_{\Sigma_{p\mathfrak{r}}}(F,W_M^*)} \;=\; 0.$$

Proof. Fix a finite Galois extension L of $F(\boldsymbol{\mu}_M, (\mathcal{O}_K^\times)^{1/M}, W_M)$ such that the restrictions to L of $\kappa_{[F,\mathfrak{r},M]}$ and of $\mathcal{S}_{\Sigma_p}(F, W_M^*)$ are zero ($\mathcal{S}_{\Sigma_p}(F, W_M^*)$ is finite by Lemma 1.5.7, so such an extension exists). Let $\mathcal{N}$ be the ideal

of Definition 2.1.1 corresponding to $\mathbf{c}$. Choose a prime $\mathfrak{q}$ of K prime to $\mathfrak{r}\mathcal{N}$ (and a prime $\mathfrak{Q}$ of $\bar{K}$ above $\mathfrak{q}$) such that $\mathrm{Fr}_{\mathfrak{q}} = \gamma$ on L.

We have $\mathrm{Ev}^*_{F,M}(\gamma) = \mathrm{Ev}^*_{F,M}(\mathrm{Fr}_{\mathfrak{q}}) = \mathrm{Ev}^*_{\mathfrak{q},f}$. Since $\mathfrak{q} \in \mathcal{R}_{F,M,\mathfrak{r}}$, Theorems 7.2.10 and 4.5.1 show that

$$\begin{aligned}\widetilde{\mathrm{Ev}}_{F,M}(\gamma)(\kappa_{[F,\mathfrak{r},M]}) &= \widetilde{\mathrm{Ev}}_{\mathfrak{q},f}(\kappa_{[F,\mathfrak{r},M]}) \\ &= \widetilde{\mathrm{Ev}}_{\mathfrak{q},s}(\kappa_{[F,\mathfrak{r}\mathfrak{q},M]}) \in \widetilde{\mathrm{Ev}}_{\mathfrak{q},s}(\mathcal{S}^{\Sigma_{p\mathfrak{r}\mathfrak{q}}}(F, W_M)).\end{aligned}$$

Now the corollary follows from Theorem 7.2.11. □

7.3. Proof of Theorem 2.3.2

In this section we will prove Theorem 2.3.2. The general idea is that if $\mathbf{c} \notin H^1_\infty(K,T)_{\mathrm{tors}}$, then we can use Corollary 7.2.12 to construct a nonzero annihilator of X_∞, and hence X_∞ is Λ-torsion.

Lemma 7.3.1. *Suppose $K \subset_f F \subset K_\infty$ and $M \in \mathcal{O}$ is nonzero. Let $\mathrm{res}_{S,\infty}$ denote the restriction map*

$$\mathcal{S}_{\Sigma_p}(F, W^*) \longrightarrow \mathcal{S}_{\Sigma_p}(K_\infty, W^*)^{G_F}.$$

If v is a place of F, fix an extension w of v to K_∞ and define

$$B_v = \begin{cases} H^1(K_{\infty,w}/F_v, (W^*)^{G_{K_{\infty,w}}}) & \text{if } v \mid p, \\ H^1_{\mathrm{ur}}(F_v, W^*)/H^1_f(F_v, W^*) & \text{if } v \nmid p. \end{cases}$$

Then there are a submodule B of $\oplus_{v|\mathcal{N}} B_v$ and an exact sequence

$$B \longrightarrow \mathrm{coker}(\mathrm{res}_{S,\infty}) \longrightarrow H^2(K_\infty/F, (W^*)^{G_{K_\infty}}).$$

Proof. Let res_∞ denote the restriction map $H^1(F, W^*) \to H^1(K_\infty, W^*)^{G_F}$. The inflation-restriction exact sequence shows that the cokernel of res_{K_∞} is isomorphic to a subgroup of $H^2(K_\infty/F, (W^*)^{G_{K_\infty}})$, and hence the same is true for the cokernel of

$$\mathrm{res}^{-1}_{K_\infty}(\mathcal{S}_{\Sigma_p}(K_\infty, W^*)^{G_F}) \xrightarrow{\mathrm{res}_{K_\infty}} \mathcal{S}_{\Sigma_p}(K_\infty, W^*)^{G_F}.$$

Since K_∞/F is unramified outside primes above p, we have

$$\mathrm{res}^{-1}_{K_\infty}(\mathcal{S}_{\Sigma_p}(K_\infty, W^*)^{G_F}) \subset \mathcal{S}^{\Sigma_{p\mathcal{N}}}(F, W^*)$$

so the cokernel of the inclusion

$$\mathcal{S}_{\Sigma_p}(F, W^*) \hookrightarrow \mathrm{res}^{-1}_{K_\infty}(\mathcal{S}_{\Sigma_p}(K_\infty, W^*)^{G_F})$$

injects naturally into $\oplus_{v|\mathcal{N}} B_v$. This proves the lemma. □

Lemma 7.3.2. *The Λ-module X_∞ is finitely generated.*

Proof. Let $\mathcal{J}$ denote the augmentation ideal in Λ. Then $X_\infty/\mathcal{J}X_\infty = \mathrm{Hom}(\mathcal{S}_{\Sigma_p}(K_\infty, W^*)^{G_K}, \mathbf{D})$. By Nakayama's Lemma, to prove the lemma we need only show that $\mathrm{Hom}(\mathcal{S}_{\Sigma_p}(K_\infty, W^*)^{G_K}, \mathbf{D})$ is finitely generated over $\mathcal{O}$.

Let $\mathrm{res}_{\mathcal{S},\infty}$ denote the restriction map

$$\mathcal{S}_{\Sigma_p}(K, W^*) \longrightarrow \mathcal{S}_{\Sigma_p}(K_\infty, W^*)^{G_K}.$$

By Lemmas 7.3.1 and 1.3.5, $\mathrm{Hom}(\mathrm{coker}(\mathrm{res}_{\mathcal{S},\infty}), \mathbf{D})$ is finitely generated over $\mathcal{O}$. Lemma 1.5.7(iii) shows that $\mathrm{Hom}(\mathcal{S}_{\Sigma_p}(K, W^*), \mathbf{D})$ is also finitely generated over $\mathcal{O}$, and the lemma follows. □

Recall that $\Omega_\infty = K_\infty(\mathbf{1})K(W, \boldsymbol{\mu}_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$.

Lemma 7.3.3. *Suppose X_∞ is not a torsion Λ-module. Define*

$$J = \{\gamma \in \tau G_{\Omega_\infty} : \mathrm{Ev}^*(\gamma) \notin (X_\infty)_{\mathrm{tors}}\}.$$

Then the subgroup of G_K generated by J contains a nonempty open subgroup of G_{Ω_∞}.

Proof. Corollary C.2.2 (with $F = K_\infty$) shows that $H^1(\Omega_\infty/K_\infty, W^*)$ is a torsion Λ-module. Therefore if X_∞ is not a torsion Λ-module, Lemma 7.2.4(iii) shows that there is a $\gamma_0 \in G_{\Omega_\infty}$ such that $\mathrm{Ev}^*(\gamma_0) \notin (X_\infty)_{\mathrm{tors}}$. Then either τ or $\tau\gamma_0$ belongs to J, so J is nonempty.

Since X_∞ is finitely generated by Lemma 7.3.2, the submodule $(X_\infty)_{\mathrm{tors}}$ is closed in X_∞. The map Ev^* is continuous, so

$$J = (\mathrm{Ev}^*)^{-1}(X_\infty - (X_\infty)_{\mathrm{tors}}) \cap \tau G_{\Omega_\infty}$$

is open in τG_{Ω_∞}, and the lemma follows. □

Proof of Theorem 2.3.2. Let $\mathbf{c}$ be the Euler system of Theorem 2.3.2. We will show, under the assumption that X_∞ is not a torsion Λ-module, that $\mathbf{c}_{K,\infty} \in H^1_\infty(K, T)_{\mathrm{tors}}$.

Suppose that X_∞ is not a torsion Λ-module. Choose a γ in the set J of Lemma 7.3.3, i.e., $\gamma \in \tau G_{\Omega_\infty}$ such that $\mathrm{Ev}^*(\gamma) \notin (X_\infty)_{\mathrm{tors}}$.

Suppose $K \subset_{\mathrm{f}} F \subset K_\infty$ and M is a power of p. Let $\kappa_{[F,M]} = \kappa_{[F,\mathbf{1},M]}$ be the derivative class constructed in §4.4. By Corollary 7.2.12 (with $\mathfrak{r} = \mathbf{1}$, the trivial ideal),

$$a_\tau \widetilde{\mathrm{Ev}}_{F,M}(\gamma)(\kappa_{[F,M]})\ \mathrm{Ev}^*_{F,M}(\gamma) = 0.$$

By definition the map $\widetilde{\mathrm{Ev}}_{F,M}(\gamma)$ factors through restriction to K_∞, and for every $F \subset_{\mathrm{f}} F'$ we have

$$(\kappa_{[F,M]})_{F'} = (\mathrm{Cor}_{F'/F}\kappa_{[F',M]})_{F'} = \sum_{\rho \in \mathrm{Gal}(F'/F)} \rho\kappa_{[F',M]}.$$

Hence $\widetilde{\mathrm{Ev}}_{F',M}(\gamma)(\kappa_{[F',M]}) \in \Lambda_{F',M}$ restricts to $\widetilde{\mathrm{Ev}}_{F,M}(\gamma)(\kappa_{[F,M]}) \in \Lambda_{F,M}$. Thus $\varprojlim_{F,M} \widetilde{\mathrm{Ev}}_{F,M}(\gamma)(\kappa_{[F,M]}) \in \Lambda$ and

$$a_\tau\left(\varprojlim_{F,M} \widetilde{\mathrm{Ev}}_{F,M}(\gamma)(\kappa_{[F,M]})\right)\,\mathrm{Ev}^*(\gamma) \;=\; 0.$$

Since $\mathrm{Ev}^*(\gamma) \notin (X_\infty)_{\mathrm{tors}}$ it follows that $\varprojlim \widetilde{\mathrm{Ev}}_{F,M}(\gamma)(\kappa_{[F,M]}) = 0$. This holds for every $\gamma \in J$, so Lemma 7.3.3 shows that it holds for every γ in a nonempty open subgroup of G_{Ω_∞}. Since a nonempty open subgroup has finite index, and Λ is torsion-free, we conclude that for every F, every M, and *every* $\gamma \in G_{\Omega_\infty}$, we have $\mathrm{Ev}_{F,M}(\gamma)(\kappa_{[F,M]}) = 0$. Equivalently, writing $(\kappa_{[F,M]})_{K_\infty}$ for the image of $\kappa_{[F,M]}$ in $H^1(K_\infty, W)$, we have

$$\mathrm{Ev}(\gamma)((\kappa_{[F,M]})_{K_\infty}) \;=\; 0. \tag{7.5}$$

We will show that this is not compatible with the assumption that

$$\mathbf{c}_{K,\infty} \;=\; \{\mathbf{c}_F\}_F \;\notin\; H^1_\infty(K,T)_{\mathrm{tors}}.$$

Recall that $\mathrm{Ann}_\Lambda(B)$ denotes the annihilator of a Λ-module B. By Lemma 7.2.4(i), it follows from (7.5) that

$$a_\tau \mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W))(\kappa_{[F,M]})_{K_\infty} \;=\; 0.$$

The inflation-restriction exact sequence shows that the kernel of the restriction map $H^1(F,W) \to H^1(K_\infty, W)$ is $H^1(K_\infty/F, W^{G_{K_\infty}})$ which (since K_∞/K is abelian) is annihilated by $\mathrm{Ann}_\Lambda(W^{G_{K_\infty}})$. By Lemma 1.2.2(i), the kernel of the natural map $H^1(F, W_M) \to H^1(F,W)$ is annihilated by $m_F = [W^{G_F} : (W^{G_F})_{\mathrm{div}}]$. Hence for every F and M, we have

$$m_F \mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W))\mathrm{Ann}_\Lambda(W^{G_{K_\infty}})\kappa_{[F,M]} \;=\; 0.$$

By Lemma 4.4.13(i), $\kappa_{[F,M]}$ is the image of $\mathbf{c}_F$ under the injection

$$H^1(F,T)/MH^1(F,T) \;\hookrightarrow\; H^1(F, W_M).$$

It follows that $m_F \mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W))\mathrm{Ann}_\Lambda(W^{G_{K_\infty}})\mathbf{c}_F$ is divisible in $H^1(F,T)$, so by Proposition B.2.4,

$$m_F \mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W))\mathrm{Ann}_\Lambda(W^{G_{K_\infty}})\mathbf{c}_F \;=\; 0.$$

Lemma 1.2.2(ii) shows that the kernel of multiplication by m_F in $H^1(F,T)$ is annihilated by $\mathrm{Ann}_\Lambda(W^{G_{K_\infty}})$, so for every $K \subset_{\mathrm{f}} F \subset K_\infty$,

$$\mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W))\mathrm{Ann}_\Lambda(W^{G_{K_\infty}})^2\mathbf{c}_F \;=\; 0.$$

But this annihilator of $\mathbf{c}_F$ is independent of F, and by Corollary C.2.2 applied with $F = K_\infty$, it is nonzero as well. Thus we have found a nonzero annihilator of $\mathbf{c}_{K,\infty} \in H^1_\infty(K,T)$. This contradicts the assumption that $\mathbf{c}_{K,\infty} \notin H^1_\infty(K,T)_{\mathrm{tors}}$, and completes the proof. □

We also record the following lemma for later use.

Lemma 7.3.4. *Suppose $\Gamma \cong \mathbf{Z}_p$, and either K is totally real and Leopoldt's conjecture holds for K, or K is imaginary quadratic.*

(i) *If G_{K_∞} acts trivially on T, then $X_\infty/\mathrm{Ann}_\Lambda(T)X_\infty$ is finite.*
(ii) *If G_{K_∞} acts trivially on $T(-1)$, then $X_\infty/\mathrm{Ann}_\Lambda(T(-1))X_\infty$ is finite.*

Proof. Recall that $T(-1) = T \otimes \mathcal{O}_{\varepsilon_{\mathrm{cyc}}^{-1}}$. We have assumed (Hyp$(K_\infty, V)$) that V is an irreducible G_{K_∞}-representation, so the situations (i) and (ii) can only arise if $\mathrm{rank}_{\mathcal{O}} T = 1$ and T is a twist of $\mathcal{O}$ or $\mathcal{O}(1)$, respectively, by a character of Γ.

Suppose ρ is a character of Γ. If we replace T by its twist $T \otimes \rho$, then Corollary 6.2.2 shows that X_∞ is replaced by $X_\infty \otimes \rho$. Also $\mathrm{Ann}_\Lambda(T)$ is replaced by $\mathrm{Tw}_{\rho^{-1}}(\mathrm{Ann}_\Lambda(T))$ by Lemma 6.1.2(ii) (where $\mathrm{Tw}_\rho : \Lambda \to \Lambda$ is the map of Definition 6.1.1 induced by $\gamma \mapsto \rho^{-1}(\gamma)\gamma$ on Γ), and similarly for $\mathrm{Ann}_\Lambda(T(-1))$. It follows easily that $X_\infty/\mathrm{Ann}_\Lambda(T)X_\infty$ and $X_\infty/\mathrm{Ann}_\Lambda(T(-1))X_\infty$ remain unchanged as $\mathcal{O}$-modules. Thus both assertions of the lemma are invariant under twisting by characters of Γ, so we may assume that $T = \mathcal{O}$ for (i) and $T = \mathcal{O}(1)$ for (ii). Then in both cases we are trying to show that $X_\infty/\mathcal{J}X_\infty$ is finite, where $\mathcal{J}$ denotes the augmentation ideal of Λ. Without loss of generality we may also suppose that $\mathcal{O} = \mathbf{Z}_p$.

Suppose first that $T = \mathbf{Z}_p(1)$. Then $W^* = \mathbf{Q}_p/\mathbf{Z}_p$ and $H^1(K_\infty, W^*) = \mathrm{Hom}(G_{K_\infty}, \mathbf{Q}_p/\mathbf{Z}_p)$, so by Proposition 1.6.1 we have $X_\infty = \mathrm{Gal}(L_\infty/K_\infty)$, where L_∞ is the maximal everywhere unramified abelian p-extension of K_∞ in which all primes above p split completely. A standard Iwasawa theory argument ([**Iw3**] §3.1) now shows that $X_\infty/\mathcal{J}X_\infty = \mathrm{Gal}(L/K_\infty)$ where L is the maximal abelian extension of K in L_∞, and that this Galois group is finitely generated.

If K is totally real and Leopoldt's conjecture holds for K, then K has no extension with Galois group $\mathbf{Z}_p^2$, so L/K_∞ is finite. If K is imaginary quadratic then K has a unique extension with Galois group $\mathbf{Z}_p^2$, but no prime above p is infinitely split in this extension, so again L/K_∞ is finite. This proves the lemma in this case.

Now suppose $T = \mathbf{Z}_p$, so $W^* = \boldsymbol{\mu}_{p^\infty}$. It follows directly from Lemma 7.3.1 in this case that the map

$$\mathcal{S}_{\Sigma_p}(K, \boldsymbol{\mu}_{p^\infty}) \longrightarrow \mathcal{S}_{\Sigma_p}(K_\infty, \boldsymbol{\mu}_{p^\infty})^{G_K} = \mathrm{Hom}(X_\infty/\mathcal{J}X_\infty, \mathbf{Q}_p/\mathbf{Z}_p)$$

has finite cokernel. Since Leopoldt's conjecture holds for K, Corollary 1.6.5 shows that $\mathcal{S}_{\Sigma_p}(K, \boldsymbol{\mu}_{p^\infty})$ is finite. This completes the proof. □

7.4. The Kernel and Cokernel of the Restriction Map

For the remainder of this chapter we will assume that Assumptions 7.1.4 and 7.1.5 are satisfied. Since we have now proved Theorem 2.3.2, we can do this with no loss of generality (see Remark 7.1.6).

In particular, if $K \subset_f F \subset K_\infty$ and λ is a prime of F dividing $\mathcal{N}$, then it follows from Assumption 7.1.5 that W^{G_F}, $(W^*)^{G_F}$, $W^{G_{F_\lambda}}$, and $(W^*)^{G_{F_\lambda}}$ are all finite.

Definition 7.4.1. We define several ideals of Λ which will play a role in the proofs below. Recall that $\mathrm{Ann}_\Lambda(B)$ denotes the annihilator in Λ of a Λ-module B. Define

$$\mathcal{A}_{\mathrm{glob}} = \begin{cases} \mathrm{Ann}_\Lambda(W^{G_{K_\infty}}) & \text{if } \mathrm{rank}_{\mathbf{Z}_p}\Gamma > 1, \\ \mathrm{Ann}_\Lambda(W^{G_{K_\infty}}/(W^{G_{K_\infty}})_{\mathrm{div}}) & \text{if } \Gamma = \mathbf{Z}_p. \end{cases}$$

If v is a place of K, fix an extension w of v to $\bar{K}$, let D_v be the decomposition group of v in Γ, let $\mathcal{I}_w$ be the inertia group of w in G_K, and let

$$K_{\infty,w} = \cup_{K \subset_f F \subset K_\infty} F_w.$$

Define

$$\mathcal{A}_v = \begin{cases} \mathrm{Ann}_{\mathcal{O}[[D_v]]}(W^{G_{K_{\infty,w}}}) & \text{if } v \mid p \text{ and } \mathrm{rank}_{\mathbf{Z}_p} D_v > 1, \\ \mathrm{Ann}_{\mathcal{O}[[D_v]]}(W^{G_{K_{\infty,w}}}/(W^{G_{K_{\infty,w}}})_{\mathrm{div}}) & \text{if } v \mid p \text{ and } D_v = \mathbf{Z}_p, \\ \mathrm{Ann}_{\mathcal{O}[[D_v]]}(H^1(K_{\infty,w}^{\mathrm{ur}}/K_{\infty,w}, W^{\mathcal{I}_v}/(W^{\mathcal{I}_v})_{\mathrm{div}})) & \text{if } v \nmid p, \end{cases}$$

$$\mathcal{A}_{\mathcal{N}} = \prod_{v \mid \mathcal{N}} \mathcal{A}_v \Lambda.$$

We define $\mathcal{A}^*_{\mathrm{glob}}$, $\mathcal{A}^*_v$, and $\mathcal{A}^*_{\mathcal{N}}$ in exactly the same way with W replaced by W^*.

Lemma 7.4.2. *The ideals $\mathcal{A}_{\mathrm{glob}}$, $\mathcal{A}_{\mathcal{N}}$, $\mathcal{A}^*_{\mathrm{glob}}$, and $\mathcal{A}^*_{\mathcal{N}}$ defined above have height at least two in Λ.*

Proof. This is clear from the definitions of these ideals. □

Lemma 7.4.3. *Suppose $K \subset_f F \subset K_\infty$ and $i \geq 1$.*

(i) *$H^i(K_\infty/F, W^{G_{K_\infty}})$ is finite and annihilated by $\mathcal{A}_{\mathrm{glob}}$.*
(ii) *$H^i(K_\infty/F, (W^*)^{G_{K_\infty}})$ is finite and annihilated by $\mathcal{A}^*_{\mathrm{glob}}$.*
(iii) *If v is a prime of K above p and w is a prime of K_∞ above v, then $H^i(K_{\infty,w}/F_w, (W^*)^{G_{K_{\infty,w}}})$ is finite and annihilated by $\mathcal{A}^*_v$.*

Proof. Let $W' = W^{G_{K_\infty}}$, $(W^*)^{G_{K_\infty}}$, or $(W^*)^{G_{K_{\infty,w}}}$, and $G = \mathrm{Gal}(K_\infty/F)$, $\mathrm{Gal}(K_\infty/F)$, or $\mathrm{Gal}(K_{\infty,w}/F_w)$, respectively. By Assumption 7.1.5, we can choose a $\gamma \in G_{F_w} \subset G_F$ such that $T^{\gamma=1} = (T^*)^{\gamma=1} = 0$. Let $\bar{\gamma} \in \Gamma$ denote the restriction of γ to K_∞.

Since Γ is abelian, the annihilator of W' annihilates $H^i(G, W')$ for every i. If $f(x) = \det(1 - \gamma x | T \oplus T^*) \in \mathcal{O}[x]$, then the Cayley-Hamilton theorem shows that $f(\bar{\gamma}^{-1})$ annihilates W', so in particular $f(\bar{\gamma}^{-1})$ annihilates $H^i(G, W')$. Since G acts trivially on $H^i(G, W')$, it follows that $f(1)$ annihilates $H^i(G, W')$. Our hypothesis on γ ensures that $f(1) \neq 0$, so it follows without difficulty (since G is finitely generated and W' is co-finitely generated) that $H^i(G, W')$ is finite.

This proves the finiteness in all three cases, and the annihilation when $\mathrm{rank}_{\mathbf{Z}_p}(G) > 1$. Suppose now that $G \cong \mathbf{Z}_p$, and use the exact sequences

$$H^i(G, W'_{\mathrm{div}}) \longrightarrow H^i(G, W') \longrightarrow H^i(G, W'/W'_{\mathrm{div}}) \longrightarrow H^{i+1}(G, W'_{\mathrm{div}}).$$

If $i > 1$ then $H^i(G, W'_{\mathrm{div}}) = 0$ because G has cohomological dimension 1, and if σ is a topological generator of G then (Lemma B.2.8)

$$H^1(G, W'_{\mathrm{div}}) \cong W'_{\mathrm{div}}/(\sigma - 1)W'_{\mathrm{div}} = 0$$

because $W'_{\mathrm{div}}/(\sigma-1)W'_{\mathrm{div}}$ is a quotient of $W'_{\mathrm{div}}/(\bar{\gamma}-1)W'_{\mathrm{div}}$. Thus for every $i > 0$ we have

$$H^i(G, W') \cong H^i(G, W'/W'_{\mathrm{div}}),$$

so we see that the annihilator of W'/W'_{div} annihilates $H^i(G, W')$. □

Proposition 7.4.4. *Suppose $K \subset_{\mathrm{f}} F \subset K_\infty$ and M is a power of p.*

(i) *The kernel of the restriction map*

$$H^1(F, W) \longrightarrow H^1(K_\infty, W)^{G_F}$$

is finite and is annihilated by $\mathcal{A}_{\mathrm{glob}}$.

(ii) *The kernel of the natural map*

$$H^1(F, W_M) \longrightarrow H^1(F, W)_M$$

is finite with order bounded independently of M, and is annihilated by $\mathrm{Ann}_\Lambda(W^{G_{K_\infty}})$.

(iii) *The cokernel of the restriction map*

$$\mathcal{S}_{\Sigma_p}(F, W^*) \longrightarrow \mathcal{S}_{\Sigma_p}(K_\infty, W^*)^{G_F}$$

is finite and is annihilated by $\mathcal{A}^*_{\mathrm{glob}}\mathcal{A}^*_{\mathcal{N}}$.

(iv) *If $\mathcal{S}_{\Sigma_p}(K_\infty, W^*)^{G_F}$ is finite, then there is a power M_F of p such that if $M \geq M_F$ is a power of p, then $\mathcal{A}^*_{\mathrm{glob}}\mathcal{A}^*_{\mathcal{N}}$ annihilates the cokernel of the natural map*

$$\mathcal{S}_{\Sigma_p}(F, W^*_M) \longrightarrow \mathcal{S}_{\Sigma_p}(K_\infty, W^*)^{G_F}.$$

(v) *The cokernel of the natural map*

$$\mathcal{S}_{\Sigma_p}(F, W_M^*) \longrightarrow \mathcal{S}_{\Sigma_p}(F, W^*)_M$$

is finite and bounded independently of M.

Proof. The inflation-restriction exact sequence shows that the kernel of the restriction map in (i) is $H^1(K_\infty/F, W^{G_{K_\infty}})$, so (i) follows from Lemma 7.4.3(i). By Lemma 1.2.2(i), the kernel (ii) is W^{G_F}/MW^{G_F}, which in turn is a quotient of $W^{G_F}/(W^{G_F})_{\mathrm{div}}$, and (ii) follows. Assertion (iii) is immediate from Lemmas 7.3.1, 7.4.3, and 1.3.5(iii).

Suppose further that $\mathcal{S}_{\Sigma_p}(K_\infty, W^*)^{G_F}$ is finite. Since

$$\mathcal{S}_{\Sigma_p}(F, W^*) = \varinjlim \mathcal{S}_{\Sigma_p}(F, W_M^*),$$

we can choose M_F so that the image of $\mathcal{S}_{\Sigma_p}(F, W_{M_F}^*)$ in $H^1(K_\infty, W^*)$ contains the image of $\mathrm{res}_{K_\infty}(\mathcal{S}_{\Sigma_p}(F, W^*))$. With this choice (iv) follows from (iii).

By Lemma 1.5.4, the map $\mathcal{S}^{\Sigma_p}(F, W_M^*) \to \mathcal{S}^{\Sigma_p}(F, W^*)_M$ is surjective. Thus the cokernel in (v) is isomorphic to a subquotient of

$$\oplus_{w|p} \ker\left(H^1(F_w, W_M^*) \longrightarrow H^1(F_w, W^*)\right).$$

For each w dividing p, Lemma 1.2.2(i) shows that the above kernel is

$$(W^*)^{G_{F_w}}/M(W^*)^{G_{F_w}},$$

which is a quotient of the finite group $(W^*)^{G_{F_w}}/((W^*)^{G_{F_w}})_{\mathrm{div}}$ and hence is bounded independently of M. This proves (v). □

7.5. Galois Equivariance of the Evaluation Maps

For the proofs of Propositions 7.1.7 and 7.1.9 in the following sections, it would be convenient if G_{Ω_∞} were a Λ-module and Ev and Ev^* were Λ-module homomorphisms. Unfortunately this makes no sense, since G_{Ω_∞} is not a Λ-module. We will get around this by defining an action of a subring of Λ on a quotient of G_{Ω_∞}, and Ev and Ev^* will behave well with respect to this action.

Proposition 7.5.1. *There are a subgroup* Γ_0 *of finite index in* Γ, *characters* $\chi, \chi^* : \Gamma_0 \to \mathcal{O}^\times$, *an abelian extension* L *of* Ω_∞, *and an action of* $\mathbf{Z}_p[[\Gamma_0]]$ *on* $\mathrm{Gal}(L/\Omega_\infty)$ *such that*

(i) Ev *and* Ev^* *on* G_{Ω_∞} *factor through* $\mathrm{Gal}(L/\Omega_\infty)$,

(ii) *if* $\eta \in \Gamma_0$ *and* $\gamma \in \mathrm{Gal}(L/\Omega_\infty)$ *then*

$$\mathrm{Ev}(\gamma^\eta) = \chi(\eta)\eta(\mathrm{Ev}(\gamma)), \quad \mathrm{Ev}^*(\gamma^\eta) = \chi^*(\eta)\eta(\mathrm{Ev}^*(\gamma)).$$

Proof. Let L be the maximal abelian p-extension of

$$K_\infty(\boldsymbol{\mu}_{p^\infty}, W) \;=\; K_\infty(\boldsymbol{\mu}_{p^\infty}, W^*) \;=\; K_\infty(W, W^*).$$

Then $\Omega_\infty \subset L$, and every cocycle in $H^1(K_\infty, W)$ or in $H^1(K_\infty, W^*)$ vanishes on G_L, so (i) is satisfied.

Consider the diagram of fields in Figure 2. By Proposition C.1.7,

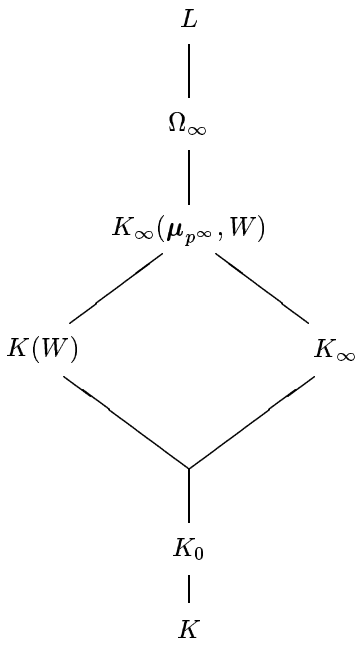


FIGURE 2

there is a finite extension K_0 of K in $K(W) \cap K_\infty$ such that the center of $\mathrm{Gal}(K(W)/K)$ maps onto $\mathrm{Gal}((K(W) \cap K_\infty)/K_0)$. Define

$$\Gamma_0 \;=\; \mathrm{Gal}(K_\infty/K_0).$$

Fix once and for all a set of independent topological generators $\{\gamma_1, \ldots, \gamma_d\}$ of Γ_0, and for $1 \le i \le d$ fix a lift $\tilde{\gamma}_i \in \mathrm{Gal}(K_\infty(\boldsymbol{\mu}_{p^\infty}, W)/K_0)$ of γ_i such that the restriction of $\tilde{\gamma}_i$ to $K(W)$ belongs to the center of $\mathrm{Gal}(K(W)/K)$. Since $K_\infty(\boldsymbol{\mu}_{p^\infty}, W)$ is the compositum of $K(W)$ with an abelian extension of K, each $\tilde{\gamma}_i$ belongs to the center of $\mathrm{Gal}(K_\infty(\boldsymbol{\mu}_{p^\infty}, W)/K)$. Therefore these choices extend by multiplicativity to define a homomorphism

$$\Gamma_0 \longrightarrow \mathrm{Gal}(K_\infty(\boldsymbol{\mu}_{p^\infty}, W)/K_0),$$

whose image lies in the center of $\mathrm{Gal}(K_\infty(\boldsymbol{\mu}_{p^\infty}, W)/K)$, which is a section for the projection map $\mathrm{Gal}(K_\infty(\boldsymbol{\mu}_{p^\infty}, W)/K_0) \to \Gamma_0$. We will denote this map by $\gamma \mapsto \tilde{\gamma}$, and we will use this map to define an action of Γ_0 on $\mathrm{Gal}(L/\Omega_\infty)$: for $\gamma \in \mathrm{Gal}(L/\Omega_\infty)$ and $\eta \in \Gamma_0$, define

$$\gamma^\eta \;=\; \tilde{\eta}\gamma\tilde{\eta}^{-1}.$$

This definition extends to give an action of $\mathbf{Z}_p[[\Gamma_0]]$ on $\mathrm{Gal}(L/\Omega_\infty)$. It is not canonical, since it depends on our choice of the $\tilde{\gamma}_i$.

By Lemma C.1.6, since V is assumed to be irreducible, every element of the center of $\mathrm{Gal}(K(W)/K)$ acts on W by a scalar in $\mathcal{O}^\times$. Thus the choice above defines a character

$$\chi \,:\, \Gamma_0 \longrightarrow \mathcal{O}^\times, \quad \chi(\eta) \;=\; \tilde{\eta} \in \mathrm{Aut}(W).$$

Similarly, if $\eta \in \Gamma_0$ then $\tilde{\eta}$ belongs to the center of $\mathrm{Gal}(K(W^*)/K)$ so we get a second character

$$\chi^* \,:\, \Gamma_0 \longrightarrow \mathcal{O}^\times, \quad \chi^*(\eta) \;=\; \tilde{\eta} \in \mathrm{Aut}(W^*).$$

Suppose that $c \in H^1(K_\infty, W)$, that $\gamma \in \mathrm{Gal}(L/\Omega_\infty)$, and that $\eta \in \Gamma_0$. Since $\mathrm{Ev}(\gamma) \in \mathrm{Hom}(H^1(K_\infty, W), \mathbf{D})$, we have

$$\begin{aligned}(\eta\mathrm{Ev}(\gamma))(c) \;=\; \mathrm{Ev}(\gamma)(\eta^{-1}c) \;&=\; \bar{\theta}((\eta^{-1}c)(\gamma)) \\ &=\; \bar{\theta}(\tilde{\eta}^{-1}(c(\gamma^\eta))) \;=\; \chi(\eta^{-1})\mathrm{Ev}(\gamma^\eta)(c).\end{aligned}$$

In other words,

$$\mathrm{Ev}(\gamma^\eta) \;=\; \chi(\eta)\eta(\mathrm{Ev}(\gamma)),$$

and similarly with Ev^* and χ^*. This proves (ii). □

Recall the involution $\eta \mapsto \eta^\bullet$ of Λ given by Definition 7.2.3.

Proposition 7.5.2. *Suppose X' is a Λ-submodule of X_∞ and X_∞/X' is pseudo-null. Then there is an ideal $\mathcal{A}_0$ of height at least two in Λ such that for every $K \subset_{\mathrm{f}} F \subset K_\infty$,*

$$\begin{aligned}\mathcal{A}_0 a_\tau \mathrm{Ann}_\Lambda(W^{G_{K_\infty}})^\bullet \mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W))^\bullet \mathrm{Hom}(H^1(F, W_M), \mathcal{O}/M\mathcal{O}) \\ \subset\; \mathcal{O}\mathrm{Ev}_{F,M}((\mathrm{Ev}^*)^{-1}(X') \cap G_{\Omega_\infty}).\end{aligned}$$

In other words, if ψ belongs to

$$\mathcal{A}_0 a_\tau \mathrm{Ann}_\Lambda(W^{G_{K_\infty}})^\bullet \mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W))^\bullet \mathrm{Hom}(H^1(F, W_M), \mathcal{O}/M\mathcal{O})$$

then there are $\gamma_1, \ldots, \gamma_k \in G_{\Omega_\infty}$ and $c_1, \ldots, c_k \in \mathcal{O}$ such that $\mathrm{Ev}^*(\gamma_i) \in X'$ for every i and

$$\sum_{i=1}^{k} c_i \mathrm{Ev}_{F,M}(\gamma_i) \;=\; \psi.$$

Proof. Let Γ_0, L, χ, and χ^* be as in Proposition 7.5.1. We define

$$\mathrm{Tw}_\chi : \mathcal{O}[[\Gamma_0]] \longrightarrow \mathcal{O}[[\Gamma_0]] \text{ by } \gamma \mapsto \chi(\gamma)\gamma$$

and similarly for Tw_{χ^*}, and then Proposition 7.5.1 shows that for every $\eta \in \mathbf{Z}_p[[\Gamma_0]]$ and $\gamma \in \mathrm{Gal}(L/\Omega_\infty)$,

$$\mathrm{Ev}(\gamma^\eta) = \mathrm{Tw}_\chi(\eta)(\mathrm{Ev}(\gamma)), \quad \mathrm{Ev}^*(\gamma^\eta) = \mathrm{Tw}_{\chi^*}(\eta)(\mathrm{Ev}^*(\gamma)). \tag{7.6}$$

Note that a pseudo-null Λ-module is also pseudo-null as a $\mathbf{Z}_p[[\Gamma_0]]$-module, and conversely if $\mathcal{A}$ is an ideal of $\mathbf{Z}_p[[\Gamma_0]]$ of height at least two then $\mathcal{A}\Lambda$ is an ideal of Λ of height at least two.

Define

$$\mathcal{A} = \mathrm{Tw}_{\chi^*}^{-1}(\mathrm{Ann}_{\mathcal{O}[[\Gamma_0]]}(X_\infty/X')) \cap \mathbf{Z}_p[[\Gamma_0]].$$

Since X_∞/X' is assumed to be a pseudo-null Λ-module, $\mathcal{A}$ is an ideal of height at least two in $\mathbf{Z}_p[[\Gamma_0]]$. By (7.6),

$$\mathrm{Ev}^*(\mathcal{A}\mathrm{Gal}(L/\Omega_\infty)) = \mathrm{Tw}_{\chi^*}(\mathcal{A})\mathrm{Ev}^*(G_{\Omega_\infty}) \subset X',$$

and by (7.6) and Lemma 7.2.4(ii), for every $K \subset_{\mathrm{f}} F \subset K_\infty$,

$$\begin{aligned}\mathcal{O}\mathrm{Ev}(\mathcal{A}\mathrm{Gal}(L/\Omega_\infty)) &= \mathcal{O}\mathrm{Tw}_\chi(\mathcal{A})\mathrm{Ev}(G_{\Omega_\infty}) \\ &\supset \mathrm{Tw}_\chi(\mathcal{A})a_\tau \mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W))^\bullet \mathrm{Hom}(H^1(K_\infty, W), \mathbf{D}).\end{aligned}$$

By Proposition 7.4.4(i) and (ii), the image of the composition

$$\begin{aligned}\mathrm{Hom}(H^1(K_\infty, W), \mathbf{D}) &\longrightarrow \mathrm{Hom}(H^1(F, W), \mathbf{D}) \\ &\longrightarrow \mathrm{Hom}(H^1(F, W_M), \mathcal{O}/M\mathcal{O})\end{aligned}$$

contains

$$\mathcal{A}_{\mathrm{glob}}^\bullet \mathrm{Ann}_\Lambda(W^{G_{K_\infty}})^\bullet \mathrm{Hom}(H^1(F, W_M), \mathcal{O}/M\mathcal{O}).$$

Combining these inclusions, we see that the proposition holds with

$$\mathcal{A}_0 = \mathcal{A}_{\mathrm{glob}}^\bullet \mathrm{Tw}_\chi(\mathcal{A}),$$

which has height at least two by Lemma 7.4.2. □

Remark 7.5.3. When $\Gamma \cong \mathbf{Z}_p$ there is a simpler proof of Proposition 7.5.2, which does not rely on the noncanonical construction of Proposition 7.5.1. In that case X_∞/X' is finite, so $(\mathrm{Ev}^*)^{-1}(X') \cap G_{\Omega_\infty}$ has finite index in G_{Ω_∞}, so by Lemma 7.2.4(ii), $\mathcal{O}\mathrm{Ev}((\mathrm{Ev}^*)^{-1}(X') \cap G_{\Omega_\infty})$ contains a subgroup of finite index (not *a priori* a Λ-submodule) of

$$a_\tau \mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W))^\bullet \mathrm{Hom}(H^1(K_\infty, W), \mathbf{D}).$$

But every subgroup of finite index contains a submodule of finite index, and hence there is a $j \geq 0$ such that

$$\mathcal{M}^j a_\tau \mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W))^\bullet \mathrm{Hom}(H^1(K_\infty, W), \mathbf{D}) \subset \mathcal{O}\mathrm{Ev}((\mathrm{Ev}^*)^{-1}(X') \cap G_{\Omega_\infty})$$

where we recall that $\mathcal{M}$ is the maximal ideal of Λ. By Proposition 7.4.4(i) and (ii), $\mathcal{A}_{\mathrm{glob}}^\bullet \mathrm{Ann}_\Lambda(W^{G_{K_\infty}})^\bullet$ annihilates the cokernel of the map

$$\mathrm{Hom}(H^1(K_\infty, W), \mathbf{D}) \longrightarrow \mathrm{Hom}(H^1(F, W_M), \mathbf{D}),$$

so the proposition is satisfied with $\mathcal{A}_0 = \mathcal{M}^j \mathcal{A}_{\mathrm{glob}}^\bullet$

7.6. Proof of Proposition 7.1.7

Proposition 7.1.7 is very easy to prove in the following (fairly common, see the examples of Chapter 3) special case. Suppose that hypotheses $\mathrm{Hyp}(K_\infty, T)$ are satisfied (so $a_\tau = 1$ by Lemma 7.1.3(ii)), $\mathcal{O} = \mathbf{Z}_p$, and $H^1(\Omega_\infty/K_\infty, W^*) = 0$. Use (7.1) to choose a sequence $z_1, \ldots, z_r \in X_\infty$ such that $\oplus \Lambda z_i \cong \oplus \Lambda/f_i\Lambda$. By Lemma 7.2.4(iii), under our assumptions we have

$$\mathrm{Ev}^*(\tau G_{\Omega_\infty}) = \mathrm{Ev}^*(\tau) + \mathrm{Ev}^*(G_{\Omega_\infty}) = \mathrm{Ev}^*(\tau) + X_\infty = X_\infty,$$

so Proposition 7.1.7 holds with these z_i and with $\mathfrak{g}_i = f_i\Lambda$.

The rest of this section is devoted to the proof of Proposition 7.1.7 in the general case, which unfortunately is more complicated.

We say that two ideals $\mathcal{A}$ and $\mathcal{B}$ of Λ are relatively prime if $\mathcal{A} + \mathcal{B}$ has height at least two.

Lemma 7.6.1. *The ideal* $\mathrm{char}(X_\infty)$ *is relatively prime to each of the ideals*

$$\mathrm{Ann}_\Lambda(W^{G_{K_\infty}}), \quad \mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W)), \quad \mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W^*))^\bullet.$$

Proof. The proofs for all three ideals are similar. If $W^{G_{K_\infty}}$ is finite or if $\mathrm{rank}_{\mathbf{Z}_p}(\Gamma) > 1$ then $\mathrm{Ann}_\Lambda(W^{G_{K_\infty}})$ has height at least two and the first assertion holds trivially. We have assumed that V is irreducible over G_{K_∞}, so if $W^{G_{K_\infty}}$ is infinite then G_{K_∞} acts trivially on T. Thus (using hypothesis $\mathrm{Hyp}(K_\infty/K)$) the first assertion follows from Lemma 7.3.4(i).

The other two assertions follow similarly, using Lemma 7.3.4 and Corollary C.2.2. We sketch briefly the proof for the third ideal.

Corollary C.2.2 applied to T^*, with $F = K_\infty$ and $\Omega = \Omega_\infty$, gives three cases. In case (i), $H^1(\Omega_\infty/K_\infty, W^*)$ is finite, so $\mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W^*))$ has height at least two, and hence is relatively prime to everything. In case (ii) (resp. (iii)), G_K acts on T^* via a character ρ of Γ (resp. $\varepsilon_{\mathrm{cyc}}\rho$), and $H^1(\Omega_\infty/K_\infty, W^*)$ has a subgroup C of finite index on which G_K acts via ρ.

Then G_K acts on T via $\varepsilon_{\rm cyc}\rho^{-1}$ (resp. ρ^{-1}), so $\mathrm{Ann}_\Lambda(C)^\bullet \supset \mathrm{Ann}_\Lambda(T(-1))$ (resp. $\mathrm{Ann}_\Lambda(C)^\bullet \supset \mathrm{Ann}_\Lambda(T)$). Since

$$\mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W^*)) \supset \mathrm{Ann}_\Lambda(C)\mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W^*)/C)$$

and $H^1(\Omega_\infty/K_\infty, W^*)/C$ is finite, the lemma in this case follows from Lemma 7.3.4. □

Lemma 7.6.2. *Suppose B is a torsion Λ-module and $x, y \in B$. Suppose further that $g_x, g_y \in \Lambda$ are such that $\mathrm{Ann}_\Lambda(x) \subset g_x\Lambda$ and $\mathrm{Ann}_\Lambda(y) \subset g_y\Lambda$. Then there is an $n \in \mathbf{Z}$ such that*

$$\mathrm{Ann}_\Lambda(x + ny) \subset [g_x, g_y]\Lambda$$

where $[g_x, g_y]$ denotes the least common multiple of g_x and g_y.

Proof. Suppose $\mathfrak{P}$ is a (height-one) prime divisor of $[g_x, g_y]$, and define

$$S_\mathfrak{P} = \{n \in \mathbf{Z} : \mathrm{Ann}_\Lambda(x + ny) \not\subset \mathfrak{P}^{\mathrm{ord}_\mathfrak{P}[g_x, g_y]}\}.$$

Recall that $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}$. We will show that $S_\mathfrak{P}$ has at most one element if $\mathfrak{P} \neq \mathfrak{p}\Lambda$, and $S_\mathfrak{P}$ is contained in a congruence class modulo p if $\mathfrak{P} = \mathfrak{p}\Lambda$. Then it will follow that $\mathbf{Z} - \cup_\mathfrak{P} S_\mathfrak{P}$ is nonempty, and every n in this set satisfies the conclusion of the lemma.

Suppose $n, m \in S_\mathfrak{P}$, and let $\mathcal{A} = \mathrm{Ann}_\Lambda(x + ny) \cap \mathrm{Ann}_\Lambda(x + my)$. Then $\mathcal{A} \not\subset \mathfrak{P}^k$, where $k = \mathrm{ord}_\mathfrak{P}[g_x, g_y]$. But $(n - m)\mathcal{A}$ annihilates both y and x, so $(n - m)\mathcal{A} \subset \mathfrak{P}^k$ and we conclude that $n - m \in \mathfrak{P}$. If $\mathfrak{P} \neq \mathfrak{p}\Lambda$ it follows that $n = m$, and if $\mathfrak{P} = \mathfrak{p}\Lambda$ then $n \equiv m \pmod{p}$. This completes the proof. □

Lemma 7.6.3. *Suppose B is a finitely generated torsion Λ-module, and B is pseudo-isomorphic to $\oplus_{i=1}^k \Lambda/h_i\Lambda$, where $h_{i+1} \mid h_i$ for $1 \leq i < k$. Suppose we are given a subring Λ_0 of Λ such that Λ is finitely generated as a Λ_0-module, a Λ_0-submodule $B_0 \subset B$, and an element $t \in B$ such that t and B_0 generate B over Λ. Then there are elements $x_1 \in t + B_0$ and $x_2, \ldots, x_k \in B_0$ such that*

(i) *$\Lambda x_1 \cong \Lambda/\mathfrak{h}_1$ where $\mathfrak{h}_1 \subset h_1\Lambda$ and $h_1\Lambda/\mathfrak{h}_1$ is pseudo-null,*
(ii) *if $2 \leq j \leq k$ there is a split exact sequence*

$$0 \longrightarrow \sum_{i=1}^{j-1} \Lambda x_i \longrightarrow \sum_{i=1}^{j} \Lambda x_i \longrightarrow \Lambda/h_j\Lambda \longrightarrow 0.$$

If $t = 0$ then we can replace (i) *by*

(i′) *$\Lambda x_1 \cong \Lambda/h_1\Lambda$, i.e.,* (ii) *holds for $j = 1$ as well.*

Proof. We will prove the lemma by induction on k.

If $\mathcal{A}$ is an ideal of Λ then $\mathrm{char}(\Lambda/\mathcal{A})$ is the unique principal ideal containing $\mathcal{A}$ with pseudo-null quotient. For every $x \in B_0$ write

$$\mathcal{A}_x \;=\; \mathrm{char}(\Lambda/\mathrm{Ann}_\Lambda(x)).$$

By Lemma 7.6.2 (applied successively with $x = t$ and y running through a sequence of elements of B_0) we can choose $x_1 \in t + B_0$ such that $\mathcal{A}_{x_1} \subset \mathcal{A}_x$ for every $x \in t + B_0$. Since t and B_0 generate B over Λ, we must have $\mathcal{A}_{x_1} = h_1\Lambda$, so (i) is satisfied. This proves the lemma when $k = 1$ and $t \neq 0$.

If $t = 0$ then choose $g \in \Lambda_0$, prime (in Λ) to h_1, which annihilates the pseudo-null Λ-module $h_1\Lambda/\mathrm{Ann}_\Lambda(x_1)$, and replace x_1 by gx_1. This element has annihilator exactly $h_1\Lambda$, so this completes the proof when $k = 1$.

If $k > 1$, choose x_1 as above. Let $B' = B/\Lambda x_1$, let B'_0 be the image of B_0 in B', and let $t' = 0$. Then B' is pseudo-isomorphic to $\oplus_{i=2}^k \Lambda/h_i\Lambda$, so by the induction hypothesis (in the "$t = 0$" case) we can choose $\bar{x}_2, \ldots, \bar{x}_k \in B'_0$ leading to split exact sequences

$$0 \longrightarrow \sum_{i=2}^{j-1} \Lambda\bar{x}_i \longrightarrow \sum_{i=2}^{j} \Lambda\bar{x}_i \longrightarrow \Lambda/h_j\Lambda \longrightarrow 0$$

if $2 \leq j \leq k$.

Now for $i \geq 2$ choose x_i to be any lift of $\bar{x}_i$ to B_0. We claim the lemma is satisfied with this choice of $x_1, \ldots, x_k$. It will suffice to check that the exact sequences

$$0 \longrightarrow \Lambda x_1 \longrightarrow \sum_{i=1}^{j} \Lambda x_i \longrightarrow \sum_{i=2}^{j} \Lambda\bar{x}_i \longrightarrow 0 \tag{7.7}$$

split for $2 \leq j \leq k$.

Let $\mathfrak{h} = \mathrm{Ann}_\Lambda(B)$. Then $\mathfrak{h} \subset h_1\Lambda$ and $h_1^{-1}\mathfrak{h}$ is pseudo-null. By our induction hypothesis we can choose elements $\bar{y}_2, \ldots, \bar{y}_k \in \sum_{i=2}^k \Lambda\bar{x}_i$ such that $\Lambda\bar{y}_i \cong \Lambda/h_i\Lambda$ for each i and $\sum_{i=2}^k \Lambda\bar{y}_i = \sum_{i=2}^k \Lambda\bar{x}_i$. Let y_i be a lift of $\bar{y}_i$ to $\sum_{i=1}^k \Lambda x_i$.

For each i we have $h_iy_i \in \Lambda x_1$, say $h_iy_i = c_ix_1$. Then $h_i^{-1}\mathfrak{h}$ annihilates c_ix_1, i.e., $c_i\mathfrak{h} \subset h_i\mathfrak{h}_1$, and we conclude that h_i divides c_i. Now the map $\bar{y}_i \mapsto y_i - (c_i/h_i)x_1$ gives a splitting of (7.7). □

Proof of Proposition 7.1.7. Recall that we have a pseudo-isomorphism

$$\oplus_{i=1}^r \Lambda/f_i\Lambda \longrightarrow X_\infty.$$

Define a Λ-submodule

$$X_0 \;=\; \Lambda\mathrm{Ev}^*(\tau) + \Lambda\mathrm{Ev}^*(G_{\Omega_\infty}) \;\subset\; X_\infty.$$

Then $X_\infty \supset X_0 \supset X_0 \cap a_\tau X_\infty$, and Lemmas 7.2.4(iii) and 7.6.1 show that $(a_\tau X_\infty)/(X_0 \cap a_\tau X_\infty)$ is pseudo-null. Thus we can find a new injective pseudo-isomorphism

$$\oplus_{i=1}^r \Lambda/g_i\Lambda \longrightarrow X_0$$

with elements $g_i \in \Lambda$ satisfying, for every i,

$$g_{i+1} \mid g_i, \qquad f_i \mid a_\tau g_i, \qquad g_i \mid f_i.$$

Apply Lemma 7.6.3 with $B = X_0$, with $h_i = g_i$, with $B_0 = \mathrm{Ev}^*(G_{\Omega_\infty})$, and with $t = \mathrm{Ev}^*(\tau)$ to produce a sequence $x_1, \dots, x_n \in X_0$. (Note that B_0 satisfies the hypotheses of Lemma 7.6.3 with $\Lambda_0 = \mathrm{Tw}_{\chi^*}(\mathbf{Z}_p[[\Gamma_0]])$, where Γ_0 and χ^* are as in Proposition 7.5.1, and Tw_{χ^*} is as in the proof of Proposition 7.5.2.) Define $z_1 = x_1 \in \mathrm{Ev}^*(\tau G_{\Omega_\infty})$ and $\mathfrak{g}_1 = \mathfrak{h}_1$. For $2 \le i \le r$ define

$$z_i = x_1 + x_i \in \mathrm{Ev}^*(\tau G_{\Omega_\infty}), \quad \mathfrak{g}_i = g_i\Lambda.$$

Lemma 7.6.3 shows that $\mathrm{char}(\sum \Lambda z_i) = \prod g_i = \mathrm{char}(X_0)$, so $X_0/\sum \Lambda z_i$ is pseudo-null. The other conclusions of Proposition 7.1.7 for these z_i and $\mathfrak{g}_i$ also follow immediately from Lemma 7.6.3. □

Corollary 7.6.4. *Suppose $z_1, \dots, z_k$ and $\mathfrak{g}_1, \dots, \mathfrak{g}_k$ are as in Proposition 7.1.7. If $1 \le k \le r$ then $\sum_{i=1}^k \Lambda z_i \cong \oplus_{i=1}^k \Lambda/\mathfrak{g}_i$ and $\sum_{i=1}^k \Lambda z_i$ is a direct summand of $\sum_{i=1}^r \Lambda z_i$.*

Proof. This follows easily by induction on k from Proposition 7.1.7(iii). □

7.7. Proof of Proposition 7.1.9

In this section we will prove Proposition 7.1.9, and thereby complete the proof of Theorems 2.3.3 and 2.3.4 begun in §7.1. Keep the notation of §7.1. In particular recall that

$$Z_\infty = \sum_{i=1}^r \Lambda z_i \cong \bigoplus_{i=1}^r \Lambda/\mathfrak{g}_i \subset X_\infty$$

where the z_i and $\mathfrak{g}_i$ are given by Proposition 7.1.7.

If $\boldsymbol{\sigma}$ is a Selmer sequence of length k, as in Definition 7.1.8, define

$$Z_{\boldsymbol{\sigma}} = \sum_{i=1}^k \Lambda \mathrm{Ev}^*(\sigma_i) \subset Z_\infty.$$

Lemma 7.7.1. *If $\boldsymbol{\sigma}$ is a Selmer sequence of length k, then*

$$Z_{\boldsymbol{\sigma}} \cong \oplus_{i=1}^k \Lambda/\mathfrak{g}_i\Lambda$$

and $Z_{\boldsymbol{\sigma}}$ is a direct summand of Z_∞. If $k < r$ and $\boldsymbol{\sigma}'$ is a Selmer sequence of length $k+1$ extending $\boldsymbol{\sigma}$, then $Z_{\boldsymbol{\sigma}'}/Z_{\boldsymbol{\sigma}} \cong \Lambda/\mathfrak{g}_{k+1}$.

Proof. Let $Y_k = \sum_{i=1}^{k} \Lambda z_i$. By Corollary 7.6.4, $Y_k \cong \oplus_{i=1}^{k} \Lambda/\mathfrak{g}_i$ and there is a complementary submodule $Y_k' \subset Z_\infty$ such that $Y_k \oplus Y_k' = Z_\infty$. The image of $Z_{\boldsymbol{\sigma}} + Y_k'$ in $Z_\infty/\mathcal{M}Z_\infty$ contains the image of $Y_k + Y_k' = Z_\infty$, so by Nakayama's Lemma $Z_{\boldsymbol{\sigma}} + Y_k' = Z_\infty$. We will show that $Z_{\boldsymbol{\sigma}} \cap Y_k' = 0$, and thus $Z_\infty = Z_{\boldsymbol{\sigma}} \oplus Y_k'$ and

$$Z_{\boldsymbol{\sigma}} \cong Z_\infty/Y_k' \cong Y_k \cong \oplus_{i=1}^{k} \Lambda/\mathfrak{g}_i\Lambda.$$

If $k < r$ and $\boldsymbol{\sigma}'$ extends $\boldsymbol{\sigma}$, we can repeat the argument above with k replaced by $k+1$. We can choose Y_{k+1}' to be contained in Y_k', and then $Y_k'/Y_{k+1}' \cong \Lambda/\mathfrak{g}_{k+1}$ and

$$Z_{\boldsymbol{\sigma}'} \oplus Y_{k+1}' = Z_\infty = Z_{\boldsymbol{\sigma}} \oplus Y_k',$$

so

$$Z_{\boldsymbol{\sigma}'} \cong Z_{\boldsymbol{\sigma}} \oplus Y_k'/Y_{k+1}' = Z_{\boldsymbol{\sigma}} \oplus \Lambda/\mathfrak{g}_{k+1}.$$

It remains to show that $Z_{\boldsymbol{\sigma}} \cap Y_k' = 0$. For $1 \le i \le k$ write

$$\mathrm{Ev}^*(\sigma_i) = z_i + v_i + w_i$$

where $v_i \in \mathcal{M}Y_k$ and $w_i \in \mathcal{M}Y_k'$. Suppose

$$\sum_{i=1}^{k} a_i \mathrm{Ev}^*(\sigma_i) \in Y_k'$$

with $a_i \in \Lambda$; we need to show that $\sum_{i=1}^{k} a_i \mathrm{Ev}^*(\sigma_i) = 0$.

Projecting into Y_k we see that

$$\sum_{i=1}^{k} a_i(z_i + v_i) = 0. \tag{7.8}$$

Using Proposition 7.1.7(iii) (see also Corollary 7.6.4), fix $y_1, \ldots, y_k \in Y_k$ so that for $1 \le i \le k$ we have

$$Y_i = \sum_{j=1}^{i} \Lambda z_j = \bigoplus_{j=1}^{i} \Lambda y_j$$

and $\Lambda y_i \cong \Lambda/\mathfrak{g}_i$. We can rewrite (7.8) in matrix form, using these generators, as

$$(a_1, \ldots, a_k)A \in (\mathfrak{g}_1\Lambda, \ldots, \mathfrak{g}_k\Lambda)$$

where A is a $k \times k$ matrix with entries in Λ. Modulo $\mathcal{M}$, we see that A is lower-triangular with invertible diagonal entries (since $z_i \in Y_i$, and the projection of z_i generates $Y_i/Y_{i-1} = \Lambda y_i$, and the v_i vanish modulo $\mathcal{M}$). Therefore A is invertible, and, since $\mathfrak{g}_i \subset \mathfrak{g}_k$ for every $i \le k$, we conclude

that $a_i \in \mathfrak{g}_k$ for every i. But $\mathfrak{g}_k$ annihilates Y'_k because $\mathfrak{g}_k \subset \mathfrak{g}_i$ for $i \geq k$, so we deduce that

$$\sum_{i=1}^{k} a_i \mathrm{Ev}^*(\sigma_i) \;=\; \sum_{i=1}^{k} a_i w_i \;=\; 0.$$

This completes the proof of the lemma. □

Lemma 7.7.2. *For every Selmer sequence $\boldsymbol{\sigma}$, every power M of p, and every $K \subset_{\mathrm{f}} F \subset K_\infty$, the ideal $\mathrm{Ann}_\Lambda(X_\infty/Z_\infty)$ annihilates the kernel of the map*

$$Z_{\boldsymbol{\sigma}} \otimes \Lambda_{F,M} \longrightarrow X_\infty \otimes \Lambda_{F,M}.$$

Proof. By Lemma 7.7.1, $Z_{\boldsymbol{\sigma}}$ is a direct summand of Z_∞, so $Z_{\boldsymbol{\sigma}} \otimes \Lambda_{F,M}$ injects into $Z_\infty \otimes \Lambda_{F,M}$. If $\mathcal{J}_{F,M}$ is the kernel of the map $\Lambda \to \Lambda_{F,M}$, then the kernel of $Z_\infty \otimes \Lambda_{F,M} \to X_\infty \otimes \Lambda_{F,M}$ is $(Z_\infty \cap \mathcal{J}_{F,M} X_\infty)/\mathcal{J}_{F,M} Z_\infty$, which is annihilated by $\mathrm{Ann}_\Lambda(X_\infty/Z_\infty)$. This proves the lemma. □

For the rest of this section fix a field F such that $K \subset_{\mathrm{f}} F \subset K_\infty$. By Assumption 7.1.4, $\Lambda_F/f_1\Lambda_F$ is finite. Fix a power of N_F of p such that $N_F \geq |\Lambda_F/f_1\Lambda_F|$ and such that N_F is at least as large as the integer M_F of Proposition 7.4.4(iv).

Let $\mathcal{B}_0 = (\mathcal{A}^*_{\mathrm{glob}})^\bullet (\mathcal{A}^*_{\mathcal{N}})^\bullet \mathrm{Ann}_\Lambda(X_\infty/Z_\infty)$.

Corollary 7.7.3. *If $\boldsymbol{\sigma}$ is a Selmer sequence and $M \geq N_F$ is a power of p, then $\mathcal{B}_0$ annihilates the kernel of the natural map*

$$Z_{\boldsymbol{\sigma}} \otimes \Lambda_{F,M} \longrightarrow \mathrm{Hom}(\mathcal{S}_{\Sigma_p}(F, W^*_M), \mathcal{O}/M\mathcal{O}).$$

Proof. The map in question is the composition

$$Z_{\boldsymbol{\sigma}} \otimes \Lambda_{F,M} \longrightarrow X_\infty \otimes \Lambda_{F,M} \longrightarrow \mathrm{Hom}(\mathcal{S}_{\Sigma_p}(F, W^*_M), \mathcal{O}/M\mathcal{O}).$$

It follows from Assumption 7.1.4 that $\mathcal{S}_{\Sigma_p}(K_\infty, W^*)^{G_F}$ is finite, so the corollary follows from Proposition 7.4.4(iv) and Lemma 7.7.2. □

If $\mathfrak{r} \in \mathcal{R}$, recall that $\Sigma_{p\mathfrak{r}}$ denotes the set of primes of K dividing $p\mathfrak{r}$. Recall also from Definition 7.1.8 that $\Pi(k, F, M)$ is the set of Kolyvagin sequences of length k for F and M, and if $\boldsymbol{\pi} = (\mathfrak{q}_1, \ldots, \mathfrak{q}_k) \in \Pi(k, F, M)$ then $\mathfrak{r}(\boldsymbol{\pi}) = \prod_{i=1}^k \mathfrak{q}_i$.

Lemma 7.7.4. *Suppose that M is a power of p, that $\boldsymbol{\sigma}$ is a Selmer sequence of length k, and that $\boldsymbol{\pi}$ is a Kolyvagin sequence corresponding to $\boldsymbol{\sigma}$. Then the map of Corollary 7.7.3 factors through a surjective map*

$$Z_{\boldsymbol{\sigma}} \otimes \Lambda_{F,M} \twoheadrightarrow \mathrm{Hom}(\mathcal{S}_{\Sigma_p}(F, W^*_M)/\mathcal{S}_{\Sigma_{p\mathfrak{r}(\boldsymbol{\pi})}}(F, W^*_M), \mathcal{O}/M\mathcal{O}).$$

Proof. Write $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_k)$ and $\boldsymbol{\pi} = (\mathfrak{q}_1, \dots, \mathfrak{q}_k)$. The image of $Z_{\boldsymbol{\sigma}}$ in $\mathrm{Hom}(\mathcal{S}_{\Sigma_p}(F, W_M^*), \mathcal{O}/M\mathcal{O})$ is $\mathrm{Hom}(\mathcal{S}_{\Sigma_p}(F, W_M^*)/B, \mathcal{O}/M\mathcal{O})$, where

$$B = \bigcap_{\substack{1 \le i \le k \\ \gamma \in \mathrm{Gal}(\overline{F}/K)}} \ker(\mathrm{Ev}_{F,M}^*(\sigma_i)^\gamma) = \bigcap_{\substack{\mathcal{Q} \text{ of } F \\ \mathcal{Q} \mid \prod_i \mathfrak{q}_i}} \ker(\mathrm{Ev}_{F,M}^*(\mathrm{Fr}_{\mathcal{Q}})).$$

Each $\mathrm{Ev}_{F,M}^*(\mathrm{Fr}_{\mathcal{Q}})$ factors through a map which is injective on $H_f^1(F_{\mathcal{Q}}, W_M^*)$, so the right-hand intersection is exactly $\mathcal{S}_{\Sigma_{p\mathfrak{r}(\boldsymbol{\pi})}}(F, W_M^*)$. □

Proposition 7.7.5. *Suppose that $1 \le k \le r$, that $M \ge N_F$ is a power of p, and that $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \dots, \boldsymbol{\pi}_k) \in \Pi(k, F, M)$. Let $\Sigma = \Sigma_{p\mathfrak{r}(\boldsymbol{\pi})}$ and $\mathfrak{q} = \mathfrak{q}_k$. Then with $\mathcal{B}_0$ as defined before Corollary 7.7.3,*

$$a_\tau \mathcal{B}_0 \widetilde{\mathrm{Ev}}_{\mathfrak{q},s}(\mathcal{S}^\Sigma(F, W_M)) \subset \mathfrak{g}_k \Lambda_{F,M}.$$

Proof. Fix M, k, and $\boldsymbol{\pi}$ as in the statement of the proposition. Let $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_k)$ be a Selmer sequence corresponding to $\boldsymbol{\pi}$ and let $\boldsymbol{\sigma}' = (\sigma_1, \dots, \sigma_{k-1})$ and $\Sigma' = \Sigma - \{\mathfrak{q}\}$.

Consider the commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
Z_{\boldsymbol{\sigma}'} \otimes \Lambda_{F,M} & \longrightarrow & \mathrm{Hom}(\mathcal{S}_{\Sigma_p}(F, W_M^*)/\mathcal{S}_{\Sigma'}(F, W_M^*), \mathcal{O}/M\mathcal{O}) \\
\downarrow & & \downarrow \\
Z_{\boldsymbol{\sigma}} \otimes \Lambda_{F,M} & \longrightarrow & \mathrm{Hom}(\mathcal{S}_{\Sigma_p}(F, W_M^*), \mathcal{O}/M\mathcal{O}) \\
\downarrow & & \downarrow \\
(Z_{\boldsymbol{\sigma}}/Z_{\boldsymbol{\sigma}'}) \otimes \Lambda_{F,M} & \xrightarrow{j} & \mathrm{Hom}(\mathcal{S}_{\Sigma'}(F, W_M^*), \mathcal{O}/M\mathcal{O}) \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}$$

The left-hand column is exact by Lemma 7.7.1, and the top horizontal map is the surjection of Lemma 7.7.4. Applying the snake lemma, Corollary 7.7.3 shows that $\ker(j)$ is annihilated by $\mathcal{B}_0$. The image of j is generated by $\mathrm{Ev}_{F,M}^*(\sigma_k)|_{\mathcal{S}_{\Sigma'}(F, W_M^*)} = \mathrm{Ev}_{\mathfrak{q},f}^*|_{\mathcal{S}_{\Sigma'}(F, W_M^*)}$, and $Z_{\boldsymbol{\sigma}}/Z_{\boldsymbol{\sigma}'} \cong \Lambda/\mathfrak{g}_k\Lambda$. Hence

$$\mathcal{B}_0 \mathrm{Ann}_{\Lambda_{F,M}}(\mathrm{Ev}_{\mathfrak{q},f}^*|_{\mathcal{S}_{\Sigma'}(F, W_M^*)}) \subset \mathfrak{g}_k \Lambda_{F,M}.$$

By Theorem 7.2.11, $a_\tau \widetilde{\mathrm{Ev}}_{\mathfrak{q},s}(\mathcal{S}^\Sigma(F, W_M))$ annihilates $\mathrm{Ev}_{\mathfrak{q},f}^*|_{\mathcal{S}_{\Sigma'}(F, W_M^*)}$. This proves the proposition. □

Recall that we have fixed a field F. If M is a power of p and $\mathfrak{r} \in \mathcal{R}_{F,M}$, we will write simply $\kappa_{[\mathfrak{r},M]}$ for $\kappa_{[F,\mathfrak{r},M]}$, and $\langle \kappa_{[\mathfrak{r},M]} \rangle$ for the $\Lambda_{F,M}$-submodule $\Lambda_{F,M}\kappa_{[\mathfrak{r},M]}$ of $H^1(F, W_M)$.

Recall also that the map $\widetilde{\mathrm{Ev}}_{\mathfrak{q},s}$ depends on M; we now need to record that dependence. For every power M of p, instead of just $\widetilde{\mathrm{Ev}}_{\mathfrak{q},s}$ we will write $\widetilde{\mathrm{Ev}}_{\mathfrak{q},s,M} : H^1(F, W_M) \to \Lambda_{F,M}$ for the singular evaluation map of Definitions 7.2.5 and 7.2.9.

Corollary 7.7.6. *Suppose $1 \le k \le r$ and M is a power of p. Suppose further that $\boldsymbol{\pi} = (\mathfrak{q}_1, \ldots, \mathfrak{q}_k) \in \Pi(k, F, MN_F)$, and let $\mathfrak{q} = \mathfrak{q}_k$ and $\mathfrak{r} = \mathfrak{r}(\boldsymbol{\pi})$. If $\eta \in a_\tau^2 \mathcal{B}_0$ then*

$$\eta\widetilde{\mathrm{Ev}}_{\mathfrak{q},s,M}|_{\langle \kappa_{[\mathfrak{r},M]} \rangle} \in f_k \mathrm{Hom}_\Lambda(\langle \kappa_{[\mathfrak{r},M]} \rangle, \Lambda_{F,M}).$$

Proof. Let $M' = MN_F$ and $\Sigma = \Sigma_{p\mathfrak{r}(\boldsymbol{\pi})}$. By Propositions 7.7.5 and 7.1.7(ii),

$$\eta\widetilde{\mathrm{Ev}}_{\mathfrak{q},s,M'} : \mathcal{S}^\Sigma(F, W_{M'}) \longrightarrow a_\tau \mathfrak{g}_k \Lambda_{F,M'} \subset f_k \Lambda_{F,M'}.$$

Since $f_k \mid N_F$ in Λ_F, there is a well-defined "division by f_k" map

$$f_k \Lambda_{F,M'} \longrightarrow \Lambda_{F,M}$$

which sends $f_k g$ to $g \pmod{M}$ for every g. Let $\psi' : \mathcal{S}^\Sigma(F, W_{M'}) \to \Lambda_{F,M}$ be the composition of $\eta\widetilde{\mathrm{Ev}}_{\mathfrak{q},s,M'}$ with this division map.

Let $\iota_{N_F,M'}$ and $\iota_{M',M}$ be the natural maps in the exact cohomology sequence

$$H^1(F, W_{N_F}) \xrightarrow{\iota_{N_F,M'}} H^1(F, W_{M'}) \xrightarrow{\iota_{M',M}} H^1(F, W_M).$$

If we identify Λ_{F,N_F} with $M\Lambda_{F,M'}$, we have

$$\widetilde{\mathrm{Ev}}_{\mathfrak{q},s,N_F} = \widetilde{\mathrm{Ev}}_{\mathfrak{q},s,M'} \circ \iota_{N_F,M'}.$$

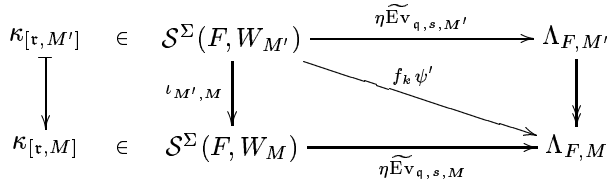
Applying Propositions 7.7.5 and 7.1.7(ii) again we see that

$$\eta\widetilde{\mathrm{Ev}}_{\mathfrak{q},s,N_F}(\mathcal{S}^\Sigma(F, W_{N_F})) \subset f_k \Lambda_{F,N_F},$$

and it follows that $\psi' \circ \iota_{N_F,M'} = 0$. Therefore ψ' factors through $\iota_{M',M}$, i.e.,

$$\psi' = \psi \circ \iota_{M',M} \text{ where } \psi \in \mathrm{Hom}_\Lambda(\iota_{M',M}(\mathcal{S}^\Sigma(F, W_{M'})), \Lambda_{F,M}).$$

Using Lemma 4.4.13(iii) and Theorem 4.5.1, we also have a diagram



It follows that

$$f_k\psi(\kappa_{[\mathfrak{r},M]}) \;=\; f_k\psi'(\kappa_{[\mathfrak{r},M']}) \;=\; \eta\widetilde{\mathrm{Ev}}_{\mathfrak{q},s}(\kappa_{[\mathfrak{r},M]}),$$

and so $\eta\widetilde{\mathrm{Ev}}_{\mathfrak{q},s,M} = f_k\psi$ on $\langle\kappa_{[\mathfrak{r},M]}\rangle$. □

The following is a precise version of Proposition 7.1.9. Define

$$\mathcal{B} \;=\; a_\tau^4\mathcal{A}_0^\bullet\mathcal{B}_0\mathrm{Ann}_\Lambda(W^{G\kappa_\infty})\mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty,W))$$

where $\mathcal{A}_0$ is the ideal of Proposition 7.5.2 applied with

$$X' = \{x \in X_\infty : a_\tau x \in \mathcal{M}Z_\infty\}$$

(so by Proposition 7.1.7(iv), X_∞/X' is pseudo-null) and $\mathcal{B}_0$ is as defined before Corollary 7.7.3.

Proposition 7.7.7. *If $M \geq N_F$ is a power of p and $0 \leq k < r$, then*

$$\mathcal{B}\Psi(k,F,N_FM)\Lambda_{F,M} \;\subset\; f_{k+1}\Psi(k+1,F,M).$$

Proof. Let $M' = N_FM$. Fix a Kolyvagin sequence $\boldsymbol{\pi} \in \Pi(k,F,M')$, let $\mathfrak{r} = \mathfrak{r}(\boldsymbol{\pi})$, and fix $\psi : \langle\kappa_{[\mathfrak{r},M']}\rangle \to \Lambda_{F,M'}$. We need to show that

$$\mathcal{B}\psi(\kappa_{[\mathfrak{r},M']})\Lambda_{F,M} \;\subset\; f_{k+1}\Psi(k+1,F,M).$$

The idea of the proof is as follows. Ideally, we would like to find $\gamma \in \tau G_{\Omega_\infty}$ such that

(a) $\mathrm{Ev}^*(\gamma) \in z_{k+1} + \mathcal{M}Z_\infty$,
(b) $\widetilde{\mathrm{Ev}}_{F,M'}(\gamma) = \psi$ on $\langle\kappa_{[\mathfrak{r},M']}\rangle$,

and choose a prime $\mathfrak{q}$ whose Frobenius on a suitable extension of F is γ. If we can do this then (a) says we can use $\mathfrak{q}$ to extend $\boldsymbol{\pi}$ to a Kolyvagin sequence of length $k+1$, (b) combined with Theorem 7.2.10 shows that $\psi(\kappa_{[\mathfrak{r},M']}) = \widetilde{\mathrm{Ev}}_{\mathfrak{q},s,}(\kappa_{[\mathfrak{r}\mathfrak{q},M']})$, and Corollary 7.7.6 shows that the restriction of $\widetilde{\mathrm{Ev}}_{\mathfrak{q},s}$ to $\langle\kappa_{[\mathfrak{r}\mathfrak{q},M]}\rangle$ is (almost) divisible by f_{k+1}.

Unfortunately, conditions (a) and (b) on γ may not be independent, and it may not be possible to satisfy them simultaneously. Instead, we will use Proposition 7.5.2 to find a finite set of elements $\{\gamma_i\}$ such that $\mathrm{Ev}^*(\gamma_i) \in \mathcal{M}Z_\infty$ and such that, instead of (b), a "small multiple" of ψ is a linear combination of the $\widetilde{\mathrm{Ev}}_{F,M'}(\gamma_i)$.

We now return to the proof. Let $\psi_0 \in \mathrm{Hom}(\langle \kappa_{[\mathfrak{r},M']}\rangle, \mathcal{O}/M'\mathcal{O})$ be the homomorphism corresponding to ψ under the isomorphism of Lemma 7.2.7(i). If

$$\eta \in \mathcal{A}_0 a_\tau^2 \mathrm{Ann}_\Lambda(W^{G_{K_\infty}})^\bullet \mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty, W))^\bullet,$$

then by Proposition 7.5.2 (applied with X' as defined just before the statement of this proposition) there are $\gamma_1, \dots, \gamma_j \in G_{\Omega_\infty}$ and $c_1, \dots, c_j \in \mathcal{O}$ such that $\mathrm{Ev}^*(\gamma_i) \in (\mathcal{M}Z_\infty)$ for every i and

$$\sum_{i=1}^{j} c_i \mathrm{Ev}_{F,M'}(\gamma_i)|_{\langle \kappa_{[\mathfrak{r},M']}\rangle} = \eta\psi_0. \tag{7.9}$$

Fix i such that $1 \le i \le j$. Let $\boldsymbol{\sigma}$ be a Selmer sequence corresponding to $\boldsymbol{\pi}$. Choose $\delta \in \tau G_{\Omega_\infty}$ such that $\mathrm{Ev}^*(\delta) = z_{k+1}$ (by Proposition 7.1.7(i)), and define two Selmer sequences $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}''$ of length $k+1$ extending $\boldsymbol{\sigma}$ by $\sigma'_{k+1} = \delta$ and $\sigma''_{k+1} = \delta\gamma_i$. (These are Selmer sequences because $\mathrm{Ev}^*(\delta) = z_{k+1}$ and $\mathrm{Ev}^*(\gamma_i) \in \mathcal{M}Z_\infty$.) Fix primes $\mathfrak{q}', \mathfrak{q}''$ of K such that, for an appropriate choice of primes above $\mathfrak{q}'$ and $\mathfrak{q}''$, we have

$$\mathrm{Fr}_{\mathfrak{q}'} = \sigma'_{k+1}, \quad \mathrm{Fr}_{\mathfrak{q}''} = \sigma''_{k+1} \text{ on } L$$

where L is a finite Galois extension of F containing $F(W_{M'}, \boldsymbol{\mu}_{M'}, (\mathcal{O}_K^\times)^{1/M'})$ and such that the restriction to L of every element of the finite groups (see Lemma 1.5.7) $\mathcal{S}_{\Sigma_p}(F, W^*_{M'})$ and $\mathcal{S}^{\Sigma_{p\mathfrak{r}}}(F, W_{M'})$ is zero.

We define two Kolyvagin sequences $\boldsymbol{\pi}', \boldsymbol{\pi}'' \in \Pi(k+1, F, M')$ extending $\boldsymbol{\pi}$ by setting $\mathfrak{q}'_{k+1} = \mathfrak{q}'$ and $\mathfrak{q}''_{k+1} = \mathfrak{q}''$. By Corollary 7.7.6, if $\eta' \in a_\tau^2 \mathcal{B}_0$ we can choose

$$\psi' \in \mathrm{Hom}_\Lambda(\langle \kappa_{[\mathfrak{r}\mathfrak{q}',M]}\rangle, \Lambda_{F,M}), \quad \psi'' \in \mathrm{Hom}_\Lambda(\langle \kappa_{[\mathfrak{r}\mathfrak{q}'',M]}\rangle, \Lambda_{F,M})$$

so that

$$\begin{aligned} f_{k+1}\psi'(\kappa_{[\mathfrak{r}\mathfrak{q}',M]}) &= \eta' \widetilde{\mathrm{Ev}}_{\mathfrak{q}',s,M}(\kappa_{[\mathfrak{r}\mathfrak{q}',M]}), \\ f_{k+1}\psi''(\kappa_{[\mathfrak{r}\mathfrak{q}'',M]}) &= \eta' \widetilde{\mathrm{Ev}}_{\mathfrak{q}'',s,M}(\kappa_{[\mathfrak{r}\mathfrak{q}'',M]}). \end{aligned}$$

Therefore, using Theorem 7.2.10 for the third equality below,

$$\begin{aligned} &\eta' \widetilde{\mathrm{Ev}}_{F,M'}(\gamma_i)(\kappa_{[\mathfrak{r},M']}) \\ &\quad = \eta' \widetilde{\mathrm{Ev}}_{F,M'}(\sigma''_{k+1})(\kappa_{[\mathfrak{r},M']}) - \eta' \widetilde{\mathrm{Ev}}_{F,M'}(\sigma'_{k+1})(\kappa_{[\mathfrak{r},M']}) \\ &\quad = \eta' \widetilde{\mathrm{Ev}}_{F,M'}(\mathrm{Fr}_{\mathfrak{q}''})(\kappa_{[\mathfrak{r},M']}) - \eta' \widetilde{\mathrm{Ev}}_{F,M'}(\mathrm{Fr}_{\mathfrak{q}'})(\kappa_{[\mathfrak{r},M']}) \\ &\quad = \eta' \widetilde{\mathrm{Ev}}_{\mathfrak{q}'',s,M'}(\kappa_{[\mathfrak{r}\mathfrak{q}'',M']}) - \eta' \widetilde{\mathrm{Ev}}_{\mathfrak{q}',s,M'}(\kappa_{[\mathfrak{r}\mathfrak{q}',M']}) \\ &\quad \equiv f_{k+1}(\psi''(\kappa_{[\mathfrak{r}\mathfrak{q}'',M]}) - \psi'(\kappa_{[\mathfrak{r}\mathfrak{q}',M]})) \pmod{M} \\ &\quad \in f_{k+1}\Psi(k+1, F, M). \end{aligned}$$

By (7.9) and Lemma 7.2.7(ii),

$$\sum_{i=1}^{j} c_i \widetilde{\mathrm{Ev}}_{F,M'}(\gamma_i) \;=\; \widetilde{\eta\psi_0} \;=\; \eta^\bullet\psi,$$

so we conclude that

$$\eta^\bullet\eta'\psi(\kappa_{[\mathfrak{r},M']})\Lambda_{F,M} \;\subset\; f_{k+1}\Psi(k+1,F,M).$$

As η and η' vary, the products $\eta^\bullet\eta'$ generate $\mathcal{B}$, and the Proposition is proved. □

Proof of Proposition 7.1.9. Observe that

- $\mathcal{A}_0$, $\mathcal{A}^*_{\mathrm{glob}}$, and $\mathcal{A}^*_{\mathcal{N}}$ have height at least two by Lemma 7.4.2 and Proposition 7.5.2,
- $\mathrm{Ann}_\Lambda(H^1(\Omega_\infty/K_\infty,W))$ and $\mathrm{Ann}_\Lambda(W^{G_{K_\infty}})$ are prime to $\mathrm{char}(X_\infty)$ by Lemma 7.6.1, and
- $\mathrm{Ann}_\Lambda(X_\infty/Z_\infty)$ contains the product of a_τ and an ideal of height at least two by Proposition 7.1.7(iv).

An ideal of height at least two necessarily contains an element relatively prime to $\mathrm{char}(X_\infty)$ (since $\mathrm{char}(X_\infty) \neq 0$ by Theorem 2.3.2), so the ideal $\mathcal{B}$ defined before the statement of Proposition 7.7.7 contains the product of a_τ^5 and an element h of Λ prime to $\mathrm{char}(X_\infty)$. Now Proposition 7.1.9 follows from Proposition 7.7.7. □

CHAPTER 8

Euler Systems and p-adic L-functions

So far we have discussed at length how an Euler system for a p-adic representation T of G_K controls the Selmer groups $\mathcal{S}(K, W^*)$ and $\mathcal{S}(K_\infty, W^*)$. This raises several natural questions which we have not yet touched on.

- Except for the examples in Chapter 3, we have not discussed at all how to produce Euler systems. For which representations do (nontrivial) Euler systems exist?
- If there is a nontrivial Euler system $\mathbf{c}$ for T, then there are infinitely many such (for example, we can act on $\mathbf{c}$ by elements of $\mathcal{O}[[G_K]]$). Is there a "best" Euler system?
- Conjecturally, Selmer groups should be related to L-functions and their special values. Is there an Euler system related to an L-function attached to T?

In this chapter we will sketch a picture which gives a highly conjectural partial answer to these questions, by describing a fundamental connection between Euler systems and (p-adic) L-functions. This general picture will rest on several layers of conjectures, but nonetheless there are several known examples (such as the ones in Chapter 3) where the connection is proved.

The connection is made via the work of Perrin-Riou [**PR2**], [**PR4**]. Briefly, for certain p-adic representations T of $G_{\mathbf{Q}}$, and subject to some vast but plausible conjectures, Perrin-Riou shows how to view the p-adic L-functions attached to twists of T by characters of conductor m as elements in $H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_m), T)$ (or more precisely, in the tensor product of $H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_m), T)$ with the field of fractions of Λ). As we will see below in §8.3, these cohomology classes (if they exist) satisfy the distribution relation defining an Euler system for T. In other words, Perrin-Riou's conjectural elements form an Euler system, and since they arise from p-adic L-functions, Theorems 2.2.10 and 2.3.8 then relate the Selmer groups $\mathcal{S}(\mathbf{Q}, W^*)$ and $\mathcal{S}(\mathbf{Q}_\infty, W^*)$ to L-values.

8.1. The Setting

For this chapter we will assume

- $K = \mathbf{Q}$, i.e., T is a p-adic representation of $G_{\mathbf{Q}}$,
- the scalar ring $\mathcal{O}$ is $\mathbf{Z}_p$.

The first assumption is not too serious a restriction, as in general one could consider the induced representation $\mathrm{Ind}_{K/\mathbf{Q}}T$. The second is completely unimportant, and is made only for notational convenience.

Following Perrin-Riou [**PR4**], we will also make the more serious assumptions that $V = T \otimes \mathbf{Q}_p$ is the p-adic realization of a "motivic structure" in the sense of [**FPR**] Chapter III, that T corresponds to an integral structure on this motive, and that the representation V is crystalline at p.

We let $\boldsymbol{\mathcal{D}}(V)$ denote Fontaine's filtered vector space attached to V, i.e.,

$$\boldsymbol{\mathcal{D}}(V) \;=\; (V \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}})^{G_{\mathbf{Q}_p}}.$$

By definition,

$$V \text{ is crystalline} \quad\Longleftrightarrow\quad dim_{\mathbf{Q}_p}\boldsymbol{\mathcal{D}}(V) \;=\; \dim_{\mathbf{Q}_p} V.$$

Suppose F is an abelian extension of $\mathbf{Q}$, unramified at p. Then F has $[F:\mathbf{Q}]$ distinct embeddings into B_{cris} and we also define

$$\boldsymbol{\mathcal{D}}_F(V) \;=\; \boldsymbol{\mathcal{D}}(\oplus_{F\hookrightarrow B_{\mathrm{cris}}} V) \;\cong\; \boldsymbol{\mathcal{D}}(\mathrm{Ind}_{F/\mathbf{Q}}V)$$

where $G_{\mathbf{Q}}$ acts on $\oplus_{F\hookrightarrow B_{\mathrm{cris}}}V$ by acting both on V and on the set of embeddings.

Suppose E is a finite extension of $\mathbf{Q}_p$, with ring of integers $\mathcal{O}_E$, and $\chi : \mathrm{Gal}(F/\mathbf{Q}) \to E^\times$ is a character. Let $\mathcal{O}_\chi$ denote a free rank-one $\mathcal{O}_E$-module (with a fixed generator) on which $\mathrm{Gal}(F/\mathbf{Q})$ acts via χ, and write $T\otimes\chi$ for $T\otimes\mathcal{O}_\chi$ and $V\otimes\chi$ for $V\otimes\mathcal{O}_\chi$. Let

$$\epsilon_\chi \;=\; \sum_{\gamma\in\mathrm{Gal}(F/\mathbf{Q})} \chi(\gamma)\gamma^{-1} \in \mathcal{O}_E[\mathrm{Gal}(F/\mathbf{Q})].$$

Lemma 8.1.1. (i) *There is a natural identification*

$$\boldsymbol{\mathcal{D}}_F(V) \cong F\otimes_{\mathbf{Q}} \boldsymbol{\mathcal{D}}(V).$$

(ii) *Each choice of embedding $F\hookrightarrow B_{\mathrm{cris}}$ induces an isomorphism*

$$\boldsymbol{\mathcal{D}}(V\otimes\chi) \;\cong\; \epsilon_{\chi^{-1}}(E\otimes_{\mathbf{Q}_p}\boldsymbol{\mathcal{D}}_F(V))$$

where we let $\mathrm{Gal}(F/\mathbf{Q})$ *act on* $\boldsymbol{\mathcal{D}}_F(V)$ *via its action on* F *in* (i).

Proof. We have

$$\boldsymbol{\mathcal{D}}_F(V) \;=\; (\oplus_{j:F\hookrightarrow B_{\mathrm{cris}}} V\otimes B_{\mathrm{cris}})^{G_{\mathbf{Q}_p}},$$

so there is a natural embedding

$$F\otimes_{\mathbf{Q}}\boldsymbol{\mathcal{D}}(V) \;\hookrightarrow\; \boldsymbol{\mathcal{D}}_F(V) \;\text{ given by }\; \alpha\otimes d \mapsto \oplus_j(j(\alpha)d). \tag{8.1}$$

In general

$$\dim_{\mathbf{Q}_p} \boldsymbol{\mathcal{D}}_F(V) \leq \dim_{\mathbf{Q}_p}(\mathrm{Ind}_{F/\mathbf{Q}}V) = \dim_{\mathbf{Q}_p}(F \otimes_{\mathbf{Q}} \boldsymbol{\mathcal{D}}(V)),$$

so we conclude that equality must hold (i.e., $\mathrm{Ind}_{F/\mathbf{Q}}V$ is crystalline), and the map (8.1) is an isomorphism. This proves (i).

For (ii), let $(E \otimes V \otimes B_{\mathrm{cris}})^{\chi^{-1}}$ be the subspace of $E \otimes_{\mathbf{Q}_p} V \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}}$ on which $G_{\mathbf{Q}_p}$ (acting on V and B_{cris}, not on E) acts via χ^{-1}. An embedding $j : F \hookrightarrow B_{\mathrm{cris}}$ induces an embedding $E \otimes F \hookrightarrow E \otimes B_{\mathrm{cris}}$, and hence (using (i)) an isomorphism

$$\epsilon_{\chi^{-1}}(E \otimes \boldsymbol{\mathcal{D}}_F(V)) \;=\; \epsilon_{\chi^{-1}}(E \otimes F) \otimes \boldsymbol{\mathcal{D}}(V) \xrightarrow{\sim} (E \otimes V \otimes B_{\mathrm{cris}})^{\chi^{-1}}.$$

Our fixed generator of $\mathcal{O}_\chi$ induces an isomorphism from $(E \otimes V \otimes B_{\mathrm{cris}})^{\chi^{-1}}$ to $(V \otimes \chi \otimes B_{\mathrm{cris}})^{G_{\mathbf{Q}_p}} = \boldsymbol{\mathcal{D}}(V \otimes \chi)$. This proves (ii). □

As in Chapter 3 we let $\mathbf{Q}_\infty = \cup \mathbf{Q}_n$ denote the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$, let $\Gamma = \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$, and let $\Lambda = \mathbf{Z}_p[[\Gamma]]$ be the Iwasawa algebra. Let $\mathbf{H}$ be the extended Iwasawa algebra defined by Perrin-Riou in [**PR2**] §1, i.e., if we identify Λ with a power series ring $\mathbf{Z}_p[[X]]$ in the usual way, and we let $\mathbf{Q}_p[[X]]_r \subset \mathbf{Q}_p[[X]]$ denote the $\mathbf{Q}_p$-vector space of power series which converge on the open unit ball in $\overline{\mathbf{Q}_p}$ with growth

$$\sup_{|X|<\rho} |f(X)| \;=\; o\Big(\sup_{|X|<\rho} |\log(1+X)|^r\Big)$$

as $\rho \to 1^-$, then $\mathbf{H}$ is the Λ-algebra

$$\mathbf{H} \;=\; \Lambda \otimes_{\mathbf{Z}_p[[X]]} (\varinjlim_r \mathbf{Q}_p[[X]]_r).$$

We let $\mathbf{K}$ be the field of fractions of $\mathbf{H}$.

Suppose F is an abelian extension of $\mathbf{Q}$ unramified at p. In [**PR2**] (see also [**PR4**] §1.2) Perrin-Riou constructs[1] what she calls a "logarithme élargi", a $\mathbf{Z}_p[[\mathrm{Gal}(F\mathbf{Q}_\infty/\mathbf{Q})]]$-module homomorphism

$$\bigoplus_{v|p} \varprojlim_n H^1((F\mathbf{Q}_n)_v, T) \;\longrightarrow\; \mathbf{K} \otimes \boldsymbol{\mathcal{D}}_F(V).$$

This is a generalization of work of Coleman [**Co**], who defined such a map in the case where $T = \mathbf{Z}_p(1)$. Composing Perrin-Riou's map with the local restriction maps we obtain a $\mathbf{Z}_p[[\mathrm{Gal}(F\mathbf{Q}_\infty/\mathbf{Q})]]$-module homomorphism

$$\mathcal{L}_F : H^1_\infty(F,T) \;=\; \varprojlim_n H^1(F\mathbf{Q}_n, T) \;\longrightarrow\; \mathbf{K} \otimes \boldsymbol{\mathcal{D}}_F(V)$$

[1] Perrin-Riou's construction only deals with odd primes p. We will implicitly assume as part of the conjecture below that her construction can be extended to $p = 2$ to produce a map with similar properties.

which will be crucial in what follows. If $F' \subset F$ then there is a commutative diagram

$$\begin{array}{ccc} H^1_\infty(F,T) & \xrightarrow{\mathcal{L}_F} & \mathbf{K}\otimes\boldsymbol{\mathcal{D}}_F(V) \\ {\scriptstyle \mathrm{Cor}_{F/F'}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_{F/F'}} \\ H^1_\infty(F',T) & \xrightarrow{\mathcal{L}_{F'}} & \mathbf{K}\otimes\boldsymbol{\mathcal{D}}_{F'}(V). \end{array} \tag{8.2}$$

8.2. Perrin-Riou's p-adic L-function and Related Conjectures

Let $d = d(V) = \dim_{\mathbf{Q}_p}(V)$,

$$d_+ = d_+(V) = \dim_{\mathbf{Q}_p}(V^+) = \dim_{\mathbf{Q}_p}(V^{c=1})$$

where c is a complex conjugation in $G_{\mathbf{Q}}$, and let

$$d_- = d_-(V) = \dim_{\mathbf{Q}_p}(V^-) = \dim_{\mathbf{Q}_p}(V^{c=-1}) = d - d_+.$$

Let $\omega : G_{\mathbf{Q}} \to (\mathbf{Z}_p^\times)_{\mathrm{tors}}$ be the Teichmüller character giving the action of $G_{\mathbf{Q}}$ on $\boldsymbol{\mu}_p$ (if p is odd) or on $\boldsymbol{\mu}_4$ (if $p = 2$), and define

$$\langle\varepsilon\rangle = \omega^{-1}\varepsilon_{\mathrm{cyc}} : G_{\mathbf{Q}} \twoheadrightarrow \Gamma \xrightarrow{\sim} \begin{cases} 1+p\mathbf{Z}_p & \text{if } p \text{ is odd,} \\ 1+4\mathbf{Z}_2 & \text{if } p = 2. \end{cases}$$

Fix embeddings $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}_p}$.

Suppose that E is a finite extension of $\mathbf{Q}_p$ and $\chi : G_{\mathbf{Q}} \to E^\times$ is an *even* character of finite order, unramified at p.

Conjecture 8.2.1 (Perrin-Riou [**PR4**] §4.2). *Under the assumptions on T at the beginning of* §8.1*, if $r \in \mathbf{Z}^+$ is divisible by the conductor of χ then there is a p-adic L-function*

$$\mathbf{L}_r^{(p)}(T\otimes\chi) \in \mathbf{K}\otimes\wedge_E^{d_+}\boldsymbol{\mathcal{D}}(V^*\otimes\chi^{-1}).$$

See [**PR4**] §4.2 for the properties defining this p-adic L-function (when $p > 2$). For our purposes we only say loosely that $\mathbf{L}_r^{(p)}(T\otimes\chi)$ is defined so that for characters ρ of Γ of finite order and for sufficiently large positive integers k,

$$\begin{aligned} &\langle\varepsilon\rangle^k\rho(\mathbf{L}_r^{(p)}(T\otimes\chi)) \\ &\quad = (p\text{-Euler factor}) \times \frac{L_r(V\otimes\chi\omega^k\rho^{-1},-k)}{(\text{archimedean period})} \times (p\text{-adic period}). \end{aligned}$$

Here $L_r(V\otimes\chi\omega^k\rho^{-1},s)$ is the (conjectural) complex L-function of the representation $V\otimes\chi\omega^k\rho^{-1}$ with Euler factors at primes dividing r removed,

which has an Euler product expansion

$$\prod_{\ell \nmid r} l_\ell(V \otimes \chi\omega^k\rho^{-1}, s)^{-1}. \tag{8.3}$$

For primes $\ell \neq p$ where V is unramified, the Euler factor at ℓ is defined by

$$l_\ell(V \otimes \chi\omega^k\rho^{-1}, s) \;=\; \det(1 - \mathrm{Fr}_\ell^{-1}x | V \otimes \chi\omega^k\rho^{-1})|_{x=\ell^{-s}},$$

so in particular

$$l_\ell(V \otimes \chi\omega^k\rho^{-1}, -k) \;=\; \langle\varepsilon\rangle^k\rho\big(\det(1 - \mathrm{Fr}_\ell^{-1}x|V)|_{x=\chi^{-1}(\ell)\mathrm{Fr}_\ell}\big).$$

Hence for such ℓ, writing $P(\mathrm{Fr}_\ell^{-1}|T;x) = \det(1 - \mathrm{Fr}_\ell^{-1}x|T)$ as in §2.1, we have

$$\mathbf{L}_{r\ell}^{(p)}(T \otimes \chi) \;=\; P(\mathrm{Fr}_\ell^{-1}|T;\chi^{-1}(\ell)\mathrm{Fr}_\ell)\mathbf{L}_r^{(p)}(T \otimes \chi). \tag{8.4}$$

The following statement is in the spirit of the conjectures of Perrin-Riou in [**PR4**] §4.4, but stronger. In fact it is so strong that this formulation is certainly not true in general (see Remark 8.2.5 below). However, one can hope that it is "almost" true.

For $r \in \mathbf{Z}^+$ write $\Delta_r = \mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_r)^+/\mathbf{Q})$, where $\mathbf{Q}(\boldsymbol{\mu}_m)^+$ is the real subfield of $\mathbf{Q}(\boldsymbol{\mu}_m)$, and

$$\Lambda_r \;=\; \Lambda \otimes \mathbf{Z}_p[\Delta_r] \;=\; \mathbf{Z}_p[[\mathrm{Gal}(\mathbf{Q}_\infty(\boldsymbol{\mu}_r)^+/\mathbf{Q})]].$$

The involution $\gamma \mapsto \gamma^{-1}$ of Γ induces involutions of $\mathbf{K}$ and hence of $\mathbf{K}\otimes\wedge_E^{d_-}\boldsymbol{\mathcal{D}}(V\otimes\chi^{-1})$, for example. We denote this last involution by $f \mapsto f^\iota$.

Wishful Thinking 8.2.2. *Suppose $r \in \mathbf{Z}^+$ is prime to p. Then there is an element $\boldsymbol{\xi}_r \in \wedge_{\Lambda_r}^{d_-} H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$ such that for every finite extension E of $\mathbf{Q}_p$ and every character $\chi : \Delta_r \to E^\times$,*

$$\epsilon_\chi^{\otimes d_-}(\mathcal{L}^{\otimes d_-}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\boldsymbol{\xi}_r)) \;=\; \mathbf{L}_r^{(p)}(T^* \otimes \chi)^\iota.$$

Remark 8.2.3. In the equality above, since $d_-(V) = d_+(V^*)$, the right-hand side lies in $\mathbf{K} \otimes \wedge_E^{d_-}\boldsymbol{\mathcal{D}}(V \otimes \chi^{-1})$. On the left,

$$\mathcal{L}^{\otimes d_-}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+} \;:\; \wedge^{d_-}_{\Lambda_r} H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T) \;\longrightarrow\; \mathbf{K} \otimes \wedge^{d_-}_{\mathbf{Q}_p[\Delta_r]} \boldsymbol{\mathcal{D}}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(V)$$

is the map induced by $\mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}$. Recalling that $\epsilon_\chi = \sum \chi(\gamma)\gamma^{-1}$, we also have a map

$$\epsilon_\chi \;:\; \boldsymbol{\mathcal{D}}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(V) \;\longrightarrow\; \boldsymbol{\mathcal{D}}(V \otimes \chi^{-1})$$

from Lemma 8.1.1(ii) (our chosen embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}_p}$ gives an embedding $\mathbf{Q}(\boldsymbol{\mu}_r) \hookrightarrow B_{\mathrm{cris}}$) which induces a map

$$\epsilon_\chi^{\otimes d_-} \;:\; \mathbf{K} \otimes \wedge^{d_-}_{\mathbf{Q}_p[\Delta_r]} \boldsymbol{\mathcal{D}}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(V) \;\longrightarrow\; \mathbf{K} \otimes \wedge_E^{d_-}\boldsymbol{\mathcal{D}}(V \otimes \chi^{-1}).$$

Note that this makes sense even if $d_- = 0$, in which case $\epsilon_\chi^{\otimes 0}$ is the projection from $\mathbf{K} \otimes \mathbf{Q}_p[\Delta_r]$ to $\mathbf{K} \otimes E$ induced by χ. Thus the optimistic equality of 8.2.2 above is an identity between two elements of $\mathbf{K} \otimes \wedge_E^{d_-} \boldsymbol{\mathcal{D}}(V \otimes \chi^{-1})$.

Remark 8.2.4. The statement above is a strengthening and "extrapolation" (by introducing the level r) of the conjectures of Perrin-Riou in §4.4 of [**PR4**]. We have also rephrased the conjecture in terms of $\mathbf{L}_r^{(p)}(T^* \otimes \chi)$ instead of $\mathbf{L}_r^{(p)}(T \otimes \chi^{-1})$ by using the functional equation [**PR4**] §4.3.2, because it simplifies the formulas below.

Remark 8.2.5. One reason that the optimistic statement 8.2.2 cannot be true in general is that it asserts that the p-adic L-functions should all be "integral" in a strong sense. But the L-values can have denominators, coming from $W^{G_{\mathbf{Q}_\infty(\boldsymbol{\mu}_r)^+}}$ where $W = T \otimes (\mathbf{Q}_p/\mathbf{Z}_p)$. Inspired by the theorem of Deligne and Ribet [**DR**], by Stark's conjecture [**T5**] (where this denominator has been extensively studied), and by Perrin-Riou's [**PR4**] Conjecture 4.4.2 (and Lemme 1.3.3), one is led to the following slightly more modest assertion. We optimistically call it a conjecture in the hope that at least some similar statement is true.

Conjecture 8.2.6. *Suppose that $r \in \mathbf{Z}^+$ is prime to p, that $d_- = 1$, and that $\alpha \in \mathbf{Z}_p[[G_\mathbf{Q}]]$ annihilates $W^{G_{\mathbf{Q}_\infty(\boldsymbol{\mu}_r)^+}}$.*

Then there is an element $\boldsymbol{\xi}_r = \boldsymbol{\xi}_r^{(\alpha)} \in H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$ such that for every finite extension E of $\mathbf{Q}_p$ and every character $\chi : \Delta_r \to E^\times$,

$$\epsilon_\chi \mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\boldsymbol{\xi}_r) \;=\; \chi(\alpha)\mathbf{L}_r^{(p)}(T^* \otimes \chi)^\iota,$$

where $\chi(\alpha)$ denotes the image of α under the composition

$$\mathbf{Z}_p[[G_\mathbf{Q}]] \twoheadrightarrow \Lambda_r \cong \Lambda \otimes \mathbf{Z}_p[\Delta_r] \xrightarrow{1\otimes\chi} \Lambda \otimes E \longrightarrow \mathbf{K} \otimes E.$$

Note that if T is unramified at every prime dividing r, then

$$T^{G_{\mathbf{Q}_\infty(\boldsymbol{\mu}_r)}} = T^{G_{\mathbf{Q}_\infty}} \text{ and } W^{G_{\mathbf{Q}_\infty(\boldsymbol{\mu}_r)}} = W^{G_{\mathbf{Q}_\infty}} \tag{8.5}$$

exactly as in Lemma 4.2.5(i), since $\mathrm{Gal}(\mathbf{Q}_\infty(\boldsymbol{\mu}_r)/\mathbf{Q}_\infty)$ is generated by inertia groups which act trivially on $T^{G_{\mathbf{Q}_\infty}}$ and $W^{G_{\mathbf{Q}_\infty}}$.

8.3. Connection with Euler Systems when $d_- = 1$

Suppose that T is as above, that $d_- = 1$, that Conjectures 8.2.1 and 8.2.6 hold, and that the weak Leopoldt conjecture (see [**PR4**] §1.3) holds for T^*. For technical reasons we also assume that $T^{G_{\mathbf{Q}_\infty}} = 0$. Let N be the product of all rational primes where T is ramified.

Fix an element $\alpha \in \mathbf{Z}_p[[G_{\mathbf{Q}}]]$ which annihilates $W^{G_{\mathbf{Q}_\infty}}$. By (8.5), α annihilates $W^{G_{\mathbf{Q}_\infty(\boldsymbol{\mu}_r)^+}}$ for every $r \in \mathbf{Z}^+$ prime to Np. For such r, let

$$\boldsymbol{\xi}_r = \{\xi_{n,r}\} \in H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T), \text{ with } \xi_{n,r} \in H^1(\mathbf{Q}_n(\boldsymbol{\mu}_r)^+, T),$$

be an element satisfying the conclusion of Conjecture 8.2.6.

Proposition 8.3.1. *With hypotheses and notation as above, suppose r is prime to Np and ℓ is a prime not dividing Nrp. Then for every $n \geq 0$ we have*

$$\mathrm{Cor}_{\mathbf{Q}_n(\boldsymbol{\mu}_{r\ell})^+/\mathbf{Q}_n(\boldsymbol{\mu}_r)^+}\xi_{n,r\ell} = P(\mathrm{Fr}_\ell^{-1}|T^*; \mathrm{Fr}_\ell^{-1})\xi_{n,r}$$

where $P(\mathrm{Fr}_\ell^{-1}|T^*; x) = \det(1 - \mathrm{Fr}_\ell^{-1}x|T^*) \in \mathbf{Z}_p[x]$.

Proof. Suppose that E is large enough so that all characters of Δ_r into $\overline{\mathbf{Q}}_p^\times$ take values in $E^\times$, and that $\chi : \Delta_r \to E^\times$ is a (necessarily even) character. Write χ' for the character $\Delta_{r\ell} \twoheadrightarrow \Delta_r \xrightarrow{\chi} E^\times$. Using (8.2) and the definition of $\boldsymbol{\xi}_{r\ell}$ we see that

$$\begin{aligned}\epsilon_\chi \mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\mathrm{Cor}_{\mathbf{Q}(\boldsymbol{\mu}_{r\ell})^+/\mathbf{Q}(\boldsymbol{\mu}_r)^+}\boldsymbol{\xi}_{r\ell}) &= \epsilon_\chi \mathrm{Tr} \circ \mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_{r\ell})^+}(\boldsymbol{\xi}_{r\ell}) \\ &= \epsilon_{\chi'}\mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_{r\ell})^+}(\boldsymbol{\xi}_{r\ell}) = \chi(\alpha)\mathbf{L}^{(p)}_{r\ell}(T^* \otimes \chi)^\iota.\end{aligned}$$

On the other hand, the definition of $\boldsymbol{\xi}_r$ shows that

$$\epsilon_\chi \mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\boldsymbol{\xi}_r) = \chi(\alpha)\mathbf{L}^{(p)}_r(T^* \otimes \chi)^\iota.$$

Replacing T by T^* in (8.4) and applying the involution ι yields

$$\mathbf{L}^{(p)}_{r\ell}(T^* \otimes \chi)^\iota = P(\mathrm{Fr}_\ell^{-1}|T^*; \chi^{-1}(\ell)\mathrm{Fr}_\ell^{-1})\mathbf{L}^{(p)}_r(T^* \otimes \chi)^\iota.$$

Combining these equalities shows that

$$\epsilon_\chi \mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\mathrm{Cor}_{\mathbf{Q}(\boldsymbol{\mu}_{r\ell})^+/\mathbf{Q}(\boldsymbol{\mu}_r)^+}\boldsymbol{\xi}_{r\ell}) = \epsilon_\chi \mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(P(\mathrm{Fr}_\ell^{-1}|T^*; \mathrm{Fr}_\ell^{-1})\boldsymbol{\xi}_r)$$

for every χ, and therefore since $\sum_\chi \epsilon_\chi = [\mathbf{Q}(\boldsymbol{\mu}_r)^+ : \mathbf{Q}] \in E[\Delta_r]^\times$, we conclude that

$$\mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\mathrm{Cor}_{\mathbf{Q}(\boldsymbol{\mu}_{r\ell})^+/\mathbf{Q}(\boldsymbol{\mu}_r)^+}\boldsymbol{\xi}_{r\ell}) = \mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(P(\mathrm{Fr}_\ell^{-1}|T^*; \mathrm{Fr}_\ell^{-1})\boldsymbol{\xi}_r).$$

It remains only to show that, under our hypotheses, $\mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}$ is injective. Recall that $\mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}$ is the composition

$$H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T) \longrightarrow \bigoplus_{v|p} \varprojlim_n H^1(\mathbf{Q}_n(\boldsymbol{\mu}_r)^+_v, T) \longrightarrow \mathbf{K} \otimes \boldsymbol{\mathcal{D}}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(V). \tag{8.6}$$

The weak Leopoldt conjecture, which we have assumed, implies that ([**PR4**] (1.4.2) and Corollary B.3.6) the restriction map

$$H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T) \longrightarrow \bigoplus_{q|Np} \varprojlim_n \oplus_{v|q} H^1(\mathbf{Q}_n(\boldsymbol{\mu}_r)^+_v, T)$$

is injective. By Proposition A.2.3 of [**PR4**], $\varprojlim \oplus_{v|q} H^1(\mathbf{Q}_n(\boldsymbol{\mu}_r)_v^+, T)$ is a torsion Λ-module if $q \neq p$. Therefore the kernel of the first map of (8.6) is a torsion Λ-module, and the definition of the second map ([**PR4**] §1.2.5) shows that its kernel is torsion as well. But by [**PR4**] Lemme 1.3.3, the Λ-torsion submodule of $H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$ is $T^{G_{\mathbf{Q}_\infty(\boldsymbol{\mu}_r)^+}}$, which is $T^{G_{\mathbf{Q}_\infty}}$ by (8.5), and by our hypothesis this is zero. Thus $\mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}$ is injective and the proposition follows. □

Corollary 8.3.2. *With notation as above, the collection*

$$\{\xi_{n,r} \in H^1(\mathbf{Q}_n(\boldsymbol{\mu}_r)^+, T) : n \geq 0 \text{ and } r \text{ prime to } Np\}$$

defines an Euler system for $(T, \mathbf{Q}_\infty(\boldsymbol{\mu}')^+, Np)$ *in the sense of Definition* 2.1.1 *and Remark* 2.1.3*, where* $\boldsymbol{\mu}'$ *is the group of all roots of unity of order prime to* Np*.*

Proof. This is immediate from Proposition 8.3.1. □

Remark 8.3.3. There is another way to think about the existence of Euler systems when $d_- = 1$, in terms of complex L-functions. Namely, the Euler product (8.3) for $L(V^*, s)$ converges (conjecturally), and hence is nonzero, if s is a sufficiently large positive integer. This allows us to read off the value of $\mathrm{ord}_{s=-k} L(V, s)$ for large positive integers k in terms of the Γ-factors in the functional equation relating $L(V, s)$ and $L(V^*, s)$. Subject to standard conjectures, one can show in this way that

$$\mathrm{ord}_{s=0} L(V \otimes \langle\varepsilon\rangle^{-k}\rho, s) \;=\; d_-$$

for all sufficiently large positive integers k and all characters ρ of finite order of $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$.

Fix one such k. The Beilinson and Bloch-Kato conjectures then predict that the leading term in the Taylor expansion of $L(V \otimes \langle\varepsilon\rangle^{-k}\rho, s)$ at 0 can be expressed in terms of, among other things, a $d_- \times d_-$ regulator. When $d_- = 1$, this predicts the existence of certain special elements, and one can hope that these elements produce an Euler system for $T \otimes \langle\varepsilon\rangle^{-k}$.

By Theorem 6.3.5, an Euler system for $T \otimes \langle\varepsilon\rangle^{-k}$ can then be twisted to produce an Euler system for T.

Remark 8.3.4. In the next section we consider the example $T = \mathbf{Z}_p(1)$, which has $d = d^- = 1$. Other interesting examples are when T is the Tate module of a modular elliptic curve (as in §3.5), which has $d^- = d^+ = 1$, or when T is the symmetric square of the Tate module of an elliptic curve (as in §3.6), which has $d^- = 1$ and $d^+ = 2$.

8.4. Example: Cyclotomic Units

In this section we discuss the example $T = \mathbf{Z}_p(1)$. The results in this section were all worked out by Perrin-Riou in [**PR3**]. The main ideas of the computation go back to Iwasawa.

We suppose for this section that $p > 2$. We will show that Conjecture 8.2.6 is true for $\mathbf{Z}_p(1)$, and in the process we will show that the Euler system of cyclotomic units discussed in §3.2 arises in the way described in the previous section. Note that $d_-(\mathbf{Q}_p(1)) = d(\mathbf{Q}_p(1)) = 1$ and $d_+(\mathbf{Q}_p(1)) = 0$.

For every $r \in \mathbf{Z}^+$ prime to p, and $n \geq 0$, let

$$\tilde{\mathbf{c}}_{p^n r} = \mathbf{N}_{\mathbf{Q}(\boldsymbol{\mu}_{rp^{n+1}})/\mathbf{Q}_n(\boldsymbol{\mu}_r)^+}(\zeta_{rp^{n+1}} - 1) \in (\mathbf{Q}_n(\boldsymbol{\mu}_r)^+)^\times \subset H^1(\mathbf{Q}_n(\boldsymbol{\mu}_r)^+, \mathbf{Z}_p(1)),$$

the Euler system of §3.2, and

$$\tilde{\mathbf{c}}_{r,\infty} = \{\tilde{\mathbf{c}}_{p^n r}\}_n \in H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, \mathbf{Z}_p(1)).$$

Let

$$u_r(X) = \zeta_r(1+X)^{r^{-1}} - 1 \in (\mathbf{Z}[\boldsymbol{\mu}_r] \otimes \mathbf{Z}_p)[[X]]$$

and

$$h_r(X) = \prod_{\beta \in \boldsymbol{\mu}_{p-1} \subset \mathbf{Z}_p^\times} u_r((1+X)^\beta - 1)\bar{u}_r((1+X)^\beta - 1)$$

where $\bar{u}_r(X) = \zeta_r^{-1}(1+X)^{r^{-1}} - 1$. Then h_r is the "Coleman power series" attached to $\tilde{\mathbf{c}}_{r,\infty}$, i.e., for every $n \geq 0$,

$$h_r^{\mathrm{Fr}_p^{-n-1}}(\zeta_{p^{n+1}} - 1) = \tilde{\mathbf{c}}_{p^n r}.$$

The p-adic L-functions $\mathbf{L}_r^{(p)}(\mathbf{Z}_p \otimes \chi)$ that arise below are the Kubota-Leopoldt p-adic L-functions, so their existence does not rely on any conjectures. The following proposition is essentially due to Iwasawa and Coleman, but we present it in the language of Perrin-Riou, following [**PR3**].

Proposition 8.4.1. *Suppose $r \geq 1$ and E is a finite extension of $\mathbf{Q}_p$. For every character $\chi : \Delta_r \to E^\times$, we have*

$$\epsilon_\chi \mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\tilde{\mathbf{c}}_{r,\infty}) = 2\mathbf{L}_r^{(p)}(\mathbf{Z}_p \otimes \chi)^\iota.$$

Proof. Suppose first that $r > 1$. By [**PR3**] §1.8, §3.1 (or [**PR2**] §4.1.3) and [**Iw2**],

$$\mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\tilde{\mathbf{c}}_{r,\infty}) \in \Lambda \otimes \boldsymbol{\mathcal{D}}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\mathbf{Q}_p(1)) = \mathbf{Q}(\boldsymbol{\mu}_r)^+ \otimes \Lambda \otimes \boldsymbol{\mathcal{D}}(\mathbf{Q}_p(1)), \quad (8.7)$$

$$\mathbf{L}_r^{(p)}(\mathbf{Z}_p \otimes \chi) \in \Lambda \otimes \boldsymbol{\mathcal{D}}(\mathbf{Q}_p(1) \otimes \chi^{-1}) = \epsilon_\chi(\Lambda \otimes \boldsymbol{\mathcal{D}}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\mathbf{Q}_p(1))), \quad (8.8)$$

where the equalities use Lemma 8.1.1. Let e_{-1} denote the canonical generator of the one-dimensional vector space $\mathcal{D}(\mathbf{Q}_p(1))$, and define

$$\mathcal{H}_r(X) \;=\; \log h_r(X) - \frac{1}{p}\log h_r^{\mathrm{Fr}_p}((1+X)^p - 1).$$

From the definition in [**PR4**] §1.2.5 (see also [**PR3**] §1.3 and §3.1.4), we see that

$$\mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\tilde{\mathbf{c}}_{r,\infty}) \;=\; \mathcal{F}_r e_{-1}$$

where $\mathcal{F}_r \in \mathbf{Q}(\boldsymbol{\mu}_r)^+ \otimes \Lambda$ is such that for every $k \geq 1$,

$$\langle\varepsilon\rangle^k(\mathcal{F}_r) \;=\; (D^k\mathcal{H}_r)(\zeta_p - 1)$$

where D is the derivation $(1+X)\frac{d}{dX}$. Thus if $\chi : \Delta_r \to E^\times$ is a character, then

$$\epsilon_\chi\mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\tilde{\mathbf{c}}_{r,\infty}) \;=\; \mathcal{F}_{r,\chi}e_{-1}$$

where $\mathcal{F}_{r,\chi} \in \mathbf{Q}(\boldsymbol{\mu}_r)^+ \otimes \Lambda \otimes E$ is such that for every $k \geq 1$,

$$\langle\varepsilon\rangle^k(\mathcal{F}_{r,\chi}) \;=\; \sum_{\gamma\in\Delta_r} \chi^{-1}(\gamma)(D^k\mathcal{H}_r^\gamma)(\zeta_p - 1).$$

Therefore by Lemma D.2.2, if $k \geq 2$ we have

$$\langle\varepsilon\rangle^k(\mathcal{F}_{r,\chi}) = 2\Gamma(k)(-2\pi i)^{-k}L(\chi^{-1}\omega^k, k) \times \begin{cases} -\chi(p)p^k & \text{if } (p-1)\nmid k, \\ 1 - p^{k-1}\chi(p) & \text{if } (p-1)\mid k, \end{cases}$$

so by the formulas in [**PR4**] §4.2 and §4.3.3 we see that for $k \geq 2$,

$$\begin{aligned}\langle\varepsilon\rangle^k(\epsilon_\chi\mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\tilde{\mathbf{c}}_{r,\infty})) \;&=\; \langle\varepsilon\rangle^k(\mathcal{F}_{r,\chi}e_{-1}) \\ &=\; 2\langle\varepsilon\rangle^{-k}(\mathbf{L}_r^{(p)}(\mathbf{Z}_p\otimes\chi)) \;=\; 2\langle\varepsilon\rangle^k(\mathbf{L}_r^{(p)}(\mathbf{Z}_p\otimes\chi)^\iota)\end{aligned}$$

(the Gauss sums which appear in the formulas of [**PR4**] and [**PR3**] are not present here because we never identified $\mathbf{Q}[\mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_r)/\mathbf{Q})]$ with $\mathbf{Q}(\boldsymbol{\mu}_r)$ as in [**PR3**] §1.8). By (8.7) and (8.8), these equalities suffice to prove the proposition when $r > 1$. A similar computation shows that for every $\sigma \in G_{\mathbf{Q}}$,

$$\mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}((\sigma-1)\tilde{\mathbf{c}}_{1,\infty}) \;=\; 2(\sigma-1)\mathbf{L}_1^{(p)}(\mathbf{Z}_p)^\iota. \qquad \square$$

Corollary 8.4.2. *Conjecture* 8.2.6 *holds for* $T = \mathbf{Z}_p(1)$.

Proof. We have assumed that $p > 2$. Therefore $\boldsymbol{\mu}_{p^\infty}^{G_{\mathbf{Q}_\infty}} = \{1\}$, and for every $\alpha \in \mathbf{Z}_p[[G_{\mathbf{Q}}]]$, Proposition 8.4.1 shows that

$$\boldsymbol{\xi}_r \;=\; \frac{1}{2}\alpha\tilde{\mathbf{c}}_{r,\infty} \;\in\; H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$$

satisfies Conjecture 8.2.6. $\square$

8.5. Connection with Euler Systems when $d_- > 1$

Suppose now that T is such that d_- is greater than 1, and suppose that some version of the assertion 8.2.2 is true, i.e., suppose there is an integer N divisible by all primes where T is ramified, and an element $\alpha \in \mathbf{Z}_p[[G_{\mathbf{Q}}]]$ such that for every integer r prime to Np, there is an element

$$\boldsymbol{\xi}_r \in \wedge^{d_-}_{\Lambda_r} H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$$

satisfying

$$\epsilon_\chi^{\otimes d_-} \mathcal{L}^{\otimes d_-}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\boldsymbol{\xi}_r) = \chi(\alpha)\mathbf{L}^{(p)}_r(T^* \otimes \chi)^\iota$$

for every character χ of Δ_r. We also suppose again that the weak Leopoldt conjecture holds for T^*. In this section we will adapt an idea from [**Ru8**] §6 to construct Euler systems (elements in $H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$) from the elements $\boldsymbol{\xi}_r \in \wedge^{d_-}_{\Lambda_r} H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$.

Lemma 8.5.1. *With hypotheses and notation as above, suppose r is prime to Np and ℓ is a prime not dividing Nrp. Then*

$$\mathrm{Cor}^{\otimes d_-}_{\mathbf{Q}(\boldsymbol{\mu}_{r\ell})^+/\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\boldsymbol{\xi}_{r\ell}) - P(\mathrm{Fr}_\ell^{-1}|T^*; \mathrm{Fr}_\ell^{-1})(\boldsymbol{\xi}_r)$$

belongs to the Λ-torsion submodule of $\wedge^{d_-}_{\Lambda_r} H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$, where

$$P(\mathrm{Fr}_\ell^{-1}|T^*; x) = \det(1 - \mathrm{Fr}_\ell^{-1} x | T^*) \in \mathbf{Z}_p[x]$$

and

$$\mathrm{Cor}^{\otimes d_-}_{\mathbf{Q}(\boldsymbol{\mu}_{r\ell})^+/\mathbf{Q}(\boldsymbol{\mu}_r)^+} : \wedge^{d_-} H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_{r\ell})^+, T) \longrightarrow \wedge^{d_-} H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$$

is the map induced by corestriction.

Proof. Exactly as in the proof of Proposition 8.3.1, we deduce that

$$\mathcal{L}^{\otimes d_-}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}\big(\mathrm{Cor}^{\otimes d_-}_{\mathbf{Q}(\boldsymbol{\mu}_{r\ell})^+/\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\boldsymbol{\xi}_{r\ell})\big) = \mathcal{L}^{\otimes d_-}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}\big(P(\mathrm{Fr}_\ell^{-1}|T^*; \mathrm{Fr}_\ell^{-1})(\boldsymbol{\xi}_r)\big).$$

Also as in the proof of Proposition 8.3.1, the kernel of $\mathcal{L}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}$ is a torsion Λ-module, and so the kernel of $\mathcal{L}^{\otimes d_-}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+}$ is torsion as well. □

Suppose that r is prime to Np, and $\varphi \in \mathrm{Hom}_{\Lambda_r}(H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T), \Lambda_r)$. Then φ induces a Λ_r-module homomorphism from $\wedge^k_{\Lambda_r} H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$ to $\wedge^{k-1}_{\Lambda_r} H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$ for all $k \geq 1$ by the usual formula

$$c_1 \wedge \cdots \wedge c_k \mapsto \sum_{i=1}^{k} (-1)^{i+1} \varphi(c_i) c_1 \wedge \cdots \wedge c_{i-1} \wedge c_{i+1} \cdots \wedge c_k.$$

Iterating this construction $d_- - 1$ times gives a map

$$\wedge_{\Lambda_r}^{d_- - 1} \mathrm{Hom}_{\Lambda_r}(H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T), \Lambda_r) \longrightarrow \mathrm{Hom}(\wedge_{\Lambda_r}^{d_-} H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T), H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)). \quad (8.9)$$

If $s \mid r$ then there is a natural map

$$\mathbf{N}_{r/s} : \mathrm{Hom}_{\Lambda_r}(H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T), \Lambda_r) \longrightarrow \mathrm{Hom}_{\Lambda_s}(H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_s)^+, T), \Lambda_s)$$

induced by restriction $H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_s)^+, T) \to H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$ and the identification $\Lambda_s \cong \Lambda_r^{\mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_r)^+/\mathbf{Q}(\boldsymbol{\mu}_s)^+)}$.

Proposition 8.5.2. *With notation as above, suppose that $T^{G_{\mathbf{Q}_\infty}} = 0$ and that*

$$\mathfrak{S} = \{\mathfrak{S}_r\} \in \varprojlim_r \wedge_{\Lambda_r}^{d_- - 1} \mathrm{Hom}_{\Lambda_r}(H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T), \Lambda_r),$$

inverse limit with respect to the maps $\mathbf{N}_{r/s}^{\otimes d_- - 1}$. If r is prime to Np then $\mathfrak{S}_r(\boldsymbol{\xi}_r) \in H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$, and if further ℓ is a prime not dividing Nrp then

$$\mathrm{Cor}_{\mathbf{Q}(\boldsymbol{\mu}_{r\ell})^+/\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\mathfrak{S}_{r\ell}(\boldsymbol{\xi}_{r\ell})) = P(\mathrm{Fr}_\ell^{-1} | T^*; \mathrm{Fr}_\ell^{-1})(\mathfrak{S}_r(\boldsymbol{\xi}_r)).$$

In other words, if we write $\mathfrak{S}_r(\boldsymbol{\xi}_r) = \{\xi_{n,r}\}_n$ then the collection

$$\{\xi_{n,r} \in H^1(\mathbf{Q}_n(\boldsymbol{\mu}_r)^+, T)\}$$

is an Euler system for T (Definition 2.1.1 and Remark 2.1.3).

Proof. The proof is identical to that of Proposition 6.2 and Corollary 6.3 of [**Ru8**]. Using (8.9) it is clear that $\mathfrak{S}_r(\boldsymbol{\xi}_r) \in H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$. Suppose that $s \mid r$. If $\phi \in \mathrm{Hom}_{\Lambda_r}(H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T), \Lambda_r)$ and $c \in H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$, then $\phi(c) \in \Lambda_r$, and its image under the restriction map $\Lambda_r \to \Lambda_s$ is $(\mathbf{N}_{r/s}\phi)(\mathrm{Cor}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+/\mathbf{Q}(\boldsymbol{\mu}_s)^+}(c))$. From this it follows without difficulty that

$$\mathrm{Cor}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+/\mathbf{Q}(\boldsymbol{\mu}_s)^+}(\mathfrak{S}_r(\boldsymbol{\xi}_r)) = \mathfrak{S}_s\big(\mathrm{Cor}^{\otimes d_-}_{\mathbf{Q}(\boldsymbol{\mu}_r)^+/\mathbf{Q}(\boldsymbol{\mu}_s)^+}(\boldsymbol{\xi}_s)\big).$$

Combining this equality with Lemma 8.5.1 shows that

$$\mathrm{Cor}_{\mathbf{Q}(\boldsymbol{\mu}_{r\ell})^+/\mathbf{Q}(\boldsymbol{\mu}_r)^+}(\mathfrak{S}_{r\ell}(\boldsymbol{\xi}_{r\ell})) - P(\mathrm{Fr}_\ell^{-1} | T^*; \mathrm{Fr}_\ell^{-1})\mathfrak{S}_r(\boldsymbol{\xi}_r)$$

belongs to the Λ-torsion submodule of $H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T)$. But by [**PR4**] (Lemme 1.3.3) and (8.5) this torsion submodule is $T^{G_{\mathbf{Q}_\infty}}$, which we have assumed to be zero. □

Remark 8.5.3. Of course, Proposition 8.5.2 is only useful if we know something about the size of $\varprojlim \wedge_{\Lambda_r}^{d_- - 1} \mathrm{Hom}_{\Lambda_r}(H^1_\infty(\mathbf{Q}(\boldsymbol{\mu}_r)^+, T), \Lambda_r)$, and in particular that it is nonzero. See [**Ru8**] §6 for an example.

CHAPTER 9

Variants

In this chapter we discuss several alternatives and extensions to the definition of Euler systems we gave in Chapter 2.

9.1. Rigidity

It is tempting to remove from the definition of an Euler system the requirement that the field $\mathcal{K}$ (over whose subfields the Euler system classes are defined) contain a $\mathbf{Z}_p$-extension of K. After all, the proofs of the Theorems of §2.2 only use the derivative classes $\kappa_{[K,\mathfrak{r},M]}$ and not the $\kappa_{[F,\mathfrak{r},M]}$ for larger extensions F of K in K_∞. However, our proofs of the properties of the derivative classes $\kappa_{[K,\mathfrak{r},M]}$ very much used the fact that the Euler system class $\mathbf{c}_{K(\mathfrak{r})}$ is a "universal norm" in the extension $K_\infty(\mathfrak{r})/K(\mathfrak{r})$.

In fact, some such assumption is needed, as the following example shows. Suppose K has class number one, $\mathcal{N}$ is an ideal of K divisible by p and all primes where T is ramified, and T has the property that $P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;1) = 0$ for every $\mathfrak{q}$ not dividing $\mathcal{N}$. (For example, if T is the symmetric square of the Tate module of an elliptic curve as in §3.6 then T has this property.) Suppose further that $\mathcal{K}$ is the maximal abelian extension of K unramified outside $\mathcal{N}$ (so $\mathcal{K}$ does not contain a $\mathbf{Z}_p$-extension of K) and $\mathbf{c}$ satisfies the distribution relation in Definition 2.1.1 of an Euler system for $(T,\mathcal{K},\mathcal{N})$. Then in Definition 2.1.1, the only equations connecting $\mathbf{c}_K$ with the other $\mathbf{c}_F$ are of the form

$$\mathrm{Cor}_{F/K}\mathbf{c}_F = \prod_{\mathfrak{q}\in\Sigma(F/K)} P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;\mathrm{Fr}_\mathfrak{q}^{-1})\mathbf{c}_K = \prod_{\mathfrak{q}\in\Sigma(F/K)} P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;1)\mathbf{c}_K.$$

If $F \neq K$ then the set $\Sigma(F/K)$ of primes ramifying in F/K is nonempty, so the right-hand side will always be zero. In other words $\mathbf{c}_K$ does not appear in any nontrivial Euler system relations, so *we can replace* $\mathbf{c}_K$ *by any element at all in* $H^1(K,T)$ without disturbing the distribution relation of Definition 2.1.1. For example, the collection defined by $\mathbf{c}_F = 0$ for $F \neq K$, with $\mathbf{c}_K$ arbitrary, satisfies those relations. Since there are examples satisfying the conditions above with non-trivial Selmer groups, in this situation

one cannot hope for a theorem like Theorem 2.2.2 (or Theorem 4.5.4), in which the conclusion depends in an essential way on $\mathbf{c}_K$.

However, there are other possible ways to ensure the "rigidity" of an Euler system. Let $\mathcal{R}$ be the set of squarefree ideals of K prime to $\mathcal{N}$, as in Chapter 4. In Definition 2.1.1, we can replace condition (ii) by

(ii′) at least one of the conditions (a), (b), (c) below is satisfied:
- (a) $\mathcal{K}$ contains a $\mathbf{Z}_p^d$-extension of K in which no finite prime splits completely,
- (b) $\mathbf{c}_{K(\mathfrak{r})} \in \mathcal{S}^{\Sigma_p}(K(\mathfrak{r}), T)$ for every $\mathfrak{r} \in \mathcal{R}$, and there is a $\gamma \in G_K$ such that $T^{\gamma=1} = 0$ and $\gamma = 1$ on $K(\mathbf{1})K(\boldsymbol{\mu}_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$,
- (c) $\mathbf{c}_{K(\mathfrak{r})} \in \mathcal{S}^{\Sigma_p}(K(\mathfrak{r}), T)$ for every $\mathfrak{r} \in \mathcal{R}$, the collection $\{\mathbf{c}_{K(\mathfrak{r})}\}$ satisfies the congruence of Corollary 4.8.1, and for every prime $\mathfrak{q} \in \mathcal{R}$ and every power n of p, $\mathrm{Fr}_\mathfrak{q}^n - 1$ is injective on T.

Condition (ii′)(a) is condition (ii) of the original definition.

Under this more general definition, Theorems 2.2.2, 2.2.3, and 2.2.10 all hold, with the same conclusions as before, under the additional mild assumption that $T^{G_{K(\mathbf{1})}} = 0$. We indicate very briefly how to adapt the proofs in Chapters 4 and 5 to cover this expanded definition.

The idea is that there is a power m of p, independent of M, such that one can still construct the derivative classes $\kappa_{[K,\mathfrak{r},M]}$, and prove the local properties of §4.5, under the assumption $\mathfrak{r} \in \mathcal{R}_{K,Mm}$ rather than $\mathfrak{r} \in \mathcal{R}_{K,M}$. This additional assumption does not interfere with the proofs of the theorems of Chapter 2.

Construction of the derivative classes. Since we assumed $T^{G_{K(\mathbf{1})}} = 0$, Lemma 4.2.5(i) shows that $T^{G_{K(\mathfrak{r})}} = 0$ for every $\mathfrak{r}$. Thus if we replace $\mathbb{W}_M$ by $\mathbb{T} = \mathrm{Maps}(G_K, T)$ in Proposition 4.4.5 we get a short exact sequence

$$0 \longrightarrow \mathbb{T}^{G_{F(\mathfrak{r})}} \longrightarrow (\mathbb{T}/T)^{G_{F(\mathfrak{r})}} \xrightarrow{\delta_{F(\mathfrak{r})}} H^1(F(\mathfrak{r}), T) \longrightarrow 0.$$

Now as in Proposition 4.4.8, but using the exact sequence above instead of Proposition 4.4.7, we can find a map $\mathbf{d} : \mathbf{X}_{F(\mathfrak{r})} \to (\mathbb{T}/T)^{G_{F(\mathfrak{r})}}$ lifting $\mathbf{c}$. Projecting this map to $(\mathbb{W}_M/W_M)^{G_{F(\mathfrak{r})}}$ we can proceed exactly as in Definition 4.4.10 to define $\kappa_{[F,\mathfrak{r},M]}$.

Analogue of Theorem 4.5.1. We will use Corollary 4.6.5 instead of Theorem 4.5.1. Corollary 4.6.5 follows directly from Proposition 4.6.1, which is included as part of (ii′)(b) and (ii′)(c). (In §4.6 we used assumption (ii′)(a) and Corollary B.3.5 to prove Proposition 4.6.1.)

Analogue of Theorem 4.5.4. Theorem 4.5.4 follows directly from Lemma 4.7.3, so it will suffice to prove a form of that lemma. Suppose first that (ii′)(b) holds with an element $\gamma \in G_K$. Fix $\mathfrak{r}\mathfrak{q} \in \mathcal{R}$, a power M of p,

and a power M' of p divisible by $MP(\gamma|T;1)$. By the definition of γ, we have $P(\gamma|T;1) \neq 0$. Let $n = |\boldsymbol{\mu}_{p^\infty} \cap K|$. Choose a prime $\mathfrak{l}$ of K such that

(a) $\mathrm{Fr}_\mathfrak{l} = \gamma$ on $K(\mathbf{1})K(W_{M'}, \boldsymbol{\mu}_{nM'}, (\mathcal{O}_K^\times)^{1/(nM')})$,
(b) $\mathrm{Fr}_\mathfrak{l} = 1$ on $K(\mathfrak{r}\mathfrak{q})$,
(c) $\mathrm{Fr}_\mathfrak{l} \neq 1$ on $K(\lambda^{1/(np)})$ where $\lambda\mathcal{O}_K = \mathfrak{q}^h$ with h equal to the order of $\mathfrak{q}$ in the ideal class group of K.

(Exercise: show that these conditions can be satisfied simultaneously.) One can imitate the proof of Lemma 4.7.3 by using the extensions $K(\mathfrak{l})/K$ in place of the finite extensions of K in K_∞. Condition (a) and the definition of γ ensure that $nM' \mid [K(\mathfrak{l}) : K(\mathbf{1})]$. Condition (c) ensures that the decomposition group of $\mathfrak{q}$ has index dividing n in $\mathrm{Gal}(K(\mathfrak{l})/K)$, and therefore has order at least M'. The key point is that although $\mathbf{c}_{K(\mathfrak{r})}$ and $\mathbf{c}_{K(\mathfrak{r}\mathfrak{q})}$ are not "universal norms" from $K(\mathfrak{r}\mathfrak{l})$ and $K(\mathfrak{r}\mathfrak{q}\mathfrak{l})$ (as they would be from $K_\infty(\mathfrak{r})$ and $K_\infty(\mathfrak{r}\mathfrak{q})$), the Euler system distribution relation shows that $P(\mathrm{Fr}_\mathfrak{l}^{-1}|T^*;\mathrm{Fr}_\mathfrak{l}^{-1})\mathbf{c}_{K(\mathfrak{r})}$ is a norm from $K(\mathfrak{r}\mathfrak{l})$ and similarly with $\mathfrak{r}$ replaced by $\mathfrak{r}\mathfrak{q}$. Conditions (a) and (b) imply that in $\mathcal{O}[\mathrm{Gal}(K(\mathfrak{r}\mathfrak{q})/K)]$,

$$\begin{aligned} P(\mathrm{Fr}_\mathfrak{l}^{-1}|T^*;\mathrm{Fr}_\mathfrak{l}^{-1}) \;=\; P(\mathrm{Fr}_\mathfrak{l}^{-1}|T^*;1) &\equiv P(\gamma^{-1}|T^*;1) \pmod{M} \\ &= P(\gamma|T;1). \end{aligned}$$

Now imitating the proof of Lemma 4.7.3 one can show that, with notation as in the statement of that lemma, if $\mathfrak{r}\mathfrak{q} \in \mathcal{R}_{K,M'}$ then

$$P(\gamma|T;1)(N_\mathfrak{q}\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{r}\mathfrak{q})}) - P(\mathrm{Fr}_\mathfrak{q}^{-1}|T^*;\mathrm{Fr}_\mathfrak{q}^{-1})\gamma\hat{\mathbf{d}}(x_{F(\mathfrak{r})})) \;=\; 0 \;\in\; W_{M'}$$

This suffices to prove that $\kappa_{[F,\mathfrak{r},M]}$ and $\kappa_{[F,\mathfrak{r}\mathfrak{q},M]}$ satisfy the equality of Theorem 4.5.4.

Now suppose (ii′)(c) holds. In §4.8 we used Lemma 4.7.3 to prove the congruence of Corollary 4.8.1. Under the assumptions (ii′)(c) we can just reverse the argument to prove Lemma 4.7.3, and then Theorem 4.5.4.

Example 9.1.1 (cyclotomic units revisited)**.** With this expanded definition, we can redefine the cyclotomic unit Euler system of §3.2. Namely, for every $m > 1$ prime to p define

$$\tilde{\mathbf{c}}_m \;=\; (\zeta_m - 1)(\zeta_m^{-1} - 1) \;\in\; (\mathbf{Q}(\boldsymbol{\mu}_m)^+)^\times \;\subset\; H^1(\mathbf{Q}(\boldsymbol{\mu}_m)^+, \mathbf{Z}_p(1))$$

and set $\tilde{\mathbf{c}}_1 = 1$. This collection is not an Euler system, even under our expanded Definition 2.1.1. (If it were, then for every prime $\ell \neq p$, the class $\tilde{\mathbf{c}}_{\mathbf{Q}(\boldsymbol{\mu}_\ell)}$ would belong to $\mathcal{S}^{\Sigma_p}(\mathbf{Q}(\boldsymbol{\mu}_\ell), \mathbf{Z}_p(1))$, but this is not the case because $\tilde{\mathbf{c}}_{\mathbf{Q}(\boldsymbol{\mu}_\ell)} \notin H^1_f(\mathbf{Q}(\boldsymbol{\mu}_\ell)_\ell, \mathbf{Z}_p(1))$.)

However, suppose $\chi : G_\mathbf{Q} \to \mathcal{O}^\times$ is a nontrivial character of finite order, and its conductor f is prime to p. Then we can twist $\tilde{\mathbf{c}}$ by χ^{-1} as in Definition 2.4.1, and with the modified definition above, the collection

$\mathbf{c} = \tilde{\mathbf{c}}^{\chi^{-1}}$ *is* an Euler system for $(\mathbf{Z}_p(1) \otimes \chi^{-1}, \mathbf{Q}^{\mathrm{ab},p}, fp)$, where $\mathbf{Q}^{\mathrm{ab},p}$ is the maximal abelian extension of $\mathbf{Q}$ unramified outside p. Namely, although condition (ii′)(a) does not hold, (ii′)(b) (with $\gamma \in G_{\mathbf{Q}(\boldsymbol{\mu}_{p^\infty})}$ and $\chi(\gamma) \neq 1$) and (ii′)(c) (see Example 4.8.2) both do hold. With this Euler system we can remove one of the hypotheses from Theorem 3.2.3 and Corollary 3.2.4. With notation as in §3.2 (so L is the field cut out by χ), we have the following theorem.

Theorem 9.1.2. *Suppose $p > 2$ and χ is a nontrivial even character of conductor prime to p. Then*

$$|A_L^\chi| \;=\; [\mathcal{E}_L^\chi : \mathcal{C}_{L,\chi}].$$

Sketch of proof. If $\chi(p) \neq 1$ this is Corollary 3.2.4. So we may assume that the conductor of χ is prime to p and use the Euler system constructed above. For this Euler system, $\mathbf{c}_1$ generates $\mathcal{C}_{L,\chi}$, so exactly as in the proof of Theorem 3.2.3 we deduce from Theorem 2.2.2 that

$$|\mathcal{S}_{\Sigma_p}(\mathbf{Q}, (\mathbf{Q}_p/\mathbf{Z}_p) \otimes \chi)| \;\text{ divides }\; [\mathcal{E}_L^\chi : \mathcal{C}_{L,\chi}].$$

Unfortunately, Proposition 1.6.1 shows that

$$\mathcal{S}_{\Sigma_p}(\mathbf{Q}, (\mathbf{Q}_p/\mathbf{Z}_p) \otimes \chi) \;=\; \mathrm{Hom}(A_L^\chi/P, \mathbf{D})$$

where P is the subgroup of A_L^χ generated by the classes of primes of L above p. This is not quite what we need.

To complete the proof, we observe that the derivative classes $\kappa_{[K,r,M]}$ attached to our Euler system all lie in $\mathcal{S}^{\Sigma_r}(\mathbf{Q}, \boldsymbol{\mu}_M \otimes \chi^{-1})$, not just in $\mathcal{S}^{\Sigma_{rp}}(\mathbf{Q}, \boldsymbol{\mu}_M \otimes \chi^{-1})$ as Theorem 4.5.1 shows in the general case. (This follows from the fact that $\mathbf{c}_{\mathbf{Q}(\boldsymbol{\mu}_r)} \in \mathcal{S}(\mathbf{Q}(\boldsymbol{\mu}_r), \mathbf{Z}_p(1) \otimes \chi^{-1})$ for every r. See for example [**Ru3**] Proposition 2.4.) Therefore we can repeat the proof of Theorem 2.2.2, but using $\Sigma_0 = \emptyset$ and $\Sigma = \Sigma_r$ in Theorem 1.7.3 instead of $\Sigma_0 = \{p\}$ and $\Sigma = \Sigma_{rp}$, to conclude that

$$|A_L^\chi| \;=\; |\mathcal{S}(\mathbf{Q}, (\mathbf{Q}_p/\mathbf{Z}_p) \otimes \chi)| \;\text{ divides }\; [\mathcal{E}_L^\chi : \mathcal{C}_{L,\chi}].$$

Now the equality of the theorem follows from the analytic class number formula exactly as in Corollary 3.2.4. □

9.2. Finite Primes Splitting Completely in K_∞/K

Definition 2.1.1 of an Euler system requires a $\mathbf{Z}_p^d$-extension K_∞/K, with $K_\infty \subset \mathcal{K}$, such that no finite prime splits completely in K_∞/K.

In fact, the assumption that no prime splits completely is unnecessarily strong. We can remove this hypothesis if we assume instead that

(∗) for every prime $\mathfrak{q}$ of K which splits completely in K_∞/K, and for every finite extension F of K in $\mathcal{K}$, we have $(\mathbf{c}_F)_\mathfrak{q} \in H^1_{\mathrm{ur}}(F_\mathfrak{q}, T)$.

The set of primes which split completely in K_∞/K has density zero, so such primes do not interfere with our Tchebotarev arguments. Using this fact and (∗), the proofs in Chapters 4 through 7 work without significant modification in this more general setting.

9.3. Euler Systems of Finite Depth

Definition 9.3.1. Fix a nonzero $M \in \mathcal{O}$. An *Euler system for* W_M (or an Euler system of depth M) is a collection of cohomology classes satisfying all the properties of Definition 2.1.1 except that instead of $\mathbf{c}_F \in H^1(F,T)$ we require $\mathbf{c}_F \in H^1(F,W_M)$. An Euler system in the sense of Definition 2.1.1 can be viewed as an Euler system of infinite depth.

Remark 9.3.2. For this definition we could replace W_M by a free $\mathcal{O}/M\mathcal{O}$-module of finite rank with an action of G_K; it is not necessary that it can be written as T/MT for some T.

The construction of the derivative classes $\kappa_{[F,\mathfrak{r},M]}$ in Chapter 4 only used the images of the classes $\mathbf{c}_L$ (for various L) in $H^1(L,W_M)$. Thus if $\mathbf{c}$ is an Euler system for W_M then we can define the classes $\kappa_{[F,\mathfrak{r},M]}$ exactly as in Chapter 4.

The proof of Theorem 4.5.4 also only used the images of the Euler system classes in $H^1(L,W_M)$, so that theorem still holds for the derivative classes of an Euler system for W_M. However, the proof of Theorem 4.5.1 used the images of the Euler system classes in $H^1(L,T)$, so that proof breaks down in this setting. However, as discussed in §9.1 above (and see Remark 4.6.4), we can still prove a weaker version of Theorem 4.5.1, and this will suffice for some applications.

For example, the proofs in Chapters 4 and 5 will prove the following theorem. Keep the setting and notation of Chapter 2 (so in particular, for simplicity, $W_M = T/MT$).

Theorem 9.3.3. *Suppose $M \in \mathcal{O}$ is nonzero and $\mathbf{c}$ is an Euler system for W_M. Suppose that Hypotheses* $\mathrm{Hyp}(K,T)$ *hold, that the error terms $\mathfrak{n}_W$ and $\mathfrak{n}_W^*$ of Theorem 2.2.2 are both zero, and that $W_M^{G_K} = 0$. Let*

$$\mathfrak{a}_1 = \bigcap_{\substack{\text{primes } \mathfrak{q} \text{ of } K \\ \mathfrak{q} \nmid p}} \mathrm{Ann}_{\mathcal{O}}(W^{\mathcal{I}_\mathfrak{q}}/(W^{\mathcal{I}_\mathfrak{q}})_{\mathrm{div}})$$

and let $\mathfrak{a}_2 \subset \mathcal{O}$ be the annihilator of $\mathfrak{a}_1\mathbf{c}_K$ in $H^1(K,W_M)$. Then

$$(M/\mathfrak{a}_2)\mathfrak{a}_1 \mathcal{S}_{\Sigma_p}(K, W_M^*) = 0.$$

In particular if $\mathfrak{a}_1 \mathbf{c}_K \neq 0$ *then* $\mathcal{S}_{\Sigma_p}(K, W^*)$ *is finite.*

Remark 9.3.4. The ideal $\mathfrak{a}_1$ of Theorem 9.3.3 is finite, since $W^{\mathcal{I}}/(W^{\mathcal{I}})_{\mathrm{div}}$ is finite for all $\mathfrak{q}$ and is zero if T is unramified at $\mathfrak{q}$. See the proof of Corollary 4.6.5.

One could reformulate Theorem 9.3.3 for a general G_K-module $\bar{W}$ which is free of finite rank over $\mathcal{O}/M\mathcal{O}$, i.e., one which does not come from a "T", but one would have to redefine the Selmer group since our definition depends on T, not just on W_M.

9.4. Anticyclotomic Euler Systems

The "Euler system of Heegner points", one of Kolyvagin's original Euler systems, is not an Euler system under our Definition 2.1.1. If one tries to make the definition fit with $K = \mathbf{Q}$, the problem is that the cohomology classes (Heegner points) are not defined over abelian extensions of $\mathbf{Q}$, but rather over abelian extensions of an imaginary quadratic field which are "anticyclotomic" (and hence not abelian) over $\mathbf{Q}$. On the other hand, if one tries to make the definition fit by taking K to be an appropriate imaginary quadratic field, then the problem is that the Heegner points are not defined over large enough abelian extensions of K, but only over those which are anticyclotomic over $\mathbf{Q}$.

We will not discuss Heegner points in any detail (see instead [**Ko2**], [**Ru2**], or [**Gro2**]), but in this section we propose an expanded definition of Euler systems that will include "anticyclotomic" Euler systems such as Heegner points as examples.

Fix a number field K and a p-adic representation T of G_K as in §2.1. Suppose d is a positive integer dividing $p-1$, and $\chi : G_K \to \mathbf{Z}_p^\times$ is a character of order d. Let $K' = \bar{K}^{\ker(\chi)}$ be the cyclic extension of degree d of K cut out by χ.

For every prime $\mathfrak{q}$ of K not dividing p let $K'(\mathfrak{q})_\chi$ denote the maximal p-extension of K' inside the ray class field of K' modulo $\mathfrak{q}$, such that $\mathrm{Gal}(K'/K)$ acts on $\mathrm{Gal}(K'(\mathfrak{q})_\chi/K')$ via the character χ. Similarly, let $K'(1)_\chi$ denote the χ-part of the maximal unramified p-extension of K'.

Now suppose $\mathcal{K}'$ is an (infinite) abelian p-extension of K' and $\mathcal{N}$ is an ideal of K divisible by p, the conductor of χ, and all primes where T is ramified, and such that $\mathcal{K}'$ contains $K'(\mathfrak{q})_\chi$ for every prime $\mathfrak{q}$ of K not dividing $\mathcal{N}$.

Definition 9.4.1. A collection of cohomology classes

$$\mathbf{c} = \{\mathbf{c}_F \in H^1(F, T) : K' \subset_{\mathrm{f}} F \subset \mathcal{K}'\}$$

is a χ-*anticyclotomic Euler system* for $(T, \mathcal{K}', \mathcal{N})$ (or simply for T) if

(i) whenever $K' \subset_{\mathrm{f}} F \subset_{\mathrm{f}} F' \subset \mathcal{K}'$, then

$$\mathrm{Cor}_{F'/F}(\mathbf{c}_{F'}) = \Big(\prod_{\mathfrak{q} \in \Sigma(F'/F)} P(\mathrm{Fr}_{\mathfrak{q}}^{-1} | T^*; \mathrm{Fr}_{\mathfrak{q}}^{-1}) \Big) \mathbf{c}_F$$

where $\Sigma(F'/F)$ is the set of primes of K not dividing $\mathcal{N}$ which ramify in F' but not in F, as always $\mathrm{Fr}_{\mathfrak{q}}$ is a Frobenius of $\mathfrak{q}$ in G_K, and $P(\mathrm{Fr}_{\mathfrak{q}}^{-1} | T^*; x) = \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1} x | T^*) \in \mathcal{O}[x]$,

(ii) at least one of the following analogues of the hypotheses (ii′) of §9.1 holds:

(a) $\mathcal{K}'$ contains a $\mathbf{Z}_p^d$-extension K'_∞ of K' such that no finite prime splits completely in K'_∞/K', and $\mathrm{Gal}(K'/K)$ acts on $\mathrm{Gal}(K'_\infty/K')$ via χ,

(b) $\mathbf{c}_{K'(\mathfrak{r})_\chi} \in \mathcal{S}^{\Sigma_p}(K'(\mathfrak{r})_\chi, T)$ for every $\mathfrak{r}$, and there is a $\gamma \in G_K$ such that $\varepsilon_{\mathrm{cyc}}(\gamma) = \chi(\gamma)$, and $T^{\gamma=1} = 0$, and $\gamma^d = 1$ on $K'(1)_\chi(\boldsymbol{\mu}_{p^\infty}, (\mathcal{O}_{K'}^\times)^{1/p^\infty})$,

(c) $\mathbf{c}_{K(\mathfrak{r})} \in \mathcal{S}^{\Sigma_p}(K(\mathfrak{r}), T)$ for every $\mathfrak{r}$, the classes $\{\mathbf{c}_F\}$ satisfy the appropriate analogue of the congruence of Corollary 4.8.1, and for every $\mathfrak{q}$ not dividing $\mathcal{N}$ and every power n of p, $\mathrm{Fr}_{\mathfrak{q}}^n - 1$ is injective on T.

Remark 9.4.2. If $d = 1$, then χ is trivial, $K' = K$, and thus a χ-anticyclotomic Euler system for T is the same as an Euler system for T in the sense of Definition 2.1.1 (or §9.1).

Suppose $K = \mathbf{Q}$ and χ is an odd quadratic character, so $d = 2$. Then K' is an imaginary quadratic field and $\mathcal{K}'$ is an anticyclotomic p-extension of K'. If T is the Tate module of a modular elliptic curve, and we make the additional assumption that $\chi(q) = 1$ for every prime q dividing the conductor of χ, then the Heegner points in anticyclotomic extensions of K' give a χ-anticyclotomic Euler system for T. (One must modify the Heegner points slightly, as in §9.6 below, to obtain the correct distribution relation.) Note that in this situation we could take $\mathcal{K}'$ to contain the anticyclotomic $\mathbf{Z}_p$-extension K'_∞ of K'. However, all rational primes which are inert in K' split completely in K'_∞/K' so condition (ii)(a) of Definition 9.4.1 fails. However, both (ii)(b) and (ii)(c) hold.

Let

$$\Omega' = K'(1)_\chi(\boldsymbol{\mu}_{p^\infty}, (\mathcal{O}_{K'}^\times)^{1/p^\infty}, W).$$

For every $i \in \mathbf{Z}$ let

$$\mathrm{ind}_{\mathcal{O}}(\mathbf{c}, \chi^i) = \sup\{n : \mathbf{c}_{K'}^{\chi^i} \in \mathfrak{p}^n H^1(K', T) + H^1(K', T)_{\mathrm{tors}}\} \leq \infty,$$

where $\mathbf{c}_{K'}^{\chi^i}$ denotes the projection of $\mathbf{c}_{K'}$ into the subgroup $H^1(K',T)^{\chi^i}$ of $H^1(K',T)$ on which $\mathrm{Gal}(K'/K)$ acts via χ^i. In this setting of anticyclotomic Euler systems one can prove the following theorem.

Theorem 9.4.3. *Suppose* $\mathbf{c}$ *is a* χ*-anticyclotomic Euler system for* T*. Suppose further that* $H^1(\Omega'/K',W) = H^1(\Omega'/K',W^*) = 0$*, that* $T \otimes \Bbbk$ *is an irreducible* $\Bbbk[G_{K'}]$*-module, and that there is a* $\tau \in G_K$ *such that*

- $\varepsilon_{\mathrm{cyc}}(\tau) = \chi(\tau)$,
- τ^d *is the identity on* $K'(1)_\chi(\boldsymbol{\mu}_{p^\infty}, (\mathcal{O}_{K'}^\times)^{1/p^\infty})$,
- *if* $\zeta \in \boldsymbol{\mu}_d \subset \mathcal{O}^\times$ *then* $T/(\tau-\zeta)T$ *is free of rank one over* $\mathcal{O}$.

Then for every i,

$$\mathfrak{p}^{\mathrm{ind}_{\mathcal{O}}(\mathbf{c},\chi^i)}\mathcal{S}_{\Sigma_p}(K',W^*)^{\chi^{1-i}} = 0.$$

Remark 9.4.4. In the setting of the Heegner point Euler system mentioned in Remark 9.4.2, K' is an imaginary quadratic field, and we can take τ to be a complex conjugation. Theorem 9.4.3 then says that the "minus part" of the Heegner point in $E(K')$ controls the "plus part" of the Selmer group of E over K', and vice versa.

Sketch of proof. Given a χ-anticyclotomic Euler system and a power M of p, one can proceed exactly as in §4.4 to define derivative classes

$$\kappa_{[K',\mathfrak{r},M]} \in H^1(K',W_M)$$

for every $\mathfrak{r} \in \mathcal{R}_{K',M,\tau}$, where $\mathcal{R}_{K',M,\tau}$ is the set of squarefree ideals of K divisible only by primes $\mathfrak{q}$ such that $\mathfrak{q} \nmid \mathcal{N}$ and such that the Frobenius of $\mathfrak{q}$ in $\mathrm{Gal}(K'(1)_\chi(\boldsymbol{\mu}_M, (\mathcal{O}_{K'}^\times)^{1/M}, W_M)/K$ is (conjugate to) τ. These classes satisfy analogues of Theorems 4.5.1 and 4.5.4, and can be used along with global duality (Theorem 1.7.3) to bound the appropriate Selmer group.

The main difference between the case of trivial χ (i.e., Theorem 2.2.2) and nontrivial χ is the way the powers of χ appear in the statement of Theorem 9.4.3. This is due to the "anticyclotomic" version of Theorem 4.5.4, which states that for $\mathfrak{rq} \in \mathcal{R}_{K',M,\tau}$, we have

$$\mathrm{loc}_{\mathfrak{q}}^s(\kappa_{[K',\mathfrak{rq},M]}) = \phi_{\mathfrak{q}}^{fs}(\kappa_{[K',\mathfrak{r},M]})$$

where

$$\phi_{\mathfrak{q}}^{fs} : H_f^1(K'_{\mathfrak{q}},W_M) \longrightarrow H_s^1(K'_{\mathfrak{q}},W_M).$$

As usual we write $H_f^1(K'_{\mathfrak{q}},W_M) = \oplus_{v|\mathfrak{q}} H_f^1(K'_v,W_M)$ and similarly for $H_s^1(K'_{\mathfrak{q}},W_M)$, so that both are $\mathrm{Gal}(K'/K)$-modules. However, $\phi_{\mathfrak{q}}^{fs}$ is *not* $\mathrm{Gal}(K'/K)$-equivariant; for $\mathfrak{q} \in \mathcal{R}_{K',M,\tau}$, one can show that

$$\phi_{\mathfrak{q}}^{fs}(H_f^1(K'_{\mathfrak{q}},W_M)^{\chi^i}) \subset H_s^1(K'_{\mathfrak{q}},W_M)^{\chi^{i-1}}.$$

Thus, taking $\mathfrak{r} = 1$ and letting $\mathfrak{q}$ vary, we obtain a large collection of classes in $H^1(K', W_M)^{\chi^{i-1}}$, ramified at only one prime of K not dividing p, whose ramification is expressed in terms of $\mathbf{c}_{K'}^{\chi^i}$, and these classes can be used to annihilate classes in $\mathcal{S}_{\Sigma_p}(K', W^*)^{\chi^{1-i}}$. This is how Theorem 9.4.3 is proved. □

Remark 9.4.5. To prove an analogue of Theorem 2.2.2 and bound the order of the various components of $\mathcal{S}_{\Sigma_p}(K', W^*)$, we would need to proceed by induction as in Chapter 5. Unfortunately this is not at all straightforward, because at each step of the induction we move to a different component. We will not attempt to formulate, much less prove, such a statement here.

In the case of Kolyvagin's Euler system of Heegner points, the induction succeeds by using the fact that $T^* \cong T$. When $d > 2$ there is no obvious property to take the place of this self-duality. Also, if $d = 2$, then χ takes values ± 1, so if L is an abelian extension of K' it makes sense to ask if $\mathrm{Gal}(K'/K)$ acts on $\mathrm{Gal}(L/K')$ via χ. When $d > 2$, this only makes sense when L/K' is a p-extension. This is sufficient to discuss and work with Euler systems, but it raises the question of whether one should expect χ-anticyclotomic Euler systems with $d > 2$ to exist.

9.5. Additional Local Conditions

Inspired both by work on Stark's conjectures (see for example [**Gro1**] or [**Ru6**]) and by the connection between Euler systems and L-functions (see Chapter 8), we now allow the imposition of additional local conditions on Euler system cohomology classes.

Suppose Σ and Σ' are disjoint finite sets of places of K. If A is T, W, W_M, T^*, W^* or W_M^*, define

$$\mathcal{S}^{\Sigma}_{\Sigma'}(K, A) \;=\; \ker\left(\mathcal{S}^{\Sigma}(K, A) \to \oplus_{v \in \Sigma'} H^1(K_v, A)\right)$$

and similarly with K replaced by a finite extension. For example, $\mathcal{S}^{\Sigma}_{\Sigma'}(K, T)$ consists of all classes $c \in H^1(K, T)$ satisfying the local conditions

- $c_v \in H^1_f(K_v, T)$ if $v \notin \Sigma \cup \Sigma'$,
- $c_v = 0$ if $v \in \Sigma'$,
- no restriction for $v \in \Sigma$.

Definition 9.5.1. Suppose $\mathbf{c}$ is an Euler system for $(T, \mathcal{K}, \mathcal{N})$, and Σ is a finite set of primes of K not dividing p. We say $\mathbf{c}$ is *trivial at* Σ if $\mathbf{c}_F \in \mathcal{S}^{\Sigma_p}_{\Sigma}(F, T)$ for every F.

If an Euler system is trivial at Σ, we can use it to bound the Selmer group $\mathcal{S}^{\Sigma}_{\Sigma_p}(K, W^*)$. The proof will be the same as the original case where Σ is empty, once we have the following strengthening of Theorem 4.5.1.

Theorem 9.5.2. *Let Σ be a finite set of primes of K not dividing p. If $\mathbf{c}$ is an Euler system for T, trivial at Σ, then the derivative classes $\kappa_{[F,\mathfrak{r},M]}$ constructed in §4.4 satisfy*

$$\kappa_{[F,\mathfrak{r},M]} \in \mathcal{S}_{\Sigma}^{\Sigma_{p\mathfrak{r}}}(F, W_M).$$

Proof. By Theorem 4.5.1, we only need to show that $(\kappa_{[F,\mathfrak{r},M]})_{\mathfrak{q}} = 0$ if $\mathfrak{q} \in \Sigma$. The proof is similar to that of Theorem 4.5.1 in §4.6. We use the notation of that proof.

Fix a lift $\mathbf{d} : \mathbf{X}_{F(\mathfrak{r})} \to \mathbb{W}_M/W_M$ of $\mathbf{c}$ as in Proposition 4.4.8 and write $\mathbf{d}_{\mathfrak{q}}$ for the image of $\mathbf{d}$ in $\mathrm{Hom}(\mathbf{X}_{F(\mathfrak{r})}, \mathbb{W}_M/\mathrm{Ind}_{\mathcal{D}}^{G_K}(W_M))$ in the diagram of Lemma 4.6.7. Then $\mathbf{d}_{\mathfrak{q}}$ is a lift of $\mathbf{c}$ in the sense of Proposition 4.6.8, but so is the zero map, since $(\mathbf{c}_{F(\mathfrak{r})})_{\mathfrak{q}} = 0$. Therefore the uniqueness portion of Proposition 4.6.8 shows that $\mathbf{d}_{\mathfrak{q}} \in \mathrm{image}(\mathrm{Hom}(\mathbf{X}_{F(\mathfrak{r})}, \mathbb{W}_M^{G_{F(\mathfrak{r})}}))$, and from this it follows without difficulty, as in the proof of Theorem 4.5.1, that $(\kappa_{[F,\mathfrak{r},M]})_{\mathfrak{q}} = 0$. □

The following analogue of Theorem 2.2.2 (using the same notation) is an example of the kind of bound that comes from using an Euler system which is trivial at Σ.

Theorem 9.5.3. *Suppose that $p > 2$ and that T satisfies* $\mathrm{Hyp}(K,T)$. *Let Σ be a finite set of primes of K not dividing p. If $\mathbf{c}$ is an Euler system for T, trivial at Σ, then*

$$\ell_{\mathcal{O}}(\mathcal{S}_{\Sigma_p}^{\Sigma}(K, W^*)) \leq \mathrm{ind}_{\mathcal{O}}(\mathbf{c}) + \mathfrak{n}_W + \mathfrak{n}_W^*$$

where

$$\begin{aligned} \mathfrak{n}_W &= \ell_{\mathcal{O}}(H^1(\Omega/K, W) \cap \mathcal{S}_{\Sigma}^{\Sigma_p}(K, W)), \\ \mathfrak{n}_W^* &= \ell_{\mathcal{O}}(H^1(\Omega/K, W^*) \cap \mathcal{S}_{\Sigma_p}(K, W^*)). \end{aligned}$$

Proof. The proof is identical to that of Theorem 2.2.2, using Theorem 9.5.2 instead of Theorem 4.5.1. □

Remarks 9.5.4. There are similar analogues of the other theorems of Chapter 2, bounding $\mathcal{S}_{\Sigma_p}^{\Sigma}(K, W^*)$ and $\mathcal{S}_{\Sigma_p}^{\Sigma}(K_\infty, W^*)$.

By taking Σ to be large, we can ensure that the error term $\mathfrak{n}_W$ in Theorem 9.5.3 is small.

In the spirit of Chapter 8, if we think of Euler systems as corresponding to p-adic L-functions, then an Euler system which is trivial at Σ corresponds to a p-adic L-function with modified Euler factors at primes in Σ. As in [**Gro1**] §1 (where our Σ is denoted T), these Euler factors can be used to remove denominators from the original p-adic L-function (see Remark 8.2.5 and Conjecture 8.2.6).

9.6. Varying the Euler Factors

It may happen that one has a collection of cohomology classes satisfying distribution relations different from the ones in Definition 2.1.1. Under certain conditions one can modify the given classes to obtain an Euler system.

Return again to the setting of §2.1. We fix a number field K and a p-adic representation T of G_K. Suppose $\mathcal{K}$ is an abelian extension of K and $\mathcal{N}$ is an ideal of K divisible by p and all primes where T is ramified. If $K \subset_{\mathrm{f}} F \subset_{\mathrm{f}} F' \subset \mathcal{K}$, let $\Sigma(F'/F)$ denote the set of primes of K not dividing $\mathcal{N}$ which ramify in F'/K but not in F/K.

Lemma 9.6.1. *Suppose* $\{f_{\mathfrak{q}} \in \mathcal{O}[x] : \mathfrak{q} \nmid \mathcal{N}\}$ *and* $\{g_{\mathfrak{q}} \in \mathcal{O}[x] : \mathfrak{q} \nmid \mathcal{N}\}$ *are two collections of polynomials such that* $f_{\mathfrak{q}}(x) \equiv g_{\mathfrak{q}}(x) \pmod{\mathbf{N}(\mathfrak{q})-1}$ *for every* $\mathfrak{q}$*, and suppose* $\{\tilde{\mathbf{c}}_F \in H^1(F,T) : K \subset_{\mathrm{f}} F \subset \mathcal{K}\}$ *is a collection of cohomology classes such that if* $K \subset_{\mathrm{f}} F \subset_{\mathrm{f}} F' \subset \mathcal{K}$*, then*

$$\mathrm{Cor}_{F'/F}(\tilde{\mathbf{c}}_{F'}) \;=\; \Big(\prod_{\mathfrak{q}\in\Sigma(F'/F)} f_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\Big)\tilde{\mathbf{c}}_F.$$

Then there is a collection of classes $\{\mathbf{c}_F \in H^1(F,T) : K \subset_{\mathrm{f}} F \subset \mathcal{K}\}$ *satisfying the following properties.*

(i) *If* $K \subset_{\mathrm{f}} F \subset_{\mathrm{f}} F' \subset \mathcal{K}$*, then*

$$\mathrm{Cor}_{F'/F}(\mathbf{c}_{F'}) \;=\; \Big(\prod_{\mathfrak{q}\in\Sigma(F'/F)} g_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\Big)\mathbf{c}_F.$$

(ii) *If* $K \subset_{\mathrm{f}} F \subset \mathcal{K}$ *and* F/K *is unramified outside* $\mathcal{N}$*, then*

$$\mathbf{c}_F \;=\; \tilde{\mathbf{c}}_F.$$

(iii) *Suppose* $K \subset_{\mathrm{f}} F \subset \mathcal{K}$ *and* χ *is a character of* $\mathrm{Gal}(F/K)$ *of conductor* $\mathfrak{f}$*. If every prime which ramifies in* F/K *divides* $\mathcal{N}\mathfrak{f}$*, then*

$$\sum_{\gamma\in\mathrm{Gal}(F/K)} \chi(\gamma)\gamma\mathbf{c}_F \;=\; \sum_{\gamma\in\mathrm{Gal}(F/K)} \chi(\gamma)\gamma\tilde{\mathbf{c}}_F.$$

Proof. If $K \subset_{\mathrm{f}} F \subset \mathcal{K}$ let $\Sigma(F) = \Sigma(F/K)$, and if S is a finite set of primes of K let F_S be the largest extension of K in F which is unramified outside S and $\mathcal{N}$. If $\mathfrak{q} \nmid \mathcal{N}$ let $d_{\mathfrak{q}} = g_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1}) - f_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})$. For every F define

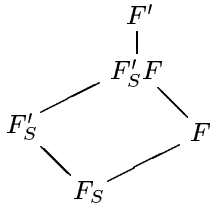
$$\mathbf{c}_F \;=\; \sum_{S\subset\Sigma(F)} \frac{\prod_{\mathfrak{q}\in\Sigma(F)-S} d_{\mathfrak{q}}}{[F:F_S]}\Big(\prod_{\mathfrak{q}\in S-\Sigma(F_S)} f_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\Big)\tilde{\mathbf{c}}_{F_S}.$$

(Let $\mathcal{I}_{\mathfrak{q}}(F/K)$ denote the inertia group of $\mathfrak{q}$ in $\mathrm{Gal}(F/K)$. Then $\mathrm{Gal}(F/F_S)$ is generated by $\{\mathcal{I}_{\mathfrak{q}} : \mathfrak{q} \in \Sigma(F) - S\}$, and $|\mathcal{I}_{\mathfrak{q}}|$ divides $(\mathbf{N}(\mathfrak{q})-1)$ in $\mathcal{O}$, so

$[F : F_S]$ divides $\prod_{\mathfrak{q}\in\Sigma(F)-S}(\mathbf{N}(\mathfrak{q})-1)$. Since $d_\mathfrak{q} \in (\mathbf{N}(\mathfrak{q})-1)\mathcal{O}[\mathrm{Gal}(F/K)]$, the fractions above belong to $\mathcal{O}[\mathrm{Gal}(F/K)]$.)

With this definition, (ii) is clear. Assertion (iii) (of which (ii) is a special case) also holds, because if every prime which ramifies in F/K divides $\mathcal{N}\mathfrak{f}$, and if S is a proper subset of $\Sigma(F)$, then χ does not factor through $\mathrm{Gal}(F_S/K)$ and so $\sum_{\gamma\in\mathrm{Gal}(F/K)}\chi(\gamma)\gamma\tilde{\mathbf{c}}_{F_S} = 0$.

For (i), observe that for every S, we have $F'_S \cap F = F_S$. Thus, using the diagram



we see that

$$\begin{aligned}\mathrm{Cor}_{F'/F}(\tilde{\mathbf{c}}_{F'_S}) &= \mathrm{Cor}_{F'_SF/F}\mathrm{Cor}_{F'/F'_SF}(\tilde{\mathbf{c}}_{F'_S}) = [F' : F'_SF]\mathrm{Cor}_{F'_SF/F}(\tilde{\mathbf{c}}_{F'_S})\\ &= \frac{[F' : F]}{[F'_S : F_S]}\mathrm{Cor}_{F'_S/F_S}(\tilde{\mathbf{c}}_{F'_S}) = \frac{[F' : F]}{[F'_S : F_S]}\Big(\prod_{\mathfrak{q}\in\Sigma(F'_S/F_S)} f_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\Big)\tilde{\mathbf{c}}_{F_S},\end{aligned}$$

and so $\mathrm{Cor}_{F'/F}(\mathbf{c}_{F'}) = \sum_{S\subset\Sigma(F')} a_S\tilde{\mathbf{c}}_{F_S}$ where

$$\begin{aligned}a_S &= \frac{\prod_{\mathfrak{q}\in\Sigma(F')-S} d_\mathfrak{q}}{[F' : F'_S]}\Big(\prod_{\mathfrak{q}\in S-\Sigma(F'_S)} f_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\Big)\frac{[F' : F]}{[F'_S : F_S]}\Big(\prod_{\mathfrak{q}\in\Sigma(F'_S/F_S)} f_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\Big)\\ &= \frac{\prod_{\mathfrak{q}\in\Sigma(F')-S} d_\mathfrak{q}}{[F : F_S]}\prod_{\mathfrak{q}\in S-\Sigma(F_S)} f_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1}).\end{aligned}$$

Since $F_S = F_{S\cap\Sigma(F)}$, we can group together those sets S which have the same intersection with $\Sigma(F)$, and we get a new expression

$$\mathrm{Cor}_{F'/F}(\mathbf{c}_{F'}) = \sum_{S\subset\Sigma(F)} b_S\tilde{\mathbf{c}}_{F_S}$$

where

$$\begin{aligned}b_S &= \sum_{S'\subset\Sigma(F'/F)} \frac{\prod_{\mathfrak{q}\in\Sigma(F')-S-S'} d_\mathfrak{q}}{[F : F_S]}\prod_{\mathfrak{q}\in S\cup S'-\Sigma(F_S)} f_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\\ &= \frac{\prod_{\mathfrak{q}\in\Sigma(F)-S} d_\mathfrak{q}}{[F : F_S]}\prod_{\mathfrak{q}\in S-\Sigma(F_S)} f_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1})\\ &\qquad\times\sum_{S'\subset\Sigma(F'/F)}\Big(\prod_{\mathfrak{q}\in\Sigma(F'/F)-S'} d_\mathfrak{q}\Big)\prod_{\mathfrak{q}\in S'} f_\mathfrak{q}(\mathrm{Fr}_\mathfrak{q}^{-1}).\end{aligned}$$

Since

$$\sum_{S'\subset\Sigma(F'/F)}\Big(\prod_{\mathfrak{q}\in\Sigma(F'/F)-S'}d_{\mathfrak{q}}\Big)\prod_{\mathfrak{q}\in S'}f_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1}) = \prod_{\mathfrak{q}\in\Sigma(F'/F)}(d_{\mathfrak{q}}+f_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})) = \prod_{\mathfrak{q}\in\Sigma(F'/F)}g_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1}),$$

we conclude that $\mathrm{Cor}_{F'/F}(\mathbf{c}_{F'}) = \prod_{\mathfrak{q}\in\Sigma(F'/F)} g_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\mathbf{c}_F$ as desired. □

Example 9.6.2. Suppose that $K = \mathbf{Q}$, that $f_q(x) = 1-x$, and that $g_q(x) = 1-q^{-1}x$. Then $f_q(x) \equiv g_q(x) \pmod{(q-1)\mathbf{Z}_p}$ for every $q \neq p$. By applying Lemma 9.6.1 with these data to the collection $\{\tilde{\mathbf{c}}'_F \in H^1(F,\mathbf{Z}_p)\}$ constructed in Definition 3.4.2, we obtain an Euler system for $\mathbf{Z}_p$.

Lemma 9.6.3. *Suppose $\{f_{\mathfrak{q}}(x) \in \mathcal{O}[x,x^{-1}] : \mathfrak{q} \nmid \mathcal{N}\}$ is a collection of polynomials, $\{u_{\mathfrak{q}} \in \mathcal{O}^\times : \mathfrak{q} \nmid \mathcal{N}\}$ is a collection of units, $d \in \mathbf{Z}$, and*

$$\{\tilde{\mathbf{c}}_F \in H^1(F,T) : K \subset_{\mathrm{f}} F \subset \mathcal{K}\}$$

is a collection of cohomology classes such that if $K \subset_{\mathrm{f}} F \subset_{\mathrm{f}} F' \subset \mathcal{K}$ then

$$\mathrm{Cor}_{F'/F}(\tilde{\mathbf{c}}_{F'}) = \Big(\prod_{\mathfrak{q}\in\Sigma(F'/F)} f_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}})\Big)\tilde{\mathbf{c}}_F.$$

For each $\mathfrak{q}$ define $g_{\mathfrak{q}}(x) = u_{\mathfrak{q}}x^d f_{\mathfrak{q}}(x^{-1}) \in \mathcal{O}[x,x^{-1}]$. Then there is a collection of classes

$$\{\mathbf{c}_F \in H^1(F,T) : K \subset_{\mathrm{f}} F \subset \mathcal{K}\}$$

such that

(i) *for all F and F' as above,*

$$\mathrm{Cor}_{F'/F}(\mathbf{c}_{F'}) = \Big(\prod_{\mathfrak{q}\in\Sigma(F'/F)} g_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\Big)\mathbf{c}_F,$$

(ii) *for every finite extension F of K unramified outside $\mathcal{N}$,*

$$\mathbf{c}_F = \tilde{\mathbf{c}}_F.$$

Proof. For every F define

$$\mathbf{c}_F = \Big(\prod_{\mathfrak{q}\in\Sigma(F/K)} u_{\mathfrak{q}}\mathrm{Fr}_{\mathfrak{q}}^{-d}\Big)\tilde{\mathbf{c}}_F$$

where we fix some Frobenius $\mathrm{Fr}_{\mathfrak{q}} \in \mathrm{Gal}(K^{\mathrm{ab}}/K)$ (previously we always had $\mathrm{Fr}_{\mathfrak{q}}$ acting through an extension unramified at $\mathfrak{q}$). Then it is easy to check that this collection has the desired properties. □

Let $P(\mathrm{Fr}_{\mathfrak{q}}^{-1}|T;x) = \det(1-\mathrm{Fr}_{\mathfrak{q}}^{-1}x|T)$.

Corollary 9.6.4. *Suppose* $\{\tilde{\mathbf{c}}_F \in H^1(F,T) : K \subset_{\mathrm{f}} F \subset \mathcal{K}\}$ *is a collection of cohomology classes such that if* $K \subset_{\mathrm{f}} F \subset_{\mathrm{f}} F' \subset \mathcal{K}$, *then*

$$\mathrm{Cor}_{F'/F}(\tilde{\mathbf{c}}_{F'}) = \Big(\prod_{\mathfrak{q} \in \Sigma(F'/F)} P(\mathrm{Fr}_{\mathfrak{q}}^{-1}|T;\mathrm{Fr}_{\mathfrak{q}}) \Big) \tilde{\mathbf{c}}_F.$$

Then there is an Euler system $\{\mathbf{c}_F\}$ *for* $(T,\mathcal{K},\mathcal{N})$ *such that for every finite extension* F *of* K *unramified outside* $\mathcal{N}$,

$$\mathbf{c}_F = \tilde{\mathbf{c}}_F.$$

Proof. This will follow directly from the previous two lemmas. For every $\mathfrak{q}$ we have

$$\begin{aligned} P(\mathrm{Fr}_{\mathfrak{q}}^{-1}|T;x^{-1}) &= \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1}x^{-1}|T) = \det(1 - \mathbf{N}(\mathfrak{q})^{-1}\mathrm{Fr}_{\mathfrak{q}}x^{-1}|T^*) \\ &= (-\mathbf{N}(\mathfrak{q}))^{-d} \det(\mathrm{Fr}_{\mathfrak{q}}|T^*)x^{-d} \det(1 - \mathbf{N}(\mathfrak{q})\mathrm{Fr}_{\mathfrak{q}}^{-1}x|T^*) \end{aligned}$$

where $d = \mathrm{rank}_{\mathcal{O}}T$. Thus if we first apply Lemma 9.6.3 with

$$f_{\mathfrak{q}} = P(\mathrm{Fr}_{\mathfrak{q}}^{-1}|T;x), \quad u_{\mathfrak{q}} = (-\mathbf{N}(\mathfrak{q}))^d \det(\mathrm{Fr}_{\mathfrak{q}}|T^*)^{-1},$$

and then apply Lemma 9.6.1 with

$$f_{\mathfrak{q}} = P(\mathrm{Fr}_{\mathfrak{q}}^{-1}|T^*;\mathbf{N}(\mathfrak{q})x), \quad g_{\mathfrak{q}} = P(\mathrm{Fr}_{\mathfrak{q}}^{-1}|T^*;x),$$

we obtain the desired Euler system. □

Remark 9.6.5. If one has a collection of cohomology classes satisfying the "wrong" distribution relation, one can either modify the classes as we did above, to get an Euler system, or else one can keep the given cohomology classes and modify the proofs in Chapters 4 through 7 instead.

APPENDIX A

Linear Algebra

Suppose for this appendix that $\mathcal{O}$ is a discrete valuation ring. Let $\ell_{\mathcal{O}}(B)$ denote the length of an $\mathcal{O}$-module B.

A.1. Herbrand Quotients

Suppose $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{O}[x]$ are nonzero.

Definition A.1.1. If S is an $\mathcal{O}[x]$-module and $\boldsymbol{\alpha}\boldsymbol{\beta}S = 0$, then

$$\boldsymbol{\alpha}S \subset S^{\boldsymbol{\beta}=0}, \quad \boldsymbol{\beta}S \subset S^{\boldsymbol{\alpha}=0},$$

and we define the (additive) Herbrand quotient

$$h(S) = \ell_{\mathcal{O}}(S^{\boldsymbol{\beta}=0}/\boldsymbol{\alpha}S) - \ell_{\mathcal{O}}(S^{\boldsymbol{\alpha}=0}/\boldsymbol{\beta}S)$$

if both lengths are finite.

Example A.1.2. If $S = \mathcal{O}[x]/\boldsymbol{\alpha}\boldsymbol{\beta}\mathcal{O}[x]$ then

$$S^{\boldsymbol{\beta}=0} = \boldsymbol{\alpha}S = \boldsymbol{\alpha}\mathcal{O}[x]/\boldsymbol{\alpha}\boldsymbol{\beta}\mathcal{O}[x], \quad S^{\boldsymbol{\alpha}=0} = \boldsymbol{\beta}S = \boldsymbol{\beta}\mathcal{O}[x]/\boldsymbol{\alpha}\boldsymbol{\beta}\mathcal{O}[x],$$

so $h(S) = 0$.

Proposition A.1.3. (i) *If S is an $\mathcal{O}[x]/\boldsymbol{\alpha}\boldsymbol{\beta}\mathcal{O}[x]$-module and $\ell_{\mathcal{O}}(S)$ is finite, then $h(S) = 0$.*

(ii) *If $0 \to S' \to S \to S'' \to 0$ is an exact sequence of $\mathcal{O}[x]/\boldsymbol{\alpha}\boldsymbol{\beta}\mathcal{O}[x]$-modules and two of the three Herbrand quotients exist, then the third exists and*

$$h(S) = h(S') + h(S'').$$

Proof. This is a standard fact about Herbrand quotients, see for example [**Se3**] §VIII.4. If $\boldsymbol{\alpha} = (x^n - 1)/(x - 1)$ and $\boldsymbol{\beta} = x - 1$, and if G is a cyclic group of order n with a generator which acts on S as multiplication by x, then

$$\hat{H}^0(G, S) = S^{\boldsymbol{\beta}=0}/\boldsymbol{\alpha}S \text{ and } \hat{H}^1(G, S) = S^{\boldsymbol{\alpha}=0}/\boldsymbol{\beta}S.$$

For completeness we sketch a proof in our more general setting.

Assertion (i) follows from the exact sequences

$$0 \longrightarrow S^{\boldsymbol{\alpha}=0} \longrightarrow S \xrightarrow{\boldsymbol{\alpha}} \boldsymbol{\alpha}S \longrightarrow 0,$$
$$0 \longrightarrow S^{\boldsymbol{\beta}=0}/\boldsymbol{\alpha}S \longrightarrow S/\boldsymbol{\alpha}S \xrightarrow{\boldsymbol{\beta}} S^{\boldsymbol{\alpha}=0} \longrightarrow S^{\boldsymbol{\alpha}=0}/\boldsymbol{\beta}S \longrightarrow 0.$$

For (ii), multiplication by $\boldsymbol{\beta}$ induces a snake lemma exact sequence

$$0 \to S'^{\boldsymbol{\beta}=0} \to S^{\boldsymbol{\beta}=0} \to S''^{\boldsymbol{\beta}=0} \xrightarrow{\psi} S'/\boldsymbol{\beta}S' \to S/\boldsymbol{\beta}S \to S''/\boldsymbol{\beta}S'' \to 0.$$

This gives rise to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{coker}(\psi) & \longrightarrow & S/\boldsymbol{\beta}S & \longrightarrow & S''/\boldsymbol{\beta}S'' & \longrightarrow & 0 \\ & & \boldsymbol{\alpha}\downarrow & & \boldsymbol{\alpha}\downarrow & & \boldsymbol{\alpha}\downarrow & & \\ 0 & \longrightarrow & S'^{\boldsymbol{\beta}=0} & \longrightarrow & S^{\boldsymbol{\beta}=0} & \longrightarrow & \ker(\psi) & \longrightarrow & 0. \end{array}$$

Applying the snake lemma again gives an exact sequence

$$0 \longrightarrow A \longrightarrow S^{\boldsymbol{\alpha}=0}/\boldsymbol{\beta}S \longrightarrow S''^{\boldsymbol{\alpha}=0}/\boldsymbol{\beta}S'' \longrightarrow S'^{\boldsymbol{\beta}=0}/\boldsymbol{\alpha}S' \longrightarrow S^{\boldsymbol{\beta}=0}/\boldsymbol{\alpha}S \longrightarrow B \longrightarrow 0$$

where A and B are defined by the exact sequence

$$0 \longrightarrow B \longrightarrow S''^{\boldsymbol{\beta}=0}/\boldsymbol{\alpha}S'' \xrightarrow{\psi} S'^{\boldsymbol{\alpha}=0}/\boldsymbol{\beta}S' \longrightarrow A \longrightarrow 0.$$

Assertion (ii) follows from these two exact sequences. □

Lemma A.1.4. *Suppose $\boldsymbol{\alpha}\boldsymbol{\beta} = \prod_{i=1}^{k} \boldsymbol{\rho}_i$ with $\boldsymbol{\rho}_i \in \mathcal{O}[x]$, and suppose further that $\boldsymbol{\rho}_i$ is relatively prime to $\boldsymbol{\beta}$ if $2 \le i \le k$. Then*

$$h(\oplus_i \mathcal{O}[x]/\boldsymbol{\rho}_i\mathcal{O}[x]) = 0.$$

Proof. For $1 \le i \le k$ let $S_i = \mathcal{O}[x]/\boldsymbol{\rho}_i\mathcal{O}[x]$. If $\boldsymbol{\rho}_i$ is relatively prime to $\boldsymbol{\beta}$ (and therefore divides $\boldsymbol{\alpha}$), we see that $S_i^{\boldsymbol{\beta}=0} = \boldsymbol{\alpha}S_i = 0$ and $S_i^{\boldsymbol{\alpha}=0} = S_i$. Thus for $i \ge 2$

$$h(S_i) = -\ell_{\mathcal{O}}(S_i/\boldsymbol{\beta}S_i) = -\ell_{\mathcal{O}}(\mathcal{O}[x]/(\boldsymbol{\beta}, \boldsymbol{\rho}_i))$$

which is finite. By Proposition A.1.3 and Example A.1.2 we conclude that the Herbrand quotient $h(S_1)$ exists as well, and that

$$h(\oplus_i \mathcal{O}[x]/\boldsymbol{\rho}_i\mathcal{O}[x]) = \sum_i h(S_i) = h(\mathcal{O}[x]/\boldsymbol{\alpha}\boldsymbol{\beta}) = 0.$$ □

A.2. p-adic Representations

Let T be a free $\mathcal{O}$-module of finite rank, and let σ be an $\mathcal{O}$-linear automorphism of T. Let $p(x) = \det(1 - \sigma^{-1}x|T) \in \mathcal{O}[x]$, and suppose further that $p(1) = 0$ (i.e., $\det(1 - \sigma|T) = 0$). Then there is a unique polynomial $q(x) \in \mathcal{O}[x]$ such that

$$p(x) = (1-x)q(x).$$

The Cayley-Hamilton theorem shows that $p(\sigma) = 0$, so T is an $\mathcal{O}[x]/p(x)$-module, with x acting via σ. Thus we are in the setting of §A.1, with $\boldsymbol{\alpha} = q(x)$ and $\boldsymbol{\beta} = x - 1$.

Let Φ denote the field of fractions of $\mathcal{O}$ and let $V = T \otimes \Phi$.

Lemma A.2.1. *Suppose that* $T \cong \oplus_i \mathcal{O}[x]/f_i(x)\mathcal{O}[x]$ *with* $f_i(x) \in \mathcal{O}[x]$, *and that* $\dim_\Phi(V/(\sigma-1)V) = 1$. *Then the Herbrand quotient* $h(T) = 0$.

Proof. We have $p(x) = \prod_i f_i(x)$. Since $\dim_\Phi(V/(\sigma-1)V) = 1$, exactly one of the $f_i(x)$ (say, f_1) is divisible by $x-1$. Thus we can apply Lemma A.1.4 to conclude that $h(T) = 0$. □

Lemma A.2.2. *There is an* $\mathcal{O}[\sigma]$*-submodule* S *of* T *such that*

(i) $\ell_{\mathcal{O}}(T/S)$ *is finite,*
(ii) $S \cong \oplus_i \mathcal{O}[x]/f_i(x)\mathcal{O}[x]$ *with* $f_1(x), \ldots, f_k(x) \in \mathcal{O}[x]$.

Proof. Since the polynomial ring $\Phi[x]$ is a principal ideal domain, V is a direct sum of cyclic $\Phi[\sigma]$-modules, and the lemma follows easily. □

Proposition A.2.3. *If* $\dim_\Phi(V/(\sigma-1)V) = 1$ *then* $h(T) = 0$.

Proof. This is immediate from Proposition A.1.3 and Lemmas A.2.1 and A.2.2. □

Lemma A.2.4. *Suppose* $\dim_\Phi(V/(\sigma-1)V) = 1$. *Then*

(i) $V^{q(\sigma)=0} = (\sigma-1)V$ *and* $V^{\sigma=1} = q(\sigma)V$,
(ii) *the map* $V/(\sigma-1)V \xrightarrow{q(\sigma)} V^{\sigma=1}$ *is an isomorphism.*

Proof. Viewing V as a $\Phi[x]$-module with x acting via σ, there is an isomorphism

$$V \cong \bigoplus_i \Phi[x]/f_i(x)^{e_i}\Phi[x]$$

where the $f_i(x) \in \Phi[x]$ are irreducible, $f_i(0) = 1$, and

$$\prod_i f_i(x)^{e_i} = p(x) = (1-x)q(x).$$

Since $\dim_\Phi(V/(\sigma-1)V) = 1$, precisely one of the f_i is $1-x$. Both assertions follow easily from this. □

Proposition A.2.5. *Suppose* $\dim_\Phi(V/(\sigma-1)V) = 1$, *and let* $W = V/T$. *Then the lengths of the following* $\mathcal{O}$*-modules are finite and equal:*

(i)	$T^{\sigma=1}/q(\sigma)T$	(iv)	$W^{\sigma=1}/q(\sigma)W$
(ii)	$T^{q(\sigma)=0}/(\sigma-1)T$	(v)	$W^{q(\sigma)=0}/(\sigma-1)W$
(iii)	$(T/(\sigma-1)T)_{\mathrm{tors}}$	(vi)	$W^{\sigma=1}/W^{\sigma=1}_{\mathrm{div}}$

where $W^{\sigma=1}_{\mathrm{div}}$ *denotes the maximal divisible* $\mathcal{O}$*-submodule of* $W^{\sigma=1}$.

Proof. Proposition A.2.3 says that $h(T) = 0$, so (i) and (ii) have the same (finite) length. Similarly, by Lemma A.2.4(i) we have $h(V) = 0$, so Proposition A.1.3(ii) shows that $h(W) = 0$ as well. Thus (iv) and (v) have the same length.

Lemma A.2.4(i) shows that $V^{q(\sigma)=0}/(\sigma-1)V = 0$, and it follows that $T^{q(\sigma)=0}/(\sigma-1)T$ is a torsion $\mathcal{O}$-module. Since $T/T^{q(\sigma)=0}$ is torsion-free we have

$$(T/(\sigma-1)T)_{\mathrm{tors}} \;=\; T^{q(\sigma)=0}/(\sigma-1)T$$

and so (ii) and (iii) are isomorphic. It follows similarly from Lemma A.2.4(i) that $q(\sigma)W = W^{\sigma=1}_{\mathrm{div}}$ and (iv) is isomorphic to (vi).

It remains to compare (i) with (v). Consider the diagram

$$\begin{array}{ccccccc}
 & & T/(\sigma-1)T & \longrightarrow & V/(\sigma-1)V & \longrightarrow & W/(\sigma-1)W \longrightarrow 0 \\
 & & \downarrow{\scriptstyle q(\sigma)} & & \downarrow{\scriptstyle q(\sigma)} & & \downarrow{\scriptstyle q(\sigma)} \\
0 & \longrightarrow & T^{\sigma=1} & \longrightarrow & V^{\sigma=1} & \longrightarrow & W^{\sigma=1}.
\end{array}$$

By Lemma A.2.4(ii), the center vertical map is an isomorphism, so the snake lemma gives (i) $\cong$ (v). □

For the next two corollaries let $W = V/T$, and if $M \in \mathcal{O}$ let W_M denote the kernel of multiplication by M on W.

Corollary A.2.6. *Suppose* $\dim_\Phi(V/(\sigma-1)V) = 1$, *and let* b *denote the common length of the modules in Proposition* A.2.5. *Then the kernel and cokernel of the map*

$$W_M/(\sigma-1)W_M \;\xrightarrow{q(\sigma)}\; W_M^{\sigma=1}$$

both have length at most $2b$.

Proof. Consider the diagram

$$\begin{array}{ccc} W/(\sigma-1)W & \xrightarrow[\phi]{q(\sigma)} & W^{\sigma=1} \\ \uparrow & & \uparrow \\ W_M/(\sigma-1)W_M & \xrightarrow[\phi_M]{q(\sigma)} & W_M^{\sigma=1}. \end{array} \tag{A.1}$$

The kernel and cokernel of ϕ are (v) and (iv) of Proposition A.2.5, respectively, and therefore both have length b. Multiplying the exact sequence

$$0 \longrightarrow W_M \longrightarrow W \xrightarrow{M} W \longrightarrow 0$$

by $\sigma - 1$ yields a snake lemma exact sequence

$$W^{\sigma=1} \xrightarrow{M} W^{\sigma=1} \longrightarrow W_M/(\sigma-1)W_M \longrightarrow W/(\sigma-1)W.$$

Therefore the kernel of the left vertical map of (A.1) is $W^{\sigma=1}/M(W^{\sigma=1})$, which is a quotient of the module (vi) of Proposition A.2.5, and hence has length at most b. Thus we conclude that $\ell_{\mathcal{O}}(\ker(\phi_M)) \leq 2b$. The exact sequence

$$0 \longrightarrow W_M^{\sigma=1} \longrightarrow W_M \xrightarrow{\sigma-1} W_M \longrightarrow W_M/(\sigma-1)W_M \longrightarrow 0$$

shows that $\ell_{\mathcal{O}}(W_M/(\sigma-1)W_M) = \ell_{\mathcal{O}}(W_M^{\sigma=1})$, so

$$\ell_{\mathcal{O}}(\operatorname{coker}(\phi_M)) = \ell_{\mathcal{O}}(\ker(\phi_M)) \leq 2b$$

as well. □

Corollary A.2.7. *Suppose*

(a) *τ is an $\mathcal{O}$-linear automorphism of W_M such that $W_M/(\tau-1)W_M$ is free of rank one over $\mathcal{O}/M\mathcal{O}$,*

(b) *$Q(x) \in (\mathcal{O}/M\mathcal{O})[x]$ is such that $(1-x)Q(x) = \det(1-\tau^{-1}x|W_M)$.*

Then the map

$$W_M/(\tau-1)W_M \xrightarrow{Q(\tau)} W_M^{\tau=1}$$

is an isomorphism.

Proof. We will show that there is an automorphism σ of T such that

(i) σ induces τ on W_M,

(ii) $T/(\sigma-1)T$ is free of rank one over $\mathcal{O}$.

Once we have done this, we will apply Proposition A.2.5 and Corollary A.2.6 with this σ. Condition (ii) shows that the module $(T/(\sigma-1)T)_{\text{tors}}$ of Proposition A.2.5(iii) is zero, so the integer b of Corollary A.2.6 is zero. It

follows from condition (i) (together with the fact that $1-x$ is not a zero-divisor in $(\mathcal{O}/M\mathcal{O})[x]$) that $q(\sigma)$ reduces to $Q(\tau)$ on W_M, so this corollary will follow from Corollary A.2.6.

It remains to find a σ satisfying (i) and (ii). Since $W_M/(\tau-1)W_M$ is free of rank one over $\mathcal{O}/M\mathcal{O}$, it follows that $W_M^{\tau=1}$ is free of rank one over $\mathcal{O}/M\mathcal{O}$ as well. Therefore we can choose a basis $\{w_1,\dots,w_d\}$ of W_M such that $\tau w_1 = w_1$, where $d = \operatorname{rank}_{\mathcal{O}} T$.

For each i fix $t_i \in T$ which reduces to w_i. By Nakayama's Lemma $\{t_1, t_2, \dots, t_d\}$ is an $\mathcal{O}$-basis of T, and we define σ on this basis by lifting the action of τ on the w_i, and requiring that $\sigma(t_1) = t_1$. Then σ is an automorphism because $\det(\sigma) \equiv \det(\tau) \pmod{M}$, and clearly (i) is satisfied. Further $\operatorname{rank}_{\mathcal{O}} T/(\sigma-1)T \geq 1$, and since $(T/(\sigma-1)T) \otimes (\mathcal{O}/M\mathcal{O}) = W_M/(\tau-1)W_M$ is a cyclic $\mathcal{O}$-module, we can apply Nakayama's Lemma again to deduce (ii). □

APPENDIX B

Continuous Cohomology and Inverse Limits

Notation. If G and T are topological groups then $\mathrm{Hom}(G,T)$ will always denote the group of *continuous* homomorphisms from G to T. We denote by $\mathrm{Maps}(G,T)$ the topological group of continuous functions (not necessarily homomorphisms) from G to T, with the compact-open topology.

B.1. Preliminaries

Since we will use it repeatedly, we record without proof the following well-known result.

Proposition B.1.1. (i) *Suppose $\{A_n\}$, $\{B_n\}$, and $\{C_n\}$ are inverse systems of topological groups and there are exact sequences*

$$0 \longrightarrow A_n \longrightarrow B_n \longrightarrow C_n \longrightarrow 0$$

for every n, compatible with the maps of the inverse systems. If the A_n are compact, then the induced sequence

$$0 \longrightarrow \varprojlim_n A_n \longrightarrow \varprojlim_n B_n \longrightarrow \varprojlim_n C_n \longrightarrow 0$$

is exact.

(ii) *If $\mathcal{O}$ is a discrete valuation ring with fraction field Φ and $\{A_n\}$ is an inverse system of finite $\mathcal{O}$-modules, then the canonical map*

$$\varinjlim_n \mathrm{Hom}(A_n, \Phi/\mathcal{O}) \longrightarrow \mathrm{Hom}(\varprojlim_n A_n, \Phi/\mathcal{O})$$

is an isomorphism.

B.2. Continuous Cohomology

For this section suppose G is a profinite group and T is a topological G-module, i.e., an abelian topological group with a continuous action of G.

Definition B.2.1. Following Tate [**T4**], we define the continuous cohomology groups $H^i(G,T)$ as follows. Let $C^i(G,T) = \mathrm{Maps}(G^i,T)$. For

every $i \geq 0$ there is a coboundary map $d_i : C^i(G,T) \to C^{i+1}(G,T)$ defined in the usual way (see for example [**Se3**] §VII.3), and we set

$$H^i(G,T) = \ker(d_i)/\mathrm{image}(d_{i-1}).$$

If $0 \to T' \to T \to T'' \to 0$ is an exact sequence and *if there is a continuous section* (again a map of sets, not necessarily a homomorphism) from $T'' \to T$, then

$$0 \longrightarrow C^i(G,T') \longrightarrow C^i(G,T) \longrightarrow C^i(G,T'') \longrightarrow 0$$

is exact for every i and there is a long exact sequence

$$\cdots \longrightarrow H^i(G,T') \longrightarrow H^i(G,T) \longrightarrow H^i(G,T'') \longrightarrow H^{i+1}(G,T') \longrightarrow \cdots .$$

Remark B.2.2. Note that if T'' is topologically discrete, as is assumed in the more "classical" formulations of profinite group cohomology, then there is always a continuous section $T'' \to T$. This is the case whenever T' is open in T. Also, if T is a finitely generated $\mathbf{Z}_p$-module or a finite-dimensional $\mathbf{Q}_p$-vector space with the usual topology, then there is a continuous section. These are the only situations in which we will use these cohomology groups.

For the situations of interest to us, the following propositions will allow us to work with the cohomology groups $H^i(G,T)$ exactly as if T were discrete. The first two are due to Tate [**T4**]; see also Jannsen [**J**].

Proposition B.2.3 ([**T4**] Corollary 2.2, [**J**] §2)**.** *Suppose $i > 0$ and $T = \varprojlim T_n$ where each T_n is a finite G-module. If $H^{i-1}(G,T_n)$ is finite for every n then*

$$H^i(G,T) = \varprojlim_n H^i(G,T_n).$$

Proposition B.2.4 ([**T4**] Proposition 2.3)**.** *If T is a finitely generated $\mathbf{Z}_p$-module and $i \geq 0$, then $H^i(G,T)$ has no divisible elements and the natural map*

$$H^i(G,T) \otimes \mathbf{Q}_p \longrightarrow H^i(G, T \otimes \mathbf{Q}_p)$$

is an isomorphism.

Proposition B.2.5. *Suppose H is a closed, normal subgroup of G.*

(i) *There is an inflation-restriction exact sequence*

$$0 \longrightarrow H^1(G/H, T^H) \longrightarrow H^1(G,T) \longrightarrow H^1(H,T).$$

(ii) *Suppose further that p is a prime, and for every G-module (resp. H-module) S of finite, p-power order, $H^1(G,S)$ and $H^2(G,S)$ (resp. $H^1(H,S)$) is finite. If T is discrete, or if T is a finitely generated $\mathbf{Z}_p$-module, or if T is a finite-dimensional $\mathbf{Q}_p$-vector space, then*

there is a Hochschild-Serre exact sequence extending the sequence in (i)

$$0 \longrightarrow H^1(G/H, T^H) \longrightarrow H^1(G,T) \longrightarrow H^1(H,T)^{G/H} \longrightarrow H^2(G/H, T^H) \longrightarrow H^2(G,T).$$

Proof. If T is discrete both assertions are standard. The proof of (i) in general is identical to the proof in this classical case.

Suppose T is a finitely generated $\mathbf{Z}_p$-module. Then for every $n \geq 0$, the quotient T/p^nT is discrete, so there is a Hochschild-Serre exact sequence for T/p^nT. Our hypotheses ensure that all the terms in this sequence are finite, and so taking the inverse limit over n and applying Proposition B.2.3 gives the exact sequence of (ii) for T.

If T is a finite-dimensional $\mathbf{Q}_p$-vector space, choose a G-stable $\mathbf{Z}_p$-lattice $T_0 \subset T$. Then by the previous case we have a Hochschild-Serre exact sequence for T_0, and tensoring with $\mathbf{Q}_p$ and using Proposition B.2.4 gives the desired exact sequence for T. □

Remark B.2.6. To apply Proposition B.2.5(ii) we need to know when a group G has the property that $H^i(G,S)$ is finite for every i and every G-module S of finite p-power order. For example, this is true whenever the pro-p-part of G is (topologically) finitely generated.

We also have the following well-known result. In the most important case $i = 1$ it follows easily from class field theory (see for example [**Se2**] Propositions II.14 and III.8). We say that a $\mathbf{Z}_p$-module is co-finitely generated if its Pontryagin dual is finitely generated.

Proposition B.2.7. *Suppose that one of the following conditions holds.*

(i) *K is a global field, K_S is a (possibly infinite) Galois extension of K unramified outside a finite set of places of K, and $G = \mathrm{Gal}(K_S/K)$,*
(ii) *K is a local field and $G = G_K$, or*
(iii) *K is a local field of residue characteristic different from p and G is the inertia group in G_K.*

If T is a G-module which is finite (resp. finitely generated over $\mathbf{Z}_p$, resp. co-finitely generated over $\mathbf{Z}_p$, resp. finite-dimensional over $\mathbf{Q}_p$) and $i \geq 0$, then $H^i(G,T)$ is finite (resp. finitely generated over $\mathbf{Z}_p$, resp. co-finitely generated over $\mathbf{Z}_p$, resp. finite-dimensional over $\mathbf{Q}_p$).

Let $\hat{\mathbf{Z}}$ denote the profinite completion of $\mathbf{Z}$.

Lemma B.2.8. *Suppose G is a topologically cyclic profinite group with infinite pro-p-part (for example, $G \cong \hat{\mathbf{Z}}$ or $G \cong \mathbf{Z}_p$), and γ is a topological*

generator of G. Suppose T is a $\mathbf{Z}_p[G_K]$-module which is either a finitely generated $\mathbf{Z}_p$-module, or a finite-dimensional $\mathbf{Q}_p$-vector space, or a discrete torsion $\mathbf{Z}_p$-module. Then

$$H^1(G,T) \cong T/(\gamma-1)T$$

with an isomorphism induced by evaluating cocycles at γ.

Proof. It is easy to see that evaluating cocycles at γ induces a well-defined, injective map $H^1(G,T) \to T/(\gamma-1)T$. It remains only to show that this map is surjective.

Using direct limits, inverse limits (Proposition B.2.3), and/or tensoring with $\mathbf{Q}_p$ (Proposition B.2.4), we can reduce this lemma to the case where T is finite. When T is finite, the lemma is well-known, see for example [**Se3**] §XIII.1. □

B.3. Inverse Limits

For this section suppose that K is a field, p is a rational prime, and T is a $\mathbf{Z}_p[G_K]$-module which is finitely generated over $\mathbf{Z}_p$.

As usual, we write $K \subset_{\mathrm{f}} F$ to indicate that F is a finite extension of K. If K_∞ is an infinite extension of K and $\{C_F : K \subset_{\mathrm{f}} F \subset K_\infty\}$ is an inverse system of abelian groups, we will write $\{c_F\}$ for a typical element of $\varprojlim C_F$ with $c_F \in C_F$.

Lemma B.3.1. *If $K \subset_{\mathrm{f}} F_1 \subset_{\mathrm{f}} F_2 \subset_{\mathrm{f}} \cdots$ and $\cup_{n=1}^{\infty} F_n = K_\infty$ then*

$$\varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^1(F,T) = \varprojlim_n H^1(F_n, T/p^nT).$$

Proof. By Proposition B.2.3 we have

$$\begin{aligned}\varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^1(F,T) = \varprojlim_n H^1(F_n,T) &= \varprojlim_n \varprojlim_m H^1(F_n, T/p^mT) \\ &= \varprojlim_n H^1(F_n, T/p^nT).\end{aligned}$$

□

Lemma B.3.2. *Suppose K_∞ is an infinite p-extension of K. Then*

$$\varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} T^{G_F} = 0$$

where the maps in the inverse system are given by the norm maps

$$\mathbf{N}_{F'/F} : T^{G_{F'}} \longrightarrow T^{G_F}$$

for $K \subset_{\mathrm{f}} F \subset_{\mathrm{f}} F' \subset K_\infty$.

Proof. Define a submodule T_0 of T by

$$T_0 \;=\; \cup_{K \subset_{\mathrm{f}} F \subset K_\infty} T^{G_F}.$$

Then T_0 is finitely generated over $\mathbf{Z}_p$ since T is, so we must have $T_0 = T^{G_{F_0}}$ for some finite extension F_0 of K in K_∞. Therefore

$$\varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} T^{G_F} \;=\; \varprojlim_{F_0 \subset_{\mathrm{f}} F \subset K_\infty} T^{G_F} \;=\; \varprojlim_{F_0 \subset_{\mathrm{f}} F \subset K_\infty} T_0$$

where the norm maps $\mathbf{N}_{F'/F}$ in the right-hand inverse system are multiplication by $[F' : F]$. Since T_0 is finitely generated over $\mathbf{Z}_p$, and for every F the index $[F' : F]$ is divisible by arbitrarily large powers of p as F' varies, this inverse limit is zero. □

If K is a finite extension of $\mathbf{Q}_\ell$ for some ℓ, let $H^1_{\mathrm{ur}}(K, T)$ denote the subgroup of unramified classes in $H^1(K, T)$, as defined in Definition 1.3.1.

Lemma B.3.3. *Suppose T is a discrete $\mathbf{Z}_p$-module, K is a finite extension of $\mathbf{Q}_\ell$ for some $\ell \neq p$, and K_∞ is an extension of K containing the unique $\mathbf{Z}_p$-extension of K. Then*

$$\varinjlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^1_{\mathrm{ur}}(F, T) \;=\; 0.$$

Proof. Without loss of generality we may assume that K_∞ is the $\mathbf{Z}_p$-extension of K, and then the general case will follow immediately. In particular K_∞/K is unramified.

If $K \subset_{\mathrm{f}} F \subset K_\infty$ then $H^1_{\mathrm{ur}}(F, T) = H^1(K^{\mathrm{ur}}/F, T^{\mathcal{I}})$ (Lemma 1.3.2(i)) where $\mathcal{I} \subset G_{K_\infty}$ is the inertia group in G_K. Thus (since T is discrete)

$$\varinjlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^1_{\mathrm{ur}}(F, T) \;=\; H^1(K^{\mathrm{ur}}/K_\infty, T^{\mathcal{I}}).$$

But the pro-p-part of $\mathrm{Gal}(K^{\mathrm{ur}}/K_\infty)$ is trivial, so this is zero. □

Proposition B.3.4. *Suppose K is a finite extension of $\mathbf{Q}_\ell$ for some $\ell \neq p$, and K_∞ is the unique $\mathbf{Z}_p$-extension of K. If $\{c_F\} \in \varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^1(F, T)$ then*

$$c_F \;\in\; H^1_{\mathrm{ur}}(F, T) \;\textit{for every } F.$$

Proof. Let $\mathcal{I} \subset G_K$ denote the inertia group. Since $\ell \neq p$, the extension K_∞/K is unramified, so $\mathcal{I}$ is also the inertia group in G_F for every $F \subset K_\infty$. Thus for $K \subset_{\mathrm{f}} F \subset K_\infty$ we have an exact sequence

$$0 \longrightarrow H^1_{\mathrm{ur}}(F, T) \longrightarrow H^1(F, T) \longrightarrow H^1(\mathcal{I}, T)^{G_F}.$$

Since $\ell \neq p$, Proposition B.2.7(iii) shows that $H^1(\mathcal{I}, T)$ is finitely generated over $\mathbf{Z}_p$. Therefore taking inverse limits with respect to F and applying Lemma B.3.2 to the G_K-module $H^1(\mathcal{I}, T)$ shows that

$$\varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^1_{\mathrm{ur}}(F, T) = \varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^1(F, T)$$

which proves the proposition. □

For the next two corollaries, suppose that K is a number field and K_∞ is an abelian extension of K satisfying

$$\mathrm{Gal}(K_\infty/K) \cong \mathbf{Z}_p^d \text{ for some } d \geq 1.$$

Corollary B.3.5. *Suppose*

$$\{c_F\} \in \varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^1(F, T).$$

If $K \subset_{\mathrm{f}} F \subset K_\infty$, and v is a prime of F not dividing p whose decomposition group in $\mathrm{Gal}(K_\infty/K)$ is infinite, then $(c_F)_v \in H^1_{\mathrm{ur}}(F_v, T)$.

Proof. Let w be a prime of K_∞ above v. Since the decomposition group of v in $\mathrm{Gal}(K_\infty/K)$ is infinite, if $K \subset_{\mathrm{f}} F \subset K_\infty$ we can find $F \subset_{\mathrm{f}} F' \subset F_\infty \subset K_\infty$ such that $\mathrm{Gal}(F_\infty/F') \cong \mathbf{Z}_p$ and w is undecomposed in F_∞/F'. Thus Proposition B.3.4 applied to the classes $\{(c_L)_w : F' \subset_{\mathrm{f}} L \subset F_\infty\}$ shows that $(c_{F'})_w \in H^1_{\mathrm{ur}}(F'_w, T)$. Since this holds for all w above v, and $\mathrm{Cor}_{F'/F}(c_{F'}) = c_F$, we deduce that $(c_F)_v \in H^1_{\mathrm{ur}}(F_v, T)$. □

Corollary B.3.6. *If S is a set of places of K containing all primes where T is ramified, all primes dividing p, all primes whose decomposition group in $\mathrm{Gal}(K_\infty/K)$ is finite, and all infinite places, then*

$$\varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^1(F, T) = \varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^1(K_S/F, T)$$

where K_S is the maximal extension of K unramified outside S.

Proof. Suppose that

$$\{c_F\} \in \varprojlim_{K \subset_{\mathrm{f}} F \subset K_\infty} H^1(F, T).$$

Fix a field F such that $K \subset_{\mathrm{f}} F \subset K_\infty$. Let $\mathcal{I} \subset G_K$ be an inertia group of a prime $\mathfrak{q}$ of K not in S. Since F/K is unramified at $\mathfrak{q}$, we see that $\mathcal{I}$ is also an inertia group of a prime $\mathcal{Q}$ of F above $\mathfrak{q}$, so by Corollary B.3.5 the restriction of c_F is zero in $H^1(\mathcal{I}, T) = \mathrm{Hom}(\mathcal{I}, T)$. It follows that every cocycle representing c_F factors through $\mathrm{Gal}(K_S/F)$, which proves the corollary. □

B.4. Induced Modules

Again we suppose that G is a profinite group, and now H is a closed subgroup of G and T is a discrete H-module (not necessarily a G-module).

Definition B.4.1. Define the induced module $\mathrm{Ind}_H(T) = \mathrm{Ind}_H^G(T)$ by

$$\mathrm{Ind}_H(T) = \{f \in \mathrm{Maps}(G,T) : f(\eta\gamma) = \eta f(\gamma) \text{ for every } \gamma \in G, \eta \in H\}.$$

We let G act on $\mathrm{Ind}_H(T)$ by

$$(gf)(\gamma) \;=\; f(\gamma g) \;\text{ for } g, \gamma \in G.$$

Since T is discrete, $\mathrm{Ind}_H(T)$ a discrete G-module.

If $H = \{1\}$, then $\mathrm{Ind}_H(T)$ is simply $\mathrm{Maps}(G,T)$. If H' is a closed subgroup of H then there is a natural inclusion $\mathrm{Ind}_H(T) \subset \mathrm{Ind}_{H'}(T)$. If T is a G-module then evaluation at 1 induces an isomorphism $\mathrm{Ind}_G(T) \xrightarrow{\sim} T$, and so there is a natural (continuous) inclusion $T \hookrightarrow \mathrm{Ind}_H(T)$, which sends $t \in T$ to the map $\gamma \mapsto \gamma t$.

Proposition B.4.2. *Suppose T, G, and H are as above, and Γ is an open subgroup of G. For every $i \geq 0$ there is an isomorphism*

$$H^i(\Gamma, \mathrm{Ind}_H(T)) \;\cong \bigoplus_{g \in H\backslash G/\Gamma} H^i(g\Gamma g^{-1} \cap H, T).$$

Proof. First suppose $i = 0$. Fix a set $S \subset G$ of double coset representatives for $H\backslash G/\Gamma$. If $f \in \mathrm{Ind}_H(T)^\Gamma$ then for every $s \in S$,

$$f(hs\gamma) \;=\; h(f(s)) \;\text{ for every } h \in H \text{ and } \gamma \in \Gamma. \tag{B.1}$$

In particular if $h \in s\Gamma s^{-1} \cap H$, then $h(f(s)) = f(s)$, so $f(s) \in T^{(s\Gamma s^{-1} \cap H)}$. Conversely, if for every $s \in S$ we have an element $f(s) \in T^{s\Gamma s^{-1} \cap H}$, then we can use (B.1) to define an element $f \in \mathrm{Ind}_H(T)^\Gamma$. This proves the proposition when $i = 0$.

Now suppose $i \geq 1$. The functor $T \rightsquigarrow \mathrm{Ind}_H(T)$ is exact on the category of discrete H-modules, so the proposition for T with $i \geq 1$ follows from the case $i = 0$ and the Leray spectral sequence comparing the functors

$$A \rightsquigarrow \bigoplus_{g \in H\backslash G/\Gamma} A^{g\Gamma g^{-1} \cap H}, \quad A \rightsquigarrow \mathrm{Ind}_H(A), \quad B \rightsquigarrow B^\Gamma$$

(see for example [**Sh**] pp. 50–51). □

Remark B.4.3. When $\Gamma = G$, Proposition B.4.2 is Shapiro's Lemma (see for example the exercise on p. 125, §VII.5 of [**Se3**]).

Corollary B.4.4. *With T, G, and H as above, for every open subgroup Γ of G there is an exact sequence*

$$0 \longrightarrow \mathrm{Ind}_H(T)^\Gamma \longrightarrow \mathrm{Ind}_{\{1\}}(T)^\Gamma \longrightarrow (\mathrm{Ind}_{\{1\}}(T)/\mathrm{Ind}_H(T))^\Gamma \longrightarrow H^1(\Gamma, \mathrm{Ind}_H(T)) \longrightarrow 0$$

Proof. By Proposition B.4.2 with $H = \{1\}$, we have $H^1(\Gamma, \mathrm{Ind}_{\{1\}}(T)) = 0$. Thus the long exact Γ-cohomology sequence of the canonical exact sequence

$$0 \longrightarrow \mathrm{Ind}_H(T) \longrightarrow \mathrm{Ind}_{\{1\}}(T) \longrightarrow \mathrm{Ind}_{\{1\}}(T)/\mathrm{Ind}_H(T) \longrightarrow 0$$

gives the exact sequence of the corollary. □

Proposition B.4.5. *Suppose K is a field, F is a finite extension of K, and T is a discrete G_K-module. Let $\mathbb{T} = \mathrm{Ind}_{\{1\}}^{G_K}(T)$. Then there is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T^{G_K} & \longrightarrow & \mathbb{T}^{G_K} & \longrightarrow & (\mathbb{T}/T)^{G_K} & \longrightarrow & H^1(K,T) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \mathrm{Res}_F \\
0 & \longrightarrow & T^{G_F} & \longrightarrow & \mathbb{T}^{G_F} & \longrightarrow & (\mathbb{T}/T)^{G_F} & \longrightarrow & H^1(F,T) \longrightarrow 0 \\
 & & \downarrow \mathbf{N}_{F/K} & & \downarrow \mathbf{N}_{F/K} & & \downarrow \mathbf{N}_{F/K} & & \downarrow \mathrm{Cor}_{F/K} \\
0 & \longrightarrow & T^{G_K} & \longrightarrow & \mathbb{T}^{G_K} & \longrightarrow & (\mathbb{T}/T)^{G_K} & \longrightarrow & H^1(K,T) \longrightarrow 0.
\end{array}$$

Proof. The horizontal sequences are the exact sequences of Corollary B.4.4 applied with $H = G_K$ and $\Gamma = G_K$ or G_F. The commutativity of the lower right square is essentially the definition of the corestriction map, and the rest of the commutativity is clear. □

B.5. Semilocal Galois Cohomology

Suppose for this section that K is a number field, that $\mathfrak{q}$ is a prime of K, that F is a finite extension of K, and that S is the set of primes of F above $\mathfrak{q}$. For every prime $\mathcal{Q} \in S$ fix a prime $\mathfrak{Q}$ of $\bar{K}$ above $\mathcal{Q}$ and let $\mathcal{I}_\mathcal{Q} \subset \mathcal{D}_\mathcal{Q} \subset G_K$ denote the inertia group and decomposition group of $\mathfrak{Q}$. Fix a prime $\mathcal{Q}_0 \in S$ and write $\mathcal{D} = \mathcal{D}_{\mathcal{Q}_0}$ and $\mathcal{I} = \mathcal{I}_{\mathcal{Q}_0}$. For every $\mathcal{Q} \in S$ fix $g_\mathcal{Q} \in G_K$ such that $\mathfrak{Q} = g_\mathcal{Q}^{-1}\mathfrak{Q}_0$. Then $\mathcal{D}_\mathcal{Q} = g_\mathcal{Q}^{-1}\mathcal{D}g_\mathcal{Q}$.

Let T be a discrete G_K-module, and let $T' \subset T$ be a subset which is a $\mathcal{D}$-submodule, i.e., $\mathcal{D}$ sends T' into itself. For every $\mathcal{Q} \in S$ we let $T'_\mathcal{Q} = g_\mathcal{Q}^{-1}T'$, and then $T'_\mathcal{Q}$ is a $\mathcal{D}_\mathcal{Q}$-module.

Proposition B.5.1. *With notation as above, if $i \geq 0$ then there is a canonical isomorphism*

$$H^i(F, \mathrm{Ind}_{\mathcal{D}}^{G_K}(T')) \cong \bigoplus_{\mathcal{Q} \in S} H^i(F_{\mathcal{Q}}, T'_{\mathcal{Q}}).$$

Proof. The map

$$\begin{array}{ccc} \mathcal{D}\backslash G_K / G_F & \longrightarrow & S \\ \mathcal{D} g G_F & \mapsto & g^{-1}\mathcal{Q}_0 \end{array}$$

is a bijection. Applying Proposition B.4.2 with $G = G_K$, with $H = \mathcal{D}$, and with $\Gamma = G_F$ yields

$$\begin{aligned} H^i(F, \mathrm{Ind}_{\mathcal{D}}^{G_K}(T')) &\cong \bigoplus_{\mathcal{Q} \in S} H^i(g_{\mathcal{Q}} G_F g_{\mathcal{Q}}^{-1} \cap \mathcal{D}, T') \\ &\cong \bigoplus_{\mathcal{Q} \in S} H^i(G_F \cap \mathcal{D}_{\mathcal{Q}}, T'_{\mathcal{Q}}) = \bigoplus_{\mathcal{Q} \in S} H^i(F_{\mathcal{Q}}, T'_{\mathcal{Q}}). \end{aligned}$$

An examination of the proof of Proposition B.4.2 shows that this isomorphism is independent of the choices of the representatives $g_{\mathcal{Q}}$. □

Corollary B.5.2. *With notation as above, there are canonical isomorphisms*

$$\begin{aligned} H^i(F, \mathrm{Ind}_{\mathcal{D}}^{G_K}(T)) &\cong \bigoplus_{\mathcal{Q} \in S} H^i(F_{\mathcal{Q}}, T), \\ H^i(F, \mathrm{Ind}_{\mathcal{D}}^{G_K}(T^{\mathcal{I}})) &\cong \bigoplus_{\mathcal{Q} \in S} H^i(F_{\mathcal{Q}}, T^{\mathcal{I}_{\mathcal{Q}}}). \end{aligned}$$

Proof. Apply Proposition B.5.1 with $T' = T$ and with $T' = T^{\mathcal{I}}$. □

Corollary B.5.3. *Suppose F is a finite Galois extension of K and T is a finitely generated $\mathbf{Z}_p$-module with a continuous action of G_K. Let $V = T \otimes \mathbf{Q}_p$.*

(i) *If $[F : K]$ is prime to p, then the restriction map induces an isomorphism*

$$H^1(K_{\mathfrak{q}}, T) \cong \left(\oplus_{\mathcal{Q}|\mathfrak{q}} H^1(F_{\mathcal{Q}}, T)\right)^{\mathrm{Gal}(F/K)}.$$

(ii) *The restriction map induces an isomorphism*

$$H^1(K_{\mathfrak{q}}, V) \cong \left(\oplus_{\mathcal{Q}|\mathfrak{q}} H^1(F_{\mathcal{Q}}, V)\right)^{\mathrm{Gal}(F/K)}.$$

Proof. For every $n \geq 0$ we have a commutative diagram

$$\begin{array}{ccc} H^1(K, \mathrm{Ind}_{\mathcal{D}}^{G_K}(T/p^nT)) & \xrightarrow{\sim} & H^1(K_{\mathfrak{q}}, T/p^nT) \\ \downarrow{\scriptstyle \mathrm{Res}_F} & & \downarrow{\scriptstyle \oplus \mathrm{Res}_{\mathcal{Q}}} \\ H^1(F, \mathrm{Ind}_{\mathcal{D}}^{G_K}(T/p^nT))^{\mathrm{Gal}(F/K)} & \xrightarrow{\sim} & \left(\oplus_{\mathcal{Q} \in S} H^1(F_{\mathcal{Q}}, T/p^nT)\right)^{\mathrm{Gal}(F/K)} \end{array}$$

where the vertical maps are restriction maps and the horizontal maps are the isomorphisms of Corollary B.5.2. The inflation-restriction sequence shows that the left-hand vertical map has kernel and cokernel annihilated by $[F : K]$, and hence the right-hand map does as well. Taking the inverse limit of the right-hand maps and applying Proposition B.2.3 shows that the restriction map

$$H^1(K_{\mathfrak{q}}, T) \longrightarrow \left(\oplus_{\mathcal{Q}\in S} H^1(F_{\mathcal{Q}}, T)\right)^{\mathrm{Gal}(F/K)}$$

has kernel and cokernel annihilated by $[F : K]$. This proves (i), and (ii) follows by tensoring with $\mathbf{Q}_p$ and using Proposition B.2.4. □

APPENDIX C

Cohomology of p-adic Analytic Groups

Suppose that F is a number field and W is a G_F-module. When analyzing a Selmer group $\mathcal{S}(F,W)$ as defined in §1.5, one frequently wants to restrict to an extension Ω of F such that the action of G_Ω on W is trivial, and then study the image of $\mathcal{S}(F,W)$ under the restriction map

$$H^1(F,W) \longrightarrow H^1(\Omega,W) = \mathrm{Hom}(G_\Omega, W).$$

In this appendix we will show (Corollary C.2.2) how to control the kernel of this restriction map.

C.1. Irreducible Actions of Compact Groups

Theorem C.1.1. *Suppose V is a finite-dimensional $\mathbf{Q}_p$-vector space, and G is a compact subgroup of $\mathrm{GL}(V)$ which acts irreducibly on V. Then $H^1(G,V) = 0$.*

The proof will be divided into a series of lemmas. For this section we fix a finite-dimensional $\mathbf{Q}_p$-vector space V and a compact subgroup G of $\mathrm{GL}(V)$ which acts irreducibly on V, as in the statement of Theorem C.1.1. Let Z denote the center of G.

Lemma C.1.2. *If $g \in Z$ and $g \neq 1$, then $g-1$ is invertible on V.*

Proof. Let $V_1 = \ker(g-1)$. Since g is in the center of G, the subspace V_1 is stable under G. Since $g \neq 1$ we have $V_1 \neq V$. Hence by our irreducibility assumption, $V_1 = 0$. □

Lemma C.1.3. *If $Z \neq \{1\}$ then $H^1(G,V) = 0$.*

Proof. Suppose that $g \in Z$ and $g \neq 1$. Let B be the closed subgroup of Z generated by g. We have an inflation-restriction exact sequence

$$0 \longrightarrow H^1(G/B, V^B) \longrightarrow H^1(G,V) \longrightarrow H^1(B,V).$$

By Lemma C.1.2 we have $V^B = 0$, and

$$H^1(B,V) \subset V/(g-1)V = 0.$$ □

Lemma C.1.4. *Suppose U is an open normal subgroup of G. Then V is completely reducible as a representation of U.*

Proof. Let V_0 denote the subspace of V generated by all irreducible U-subspaces of V. Since U is normal in G, we see that V_0 is stable under G. Clearly $V_0 \neq 0$, so the irreducibility hypothesis for G implies that $V_0 = V$. It follows easily that V is a direct sum of a finite collection of irreducible U-subspaces. □

For a general reference for the background material on p-adic Lie groups, Lie algebras, and their cohomology which we use below, see [**Laz**] or [**Bo**].

Proposition C.1.5. $\mathrm{Lie}(G)$ *is reductive.*

Proof. It follows from Lemma C.1.4 that the representation of $\mathrm{Lie}(G)$ on V is semisimple, and it is clearly also faithful. By [**Bo**] §I.6.4 Proposition 5, it follows that $\mathrm{Lie}(G)$ is reductive. □

Proof of Theorem C.1.1. The compact subgroup G of $\mathrm{GL}(V)$ is a profinite p-analytic group in the sense of [**Laz**] §III.3.2. Therefore by Lazard's Théorème [**Laz**] V.2.4.10, for every sufficiently small open normal subgroup U of G,

$$H^1(G,V) \;=\; H^1(U,V)^G \;=\; H^1(\mathrm{Lie}(G),V)^G.$$

If the center of $\mathrm{Lie}(G)$ is zero then (since $\mathrm{Lie}(G)$ is reductive by Proposition C.1.5) $\mathrm{Lie}(G)$ is semisimple, and in that case (see [**Bo**] Exercise 1(b), §I.6) $H^1(\mathrm{Lie}(G),V) = 0$. If the center of $\mathrm{Lie}(G)$ is not zero then every sufficiently small open normal subgroup U of G has nontrivial center, and then Lemmas C.1.3 and C.1.4 together show that $H^1(U,V) = 0$. Thus in either case we can conclude that $H^1(G,V) = 0$. □

Recall that Z is the center of G.

Lemma C.1.6. *Suppose that $\mathcal{O}$ is the ring of integers of a finite extension Φ of $\mathbf{Q}_p$, that V is a Φ-vector space, and that G acts Φ-linearly on V. If G contains an element g such that* $\dim_\Phi(V/(g-1)V) = 1$, *then Z acts on V via scalars in $\mathcal{O}^\times$.*

Proof. The one-dimensional subspace $\ker(g-1)$ of V is preserved by Z. Let $\chi : Z \to \mathrm{Aut}(\ker(g-1)) \cong \Phi^\times$ be the character determined by this action. Since Z is compact, $\chi(Z) \subset \mathcal{O}^\times$. Let

$$V_\chi \;=\; \{v \in V : zv = \chi(z)v \text{ for every } z \in Z\}.$$

Then V_χ is nonzero and stable under G, so the irreducibility of V implies that $V_\chi = V$. □

Proposition C.1.7. *Suppose that A is an abelian quotient of G. Then the projection of Z to A has finite cokernel.*

Proof. Let $\pi : G \twoheadrightarrow A$ be the projection map. Since A is compact, it is finitely generated.

By Proposition C.1.5, G is reductive. It follows easily that the induced map of Lie algebras sends the center of $\mathrm{Lie}(G)$ onto $\mathrm{Lie}(A)$. Therefore $[A : \pi(Z_U)]$ is finite, where Z_U is the center of a sufficiently small open normal subgroup U of G.

The finite group G/U acts on Z_U by conjugation, and we define

$$\mathbf{N}(z) = \sum_{g \in G/U} z^g$$

(writing Z_U as an additive group). Clearly $\mathbf{N}(Z_U) \subset Z$, and also (since $\ker(\pi)$ contains all commutators) $\pi(\mathbf{N}(z)) = \pi([G : U]z)$ for every $z \in Z_U$. Therefore $\pi(Z)$ contains $[G : U]\pi(Z_U)$. This completes the proof. □

C.2. Application to Galois Representations

For this section fix a (possibly infinite) Galois extension F/K of fields of characteristic different from p, and a subgroup B of $K^\times$. (In our applications K will be a number field, F will be a (possibly infinite) abelian extension of K, and B will be $\mathcal{O}_K^\times$.) Suppose that $\mathcal{O}$ is the ring of integers of a finite extension Φ of $\mathbf{Q}_p$, and V is a finite-dimensional Φ-vector space with a continuous Φ-linear action of G_K such that V is irreducible over G_F. Let $\Omega = F(\boldsymbol{\mu}_{p^\infty}, B^{1/p^\infty}, V)$, the smallest extension of F whose absolute Galois group acts trivially on $\boldsymbol{\mu}_{p^\infty}$, on B^{1/p^∞}, and on V. The result we will need for our applications to Selmer groups is the following.

Theorem C.2.1. *One of the following three situations holds.*

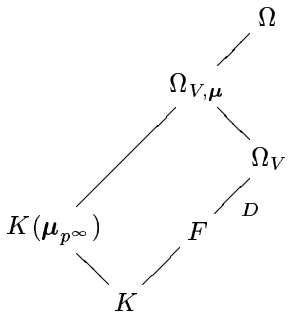
(i) $H^1(\Omega/F, V) = 0$.

(ii) *G_K acts on V via a character ρ of* $\mathrm{Gal}(F/K)$, *and* $\mathrm{Gal}(F/K)$ *acts on $H^1(\Omega/F, V)$ via ρ.*

(iii) *B is infinite, G_K acts on V via $\varepsilon_{\mathrm{cyc}}\rho$ where $\varepsilon_{\mathrm{cyc}}$ is the cyclotomic character and ρ is a character of* $\mathrm{Gal}(F/K)$, *and* $\mathrm{Gal}(F/K)$ *acts on $H^1(\Omega/F, V)$ via ρ.*

Proof. Let $\Omega_V = F(V)$, the smallest extension of F such that G_{Ω_V} acts trivially on V (so $\Omega_V = \bar{F}^H$ where $H = \ker(G_F \to \mathrm{Aut}(V))$, and Ω_V is necessarily Galois over F). Define $D = \mathrm{Gal}(\Omega_V/F)$ and $\Omega_{V,\boldsymbol{\mu}} = \Omega_V(\boldsymbol{\mu}_{p^\infty})$.

We have a diagram



The inflation-restriction exact sequence gives

$$H^1(D,V) \longrightarrow H^1(\Omega/F,V) \longrightarrow H^1(\Omega/\Omega_V,V)^D.$$

The map $D \to \mathrm{Aut}(V)$ is injective by the definition of Ω_V, so D is isomorphic to a compact subgroup of $\mathrm{GL}(V)$. We have assumed that D acts irreducibly on V, so Theorem C.1.1 shows that $H^1(D,V) = 0$. Thus we have an injection

$$H^1(\Omega/F,V) \hookrightarrow H^1(\Omega/\Omega_V,V)^D = \mathrm{Hom}(\mathrm{Gal}(\Omega/\Omega_V),V)^D.$$

If $\mathrm{Hom}(\mathrm{Gal}(\Omega/\Omega_V),V)^D = 0$ then (i) holds. We consider two cases.

Case I: $\Omega_{V,\boldsymbol{\mu}} \neq \Omega_V$. In this case $\mathrm{Gal}(\Omega_{V,\boldsymbol{\mu}}/\Omega_V)$ acts on $\mathrm{Gal}(\Omega/\Omega_{V,\boldsymbol{\mu}})$ via the (nontrivial) cyclotomic character. Let Ω_{ab} denote the maximal abelian extension of Ω_V in Ω. Then $\mathrm{Gal}(\Omega_{V,\boldsymbol{\mu}}/\Omega_V)$ acts on $\mathrm{Gal}(\Omega_{\mathrm{ab}}/\Omega_{V,\boldsymbol{\mu}})$ trivially *and* via the cyclotomic character. We deduce from this that $\mathrm{Gal}(\Omega_{\mathrm{ab}}/\Omega_{V,\boldsymbol{\mu}})$ is killed by $|\boldsymbol{\mu}_{p^\infty} \cap \Omega_V|$, which is finite since $\Omega_{V,\boldsymbol{\mu}} \neq \Omega_V$. Hence $\mathrm{Hom}(\mathrm{Gal}(\Omega_{\mathrm{ab}}/\Omega_{V,\boldsymbol{\mu}}),V) = 0$, so

$$\begin{aligned}\mathrm{Hom}(\mathrm{Gal}(\Omega/\Omega_V),V)^D &= \mathrm{Hom}(\mathrm{Gal}(\Omega_{\mathrm{ab}}/\Omega_V),V)^D \\ &= \mathrm{Hom}(\mathrm{Gal}(\Omega_{V,\boldsymbol{\mu}}/\Omega_V),V)^D = \mathrm{Hom}(\mathrm{Gal}(\Omega_{V,\boldsymbol{\mu}}/\Omega_V),V^D)\end{aligned}$$

since D (and in fact all of $\mathrm{Gal}(\Omega_V/K)$) acts trivially on $\mathrm{Gal}(\Omega_{V,\boldsymbol{\mu}}/\Omega_V)$. Since D acts irreducibly on V, either $V^D = 0$ or V is one-dimensional with trivial action of G_F. Therefore (i) or (ii) is satisfied in this case.

Case II: $\Omega_{V,\boldsymbol{\mu}} = \Omega_V$. In this case $\boldsymbol{\mu}_{p^\infty} \subset \Omega_V$, so $\mathrm{Gal}(\Omega/\Omega_V)$ is abelian and $\mathrm{Gal}(\Omega_V/K)$ acts on $\mathrm{Gal}(\Omega/\Omega_V)$ via the cyclotomic character. Thus

$$\mathrm{Hom}(\mathrm{Gal}(\Omega/\Omega_V),V)^D = \mathrm{Hom}(\mathrm{Gal}(\Omega/\Omega_V),V^{\varepsilon_{\mathrm{cyc}}})$$

where $V^{\varepsilon_{\mathrm{cyc}}}$ denotes the subspace of V on which D (and hence G_F) acts via $\varepsilon_{\mathrm{cyc}}$. Again, since D acts irreducibly on V, either $V^{\varepsilon_{\mathrm{cyc}}} = 0$ or V is one-dimensional with G_F acting via $\varepsilon_{\mathrm{cyc}}$. Therefore (i) or (iii) is satisfied in this case. □

Corollary C.2.2. *Suppose T is a finitely generated $\mathcal{O}$-submodule of V which is stable under G_K, and let $W = V/T$. Then one of the following three situations holds.*

(i) *$H^1(\Omega/F, W)$ is finite.*

(ii) *G_K acts on T via a character ρ of $\mathrm{Gal}(F/K)$, and $H^1(\Omega/F, W)$ has a subgroup of finite index on which $\mathrm{Gal}(F/K)$ acts via ρ.*

(iii) *B is infinite, G_K acts on T via $\varepsilon_{\mathrm{cyc}}\rho$ where $\varepsilon_{\mathrm{cyc}}$ is the cyclotomic character and ρ is a character of $\mathrm{Gal}(F/K)$, and $H^1(\Omega/F, W)$ has a subgroup of finite index on which $\mathrm{Gal}(F/K)$ acts via ρ.*

Proof. From Proposition B.2.4 and the exact sequence

$$H^1(\Omega/F, V) \to H^1(\Omega/F, W) \to H^2(\Omega/F, T) \to H^2(\Omega/F, V)$$

we see that the cokernel of $H^1(\Omega/F, V) \to H^1(\Omega/F, W)$ is $H^2(\Omega/F, T)_{\mathrm{tors}}$. Since $\mathrm{Gal}(\Omega/F)$ is (topologically) finitely generated, $H^2(\Omega/F, T)_{\mathrm{tors}}$ is finite. Now the corollary is immediate from Theorem C.2.1. □

APPENDIX D

p-adic Calculations in Cyclotomic Fields

In this appendix we carry out some p-adic calculations in cyclotomic fields which are used in examples in Chapters 3 and 8. Everything here is basically well-known, due originally to Iwasawa and Coleman.

For every $n \in \mathbf{Z}^+$ fix a primitive n-th root of unity ζ_n such that $\zeta_{mn}^n = \zeta_m$ for every m and n. By slight abuse of notation, for every n we will write $\mathbf{Z}_p[\boldsymbol{\mu}_n] = \mathbf{Z}[\boldsymbol{\mu}_n] \otimes \mathbf{Z}_p$, the p-adic completion of $\mathbf{Z}[\boldsymbol{\mu}_n]$, and similarly $\mathbf{Q}_p(\boldsymbol{\mu}_n) = \mathbf{Q}(\boldsymbol{\mu}_n) \otimes \mathbf{Q}_p$.

Define

$$\log : \mathbf{Z}_p[\boldsymbol{\mu}_n][[X]]^\times = \mathbf{Z}_p[\boldsymbol{\mu}_n]^\times \times (1 + X\mathbf{Z}_p[\boldsymbol{\mu}_n][[X]]) \longrightarrow \mathbf{Q}_p(\boldsymbol{\mu}_n)[[X]]$$

to be the usual p-adic logarithm on $\mathbf{Z}_p[\boldsymbol{\mu}_n]^\times$ and the power series expansion of $\log(1 + Xf(X))$ on $1 + X\mathbf{Z}_p[\boldsymbol{\mu}_n][[X]]$. If $\alpha \in \mathbf{Z}_p$ define

$$[\alpha](X) = (1+X)^\alpha - 1 \in X\mathbf{Z}_p[[X]].$$

Let D be the derivation $(1+X)\frac{d}{dX}$ of $\overline{\mathbf{Q}_p}[[X]]$. Then for every $\alpha \in \mathbf{Z}_p$ and $g \in \overline{\mathbf{Q}_p}[[X]]$, we have

$$D[\alpha] = \alpha \cdot ([\alpha](X) + 1) \text{ and } D(g \circ [\alpha]) = \alpha \cdot (Dg) \circ [\alpha].$$

If m is prime to p let Fr_p be the Frobenius of p in $\mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_m)/\mathbf{Q})$, i.e., the automorphism which sends ζ_m to ζ_m^p. We let Fr_p act on $\mathbf{Q}_p(\boldsymbol{\mu}_m)[[X]]$ by acting on the power series coefficients.

D.1. Local Units in Cyclotomic Fields

In this section we will construct, for every positive integer n, a homomorphism $\lambda_n : \mathbf{Z}_p[\boldsymbol{\mu}_n]^\times \to \mathbf{Z}_p$. These maps are used in §3.4 to construct an Euler system for the trivial representation $\mathbf{Z}_p$.

Fix an integer m prime to p. Define

$$f_m(X) = m\zeta_m[m^{-1}](X) - \frac{m\zeta_m^p}{|(\mathbf{Z}_p^\times)_{\mathrm{tors}}|} \sum_{\beta \in (\mathbf{Z}_p^\times)_{\mathrm{tors}}} \frac{[m^{-1}\beta](X)}{\beta}$$

$$\in |(\mathbf{Z}_p^\times)_{\mathrm{tors}}|^{-1} \mathbf{Z}_p[\boldsymbol{\mu}_m][[X]]$$

and

$$\mathcal{G}_m(X) = \zeta_m \log(1+X) - m\sum_{i=1}^{\infty} p^i \zeta_m^{p^{-i}} + \sum_{i=0}^{\infty}\Big(\frac{f_m^{\mathrm{Fr}_p^i}([p^i](X))}{p^i} - (\zeta_m^{p^i} - \zeta_m^{p^{i+1}})\log(1+X)\Big).$$

Lemma D.1.1(i) below shows that the latter sum converges to an element of $\mathbf{Q}_p(\boldsymbol{\mu}_m)[[X]]$, and a direct computation shows that

$$D\mathcal{G}_m(X) = \zeta_m + \sum_{i=0}^{\infty}\Big(\zeta_m^{p^i}[m^{-1}p^i](X) - \frac{\zeta_m^{p^{i+1}}}{|(\mathbf{Z}_p^\times)_{\mathrm{tors}}|}\sum_{\beta\in(\mathbf{Z}_p^\times)_{\mathrm{tors}}}[m^{-1}\beta p^i](X)\Big). \tag{D.1}$$

Lemma D.1.1. (i) *$\mathcal{G}_m(X) \in \mathbf{Q}_p(\boldsymbol{\mu}_m)[[X]]$, i.e., the sum in the definition of $\mathcal{G}_m(X)$ converges.*

(ii) *There is a unique $g_m(X) \in 1 + (p,X)\mathbf{Z}_p[\boldsymbol{\mu}_m][[X]]$ such that*

$$\log(g_m(X)) = \mathcal{G}_m(X).$$

(iii) *If ℓ is a prime different from p, then*

$$\mathrm{Tr}_{\mathbf{Q}_p(\boldsymbol{\mu}_{m\ell})/\mathbf{Q}_p(\boldsymbol{\mu}_m)} D\mathcal{G}_{\ell m}(X) = \begin{cases} -\ell D\mathcal{G}_m^{\mathrm{Fr}_\ell^{-1}}([\ell^{-1}](X)) & \text{if } \ell \nmid m, \\ 0 & \text{if } \ell \mid m. \end{cases}$$

(iv) $\sum_{\zeta\in\boldsymbol{\mu}_p} \mathcal{G}_m(\zeta(1+X)-1) = \mathcal{G}_m^{\mathrm{Fr}_p}([p](X))$.

(v) *If g_m is as in* (ii), *then* $\prod_{\zeta\in\boldsymbol{\mu}_p} g_m(\zeta(1+X)-1) = g_m^{\mathrm{Fr}_p}([p](X))$.

Proof. The first two assertions follow from Theorem 24 of [**Co**] applied with $a = -m\sum_{i=1}^{\infty} p^i\zeta_m^{p^{-i}}$, with $b = \zeta_m$, and with $f(X) = f_m(X) - (\zeta_m - \zeta_m^p)X$. Assertion (iii) follows directly from (D.1) and the fact that

$$\mathrm{Tr}_{\mathbf{Q}(\boldsymbol{\mu}_{m\ell})/\mathbf{Q}(\boldsymbol{\mu}_m)}\zeta_{m\ell} = \begin{cases} -\zeta_m^{\mathrm{Fr}_\ell^{-1}} & \text{if } \ell \nmid m, \\ 0 & \text{if } \ell \mid m. \end{cases}$$

The fourth assertion is similarly a direct computation, and then (v) follows from (iv), since log is injective on $1 + (p,X)\mathbf{Z}_p[\boldsymbol{\mu}_m][[X]]$. □

Definition D.1.2. Suppose $m \geq 1$ is prime to p, and let $N(m) = \prod_{\ell\mid m}\ell$, product over primes ℓ dividing m. Let $g_m(X) \in \mathbf{Z}_p[\boldsymbol{\mu}_m][[X]]^\times$ be as in Lemma D.1.1(ii). For $n \geq 0$ define

$$\alpha_{mp^n} = \prod_{d\mid m, N(m)\mid d}\Big(g_d^{\mathrm{Fr}_p^{-n}}(\zeta_{p^n}-1)\Big) \in \mathbf{Z}_p[\boldsymbol{\mu}_{mp^n}]^\times.$$

By Lemma D.1.1(v),

$$\mathbf{N}_{\mathbf{Q}(\boldsymbol{\mu}_{mp^{n+1}})/\mathbf{Q}(\boldsymbol{\mu}_{mp^n})}\alpha_{mp^{n+1}} = \begin{cases} \alpha_{mp^n} & \text{if } n \geq 1, \\ \alpha_m^{1-\mathrm{Fr}_p^{-1}} & \text{if } n = 0. \end{cases}$$

Suppose $\mathfrak{P}$ is a prime of $\mathbf{Q}(\boldsymbol{\mu}_m)$ above p. We will also write $\mathfrak{P}$ for the unique prime of $\mathbf{Q}(\boldsymbol{\mu}_{mp^n})$ above $\mathfrak{P}$, for every n. We let

$$\boldsymbol{\alpha}_{m,\mathfrak{P}} \in \mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_{mp^\infty})^{\mathrm{ab}}_{\mathfrak{P}}/\mathbf{Q}(\boldsymbol{\mu}_{mp^\infty})_{\mathfrak{P}})$$

be the image of $\{\alpha_{mp^n}\}_{n\geq 1}$ under the Artin map of local class field theory. Using the Kummer pairing we define

$$\lambda_{mp^n} : \mathbf{Z}_p[\boldsymbol{\mu}_{mp^n}]^\times \longrightarrow \mathbf{Z}_p$$

so that, writing $u \in \mathbf{Z}_p[\boldsymbol{\mu}_{mp^n}]^\times$ as $(u_{\mathfrak{P}}) \in \oplus_{\mathfrak{P}}\mathbf{Z}[\boldsymbol{\mu}_{mp^n}]^\times_{\mathfrak{P}}$, for every $k \geq 0$ we have

$$\prod_{\mathfrak{P}|p}(u_{\mathfrak{P}}^{p^{-k}})^{\boldsymbol{\alpha}_{m,\mathfrak{P}}-1} = \zeta_{p^k}^{m\lambda_{mp^n}(u)}.$$

The explicit reciprocity law gives the following description of the map λ_{mp^n}. Recall that D is the derivation $(1+X)\frac{d}{dX}$.

Proposition D.1.3. *If m is prime to p and $n \geq 0$ then*

$$\lambda_{mp^n}(u) = p^{-n}\mathrm{Tr}_{\mathbf{Q}_p(\boldsymbol{\mu}_{mp^n})/\mathbf{Q}_p}(x_{mp^n}\log_p(u))$$

where $\log_p$ is the usual p-adic logarithm and

$$x_{mp^n} = \begin{cases} m^{-1}\displaystyle\sum_{d|m,N(m)|d}(D\mathcal{G}_d^{\mathrm{Fr}_p^{-n}})(\zeta_{p^n}-1) & \text{if } n > 0, \\ m^{-1}\displaystyle\sum_{d|m,N(m)|d}(D\mathcal{G}_d)(0) - \frac{1}{p}(D\mathcal{G}_d^{\mathrm{Fr}_p^{-1}})(0) & \text{if } n = 0. \end{cases}$$

Proof. The formula of the proposition is the explicit reciprocity law of Wiles [**Wi**] (see also [**dS**] Theorem I.4.2) in the present situation. □

Lemma D.1.4. *For every $m \geq 1$ (not necessarily prime to p) and prime ℓ, there is a commutative diagram*

$$\begin{array}{ccc} \mathbf{Z}_p[\boldsymbol{\mu}_{m\ell}]^\times & \xrightarrow{\lambda_{m\ell}} & \\ \Big\uparrow{\scriptstyle 1 \text{ or } -\mathrm{Fr}_\ell} & & \mathbf{Z}_p \\ \mathbf{Z}_p[\boldsymbol{\mu}_m]^\times & \xrightarrow{\lambda_m} & \end{array}$$

where the vertical map is

$$\begin{cases} \text{the inclusion } \mathbf{Z}_p[\boldsymbol{\mu}_m]^\times \subset \mathbf{Z}_p[\boldsymbol{\mu}_{m\ell}]^\times & \text{if } \ell \mid m \text{ or } \ell = p, \\ -\mathrm{Fr}_\ell \text{ followed by that inclusion} & \text{if } \ell \nmid mp. \end{cases}$$

Proof. Let the x_m be as defined in Proposition D.1.3. Using Lemma D.1.1(iii) and (iv) we see that

$$\mathrm{Tr}_{\mathbf{Q}_p(\boldsymbol{\mu}_{m\ell})/\mathbf{Q}_p(\boldsymbol{\mu}_m)} x_{m\ell} = \begin{cases} x_m & \text{if } \ell \mid m \text{ or } \ell = p, \\ -\mathrm{Fr}_\ell^{-1} x_m & \text{if } \ell \nmid mp, \end{cases}$$

for every m and ℓ. Now the lemma follows from Proposition D.1.3. □

Let ω denote the Teichmüller character giving the action of $G_{\mathbf{Q}}$ on $\boldsymbol{\mu}_p$ (if p is odd) or on $\boldsymbol{\mu}_4$ (if $p = 2$).

Lemma D.1.5. *Suppose $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbf{Q}_p$, and $\chi : G_{\mathbf{Q}} \to \mathcal{O}^\times$ is a character of finite prime-to-p order. Let f be the conductor of χ, and suppose that $p^2 \nmid f$ and $\chi^{-1}\omega(p) \neq 1$ (where we view $\chi^{-1}\omega$ as a primitive Dirichlet character). Let $\Delta = \mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_f)/\mathbf{Q})$. Then $\sum_{\delta\in\Delta} \chi(\delta)\lambda_f^\delta$ generates the $\mathcal{O}$-module $\mathrm{Hom}(\mathbf{Z}_p[\boldsymbol{\mu}_f]^\times, \mathcal{O})^{\chi^{-1}}$ (the submodule of $\mathrm{Hom}(\mathbf{Z}_p[\boldsymbol{\mu}_f]^\times, \mathcal{O})$ on which Δ acts via χ^{-1}).*

Proof. Let $\lambda_{f,\chi} = \sum_{\delta\in\Delta} \chi(\delta)\lambda_f^\delta$. Write $f = mp^\epsilon$ with m prime to p and $\epsilon = 0$ or 1. Note that if $p = 2$, then we cannot have $\epsilon = 1$. Let x_f be as in Proposition D.1.3, and let y_f be the "conductor f" part of x_f, namely

$$y_f = \begin{cases} f^{-1}\big((D\mathcal{G}_f)(0) - \frac{1}{p}(D\mathcal{G}_f^{\mathrm{Fr}_p^{-1}})(0)\big) = f^{-1}(\zeta_f - \frac{1}{p}\zeta_f^{\mathrm{Fr}_p^{-1}}) & \text{if } \epsilon = 0, \\ m^{-1}(D\mathcal{G}_m^{\mathrm{Fr}_p^{-1}})(\zeta_p - 1) = m^{-1}(\zeta_f + \frac{p}{p-1}\zeta_m) & \text{if } \epsilon = 1. \end{cases}$$

By Proposition D.1.3,

$$\begin{aligned} \lambda_{f,\chi}(u) &= p^{-\epsilon} \sum_{\delta\in\Delta} \chi(\delta)\mathrm{Tr}_{\mathbf{Q}_p(\boldsymbol{\mu}_f)/\mathbf{Q}_p} x_f^\delta \log_p(u) \\ &= p^{-\epsilon} \sum_{\delta\in\Delta}\sum_{\gamma\in\Delta} \chi(\delta) x_f^{\delta\gamma} \log_p(u^\gamma) \\ &= p^{-\epsilon} \sum_{\delta\in\Delta} (\chi(\delta)x_f^\delta) \sum_{\gamma\in\Delta} (\chi^{-1}(\gamma)\log_p(u^\gamma)) \\ &= p^{-\epsilon} \sum_{\delta\in\Delta} (\chi(\delta)y_f^\delta) \sum_{\gamma\in\Delta} (\chi^{-1}(\gamma)\log_p(u^\gamma)) \\ &= \frac{1}{f}(1 - p^{-1}\chi(p)) \sum_{\delta\in\Delta} (\chi(\delta)\zeta_f^\delta) \sum_{\gamma\in\Delta} (\chi^{-1}(\gamma)\log_p(u^\gamma)). \end{aligned}$$

First suppose $p \nmid f$, so $\chi(p) \in \mathcal{O}^\times$. Let g_m be as in Lemma D.1.1(ii) and let $u = g_m(0)^{1/m} \in \mathbf{Z}_p[\boldsymbol{\mu}_f]^\times$. Then $\log_p(u) = m^{-1}\mathcal{G}_m(0) = -\sum_{i=1}^\infty p^i \zeta_m^{p^{-i}}$,

so

$$\sum_{\gamma\in\Delta}(\chi^{-1}(\gamma)\log_p(u^\gamma)) = -\sum_{i=1}^{\infty}p^i\sum_{\gamma\in\Delta}(\chi^{-1}(\gamma)\zeta_m^{\mathrm{Fr}_p^{-i}\gamma})$$
$$= -\sum_{i=1}^{\infty}p^i\chi^{-i}(p)\sum_{\gamma\in\Delta}(\chi^{-1}(\gamma)\zeta_m^{\gamma}).$$

Thus

$$\lambda_{f,\chi}(u) = \frac{1}{f}(\chi(p)-p)\sum_{i=0}^{\infty}p^i\chi^{-i-1}(p)\sum_{\delta\in\Delta}(\chi(\delta)\zeta_f^{\delta})\sum_{\gamma\in\Delta}(\chi^{-1}(\gamma)\zeta_f^{\gamma})$$
$$= \chi(-1)(\chi(p)-p)\sum_{i=0}^{\infty}p^i\chi^{-i-1}(p)\in\mathcal{O}^\times,$$

the last equality since the product of the two Gauss sums is $\chi(-1)f$.

Now suppose $p\mid f$ but $p^2\nmid f$ (so $p\neq 2$), and set

$$u = \left(g_m^{\mathrm{Fr}_p^{-1}}(\zeta_p-1)\right)^{1/m}\in\mathbf{Z}_p[\boldsymbol{\mu}_f]^\times.$$

Then

$$\log_p(u) = m^{-1}\mathcal{G}_m^{\mathrm{Fr}_p^{-1}}(\zeta_p-1)$$
$$= \Big(1-\frac{1}{p-1}\sum_{\substack{\sigma\in\Delta\\ \sigma|_{\mathbf{Q}(\boldsymbol{\mu}_m)}=\mathrm{Fr}_p}}\omega(\sigma^{-1})\sigma\Big)\left(\zeta_m^{p^{-1}}(\zeta_p^{m^{-1}}-1)\right)-\sum_{i=1}^{\infty}p^i\zeta_m^{p^{-(i+1)}}$$

so with this choice, since $\zeta_m^{p^{-1}}\zeta_p^{m^{-1}}=\zeta_f$, we have

$$\sum_{\gamma\in\Delta}\chi^{-1}(\gamma)\log_p(u^\gamma) = \Big(1-\frac{1}{p-1}\sum_{\substack{\sigma\in\Delta\\ \sigma|_{\mathbf{Q}(\boldsymbol{\mu}_m)}=\mathrm{Fr}_p}}\omega(\sigma^{-1})\chi(\sigma)\Big)\sum_{\gamma\in\Delta}(\chi^{-1}(\gamma)\zeta_f^{\gamma})$$
$$= (1-\chi\omega^{-1}(p))\sum_{\gamma\in\Delta}(\chi^{-1}(\gamma)\zeta_f^{\gamma})$$

and

$$\lambda_{f,\chi}(u) = \frac{1}{f}(1-\chi\omega^{-1}(p))\sum_{\delta\in\Delta}(\chi(\delta)\zeta_f^{\delta})\sum_{\gamma\in\Delta}(\chi^{-1}(\gamma)\zeta_f^{\gamma})$$
$$= \chi(-1)(1-\chi\omega(p)^{-1})\in\mathcal{O}^\times.$$

In either case the p-adic logarithm shows that $\mathrm{Hom}(\mathbf{Z}_p[\boldsymbol{\mu}_f]^\times,\mathcal{O})^{\chi^{-1}}$ is a rank-one $\mathcal{O}$-module, which is clearly torsion-free and hence free. The formulas above show that

$$\lambda_{f,\chi}\notin\mathfrak{p}\mathrm{Hom}(\mathbf{Z}_p[\boldsymbol{\mu}_f]^\times,\mathcal{O})^{\chi^{-1}}$$

where $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}$, and the lemma follows. □

D.2. Cyclotomic Units

For this section suppose that $m > 1$ and m is prime to p. Fix an embedding $\overline{\mathbf{Q}_p} \subset \mathbf{C}$ and let $\zeta_n = e^{2\pi i/n}$ for every $n \in \mathbf{Z}^+$. Define

$$u_m(X) = \zeta_m(1+X)^{m^{-1}} - 1 \in \mathbf{Z}_p[\boldsymbol{\mu}_m][[X]].$$

Lemma D.2.1. *Suppose $m > 1$ and m is prime to p. Suppose $n \geq 0$ and $\gamma \in \mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_{mp^n})/\mathbf{Q})$, and choose $b \in \mathbf{Z}$ such that $\zeta_{mp^n}^{\gamma} = \zeta_{mp^n}^{b}$. Then for every $k \geq 2$,*

$$\begin{aligned}(D^k \log u_m^{\mathrm{Fr}_p^{-n}\gamma})(\zeta_{p^n}^{\gamma} - 1) \\ = (-1)^{k-1}\Gamma(k)(2\pi i)^{-k}p^{nk}(\zeta(b, mp^n; k) + (-1)^k\zeta(-b, mp^n; k))\end{aligned}$$

where $\zeta(a, r; s)$ is the partial Riemann zeta function $\sum_{0<j\equiv a \,(\mathrm{mod}\ r)} j^{-s}$.

Proof. Since $m > 1$ and m is prime to p, we see that $u_m(0) \in \mathbf{Z}_p[\boldsymbol{\mu}_m]^{\times}$. Therefore $u_m^{\mathrm{Fr}_p^{-n}\gamma} \in \mathbf{Z}_p[\boldsymbol{\mu}_m][[X]]^{\times}$ and $\log u_m^{\mathrm{Fr}_p^{-n}\gamma}$ is defined. Thus

$$\begin{aligned}(D^k \log u_m^{\mathrm{Fr}_p^{-n}\gamma})(\zeta_{p^n}^{\gamma} - 1) &= D^{k-1}\left.\frac{(1+X)(u_m^{\mathrm{Fr}_p^{-n}\gamma})'(X)}{u_m^{\mathrm{Fr}_p^{-n}\gamma}(X)}\right|_{X=\zeta_{p^n}^{b}-1} \\ &= D^{k-1}\left.\frac{m^{-1}\zeta_m^{bp^{-n}}(1+X)^{m^{-1}}}{\zeta_m^{bp^{-n}}(1+X)^{m^{-1}} - 1}\right|_{X=\zeta_{p^n}^{b}-1}.\end{aligned}$$

Substituting $e^Z = (1+X)^{m^{-1}}$ and $m^{-1}\frac{d}{dZ} = (1+X)\frac{d}{dX}$, this becomes

$$\begin{aligned}(D^k \log u_m^{\mathrm{Fr}_p^{-n}\gamma})(\zeta_{p^n}^{\gamma} - 1) &= m^{-k}\frac{d^{k-1}}{dZ^{k-1}}\left.\frac{\zeta_m^{bp^{-n}}e^Z}{\zeta_m^{bp^{-n}}e^Z - 1}\right|_{e^Z=\zeta_{p^n}^{bm^{-1}}} \\ &= m^{-k}\frac{d^{k-1}}{dZ^{k-1}}\left.\frac{e^Z}{e^Z-1}\right|_{Z=\frac{2\pi i b}{mp^n}}.\end{aligned}$$

By [**Al**] equation (10), p. 187 (or just observe that the difference is a bounded entire function which vanishes at 0)

$$\frac{e^Z}{e^Z-1} = \frac{1}{2} + \frac{1}{Z} + \sum_{n=1}^{\infty}\left(\frac{1}{Z-2\pi i n} + \frac{1}{Z+2\pi i n}\right).$$

Thus for $k \geq 2$, $r > 1$, and $c \in \mathbf{Z} - r\mathbf{Z}$, we have

$$\begin{aligned}\left.\frac{d^{k-1}}{dZ^{k-1}}\frac{e^Z}{e^Z-1}\right|_{Z=\frac{2\pi i c}{r}} &= (-1)^{k-1}(k-1)!(2\pi i)^{-k}r^k\sum_{n\in\mathbf{Z}}\frac{1}{(c+nr)^k} \\ &= (-1)^{k-1}\Gamma(k)(2\pi i)^{-k}r^k(\zeta(c, r; k) + (-1)^k\zeta(-c, r; k)).\end{aligned}$$

Combining these formulas proves the lemma. □

Define

$$h_m(X) \;=\; \prod_{\beta\in(\mathbf{Z}_p^\times)_{\mathrm{tors}}} u_m((1+X)^\beta-1)\bar{u}_m((1+X)^\beta-1)$$

where $\bar{u}_m(X)=1-\zeta_m^{-1}(1+X)^{m^{-1}}$, and define

$$\mathcal{H}_m(X) \;=\; \log h_m(X) - \frac{1}{p}\log h_m^{\mathrm{Fr}_p}((1+X)^p-1).$$

For every $n>1$ write $\Delta_n=\mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_n)/\mathbf{Q})$ and $\Delta_n^+=\mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_n)^+/\mathbf{Q})$.

Lemma D.2.2. *Suppose $p>2$, and let ω be the Teichmüller character giving the action of $G_{\mathbf{Q}}$ on $\boldsymbol{\mu}_p$. Suppose $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbf{Q}_p$, and $\chi: G_{\mathbf{Q}}\to\mathcal{O}^\times$ is a nontrivial even character of finite order, unramified at p. If m is the conductor of χ then*

$$\sum_{\gamma\in\Delta_m^+}\chi^{-1}(\gamma)D^k\mathcal{H}_m^\gamma(\zeta_p-1)$$
$$= \;2\Gamma(k)(-2\pi i)^{-k}L(\chi^{-1}\omega^k,k)\times\begin{cases}-\chi(p)p^k & \text{if } p-1\nmid k,\\ 1-p^{k-1}\chi(p) & \text{if } p-1\mid k.\end{cases}$$

Proof. We have

$$D^k\mathcal{H}_m^\gamma(\zeta_p-1) \;=\; D^k\log h_m^\gamma(\zeta_p-1) - p^{k-1}D^k\log h_m^{\mathrm{Fr}_p\gamma}(0).$$

If $\zeta=\zeta_p$ or $\zeta=1$, then

$$D^k\log h_m^\gamma(\zeta-1) = \sum_{\beta\in(\mathbf{Z}_p^\times)_{\mathrm{tors}}}\beta^kD^k\log u_m^\gamma(\zeta^\beta-1)+\beta^kD^k\log\bar{u}_m^\gamma(\zeta^\beta-1)$$
$$= \sum_{\sigma\in\mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_{mp})/\mathbf{Q}(\boldsymbol{\mu}_m)^+)}\omega^k(\sigma)D^k\log u_m^{\gamma\sigma}(\zeta^\sigma-1).$$

Thus by Lemma D.2.1, writing $L_r(\chi^{-1}\omega^k,s)$ for the Dirichlet L-function with Euler factors for primes dividing r removed,

$$\sum_{\gamma\in\Delta_m^+}\chi^{-1}(\gamma)D^k\mathcal{H}_m^\gamma(\zeta_p-1)$$
$$= \sum_{\gamma\in\Delta_{mp}}\chi^{-1}\omega^k(\gamma)\Big(D^k\log u_m^\gamma(\zeta_p^\gamma-1)-p^{k-1}D^k\log u_m^{\mathrm{Fr}_p\gamma}(0)\Big)$$
$$= (-1)^{k-1}\Gamma(k)(2\pi i)^{-k}p^k(1+(-1)^k\chi^{-1}\omega^k(-1))\chi(p)L_{mp}(\chi^{-1}\omega^k,k)$$
$$-\;p^{k-1}(-1)^{k-1}\Gamma(k)(2\pi i)^{-k}(1+(-1)^k)\chi(p)L_m(\chi^{-1},k)\sum_{\gamma\in\Delta_p}\omega^k(\gamma).$$

Note that $\chi^{-1}\omega^k(-1) = (-1)^k$. If $(p-1) \mid k$ then $\omega^k = 1$, and the formula above simplifies to

$$2\Gamma(k)(-2\pi i)^{-k}\chi(p)L(\chi^{-1},k)(-p^k(1-\chi^{-1}(p)p^{-k}) + (p-1)p^{k-1}) \\ = 2\Gamma(k)(-2\pi i)^{-k}L(\chi^{-1},k)(1-p^{k-1}\chi(p)).$$

If $(p-1) \nmid k$ then $\sum_{\gamma\in\Delta_p} \omega^k(\gamma) = 0$, so in that case

$$\sum_{\gamma\in\Delta_m^+} \chi^{-1}(\gamma)D^k\mathcal{H}_m^\gamma(\zeta_p - 1) = -2\Gamma(k)(-2\pi i)^{-k}p^k\chi(p)L(\chi^{-1}\omega^k,k). \qquad \square$$

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