This is the body of a letter that I (Ken Ribet) wrote to J.-F. Mestre in early November, 1987. I did not publish the theorems in this letter because I expected that they would be incorporated into an article by Mestre and J. Oesterlé on the "graph method" for calculating eigenvalues of Hecke operators on spaces of cusp forms. My understanding is that the formulas in this letter have not appeared in print and that they retain some interest. I am happy to make them available at this time (May, 2001). My formulas have been generalized to some extent by William A. Stein, who calculates component groups of optimal abelian variety quotients of Jacobians of modular curves.

I return, first, to the notations you had on the board: p is a prime number, M is prime to p, N is the product pM. We suppose given an elliptic curve E, plus a map $\pi : X_o(N) \to E$, over $\mathbf{Q}$. We assume that the induced map on Albanese varieties $\pi_* : J_o(N) \to E$ has connected kernel. Finally, we assume that E has bad (hence multiplicative) reduction at p. This situation occurs, in particular, if E has conductor N and is given as a "strong (Taniyama-)Weil curve."

Next, we have some notation relative to the Néron models of E and $J = J_o(N)$ at p. I will let X be the character group of the torus attached to $J_{\mathbf{F}_p}$, so that X is the group of degree-0 divisors on the set $\mathcal{C}$ of isomorphism classes of supersingular elliptic curves over $\overline{\mathbf{F}}_p$, which are enhanced by $\Gamma_o(M)$ structures. A typical element of X is a sum $\sum \lambda_C(C)$, where the λ_C are integers which sum to 0. Let X_E be the character group associated with E. We chose a generator of this cyclic group. The natural map $\pi^* : X_E \to X$ then sends this generator to a certain element $\sum \lambda_C(C)$ of X. We can identify X_E with the group of multiples of this sum. There is a second natural map $\pi_* : X \to X_E$. The endomorphism of X_E gotten by following π^* by π_* is multiplication by $n = \deg(\pi)$.

There are natural symmetric $\mathbf{Z}$-valued pairings $\langle\, ,\, \rangle$ and $\langle\, ,\, \rangle_E$ on X and X_E. These are the *monodromy pairings* considered by Grothendieck in SGA7. The component groups Φ and Ψ attached to J and E are then the cokernels of the maps

$$X \to \mathrm{Hom}(X, \mathbf{Z})$$

and

$$X_E \to \mathrm{Hom}(X_E, \mathbf{Z})$$

deduced from these pairings. The map

$$\pi_* : \Phi \to \Psi$$

that π induces on component groups can be seen in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X & \to & \mathrm{Hom}(X, \mathbf{Z}) & \to & \Phi & \to & 0 \\ & & \downarrow \pi_* & & \downarrow \beta & & \downarrow \pi_* & & \\ 0 & \to & X_E & \to & \mathrm{Hom}(X_E, \mathbf{Z}) & \to & \Psi & \to & 0, \end{array} \qquad (1)$$

in which β is $\mathrm{Hom}(\pi^*, \mathbf{Z})$.

Lemma 1 *The map $\pi_* : X \to X_E$ is surjective.*

Proof. The map (in characteristic 0) $\pi^* : E \to J$ is injective, because its transpose π_* has connected kernel, by assumption. In particular, the map $E[\ell] \to J[\ell]$ is injective, for all prime numbers ℓ. The map which π^* induces on tori

$$E^{\mathrm{toric}} \to J^{\mathrm{toric}}$$

is then injective (in the appropriate sense), as one sees by lifting these tori to $\mathbf{Z}_p$. This injectivity corresponds to the surjectivity asserted by the lemma. ∎

Formula 1 *The cokernel of $\pi_* : \Phi \to \Psi$ has order $\gcd(\ldots, \lambda_C, \ldots)$.*

Proof. The cokernel in question is

$$\mathrm{Hom}(X_E, \mathbf{Z})/\{X_E + \beta(\mathrm{Hom}(X, \mathbf{Z})\}.$$

By the lemma, we have $X_E \subseteq \beta(\mathrm{Hom}(X, \mathbf{Z})$, since $\mathrm{Hom}(X, \mathbf{Z})$ contains X. Therefore the cokernel is simply

$$\mathrm{Hom}(X_E, \mathbf{Z})/\beta(\mathrm{Hom}(X, \mathbf{Z}).$$

After some fooling with Ext groups, we see that this quotient map be identified with the torsion subgroup of X/X_E [1]. However, the gcd in the formula is the order of the torsion subgroup of X/X_E. ∎

[1] Actually, this claim is wrong; I think that you get instead the *dual* of the indicated torsion group

Formula 2 *We have*

$$\operatorname{card}(\Psi) = \gcd_{C,C'} \; (\lambda_C w_C - \lambda_{C'} w_{C'}),$$

where w_C *is the integer* $\mathrm{Aut}(C)/2$*, as in your notation.*

Proof. Consider the map

$$\gamma : X \to \mathrm{Hom}(X_E, \mathbf{Z}) \tag{2}$$

gotten by composing

$$\pi_* : X \to X_E$$

and the map

$$X_E \to \mathrm{Hom}(X_E, \mathbf{Z})$$

which is $\langle\ ,\ \rangle_E$. By the lemma, its cokernel is Ψ. To obtain the formula, we notice that the image of γ is generated by the elements $\gamma(D - D')$ for $D, D' \in \mathcal{C}$. For D and D' in $\mathcal{C}$, $\gamma(D - D')$ maps the generator $\sum \lambda_C(C)$ of X_E to $\lambda_D w_D - \lambda_{D'} w_{D'}$, since $\langle\ ,\ \rangle$ is deduced by restriction from the pairing $C, C' \mapsto \delta_{C,C'} w_C$ on $\mathbf{Z}^{\mathcal{C}}$. It should be remarked here that we are using the compatibility between $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle$ which is summarized by the relation

$$\langle \xi, \pi_*(\eta) \rangle_E = \langle \pi^*(\xi), \eta \rangle \tag{3}$$

for $\xi \in X_E$ and $\eta \in X$. ∎

Formula 3 *We have*

$$n \operatorname{card}(\Psi) = \sum \lambda_C^2 w_C.$$

Proof. Let g be our chosen generator of X_E. Then

$$\operatorname{card}(\Psi) = \langle g, g \rangle_E,$$

so that

$$n \operatorname{card}(\Psi) = \langle \pi_* \pi^* g, g \rangle_E = \langle \pi^* g, \pi^* g \rangle.$$

The quantity $\langle \pi^* g, \pi^* g \rangle$ visibly coincides with the sum $\sum \lambda_C^2 w_C$. ∎

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