The Weierstrass approximation theorem for maps between real algebraic varieties

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1 Introduction

Throughout this paper the term *real algebraic variety* designates a locally ringed space isomorphic to an irreducible Zariski locally closed subset of $\mathbb{P}^n(\mathbb{R})$, for some n, endowed with the Zariski topology and the sheaf of $\mathbb{R}$-valued regular functions. Morphisms between real algebraic varieties will be called *regular maps*. Every real algebraic variety carries also the Euclidean topology, that is, the topology induced by the usual metric topology on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions related to real algebraic varieties will refer to the Euclidean topology.

Given two real algebraic varieties X and Y, we regard the set $\mathcal{R}(X,Y)$ of all regular maps from X into Y as a subset of the space $\mathcal{C}(X,Y)$ of all continuous maps from X into Y, endowed with the compact-open topology. If X and Y are nonsingular, then we also regard $\mathcal{R}(X,Y)$ as a subset of the space $\mathcal{C}^\infty(X,Y)$ of all $\mathcal{C}^\infty$ maps from X into Y, endowed with the $\mathcal{C}^\infty$ compact-open topology (the weak $\mathcal{C}^\infty$ topology in the terminology used in [6]).

Theorem 1.1 *Let X be a compact nonsingular real algebraic curve and let Y be a nonsingular real algebraic variety. If Y is rational, then $\mathcal{R}(X,Y)$ is dense in $\mathcal{C}^\infty(X,Y)$.*

Recall that an n-dimensional real algebraic variety is said to be *rational* if it is birationally equivalent to $\mathbb{P}^n(\mathbb{R})$, that is, if it contains a nonempty Zariski open subvariety isomorphic to a Zariski open subvariety of $\mathbb{P}^n(\mathbb{R})$.

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Of course, for $Y = \mathbb{R}^n$, the statement follows from the classical Weierstrass approximation theorem. A special case of Theorem 1.1, where Y is a real Grassmann variety, is in [2, Théorème 12.5.1, Corollaire 13.3.2]. In general, the curve X cannot be replaced by a higher dimensional real algebraic variety, even if $Y = \mathbb{P}^1(\mathbb{R})$. Indeed, if M is any closed connected orientable $\mathcal{C}^\infty$ manifold of dimension at least 2, whose first Betti number is different from 0, then there exists a nonsingular real algebraic variety X diffeomorphic to M such that $\mathcal{R}(X, \mathbb{P}^1(\mathbb{R}))$ is not dense in $\mathcal{C}^\infty(X, \mathbb{P}^1(\mathbb{R}))$ [4, Corollary 1.7].

We do not know whether or not the assumption of rationality of Y can be relaxed. Theorem 1.2 below implies that by relaxing it too much we would be led to non-density results.

Let Y be a compact nonsingular real algebraic variety. It follows from the resolution of singularities theorem [5, 1] that there exists a nonsingular projective complexification V of Y. By this we mean that V is a nonsingular projective complex algebraic variety defined over $\mathbb{R}$ such that the set $V(\mathbb{R})$ of real points of V is Zariski dense in V, and $V(\mathbb{R})$, viewed in the usual way as a real algebraic variety, is isomorphic to Y. Any two nonsingular projective complexifications of Y are birationally equivalent and hence have the same first Betti number, which we denote by $b_1(Y, \mathbb{C})$. Obviously, $b_1(Y, \mathbb{C}) = 0$ if Y is rational.

Theorem 1.2 *Let X be a real algebraic variety of positive dimension and let Y be a compact nonsingular real algebraic variety. If $\mathcal{R}(X, Y)$ is dense in $\mathcal{C}(X, Y)$, then $b_1(Y, \mathbb{C}) = 0$.*

Condition $b_1(Y, \mathbb{C}) = 0$ is by far too weak to insure validity of Theorem 1.1 for Y. Indeed, let $H_1^{\mathrm{alg}}(Y, \mathbb{Z}/2)$ be the subgroup of $H_1(Y, \mathbb{Z}/2)$ generated by the homology classes represented by the Zariski closed real algebraic curves in Y. Let X be a compact nonsingular real algebraic curve. One easily sees that $H_1^{\mathrm{alg}}(Y, \mathbb{Z}/2) = H_1(Y, \mathbb{Z}/2)$ if $\mathcal{R}(X, Y)$ is dense in $\mathcal{C}^\infty(X, Y)$. It is known that $\dim_{\mathbb{Z}/2} H_1^{\mathrm{alg}}(Y, \mathbb{Z}/2) \le 1$ for any "general" hypersurface Y in $\mathbb{P}^3(\mathbb{R})$ of degree at least 4, cf. [4, Example 3.4(d)]. Therefore if such a surface Y is not homeomorphic to $\mathbb{P}^2(\mathbb{R})$ or the unit 2-sphere, then $\mathcal{R}(X, Y)$ cannot be dense in $\mathcal{C}^\infty(X, Y)$. Since Y is "general", its Zariski closure in $\mathbb{P}^3(\mathbb{C})$ is nonsingular and hence $b_1(Y, \mathbb{C}) = 0$ by virtue of the Lefschetz theorem.

The simplest case for which we are unable to prove or disprove Theorem 1.1 is the Fermat surface of degree 4,

$$Y = \{(x,y,z) \in \mathbb{R}^3 \mid x^4 + y^4 + z^4 = 1\},$$

that is homeomorphic to the unit 2-sphere.

The paper is organized as follows. In Section 2 we prove Theorem 1.1 in a stronger version (cf. Theorem 2.5) and derive certain more general results in which the curve X need not be compact (cf. Theorems 2.6 and 2.7). Section 3 contains a proof of Theorem 1.2.

2 Proof of Theorem 1.1 and further results

Given a $\mathcal{C}^\infty$ map $f : M \to N$ between $\mathcal{C}^\infty$ manifolds, a point x in M, and a nonnegative integer r, we denote by $j^r f(x)$ the r-jet of f at x.

Given real analytic manifolds X and Y, we denote by $\mathcal{O}(X,Y)$ the space of all analytic maps from X into Y, endowed with the $\mathcal{C}^\infty$ compact-open topology (that is, the topology induced from the ambient space $\mathcal{C}^\infty(X,Y)$). If Z is a subset of Y, let $\mathcal{O}(X,Y)_Z$ be the subset of $\mathcal{O}(X,Y)$ consisting of all maps f for which $f^{-1}(Z)$ is a finite set.

Assume now that X is compact of dimension 1 and that Z is a closed analytic submanifold of Y. Let $\pi : \tilde{Y} \to Y$ be the blowing up of Y along Z. If f is in $\mathcal{O}(X,Y)_Z$, then there exists a unique element $\tilde{f}$ of $\mathcal{O}(X,\tilde{Y})$ such that $\pi \circ \tilde{f} = f$. In general, the map $\mathcal{O}(X,Y)_Z \to \mathcal{O}(X,\tilde{Y})$, $f \to \tilde{f}$ is not continuous. However, the following holds.

Lemma 2.1 *Let f be in $\mathcal{O}(X,Y)_Z$ and let $\tilde{\mathcal{N}}$ be a neighborhood of $\tilde{f}$ in $\mathcal{O}(X,\tilde{Y})$. Let A be a finite subset of X and let s be a nonnegative integer. Then there exist a neighborhood $\mathcal{N}$ of f in $\mathcal{O}(X,Y)$ and a positive integer r such that $\mathcal{N} \subseteq \mathcal{O}(X,Y)_Z$ and for every map g in $\mathcal{N}$ satisfying $j^r g(x) = j^r f(x)$ for all x in $A \cup f^{-1}(Z)$, the map $\tilde{g}$ is in $\tilde{\mathcal{N}}$ and $j^s\tilde{g}(a) = j^s\tilde{f}(a)$ for all a in A.*

Proof. One easily sees that $\mathcal{O}(X,Y)_Z$ is an open subset of $\mathcal{O}(X,Y)$.

All the required conditions can be obtained by calculating in local coordinates. Choose a point x_0 in $f^{-1}(Z)$. Choose a neighborhood V of $f(x_0)$ in Y and an analytic diffeomorphism

$\psi : V \to \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ such that $\psi(V \cap Z) = \mathbb{R}^k \times \{0\}$ and $\psi(f(x_0)) = 0$. Choose a neighborhood U of x_0 in X and an analytic diffeomorphism $\varphi : U \to \mathbb{R}$ satisfying $\varphi(x_0) = 0$ and $f(U) \subseteq V$. Consider the composite map

$$\psi \circ f \circ \varphi^{-1} : \mathbb{R} \to \mathbb{R}^k \times \mathbb{R}^{n-k}\,,\ \psi \circ f \circ \varphi^{-1} = (f_1, \dots, f_k, f_{k+1}, \dots, f_n).$$

Since f is in $\mathcal{O}(X,Y)_Z$ and $f(x_0)$ is in Z, it follows that $f_{k+1}(0) = \cdots = f_n(0) = 0$ and at least one of the analytic functions $f_{k+1}, \dots, f_n$ is not identically equal to 0. We may select (V, ψ) in such a way that f_n has order p at 0 and the order of f_i at 0 is at least p for $i = k+1, \dots, n-1$. Furthermore, we may choose (U, φ) so that $f_n^{-1}(0) = \{0\}$. Note that now the real meromorphic functions $f_{k+1}/f_n, \dots, f_{n-1}/f_n$ are analytic on $\mathbb{R}$.

There exist a neighborhood $\tilde{V}$ of $\tilde{f}(x_0)$ in $\tilde{Y}$ and an analytic diffeomorphism $\tilde{\psi} : \tilde{V} \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ such that $\tilde{f}(U) \subseteq \tilde{V}$ and $\tilde{\psi} \circ \tilde{f} \circ \varphi^{-1} : \mathbb{R} \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ is given by

$$\tilde{\psi} \circ \tilde{f} \circ \varphi^{-1} = (f_1, \dots, f_k, f_{k+1}/f_n, \dots, f_{n-1}/f_n, f_n).$$

Let I be a bounded open interval in $\mathbb{R}$ containing 0. If $\mathcal{N}$ is a small neighborhood of f in $\mathcal{O}(X,Y)_Z$ and if g is in $\mathcal{N}$, then $g(\varphi^{-1}(I)) \subset V$. Consider the map

$$\psi \circ g \circ \varphi^{-1} : I \to \mathbb{R}^k \times \mathbb{R}^{n-k}\,,\ \psi \circ g \circ \varphi^{-1} = (g_1, \dots, g_k, g_{k+1}, \dots, g_n).$$

If $\mathcal{N}$ is sufficiently small and if r is a sufficiently large integer (in particular, $r \geq \max(p+1, s)$), and if $j^r g(x_0) = j^r f(x_0)$, then

$$\{t \in I \mid g_n(t) = 0\} = \{0\}\,,$$

the meromorphic functions $g_{k+1}/g_n, \dots, g_{n-1}/g_n$ on I are analytic, and $j^s(g_i/g_n)(0) = j^s(f_i/f_n)(0)$ for $i = k+1, \dots, n-1$. Note that $\tilde{g}(x_0) = \tilde{f}(x_0)$ and

$$\tilde{\psi} \circ \tilde{g} \circ \varphi^{-1} = (g_1, \dots, g_k, g_{k+1}/g_n, \dots g_{n-1}/g_n, g_n) \text{ on } I\,.$$

Furthermore, if $\mathcal{N}$ is small, then $\tilde{\psi} \circ \tilde{g} \circ \varphi^{-1}$ is close to $\tilde{\psi} \circ \tilde{f} \circ \varphi^{-1}|I$. Now the conclusion follows. $\square$

Let X be a compact nonsingular real algebraic curve. We say that a nonsingular real algebraic variety Y has *property* (X) if for every map f in $\mathcal{C}^\infty(X,Y)$, every neighborhood $\mathcal{U}$

of f in $\mathcal{C}^\infty(X,Y)$, every finite subset A of X, and every nonnegative integer s, there exists a map g in $\mathcal{R}(X,Y)$ such that g is in $\mathcal{U}$ and $j^s g(a) = j^s f(a)$ for all a in A. In particular, if Y has property (X), then $\mathcal{R}(X,Y)$ is dense in $\mathcal{C}^\infty(X,Y)$.

Lemma 2.2 *Let Y be a nonsingular real algebraic variety and let Z be a Zariski closed nonsingular subvariety of Y. Let $\pi : \tilde{Y} \to Y$ be the blowing up of Y along Z. Let X be a compact nonsingular real algebraic curve. If Y has property (X), then so does $\tilde{Y}$.*

Proof. Let φ be in $\mathcal{C}^\infty(X,\tilde{Y})$ and let $\mathcal{V}$ be a neighborhood of φ. Let A be a finite subset of X and let s be a nonnegative integer. One easily sees that there exists an element ψ of $\mathcal{V}$ such that $j^s\psi(a) = j^s\varphi(a)$ for all a in A and such that $\psi(C)$ is not contained in $\pi^{-1}(Z)$ for any connected component C of X. Choose an analytic map F in $\mathcal{O}(X,\tilde{Y})$, close to ψ in the $\mathcal{C}^\infty$ compact-open topology, satisfying $j^s F(a) = j^s\psi(a)$ for all a in A; such an F exists in view of [3, p. 409, Corollary 2]. By taking F sufficiently close to ψ, we may assume that F is in $\mathcal{V}$ and $F^{-1}(\pi^{-1}(Z))$ is a finite set. Then the map $f = \pi \circ F$ is in $\mathcal{O}(X,Y)_Z$ and, in the notation of Lemma 2.1, $\tilde{f} = F$.

By Lemma 2.1, there exist a neighborhood $\mathcal{N}$ of f in $\mathcal{O}(X,Y)$ and a positive integer r such that $\mathcal{N} \subseteq \mathcal{O}(X,Y)_Z$ and for every map g in $\mathcal{N}$ satisfying $j^r g(x) = j^r f(x)$ for all x in $A \cup f^{-1}(Z)$, the map $\tilde{g}$ is in $\mathcal{V}$ and $j^s\tilde{g}(a) = j^s\tilde{f}(a)$ for all a in A. Since Y has property (X), we can choose g as above that belongs to $\mathcal{R}(X,Y)$. Then $\tilde{g}$ is in $\mathcal{R}(X,\tilde{Y})$ and the conclusion follows. $\square$

Lemma 2.3 *Projective n-space $\mathbb{P}^n(\mathbb{R})$ has property (X) for every compact nonsingular real algebraic curve X.*

Proof. Recall that every real algebraic variety Z, defined as in Section 1, is isomorphic to a Zariski closed subvariety of $\mathbb{R}^m$ for some m [2, Théorème 3.4.4, Proposition 3.2.10]. Denote by ε_Z^k the trivial vector bundle on Z with total space $Z \times \mathbb{R}^k$. An algebraic subbundle η of ε_Z^k, for some k, is called a strongly algebraic vector bundle on Z (cf. [2, Chapitre 12] for the theory of strongly algebraic vector bundles). It is known that given η as above, there exists an algebraic subbundle η' of ε_Z^k such that $\eta \oplus \eta' = \varepsilon_Z^k$ [2, Théorème 12.1.7]. Obviously, if η_1 and η_2 are strongly algebraic vector bundles on Z, then $\mathrm{Hom}(\eta_1, \eta_2)$ is a strongly algebraic vector bundle on Z. If Z is compact nonsingular of dimension 1, then every $\mathcal{C}^\infty$ vector bundle of constant rank on Z is $\mathcal{C}^\infty$ isomorphic to a strongly algebraic vector bundle on Z [2, Théorème 12.5.1].

Let X be a compact nonsingular real algebraic curve and let $f : X \to \mathbb{P}^n(\mathbb{R})$ be a $\mathcal{C}^\infty$ map. Let A be a finite subset of X and let s be a nonnegative integer. Choose a strongly algebraic line bundle ξ on X and a $\mathcal{C}^\infty$ vector bundle isomorphism $\varphi : \xi \to f^*\gamma_n$ between ξ and the pullback line bundle $f^*\gamma_n$, where γ_n is the universal line bundle on $\mathbb{P}^n(\mathbb{R})$. Recall that

$$\{(\ell, e) \in \mathbb{P}^n(\mathbb{R}) \times \mathbb{R}^{n+1} \mid e \in \ell\}$$

is the total space of γ_n; here $\mathbb{P}^n(\mathbb{R})$ is regarded as the space of 1-dimensional vector subspaces of $\mathbb{R}^{n+1}$. Hence $f^*\gamma_n$ is a $\mathcal{C}^\infty$ subbundle of ε_X^{n+1} and we can define a $\mathcal{C}^\infty$ section

$$u : X \to \mathrm{Hom}(\xi, \varepsilon_X^{n+1}),$$

$$u(x)(e) = \varphi(e) \text{ for } x \text{ in } X \text{ and } e \text{ in the fiber } \xi_x \text{ of } \xi \text{ at } x.$$

Since $\mathrm{Hom}(\xi, \varepsilon_{\mathrm{X}}^{\mathrm{n}+1})$ is strongly algebraic, we have

$$\mathrm{Hom}(\xi, \varepsilon_X^{n+1}) \oplus \zeta = \varepsilon_X^N$$

for some N and some algebraic subbundle ζ of ε_X^N. Denote by

$$\pi : \varepsilon_X^N \to \mathrm{Hom}(\xi, \varepsilon_X^{n+1})$$

the projection vector bundle morphism; clearly π is regular.

Define

$$\bar{u} : X \to \varepsilon_X^N, \ \bar{u}(x) = u(x) \text{ for } x \text{ in } X.$$

By [3, p. 409, Corollary 1], there exists a regular section $w : X \to \varepsilon_X^N$, arbitrarily close in the $\mathcal{C}^\infty$ compact-open topology to $\bar{u}$, satisfying $j^s w(a) = j^s \bar{u}(a)$ for all a in A. Note that

$$v = \pi \circ w : X \to \mathrm{Hom}(\xi, \varepsilon_X^{n+1})$$

is a regular section close to u and

$$j^s v(a) = j^s u(a) \text{ for all } a \text{ in } A.$$

It follows from the definition of u that the linear map $u(x) : \xi_x \to (\varepsilon_X^{n+1})_x = \{x\} \times \mathbb{R}^{n+1}$ is injective for all x in X. Since X is compact, if v is sufficiently close to u, then the linear map $v(x) : \xi_x \to \{x\} \times \mathbb{R}^{n+1}$ is injective as well for all x in X. Define a map $g : X \to \mathbb{P}^n(\mathbb{R})$ by

$$g(x) = \rho(v(x)(\xi_x)) \text{ for } x \text{ in } X,$$

where $\rho : X \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the canonical projection. By construction, g is a regular map close to f with $j^s g(a) = j^s f(a)$ for all a in A. Thus the proof is complete. $\square$

Lemma 2.4 *Let $\varphi : Z \to Y$ be a regular map between projective nonsingular real algebraic varieties. Let C be an analytic manifold of dimension* 1 *and let $h : C \to Y$ be an analytic map. If φ is a birational map, then there exists an analytic map $H : C \to Z$ such that $\varphi \circ H = h$.*

Proof. By applying the resolution of indeterminacy theorem [5, 1] to the rational map $\varphi^{-1} : Y - - \to Z$, we obtain a sequence

$$Y_k \xrightarrow{\pi_k} Y_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y_0 = Y$$

and a regular map $\psi : Y_k \to Z$ such that π_i is the blowing up of Y_{i-1} along a Zariski closed nonsingular subvariety B_{i-1} of Y_{i-1} for $1 \leq i \leq k$, and $\varphi \circ \psi = \pi_1 \circ \cdots \circ \pi_k$. We now successively construct analytic maps $H_i : C \to Y_i$ satisfying $H_0 = h$ and $\pi_{i+1} \circ H_{i+1} = H_i$ for $0 \leq i \leq k-1$. From the start we may assume that C is connected. If the set $H_i^{-1}(B_i)$ is discrete, then H_{i+1} is uniquely determined. If $H_i^{-1}(B_i)$ is not discrete, then $H_i(C) \subseteq B_i$, and therefore H_{i+1} can easily be constructed (but is not unique). We get H by setting $H = \psi \circ H_k$. $\square$

Our next result is a stronger version of Theorem 1.1.

Theorem 2.5 *Every rational nonsingular real algebraic variety Y has property (X) for every compact nonsingular real algebraic curve X.*

Proof. It follows from the resolution of singularities theorem [5, 1] that Y is isomorphic to a Zariski open subvariety of a projective nonsingular real algebraic variety. Since X is compact, we may assume that Y itself is projective. Rationality of Y and the resolution of indeterminacy theorem [5, 1] imply then that there exist a sequence

$$Y_k \xrightarrow{\pi_k} Y_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y_0 = \mathbb{P}^n(\mathbb{R})$$

and a regular map $\varphi : Y_k \to Y$ such that π_i is the blowing up of Y_{i-1} along a Zariski closed nonsingular subvariety of Y_{i-1} for $1 \leq i \leq k$, and φ is birational. It follows from Lemmas 2.2 and 2.3 that Y_k has property (X).

Let A be a finite subset of X and let s be a nonnegative integer. Given f in $\mathcal{C}^\infty(X, Y)$, we can find an analytic map h in $\mathcal{O}(X, Y)$, arbitrarily close to f, with $j^s h(a) = j^s f(a)$ for all a in A [3, p.409, Corollary 2]. By Lemma 2.4, there exists an analytic map H in $\mathcal{O}(X, Y_k)$ satisfying $\varphi \circ H = h$. Since Y_k has property (X), we can choose a regular map G in $\mathcal{R}(X, Y_k)$ that is arbitrarily close to H and $j^s G(a) = j^s H(a)$ for all a in A. Then the regular map $g = \varphi \circ G$ in $\mathcal{R}(X, Y)$ is close to $h = \varphi \circ H$ and $j^s g(a) = j^s h(a)$ for all a in A. Thus the proof is complete. $\square$

We shall now be concerned with some generalizations of Theorem 1.1. We thank the referee, who found a mistake in our original presentation and suggested the correct one.

Let us recall that every nonsingular real algebraic curve can be regarded as a Zariski open subvariety of a uniquely determined, up to isomorphism, compact nonsingular real algebraic curve. The following observation will be repeatedly used: If X_0 is a nonempty Zariski open subvariety of a compact nonsingular real algebraic curve X, then every regular map $X_0 \to Z$, into a projective real algebraic variety Z, has a unique extension to a regular map $X \to Z$.

Theorem 2.6 *Let X and Y be as in Theorem 1.1, with Y rational, $\dim Y = 1$. For any nonempty Zariski open subvariety X_0 of X, the following conditions are equivalent:*

(a) $\mathcal{R}(X_0, Y)$ *is dense* $\mathcal{C}^\infty(X_0, Y)$;

(b) *Either* Y *is connected or* Y *is disconnected and no connected component of* X *contains two distinct connected components of* X_0.

Proof. Since Y is rational and $\dim Y = 1$, we may assume that Y is a Zariski open subvariety of $\mathbb{P}^1(\mathbb{R})$.

Suppose that (b) does not hold. Then we can find distinct connected components U_1, U_2 of X_0 and distinct connected components V_1, V_2 of Y such that the set $U_1 \cup \{a\} \cup U_2$ is connected for some point a in X. Fix orientations on X and $\mathbb{P}^1(\mathbb{R})$, and let $f : X_0 \to Y$ be a $\mathcal{C}^\infty$ map which maps U_i diffeomorphically onto V_i, where $f : U_1 \to V_1$ preserves and $f : U_2 \to V_2$ reverses the induced orientations. Then there exists a neighborhood of f in $\mathcal{C}^\infty(X_0, Y)$ containing no map that extends to a $\mathcal{C}^\infty$ map from X into $\mathbb{P}^1(\mathbb{R})$. Since every map in $\mathcal{R}(X_0, Y)$ has the latter property, $\mathcal{R}(X_0, Y)$ is not dense in $\mathcal{C}^\infty(X_0, Y)$.

If (b) is satisfied, then the set $\{F|X_0 \mid F \in \mathcal{C}^\infty(X, Y)\}$ is dense in $\mathcal{C}^\infty(X_0, Y)$, and hence condition (a) holds in view of Theorem 1.1. $\square$

To state our next result we need some prepartion. Let Y be a nonsingular real algebraic variety. By [5, 1], there exists a pair (Z, φ), where Z is a projective nonsingular real algebraic variety and $\varphi : Y \to Z$ is a regular map such that $\varphi(Y)$ is Zariski open in Z and $\varphi : Y \to \varphi(Y)$ is an isomorphism. Two connected components S and T of Y are said to be *projectively adjacent* if the intersection of the closures of $\varphi(S)$ and $\varphi(T)$ in Z is nonempty. We assert that this definition does not depend on the choice of (Z, φ). Indeed, let (Z', φ') be another pair satisfying the same conditions as (Z, φ) does. It follows from the resolution of indeterminacy theorem [5, 1], applied to the rational map $\varphi' \circ \varphi^{-1} : Z - - \to Z'$, that there exist a projective nonsingular real algebraic variety W and regular maps $\psi : W \to Z$, $\psi' : W \to Z'$ such that $\varphi' \circ \varphi^{-1} \circ \psi = \psi'$ and $\psi : \psi^{-1}(\varphi(Y)) \to \varphi(Y)$ is an isomorphism. If the intersection of the closures of $\varphi(S)$ and $\varphi(T)$ in Z is nonempty, then the set $\varphi(S) \cup \{z_0\} \cup \varphi(T)$ is connected for some point z_0 in Z. By [2, Proposition 2.5.11, Proposition 8.1.16], there exists an analytic map $\alpha : (-1, 1) \to Z$ such that $\alpha((-1, 0)) \subseteq \varphi(S)$, $\alpha((0, 1)) \subseteq \varphi(T)$, and $\alpha(0) = z_0$. Lemma

2.4 implies the existence of an analytic map $\beta : (-1,1) \to W$ satisfying $\psi \circ \beta = \alpha$. Then $(\psi' \circ \beta)((-1,0)) \subseteq \varphi'(S)$, $(\psi' \circ \beta)((0,1)) \subseteq \varphi'(T)$, and hence $(\psi' \circ \beta)(0)$ belongs to the intersection of the closures of $\varphi'(S)$ and $\varphi'(T)$ in Z'. This proves our assertion.

As an example, let us consider a variety Y obtained from $\mathbb{R}^n$ by removing finitely many parallel hyperplanes. If $n \geq 2$, then any two connected components of Y are projectively adjacent.

Theorem 2.7 *Let X and Y be as in Theorem* 1.1, *with Y rational,* $\dim Y \geq 2$. *For any nonempty Zariski open subvariety X_0 of X, the following conditions are equivalent*:

(a) $\mathcal{R}(X_0, Y)$ *is dense in* $\mathcal{C}^\infty(X_0, Y)$;

(b) *No connected component of X contains two distinct connected components of X_0 or any two connected components of Y are projectively adjacent.*

Proof. We may assume that Y is a Zariski open subvariety of a projective nonsingular real algebraic variety $\bar{Y}$.

Suppose that (b) does not hold. Then we can choose two distinct connected components U_1, U_2 of X_0 and two distinct connected components V_1, V_2 of Y such that U_1, U_2 are projectively adjacent and V_1, V_2 are not. Let f be a map in $\mathcal{C}^\infty(X_0, Y)$ satisfying $f(U_i) \subseteq V_i$ for $i = 1, 2$. Then there exists a neighborhood of f in $\mathcal{C}^\infty(X_0, Y)$ which contains no map that extends to a $\mathcal{C}^\infty$ map from X into $\bar{Y}$. Every map in $\mathcal{R}(X_0, Y)$ has such an extension property, and therefore $\mathcal{R}(X_0, Y)$ is not dense in $\mathcal{C}^\infty(X_0, Y)$.

Let us now suppose that (b) is satisfied. If no connected component of X contains two distinct connected components of X_0, then the set $\{F|X_0 \mid F \in \mathcal{C}^\infty(X, Y)\}$ is dense in $\mathcal{C}(X_0, Y)$, and hence (a) follows from Theorem 1.1.

Suppose then that any two connected components of Y are projectively adjacent. By the embedded resolution of singularities theorem [5, 1], we may assume that $\bar{Y} \backslash Y = Z_1 \cup \ldots \cup Z_r$, where the Z_i are Zariski closed nonsingular subvarieties of $\bar{Y}$ of codimension 1, which are in general position (that is, $\bar{Y} \backslash Y$ has normal crossings). For any two connected components

S and T of Y, there exists a $\mathcal{C}^\infty$ map $\sigma : (-1,1) \to \bar{Y}$ transverse to Z_i for $1 \le i \le r$, with $\sigma((-1,0)) \subseteq S$ and $\sigma((0,1)) \subseteq T$. Since $\dim Y \ge 2$, it follows that given a map f in $\mathcal{C}^\infty(X_0, Y)$ and a compact subset K of X_0, we can find a map F in $\mathcal{C}^\infty(X, \bar{Y})$ such that $F = f$ in a neighborhood of K, $F(X_0) \subseteq Y$, and F is transverse to Z_i for $1 \le i \le r$. By Theorem 2.5, there exists a regular map G in $\mathcal{R}(X, \bar{Y})$, arbitrarily close to F, with $j^1G(a) = j^1F(a)$ for all a in $X \backslash X_0$. If G is sufficiently close to F, the transversality property of F implies $G(X_0) \subseteq Y$. Thus the restriction $g = G|X_0$ belongs to $\mathcal{R}(X_0, Y)$. This proves density of $\mathcal{R}(X_0, Y)$ in $\mathcal{C}^\infty(X_0, Y)$ since K is an arbitrary compact subset of X_0. That is, (a) holds, and the proof is complete. $\square$

In connection with Theorems 2.6 and 2.7 let us recall that every compact rational nonsingular real algebraic variety is connected [2, Corollaire 10.4.6].

We conclude this section with a general remark concerning the approximation problem, which justifies the choice of topology on the space of maps. Let X and Y be as in Theorem 1.1, with Y projective, $\dim Y \ge 1$ (Y need not be rational). Let d be a metric on Y which induces the Euclidean topology. We assert that if X_0 is a nonempty Zariski open subvariety of X, different from X, then there exist a map f in $\mathcal{C}^\infty(X_0, Y)$ and a positive real number ϵ such that for every regular map g in $\mathcal{R}(X_0, Y)$, one has $d(f(x), g(x)) \ge \epsilon$ for some point x in X_0. Indeed, choose two connected components (not necessarily distinct) C_1 and C_2 of X and a point a in X such that the set $C_1 \cup \{a\} \cup C_2$ is connected. Let U be a neighborhood of a in X such that $U \backslash \{a\}$ has two connected components U_1 and U_2. Given two distinct points b_1 and b_2 in the same connected component of Y, set $\epsilon = \frac{1}{5} d(b_1, b_2)$. One easily constructs a map f in $\mathcal{C}^\infty(X_0, Y)$ satisfying $d(b_i, f(x)) < \epsilon$ for all x in U_i, $i = 1, 2$. Note that for any map $g : X_0 \to Y$ with $d(f(x), g(x)) < \epsilon$ for all x in X, we have $d(g(x_1), g(x_2)) \ge \epsilon$ for all x_i in U_i, $i = 1, 2$. Such a map g cannot be regular since it is not the restriction of a continuous map from X into Y.

3 Proof of Theorem 1.2

All complex algebraic varieties that appear in this section are assumed to be irreducible. As in Section 1, given a complex algebraic variety V defined over $\mathbb{R}$, we denote by $V(\mathbb{R})$ the set of real points of V. For x in V, we write $\bar{x}$ to denote the conjugate point of x.

If A is a complex Abelian variety defined over $\mathbb{R}$, then the connected component $A(\mathbb{R})_0$ of $A(\mathbb{R})$ containing 0 is a real analytic Lie group isomorphic to the real torus $\mathbb{R}^n/\mathbb{Z}^n$, where $n = \dim A$. Note that $a + \bar{a}$ is in $A(\mathbb{R})_0$ for every a in A. Indeed, let $\gamma : [0,1] \to A$ be a continuous map with $\gamma(0) = 0$ and $\gamma(1) = a$. Define $\delta : [0,1] \to A$ by $\delta(t) = \gamma(t) + \overline{\gamma(t)}$ for t in [0,1]. Then $\delta(0) = 0$ and $\delta(1) = a + \bar{a}$. Since δ is continuous and $\delta(t)$ is in $A(\mathbb{R})$ for all t, it follows that $a + \bar{a}$ is in $A(\mathbb{R})_0$.

Lemma 3.1 *Let $f : A \to B$ be a morphism of complex Abelian varieties defined over $\mathbb{R}$ (in particular, f is defined over $\mathbb{R}$). If $f(A) = B$, then $f(A(\mathbb{R})_0) = B(\mathbb{R})_0$.*

Proof. Let b be in $B(\mathbb{R})_0$. Since $B(\mathbb{R})_0$ is a torus, there exists b' in $B(\mathbb{R})_0$ satisfying $2b' = b$. Choose a in A such that $f(a) = b'$. Then

$$f(a + \bar{a}) = f(a) + f(\bar{a}) = f(a) + \overline{f(a)} = b' + \bar{b}' = b' + b' = b$$

and hence the conclusion follows since $a + \bar{a}$ is in $A(\mathbb{R})_0$. □

Given a set X, an Abelian group A, a positive integer n, and a map $f : X \to A$, we let $f^{(n)}$ denote the map from the n-fold Cartesian product X^n into A defined by

$$f^{(n)}(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n)$$

for all $(x_1, \ldots, x_n)$ in X^n.

Let C be a nonsingular projective complex algebraic curve defined over $\mathbb{R}$ and let c_0 be a point in $C(\mathbb{R})$. Denote by J_C the Jacobian variety of C and by $\mu : C \to J_C$ the canonical morphism corresponding to c_0 (in particular, $\mu(c_0) = 0$). Both J_C and μ are defined over $\mathbb{R}$, cf. [8].

Lemma 3.2 *With the notation as above, for every connected neighborhood N of c_0 in $C(\mathbb{R})$ there exists a positive integer $n_{C,N}$ such that*

$$\mu^{(n)}(N^n) = J_C(\mathbb{R})_0$$

for every integer $n \geq n_{C,N}$.

Proof. Let g be the genus of C. If $g = 0$, then $J_C = 0$ and the conclusion holds with $n_{C,N} = 1$.

Below we assume $g \geq 1$. By the classical result [8, Theorem 5.1], $\mu^{(g)}(C^g) = J_C$. Since the set N^g is Zariski dense in C^g, it follows that there exists a point in N^g at which $\mu^{(g)}$ has rank g, and hence the set $\mu^{(g)}(N^g)$ contains a nonempty Euclidean open subset U of $J_C(\mathbb{R})_0$. We know that $J_C(\mathbb{R})_0$ is a torus and therefore there exists a positive integer k such that the sum $U + \cdots + U$, with k terms, is equal to $J_C(\mathbb{R})_0$. The conclusion of the lemma is satisfied for $n_{C,N} = kg$. $\square$

Recall that a nonzero complex Abelian variety A defined over $\mathbb{R}$ is said to be *simple* if it does not contain an Abelian subvariety defined over $\mathbb{R}$ different from 0 and A.

In the next lemma we use the same notation as in Lemma 3.2.

Lemma 3.3 *Let A be a simple complex Abelian variety defined over $\mathbb{R}$. Let D be a Zariski neighborhood of c_0 in $C(\mathbb{R})$ and let $f : D \to A(\mathbb{R})$ be a regular map satisfying $f(c_0) = 0$. Let N be a connected neighborhood of c_0 in D. Then for every integer $n \geq n_{C,N}$, one has either $f^{(n)}(N^n) = \{0\}$ or $f^{(n)}(N^n) = A(\mathbb{R})_0$.*

Proof. Let $g : C \to A$ be a morphism defined over $\mathbb{R}$ such that $g(c) = f(c)$ for all c in D. It follows from the universal property of (J_C, μ) that there exists a morphism $\varphi : J_C \to A$ of Abelian varieties defined over $\mathbb{R}$ satisfying $\varphi \circ \mu = g$ [8, Proposition 6.1]. Since A is simple, either $\varphi = 0$ or φ is surjective. Clearly, $\varphi \circ \mu^{(n)} = g^{(n)}$ and hence the conclusion follows from Lemmas 3.1 and 3.2. $\square$

Proof of Theorem 1.2. In view of [5, 1], we may assume that Y is of the form $Y = V(\mathbb{R})$, where V is a nonsingular projective complex algebraic variety defined over $\mathbb{R}$.

Suppose that $b_1(Y, \mathbb{C}) \neq 0$. Then the first Betti number of V is different from 0 and hence the Albanese variety $\mathrm{Alb}(V)$ of V is nontrivial. Choose a point y_0 in $V(\mathbb{R})$ and let $\alpha : V \to \mathrm{Alb}(V)$ be the corresponding Albanese morphism (in particular, $\alpha(y_0) = 0$). Both $\mathrm{Alb}(V)$ and α are defined over $\mathbb{R}$ (cf. for example [9]). There exists an isogeny $\lambda : \mathrm{Alb}(V) \to A_1 \times \cdots \times A_r$ defined over $\mathbb{R}$, where $A_1, \ldots, A_r$ are simple complex Abelian varieties defined over $\mathbb{R}$, cf. [7, p.122]. Let $\pi : A_1 \times \cdots \times A_r \to A_1$ be the canonical projection and let $\varphi = \pi \circ \lambda \circ \alpha$. Let us also put $A = A_1$. Since the set $\alpha(V)$ generates $\mathrm{Alb}(V)$ as a group, it follows that the morphism $\varphi : V \to A$ is nonconstant.

Choose a nonsingular Zariski locally closed real algebraic curve Z in X. Choose a point x_0 in Z and let G be a connected neighborhood of x_0 in Z, whose closure is compact. There exists a nonsingular complex projective curve C defined over $\mathbb{R}$ such that Z is isomorphic to a Zariski open subvariety D of $C(\mathbb{R})$. Let $\psi : Z \to D$ be an isomorphism and set $\psi(x_0) = c_0$, $\psi(G) = N$. Fix a positive integer n satisfying $n \geq n_{C,N}$, where $n_{C,N}$ is as in Lemma 3.2.

Choose a small neighborhood U of y_0 in $V(\mathbb{R})$ such that the set $A(\mathbb{R})_0 \backslash \varphi^{(n)}(U^n)$ contains a nonempty open subset of $A(\mathbb{R})_0$. By shrinking U if necessary we may assume that there exists a real analytic diffeomorphism $\sigma : U \to \mathbb{R}^m$ with $\sigma(y_0) = 0$, where $m = \dim V$. Since U is Zariski dense in V and φ is continuous in the Zariski topology, it follows that φ is nonconstant on U. Choose a point y_1 in U with $\varphi(y_1) \neq 0$. Let L be the vector subspace of $\mathbb{R}^m$ generated by the vector $\sigma(y_1)$ and let $M = \sigma^{-1}(L)$. Then M is a real analytic submanifold of U containing y_0 and diffeomorphic to $\mathbb{R}$, such that φ is nonconstant on M.

Since M is diffeomorphic to $\mathbb{R}$, one can find a continuous map $f : X \to Y = V(\mathbb{R})$ such that $f(X) \subseteq M$, $f(x_0) = y_0$, and f is nonconstant on any neighborhood of x_0 in Z. We claim that f does not belong to the closure of $\mathcal{R}(X, Y)$ in $\mathcal{C}(X, Y)$. Indeed, by construction, $A(\mathbb{R})_0 \backslash (\varphi \circ f)^{(n)}(X^n)$ contains a nonempty open subset of $A(\mathbb{R})_0$ and $(\varphi \circ f)^{(n)}$ is nonconstant on G^n. On the other hand, by Lemma 3.3, if $g : X \to Y$ is a regular map, then either $(\varphi \circ g)^{(n)}(G^n) = \{0\}$ or $(\varphi \circ g)^{(n)}(G^n) = A(\mathbb{R})_0$. This implies that $(\varphi \circ g)^{(n)}$ cannot be arbitrarily close to $(\varphi \circ f)^{(n)}$, and therefore g cannot be arbitrarily close to f. $\square$

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