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*Inventiones mathematicae*


# Non-vanishing of quadratic twists of modular L-functions

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## 1. Introduction

Let $F(z) := \sum_{n=1}^{\infty} a(n)q^n \in S_{2k}(M, \chi_0)$ ($q := e^{2\pi iz}$ as usual) be a newform of weight $2k$ with trivial Nebentypus character $\chi_0$, and let $L(F, s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ be its $L$-function. If $D \neq 0$ is a fundamental discriminant, then let $\chi_D$ denote the Kronecker character for the field $\mathbb{Q}(\sqrt{D})$. The $D$-quadratic twist of $F$, denoted $F_D$, is the newform corresponding to the twist of $F$ by the character $\chi_D$. If $(D, M) = 1$, then $F_D(z) := \sum_{n=1}^{\infty} \chi_D(n)a(n)q^n$. The central critical values $L(F_D, k)$ have been the subject of much study, both because of their intrinsic interest and because of the prominent role they have played in Kolyvagin's work on the Birch and Swinnerton-Dyer Conjecture (see [B-F-H], [I], [J], [Ko], [Ma-M], [M-M1], [O-S], [P-P]).

Waldspurger proved a fundamental theorem [Théorème 1, W1] relating these central critical values to the Fourier coefficients of half-integral weight cusp forms. For notational convenience, if $D$ is a fundamental discriminant of a quadratic number field, then define $D_0$ by

$$D_0 := \begin{cases} |D| & \text{if } D \text{ is odd}, \\ |D|/4 & \text{if } D \text{ is even} \end{cases}. \tag{1}$$

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Waldspurger's theorem then guarantees the existence of a $\delta(F) \in \{\pm 1\}$, an integer $N$, and a non-zero eigenform $0 \neq g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+\frac{1}{2}}(N)$ of half-integral weight $k + \frac{1}{2}$ such that $4M|N$ and for each fundamental discriminant $D$ for which $\delta(F)D > 0$,

$$b(D_0)^2 = \begin{cases} \varepsilon_D \frac{L(F_D,k)D_0^{k-\frac{1}{2}}}{\Omega} & \text{if } (D_0, N) = 1, \\ 0 & \text{otherwise} \end{cases}, \tag{2a}$$

where $\Omega$ is some non-zero complex period of $F$ and $\varepsilon_D$ is an algebraic number. Moreover,

$$\text{the } b(D_0)\text{'s are algebraic integers in some finite extension of } \mathbb{Q}. \tag{2b}$$

(For a proof that the existence of such a $g$ is a consequence of Waldspurger's work see the beginning of §2 below.) This result is at the heart of this paper's study of the values $L(F_D, k)$.

There have been numerous papers focusing on the non-vanishing of $L(F_D, k)$. The works of Bump, Friedberg, Hoffstein [B-F-H], [F-H], Luo [H-L], M.R. Murty and V. K. Murty [M-M2], Mai [Ma-M], Ono [O1] and Waldspurger [W1] [W2], among others, guarantee the existence of infinitely many fundamental discriminants $D$ for which $L(F_D, k) \neq 0$. In this note we too focus on questions pertaining to the non-vanishing of the values $L(F_D, k)$.

**Definition.** *Let $P$ be the set of fundamental discriminants. If $\pi = \{p_1, p_2, \ldots, p_t\}$ is an arbitrary finite set of distinct primes, and if $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots \varepsilon_t) \in \{\pm 1\}^t$, then define sets of fundamental discriminants $P(r), P(\varepsilon, \pi)$, and $P(\varepsilon, \pi, r)$ by*

$$\begin{aligned} P(r) &:= \{D \in P \mid D \textit{ square-free with exactly } r \textit{ prime factors}\}, \\ P(\varepsilon, \pi) &:= \{D \in P \mid D \textit{ square-free, } \chi_D(p_i) = \varepsilon_i \textit{ for each } i\}, \\ P(\varepsilon, \pi, r) &:= \{D \in P(\varepsilon, \pi) \mid D \textit{ has exactly } r \textit{ prime factors}\}\ . \end{aligned}$$

If $F \in S_{2k}(M, \chi_0)$ is a newform, then for any $P(\varepsilon, \pi, r)$ we consider the following question.

**Question.** *How many $0 < |D| \leq X$ in $P(\varepsilon, \pi, r)$ have the property that*

$$L(F_D, k) \neq 0\ ?$$

This question for $r = 1$ (i.e., prime twists of $F$) has been asked by H. Iwaniec.

Hoffstein and Luo [H-L] have proved that there are infinitely many $D$ in $\bigcup_{r=1}^{4} P(\varepsilon, \pi, r)$ for which $L(F_D, k) \neq 0$. Inspired by some ideas in [O2], we prove a fundamental lemma that implies that under a certain (mild) condition a *positive proportion* of $D \in P(\varepsilon, \pi, r)$ have the property that $L(F_D, k) \neq 0$. We expect that this condition holds for $P(\varepsilon, \pi, 1)$ for every newform $F$ of even weight and trivial Nebentypus character. In other words, we expect that our methods always prove that there are infinitely many nonvanishing prime twists. For evidence supporting this, see Corollary 2 below and the example involving Ramanujan's Delta function.

Goldfeld conjectured that a positive proportion of $0 < |D| \leq X$ have the property that $L(F_D, k) \neq 0$ (cf. [G] and [K-S]). This has only been proved for very exceptional $F$ by James [Ja], Kohnen [K2], and Vatsal [V]. Apart from these forms, the best result to date is due to Perelli and Pomykala [P-P] who show that the number of $0 < |D| \leq X$ for which $L(F_D, k) \neq 0$ is $\gg X^{1-\varepsilon}$. Our fundamental lemma combined with a result of Friedberg and Hoffstein implies the stronger result that the number of $0 < |D| \leq X$ for which $L(F_D, k) \neq 0$ is $\gg X/\log X$.

For modular elliptic curves, these results shed light on the distribution of quadratic twists having rank zero. Recall that if $E/\mathbb{Q}$ is an elliptic curve given by

$$E : y^2 = x^3 + Ax + B \ ,$$

then $E(D)$, its $D$-quadratic twist, is the curve given by

$$E(D) : y^2 = x^3 + AD^2x + BD^3 \ .$$

Although it is widely believed that a positive proportion of twists $E(D)$ have rank zero, this is only known for special curves. Heath-Brown [HB] and Wong [Wo] obtain such results under special circumstances. When $r = 1$, the above Question is connected to the following conjecture which was brought to our attention by J. Silverman.

**Conjecture.** *If $E/\mathbb{Q}$ is an elliptic curve, then there are infinitely many primes $p$ for which either $E(p)$ or $E(-p)$ has rank zero.*

Our results follow from the following lemma.

**Fundamental Lemma.** *Let $g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+\frac{1}{2}}(N)$ be an eigenform for which*

(i) *$b(m) \neq 0$ for at least one square-free $m > 1$ coprime to $4N$,*

(ii) *the coefficients $b(n)$ are algebraic integers contained in a number field $K$,* *and let* $\delta \in \{\pm 1\}$. *Let* $v$ *be a place of* $K$ *over* 2, *and for each* $s$ *let*

$$B_s := \{\delta m/m > 1 \textit{ square-free}, \quad (m, 4N) = 1, \textit{and } \mathrm{ord}_v(b(m)) = s\} \ .$$

*Let* $s_0$ *be the smallest integer for which* $B_{s_0} \neq \emptyset$. *If* $B_{s_0} \cap P(r) \neq \emptyset$, *then*

$$\#\{m \in B_{s_0} \cap P(r)/|m| \ \leq X\} \gg \frac{X}{\log X}(\log\log X)^{r-1} \ .$$

Using this lemma we obtain the following results, valid for arbitrary $P(\varepsilon, \pi)$ and $P(\varepsilon, \pi, r)$.

**Corollary 1.** *Suppose* $F \in S_{2k}(M, \chi_0)$ *is a newform. Let* $g(z) := \sum_{n=1}^{\infty} b(n)q^n \in S_{k+\frac{1}{2}}(N)$ *be an eigenform satisfying* (2a,b). *Let* $K$ *be the extension of* $\mathbb{Q}$ *generated by the* $b(n)$'*s, and* $v$ *a place of* $K$ *over* 2. *Define* $u_0$ *by*

$$u_0 := \min\{u/\mathrm{ord}_v(b(|D|) = u \textit{ for some } D \in P(\varepsilon) \textit{ coprime to } 4N, \\ \delta(F)D > 0\} \ .$$

*If there exists a* $D_1 \in P(\varepsilon, \pi, r)$ *coprime to* $4N$, $\delta(F)D_1 > 0$, *for which* $\mathrm{ord}_v(b(|D_1|)) = u_0$, *then a positive proportion of the* $D \in P(\varepsilon, \pi, r)$ *satisfy* $L(F_D, k) \neq 0$.

**Corollary 2.** *If* $E/\mathbb{Q}$ *is an elliptic curve with conductor* $\leq 100$, *then either* $E(-p)$ or $E(p)$ *has rank zero for a positive proportion of primes* $p$.

**Corollary 3.** *If* $F \in S_{2k}(M, \chi_0)$ *is a newform, then the number of* $0 < |D| \leq X$ *in* $P(\varepsilon, \pi)$ *for which* $L(F_D, k) \neq 0$ *is* $\gg X/\log X$. *In particular, if* $E/\mathbb{Q}$ *is a modular elliptic curve, then the number of* $0 < |D| \leq X$ *for which* $E(D)$ *has rank zero is* $\gg X/\log X$.

## 2. Proofs

For each positive integer $k$, let $S_{k+\frac{1}{2}}(N)$ be the space of cusp forms of half-integral weight $k + \frac{1}{2}$ on $\Gamma_1(4N)$, and let $S_k(M)$ (resp. $S_k(M, \chi_0)$) be the space of cusp forms of weight $k$ on $\Gamma_1(M)$ (resp. $\Gamma_0(M)$ with trivial Nebentypus character $\chi_0$).

*Proof of* (*2a,b*). While Waldspurger's theorem is quite general, two technical hypotheses, H1 and H2 in the notation of [W1], intervene in an attempt to apply it to an arbitrary form $F$. However, there is a twist $F_\psi$ of $F$, satisfying these hypotheses. One can construct such a character $\psi$ as follows. Choose $\psi$ to be a product of an even character of conductor a large power of 2 and odd characters of conductor either $\ell$ or $\ell^2$ for each prime $\ell$ for which $\varepsilon_\ell(F) = -1$ ($\ell$ can equal 2). Here, $\varepsilon_\ell(F)$ is the local root number at $\ell$. That $F_\psi$ satisfies Hypothesis H1 is a consequence of the characterization of local root numbers. The large power of 2 dividing the conductor of $\psi$ ensures that $F_\psi$ satisfies Hypothesis H2. A similar construction is carried out in more detail in [Section 6, J]. Now put

$$\delta(F) = (-1)^k \psi(-1) \quad \text{and} \quad \chi = \begin{cases} \psi\left(\frac{-1}{\cdot}\right) & \text{if } \psi(-1) = -1, \\ \psi & \text{otherwise} \end{cases},$$

and apply [Théorème 1, W1] to the form $F_\psi$ and character $\chi$ (which is even by construction and satisfies $\chi^2 = \psi^2$). The existence of an $N$ and $g$ satisfying (2a) can be seen by inspecting the explicit formulae given in [W1] for the functions $c_p(n)$ (notation as in [I, 4, W1]). Moreover, $N$ can be chosen so that it is divisible by the conductor cond$(\chi)$ of $\chi$. This is just a straight-forward case-by-case analysis. The $\varepsilon_D$'s can be taken to be the root numbers $W(\chi^{-1}\chi_{-4}^k\chi_D)$ if $\delta(F) = 1$ or $W(\chi^{-1}\chi_{-4}^{k+1}\chi_D)$ if $\delta(F) = -1$. By the theory of modular symbols (cf. [M-T-T] and [Theorem 3.5.4, G-S]) there is a complex period $\Omega_0$ such that $L(F_D, k)D_0^{k-\frac{1}{2}}/\Omega_0$ is in the ring of integers of some finite extension of $\mathbb{Q}$ for any discriminant $D$ for which $\delta(F)D > 0$ and $(4M, D_0) = 1$. It is a simple consequence of the definition of the root numbers $\varepsilon_D$ and the assumption that $(D_0, N) = 1$ that $\mathrm{cond}(\chi)^{\frac{1}{2}}\varepsilon_D$ is an algebraic integer lying in a fixed, finite extension of $\mathbb{Q}$. Property (2b) follows from combining these two observations with (2a) (take $\Omega = \Omega_0 \mathrm{cond}(\chi)^{-\frac{1}{2}}$). Note that we are not making a precise claim about the nature of $N$. To do so would require much more care than is needed for either statements (2a,b) or the applications in this paper. Q.E.D.

By the theory of newforms, every $F \in S_k(M)$ can be uniquely expressed as a linear combination

$$F(z) = \sum_{i=1}^{r} \alpha_i A_i(z) + \sum_{j=1}^{s} \beta_j B_j(\delta_j z) ,$$

where $A_i(z)$ and $B_j(z)$ are newforms of weight $k$ and level a divisor of $M$, and where each $\delta_j$ is a non-trivial divisor of $M$. Let

$$F^{\mathrm{new}}(z) := \sum_{i=1}^{r} \alpha_i A_i(z) \quad \text{and} \quad F^{\mathrm{old}}(z) := \sum_{j=1}^{s} \beta_j B_j(\delta_j z)$$

be, respectively, the *new part of F* and the *old part of F*.

If $F(z) := \sum_{n=1}^{\infty} a(n)q^n \in S_k(M)$ is a newform, then the $a(n)$'s are algebraic integers and generate a finite extension of $\mathbb{Q}$, say $K_F$. If $K$ is any finite extension of $\mathbb{Q}$ containing $K_F$, and if $\mathcal{O}_v$ is the completion of the ring of integers of $K$ at any finite place $v$ with residue characteristic, say $\ell$, then by the work of Shimura, Deligne, and Serre ([Sh], [D], [D–S]) there is a (not necessarily unique) continuous representation

$$\rho_{F,v} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_v)$$

for which

(R1) $\rho_{F,v}$ is unramified at all primes $p \nmid M\ell$ .

(R2) trace $\rho_{F,v}(\mathrm{frob}_p) = a(p)$ for all primes $p \nmid M\ell$ .

Here $\mathrm{frob}_p$ denotes any Frobenius element for the prime $p$.

*Proof of Lemma.* Let $s_0$ be the smallest integer such that $B_{s_0} \neq \emptyset$, and let $m_0 > 1$ be some square-free integer coprime to $4N$ for which $\mathrm{ord}_v(b(m_0)) = s_0$. It is clear that by taking combinations of quadratic twists (and possibly twists of twists) one can find a cusp form $g'(z) := \sum_{n=1}^{\infty} b'(n)q^n$ of weight $k + \frac{1}{2}$ and level $N'$ coprime to $m_0$ for which $b'(m_0) = b(m_0)$ and, for any square-free integer $m$, $b'(m) = 0$ if $(m, 4N') \neq 1$ or $m = 1$, and otherwise $b'(m)$ is either $b(m)$ or 0. Since $g$ is an eigenform, it follows that

$$\mathrm{ord}_v(b'(n)) \geq s_0 \tag{3}$$

for all $n$. Let $G(z) := \sum_{n=1}^{\infty} c(n)q^n := g'(z) \cdot \left(1 + 2\sum_{n=1}^{\infty} q^{n^2}\right)$, so

$$c(n) = b'(n) + 2 \sum_{\substack{mx^2+y^2=n, y>0 \\ m \text{ square-free}}} b'(mx^2) \ . \tag{4}$$

Then $G$ is a cusp form of integer weight $k+1$ on $\Gamma_1(4N')$. Write $G = G^{\mathrm{new}} + G^{\mathrm{old}}$. Since $\mathrm{ord}_v(b'(m_0)) = s_0$, it follows from (3) and (4) that $c(m_0) \neq 0$. Since $m_0$ is coprime to the level of $G$, it must be that $G^{\mathrm{new}}$ is not identically zero. Write

$$G^{\mathrm{new}} = \sum_{i=1}^{h} \alpha_i f_i(z), \quad \alpha_i \neq 0 \ ,$$

where each $f_i(z) := \sum_{n=1}^{\infty} a_i(n)q^n$ is a newform of level dividing $4N'$. If $(n, 2N') = 1$, then

(5) $$c(n) = \sum_{i=1}^{h} \alpha_i a_i(n) \quad .$$

Let $L$ be a finite extension of $\mathbb{Q}$ containing $K$, the Fourier coefficients of each $f_i$, and the $\alpha_i$'s. Let $w$ be a place of $L$ over $v$, let $e$ be the ramification index of $w$ over $v$, let $\mathcal{O}_w$ be the completion of the ring of integers of $L$ at the place $w$, and let $\lambda$ be a uniformizer for $\mathcal{O}_w$. Let

(6) $$E = \max_{1 \le i \le h} |\mathrm{ord}_w(4\alpha_i)| \quad ,$$

and let $\rho_{f_i,w} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_w)$ be a representation as in the preceding discussion. Finally, let $\varepsilon : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathcal{O}^\times$ be the cyclotomic character giving the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on all $2^n$th power roots of unity. Consider the representation

$$\rho = \varepsilon \bigoplus_{i=1}^{h} \rho_{f_i,w} \mod \lambda^{E+es_0+1} \quad .$$

Write

$$m_0 = p_1 \ldots p_r, \quad p_j \text{ a prime} \quad .$$

By the Chebotarev Density Theorem, for each $j$ there are $\gg X/\log X$ primes $q$ less than $X$ for which $\rho(\mathrm{frob}_q) = \rho(\mathrm{frob}_{p_j})$. By (R2), for such a prime, $a_i(q) \equiv a_i(p_j) \mathrm{mod}\ \lambda^{E+es_0+1}$ for all $i$. Also, $q = \varepsilon(\mathrm{frob}_q) \equiv \varepsilon(\mathrm{frob}_{p_j}) = p_j \bmod 4$. It follows from these observations and the multiplicativity of the Fourier coefficients of newforms that there are $\gg \frac{X}{\log X}(\log\log X)^{r-1}$ square-free integers $\delta m = \delta q_1 \cdots q_r \in P(r)$, $m < X$, such that

$$(m, 4N') = 1 \quad \text{and } a_i(m) \equiv a_i(m_0) \bmod \lambda^{E+es_0+1} \quad .$$

For any such $m$, it follows from (5) and (6) that $c(m) \equiv c(m_0) \bmod \lambda^{es_0+1}$. By our choice of $s_0$, $\mathrm{ord}_w(c(m_0)) = es_0$, so $\mathrm{ord}_w(c(m)) = es_0$. It follows from (3) and (4) that $\mathrm{ord}_w(b'(m)) = es_0$ (equivalently, $\mathrm{ord}_v(b'(m)) = s_0$), whence $\mathrm{ord}_v(b(m)) = s_0$. In other words, $\delta m \in B_{s_0}$. This proves the lemma.

Q.E.D.

*Proof of Corollary 1.* By taking combinations of quadratic twists of $g$ (and possibly twists of twists), one obtains an eigenform $g^*(z) := \sum_{n=1}^{\infty} b^*(n)$ of level coprime to $D_1$ whose coefficients are supported on integers $m > 1$ such that $\chi_{\delta(F)m}(p_i) = \varepsilon_i$ for each $i$ and for which $b^*(|D_1|) = b(|D_1|) \neq 0$. Furthermore, $b^*(m)$ is either $b(m)$ or 0. The corollary now follows from the Fundamental Lemma applied to the eigenform $g^*(z)$ with $\delta = \delta(F)$ (note: $s_0 = u_0$).

Q.E.D.

*Proof of Corollary 2.* This result is an application of the previous corollary with $r = 1$ and $\pi = \emptyset$ and of the work of Kolyvagin [Ko]. For each isogeny class of elliptic curves over $\mathbb{Q}$, Basmaji [B] computed a basis from which a relevant eigenform $g(z) = \sum_{n=1}^{\infty} b(n)q^n$ can be constructed. The rest of the proof involves checking the condition of Corollary 1.

Q.E.D.

*Proof of Corollary 3.* Let $g(z) := \sum_{n=1}^{\infty} b(n)q^n$ be an eigenform satisfying (2a,b). By [Theorem B(i), F-H], there is a $D' \in P(\varepsilon, \pi)$ coprime to $4N$ for which $b(|D'|) \neq 0$. One now proceeds as in the proof of Corollary 1, with $D'$ playing the role of $D_1$. The application to modular elliptic curves follows from the work of Kolyvagin [Ko].

Q.E.D.

*Remark 1.* The theorem of Friedberg and Hoffstein was a critical ingredient in the proof of Corollary 3. However, the proof does not require the existence of infinitely many non-vanishing critical values, only a single suitable non-zero value.

*Remark 2.* The fundamental lemma is a result about the coefficients of an eigenform $g(z)$ of half-integral weight. However, the result can be applied in a slightly different setting. Let $M$ be an odd square-free integer, and let $F(z) \in S_{2k}(M, \chi_0)$ be a newform. Kohnen and Zagier [K1], [K-Z] have constructed an explicit cusp form $g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+\frac{1}{2}}(M)$ for which $b(n) = 0$ unless $(-1)^k n \equiv 0, 1 \pmod 4$ and for which

$$L(F_D, k) = 2^{-\nu(M)}|D|^{\frac{1}{2}-k} \frac{\pi^k}{(k-1)!} \frac{\langle F, F\rangle}{\langle g, g\rangle} |b(|D|)|^2 \tag{7}$$

for any fundamental discriminant $D$ for which $(-1)^k D > 0$ and $\chi_D(\ell) = w_\ell$, the eigenvalue of the Atkin-Lehner involution at $\ell$, for each prime $\ell$ dividing $M$. This $g(z)$ is an eigenform for operators similar to the classical Hecke operators. The conclusion and proof of the fundamental lemma applies to these forms as well.

*Example.* Let $\Delta(z) := \sum_{n=1}^{\infty} \tau(n)q^n \in S_{12}(1)$ be Ramanujan's delta function, and let $g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{\frac{13}{2}}(1, \chi_0)$ be the eigenform given in [K-Z] satisfying (7). It turns out that $g(z) \equiv \sum_{n=0}^{\infty} q^{(2n+1)^2}$ (mod 8), but modulo 16 it is

$$g(z) \equiv q + 8q^4 + 8q^5 + 9q^9 + 8q^{13} + \cdots \pmod{16}.$$

By the analog of Corollary 1 (see Remark 2), we find that $u_0 = 4$ since $b(5) \equiv 0 \pmod{8}$ but $b(5) \not\equiv 0 \pmod{16}$. Since $5 \in P(1)$, there is a positive proportion of primes $p$ for which $L(\Delta_p, 6) \neq 0$. In fact, Kohnen and Zagier [Corollary 2, K-Z] first noticed that if $p \equiv 5 \pmod{8}$ is prime, then $L(\Delta_p, 6) \neq 0$.

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## References


[B] J. Basmaji, Ein Algorithmus zur Berechnung von Hecke Operatoren und Anwendungen auf modulare Kurven, Ph.D. Thesis, Univ. Essen, 1996

[B-F-H] D. Bump, S. Friedberg, J. Hoffstein, Nonvanishing theorems for $L$-functions of modular forms and their derivatives, Invent. Math. **102**, 543–618 (1990)

[D] P. Deligne, Formes modulaires et repŕesentations $\ell$-adiques, Seminaire Bourbaki **355**, (1969)

[D-S] P. Deligne, J.-P. Serre, Formes modulaires de poids 1, Ann. Sci. École Normale Sup. $4^e$ sér. **7**, (1974)

[F-H] S. Friedberg, J. Hoffstein, Nonvanishing theorems for automorphic $L$-functions on GL(2), Ann. Math. **142**, 385–423 (1995)

[G] D. Goldfeld, Conjectures on elliptic curves over quadratic fields, Number Theory, Carbondale, Springer Lect. Notes **751**, 108–118 (1979)

[G-S] R. Greenberg, G. Stevens, On the conjecture of Mazur, Tate, and Teitelbaum, $p$-adic Monodromy and the Birch and Swinnerton-Dyer Conjecture (B. Mazur and G. Stevens, ed.)

[HB] D.R. Heath-Brown, The size of Selmer groups for the congruent number problem II Invent. Math. **118**, 331–370 (1994)

[H-L] J. Hoffstein, W. Luo, Nonvanishing of $L$-series and the combinatorial sieve, Math. Res. Let. **4**, 435–444 (1997)

[I] H. Iwaniec, On the order of vanishing of modular $L$-functions at the critical point, Sém. Thér. Nombres Bordeaux **2**, 365–376 (1990)

[Ja] K. James, $L$-series with non-zero central critical value, (preprint)

[J] N. Jochnowitz, Congruences between modular forms of half-integral weights and implications for class numbers and elliptic curves, (preprint)

[K-S] N. Katz, P. Sarnak, Zeros of zeta functions, their spacings, and their spectral nature, (preprint)

[K1] W. Kohnen, Fourier coefficients of modular forms of half-integral weight, Math. Ann. **271**, 237–268 (1985)

[K2] W. Kohnen, (preprint)

[K-Z] W. Kohnen, D. Zagier, Values of $L$-series of modular forms at the center of the critical strip, Invent. Math. **64**, 173–198 (1981)

[Ko] V. Kolyvagin, Finiteness of $E(\mathbb{Q})$ and $\text{Ш}_{E/\mathbb{Q}}$ for a subclass of Weil curves (Russian), Izv. Akad. Nauk. USSR, ser. Matem. **52**, 522–540 (1988)

[Ma-M] L. Mai, M.R. Murty, A note on quadratic twists of an elliptic curve, Elliptic curves and related topics, Ed. H. Kisilevsky and M.R. Murty,CRM **4**, 121–124 (1994)

[M-T-T] B. Mazur, J. Tate, J. Teitelbaum, On $p$-adic analogues of the conjectures of Birch and Swinnerton-Dyer, Invent. Math. **84**, 1–48 (1986)

[M-M1] M.R. Murty, V.K. Murty, Mean values of derivatives of modular $L$-series, Ann. Math. **133**, 447–475 (1991)

[M-M2] M.R. Murty, V.K. Murty, Non-vanishing of $L$-functions and applications, Birkhäuser 1997

[O1] K. Ono, Rank zero quadratic twists of modular elliptic curves, Compositio Math. **104**, 293–304 (1996)

[O2] K. Ono, Twists of elliptic curves, Compositio Math. **106**, 349–360 (1997)

[O-S] K. Ono, C. Skinner, Fourier coefficients of half-integral weight modular forms modulo $\ell$, Annals of Math., (to appear)

[P-P] A. Perelli, J. Pomykala, Averages of twisted elliptic $L$-functions, Acta Arith. **80**, 149–163 (1997)

[Sh] G. Shimura, Introduction to the arithmetic theory of automrphic functions Iwanami Shoten and Princeton Univ. Press, 1971

[V] V. Vatsal, Canonical periods and congruence formulae, (preprint)

[W1] J.-L. Waldspurger, Sur les coefficients de Fourier des formes modulaires de poids demi-entier J. Math. Pures et Appl. **60**, 375–484 (1981)

[W2] J.-L. Waldspurger, Correspondances de Shimura et quaternions, Forum Math. **3**, 219–307 (1991)

[Wo] S. Wong, Rank zero twists of elliptic curves, (preprint)