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Relative modular symbols and Rankin-Selberg convolutions

By *D. Kazhdan*, *B. Mazur* at Cambridge, and *C.-G. Schmidt* at Karlsruhe

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0. Introduction

Let $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ denote the ring of adeles of the field of rational numbers, and, for an integer $n \geqq 2$ let π and σ be cuspidal automorphic representations of $GL_n(\mathbb{A})$ and $GL_{n-1}(\mathbb{A})$, respectively. For such pairs (π, σ), Jacquet, Piatetski-Shapiro and Shalika [14] introduced an L-function $L(\pi, \sigma, s)$ which we also denote $L(\pi \otimes \sigma, s)$ to remind us that it "would be" the L-function attached to the automorphic representation $\pi \otimes \sigma$ of $GL_{n(n-1)}(\mathbb{A})$, the tensor product notation being in accordance with the Langlands dictionary, when this automorphic representation $\pi \otimes \sigma$ exists. The L-function $L(\pi, \sigma, s)$ is defined by Jacquet, Piatetski-Shapiro and Shalika as a certain adelic ("Rankin-Selberg") integral, and is shown to be an entire function of the variable s, and to satisfy a functional equation of the expected type, interchanging s and $1 - s$. The collection of functions $L(\pi, \sigma, s)$ for fixed automorphic representation π and suitably varying σ plays a key role in the "converse theorems" whose goal is to characterize an automorphic representation π in terms of the analytic character of Mellin transforms.

This well-working theory of *complex-valued* L-functions being available to us, it is natural to try to go further. Here is what we might wish to do, in this context, or in any similar automorphic context. Firstly, for fixed (π, σ) it will be of interest to us to study the package of L-functions $L(\pi \otimes \chi, \sigma, s)$ for χ ranging through all finite Dirichlet characters. Now if there are *critical values* $s = s_j$; $j = 1, 2, \ldots$ of the variable s for (π, σ) (in the sense, say, of [8]) one will wish to show that the function $\chi \mapsto L(\pi \otimes \chi, \sigma, s_j)$ (after "normalization"; i.e., after division by an appropriate "period" which depends only on j and the sign of χ) takes algebraic numbers as values, and moreover, one will wish to control the algebraic number fields containing those values, their denominators, and their dependence upon χ. It is natural, at least in certain instances, to try to organize all this information in terms of "distributions". For example, for a critical value s_j and a prime number p one can try to find two (p-adic valued) distributions $\mu^{\pm}$ on $\mathbb{Z}_p$ such that integration of a finite Dirichlet character χ of p-power conductor over the distribution $\mu^{\mathrm{sign}(\chi)}$ yields the normalized special values $L(\pi \otimes \chi, \sigma, s_j)$. Having done this, the next step is to find upper bounds for values of these distributions. In essence, such upper bounds prove analogues to the classical Kummer congruences which give us p-adic interpolational information about the special values, and lead to the construction of a corresponding theory of p-adic L-functions.

For $n = 2$ such a theory is already available, under suitable assumptions concerning the infinity components $(\pi_\infty, \sigma_\infty)$ and the p-components (π_p, σ_p). See [16], [17], and with only mild modifications such a theory is also proven for $n = 3$ [18]. The basic requirement on infinity components is that there indeed be "critical values", but more specifically that π_∞ and σ_∞ have nontrivial Lie-algebra cohomology. In both cases $n = 2$ and $n = 3$ the algebraicity (and integrality properties) of the special values is established by expressing the adelic integrals involved in terms of "modular symbols" in the homology of the appropriate symmetric space. The modular symbol structure gives one indeed a much tighter hold over the structure of "special values" insofar as it reduces many issues to relatively explicit questions regarding homology of finite simplicial complexes with coefficients in the rings of integers of algebraic number fields.

The first section of the present article sets up a topological construction providing a theory of what we call "relative modular symbols" for general n which incorporates the known constructions for $n = 2, 3$.

To put ourselves in a context for the study of "special values" we work under a very simplifying hypothesis for the pair (π, σ) of cuspidal automorphic representations of $GL_n(\mathbb{A})$ and $GL_{n-1}(\mathbb{A})$, respectively. We suppose that π_∞ and σ_∞ are both "special", which guarantees that they both occur in the cohomology of the appropriate symmetric space when we compute cohomology with constant coefficients, and which also gives us that $s = 1/2$ is the unique critical value. We also assume that p is unramified for both π and σ and make some simplifying assumptions on the nature of the twisting Dirichlet character χ. In such a context we analyze a certain product

$$P_\infty(s) \cdot L(\pi \otimes \chi, \sigma, s)$$

where $P_\infty(s)$ is an explicit entire function described as a certain integral dependent only upon the Whittaker models of π_∞ and σ_∞. In particular, since we have fixed both (π_∞ and σ_∞) to be "special" the function $P_\infty(s)$ is uniquely determined by the integer n. We express the special value $P_\infty(1/2) \cdot L(\pi \otimes \chi, \sigma, 1/2)$ directly in terms of relative modular symbols, giving that $P_\infty(1/2) \cdot L(\pi \otimes \chi, \sigma, 1/2)$ is algebraic. To obtain algebraicity results for $L(\pi \otimes \chi, \sigma, 1/2)$ itself from this method, we would be obliged to show that $P_\infty(s)$ doesn't vanish at $s = 1/2$.

Question. Is $P_\infty(1/2) \neq 0$ for all $n \geqq 2$?

An affirmative answer to the above question is indeed known for $n = 2, 3$ (for $n = 3$, see Theorem 3.8 of [18]).

In section 2 we prove an analogue (for $n \geqq 2$) of a result known in the case $n = 2$ as the "Birch Lemma". Using it, we get some control over the behavior of $L(\pi \otimes \chi, \sigma, s)$ for varying χ. In section 4 we assume an affirmative answer to the above displayed question, and construct p-adic distributions whose integral over finite Dirichlet characters (satisfying appropriate conditions) gives the desired twisted special L-values. We expect a suitably adjusted distribution to be bounded in the ordinary case as in the case $n = 3$ (see Theorem B in [18]).

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1. The topological construction of "relative modular symbols"

Let X_n denote a symmetric space associated to $GL_n(\mathbb{R})^0$, the (connected) component of $GL_n(\mathbb{R})$ consisting in all $n \times n$ real matrices with positive determinant. For concreteness we take $SO_n(\mathbb{R})$ as choice of maximal compact subgroup $K \subset GL_n(\mathbb{R})^0$, and set $X_n := GL_n(\mathbb{R})^0/K$. Let SX_n denote $SL_n(\mathbb{R})/SO_n(\mathbb{R})$ so that $g \mapsto \left(\det(g)^{-1/n} \cdot g, \log\det(g)\right)$ gives us an isomorphism

$$X_n \to SX_n \times \mathbb{R}.$$

The subgroup $SL_n(\mathbb{Z}) \subset GL_n(\mathbb{R})^0$ acts discretely (by left multiplication) on SX_n, and its natural operation on X_n is compatible with the product structure displayed above, where its action on the second factor $\mathbb{R}$ is trivial. Let Γ_n be any congruence subgroup of $SL_n(\mathbb{Z})$, and let $Y_n := \Gamma_n \backslash X_n$ and $SY_n := \Gamma_n \backslash SX_n$ denote the quotient spaces, so that we have

$$Y_n \cong SY_n \times \mathbb{R}.$$

If d_n denotes the common dimension of X_n and Y_n, then $d_n = (n^2 + n)/2$. Let H^s denote cohomology. Form

$$\mathscr{H}^s := \varinjlim_{\Gamma_n} H^s(\Gamma_n \backslash SX_n; \mathbb{C}) \cong \varinjlim_{\Gamma_n} H^s(\Gamma_n \backslash X_n; \mathbb{C})$$

on which one has a natural action of the group $GL_n(\mathbb{A}_f)$, where $\mathbb{A}_f = \mathbb{A}_{\mathbb{Q}}^f$ is the ring of finite adeles of $\mathbb{Q}$.

Any irreducible subrepresentation η of $GL_n(\mathbb{A}_f)$ in $\mathscr{H}^s$ may be written as a restricted tensor product, $\bigotimes \pi_p$, where π_p is an irreducible representation of $GL_n(\mathbb{Q}_p)$ and where the p runs through all prime numbers. There is a unique irreducible admissible representation π_∞ of $GL_n(\mathbb{R})$ such that the restricted tensor product $\pi = \pi_\infty \otimes \left\{\bigotimes_p \pi_p\right\}$ is an admissible automorphic representation of $GL_n(\mathbb{A}_{\mathbb{Q}})$. If a cuspidal admissible automorphic representation π of $GL_n(\mathbb{A}_{\mathbb{Q}})$ can be obtained in this way, one says that π '*occurs*' *in* (*s-dimensional*) *cohomology*. When $n = 2$, a cuspidal automorphic form π occurs in cohomology if and only if π is generated by a holomorphic cuspidal modular form f of weight two. More generally for any $n \geqq 2$, a cuspidal automorphic representation π of $GL_n(\mathbb{A}_{\mathbb{Q}})$ occurs in cohomology if and only if π_∞ is the "special" representation; that is, if and only if it is the infinity-component of the "$n - 1$-st symmetric power" of a classical weight two automorphic form on GL_2. This seems to be well known to the experts but seems not to be explicitly found in the literature. To do this, one must show that the infinity-component of a cuspidal automorphic form which occurs in cohomology

a) is unique (for fixed n), and

b) is indeed the "special representation".

Uniqueness, together with an explicit characterization of this infinity-component follows from Theorem 5.6 of [20]. The point here is that our representation π_∞ is non-degenerate, and, using the notation of [20], the modules $A_q(\lambda)$ are degenerate for θ-stable parabolic subalgebras q which are not the Lie algebras of Borel subgroups; one can therefore directly apply Theorem 5.6 of [20]. Once uniqueness is established, one is left with the task of showing this unique representation π_∞ to be the "special representation", as claimed. For this, one uses the characterization of the modules A_q given in section 5 of [20].

Furthermore one has complete control over the dimensions of cohomology in which a cuspidal admissible automorphic representation can occur. Specifically, let

$$e_n := (n - [n/2] - 1)/2,$$

where [-] denotes the "integer part". Put $b_n := (d_n - 1)/2 - e_n$ and $t_n := (d_n - 1)/2 + e_n$ so that

$$b_n = (n^2 - n + 2[n/2])/4$$

and

$$t_n = (n^2 + 3n - 2[n/2])/4 - 1.$$

Then no cuspidal automorphic representations occur in s-dimensional cohomology for $s < b_n$ or $s > t_n$ and if π is a cuspidal (irreducible, admissible) automorphic representation that occurs in cohomology, it will occur with multiplicity one in $\mathscr{H}^s$ for $s = b_n$ and $s = t_n$. These facts can be computed fairly directly using the explicit description of π_∞. We omit the relevant calculations, as they seem to be well known. We should signal, however, we have no reference for them in the literature. We shall say that s is in the *cuspidal range* (for GL_n) if $b_n \leqq s \leqq t_n$. We have

n	d_n	b_n	t_n
1	1	0	0
2	3	1	1
3	6	2	3
4	10	4	5
5	15	6	8
6	21	9	11
7	28	12	15.

Let

$(\pi) \qquad n = n_1 + n_2 + \cdots + n_k$

be a partition of n. Let P denote the "standard parabolic subgroup" of GL_n associated to the partition (π), i.e., P consists of upper-triangular block $k \times k$ matrices whose (i,j)-th entry is a matrix of size $n_i \times n_j$ $(1 \leqq i,j \leqq k)$. Then $P = L \cdot U$ where

$$L = GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_k}$$

is the Levi component of P and U is the unipotent part of P. We view P, L and U as algebraic groups over $\mathbb{Q}$.

We take $SO_n \cap L = SO_{n_1} \times SO_{n_2} \times \cdots \times SO_{n_k} \subset L$ as our choice of maximal compact subgroup $K \subset L$, and imbed the quotient $X := L(\mathbb{R})^0/K$ into the symmetric space $X_n = GL_n(\mathbb{R})^0/SO_n(\mathbb{R})$:

$$X = L(\mathbb{R})^0/K = X_{n_1} \times X_{n_2} \times \cdots \times X_{n_k} \hookrightarrow X_n.$$

Congruence data for the partition (π). Having already fixed the congruence subgroup $\Gamma_n \subset SL_n(\mathbb{Z})$, we now fix congruence subgroups $\Gamma_{n_i} \subset SL_{n_i}(\mathbb{Z})$ for $i = 1, \ldots, k$, and an algebraic subgroup (usually unipotent) $W \subset P$.

For each element $u \in W(\mathbb{Q})$ and $i = 1, \ldots, k$ let

$$\Gamma_{n_i,u} \subset \Gamma_{n_i} \subset SL_{n_i}(\mathbb{Z})$$

be congruence subgroups such that if

$$\Gamma_{n,u} := \Gamma_{n_1,u} \times \Gamma_{n_2,u} \times \cdots \times \Gamma_{n_k,u},$$

then $\Gamma_{n,u}$ is contained in the intersection of $u^{-1} \cdot \Gamma_n \cdot u$ with $SL_{n_1}(\mathbb{Z}) \times SL_{n_2}(\mathbb{Z}) \times \cdots \times SL_{n_k}(\mathbb{Z})$, and put

$$Y_{n_i,u} := \Gamma_{n_i,u} \backslash X_{n_i} \quad \text{for } i = 1, \ldots, k.$$

The natural map $L \to GL_n$, $\lambda \mapsto u \cdot \lambda$ induces a mapping $\Phi_u : Y_{n_1,u} \times Y_{n_2,u} \times \cdots \times Y_{n_k,u} \to Y_n$ which is proper by a result of Borel and Prasad.

Main example. Consider the above construction for the partition $n = (n-1) + 1$. Thus we imbed $GL_{n-1} \times GL_1$ in GL_n in the standard way:

$$\left(\begin{array}{c|c} GL_{n-1} & 0 \\ \hline 0 & GL_1 \end{array}\right)$$

viewing the subgroup $L := GL_{n-1} \times GL_1 \subset GL_n$ as the Levi component of the parabolic subgroup $P_{/\mathbb{Q}} \subset GL_{n/\mathbb{Q}}$,

$$P = \left(\begin{array}{c|c} GL_{n-1} & * \\ \hline 0 & GL_1 \end{array}\right),$$

and we choose W to be the group of all upper-triangular matrices.

If Γ_n and Γ_{n-1} are given congruence subgroups of SL_n and SL_{n-1}, respectively, then for any $u \in W(\mathbb{Q})$ put

$$\Gamma_{n-1,u} = u\Gamma_n u^{-1} \cap \Gamma_{n-1},$$

and $\Gamma_1 = \{1\}$ so that

$$Y_{n-1,u} = \Gamma_{n-1,u} \backslash X_{n-1} \quad \text{and} \quad Y_1 = X_1 = \mathbb{R}.$$

We have, as usual, $Y_n = \Gamma_n \backslash X_n$ and (with some apology) we put $Y_{n-1} = \Gamma_{n-1} \backslash X_{n-1}$ (hoping that the reader will forgive the incompetence of this terminology by refraining from evaluating the indices when $n = 2$). We have, for any $u \in W(\mathbb{Q})$, the natural ("finite", proper) mapping $p_u : Y_{n-1,u} \to Y_{n-1}$. For each $u \in W$ we get a proper mapping

$$\Phi_u : Y_{n-1,u} \times Y_1 = Y_{n-1,u} \times \mathbb{R} \to Y_n.$$

Let $\mathscr{H}_r(-; A)$ denote Borel-Moore homology with coefficients in a ring A. (See, e.g., [2], or [9], Ch. 19, for a summary of all we use.) Then if Γ_n is torsion-free, (Y_n is a manifold, and) $\mathscr{H}_r(Y_n; A)$ is dual to $H^r(Y_n; A)$.

Let us return now to the general partition $n = n_1 + n_2 + \cdots + n_k$, and compatible choice of "congruence data". Fix a sequence of positive integers $r_1, \ldots, r_k$, and set $r = r_1 + \cdots + r_k$. For any $u \in W(\mathbb{Q})$, the proper morphism Φ_u above induces a morphism on Borel-Moore homology groups,

$$\mathscr{H}_{r_1}(Y_{n_1,u}; A) \otimes \cdots \otimes \mathscr{H}_{r_k}(Y_{n_k,u}; A) \to \mathscr{H}_r(Y_n; A).$$

For which (nontrivial) partitions (π) is it the case that we may find a sequence of integers r_j ($j = 1, \ldots, k$) such that r_j is in the cuspidal range for GL_{n_j} ($j = 1, \ldots, k$) and $r = \sum r_j$ is in the cuspidal range for GL_n? This is an easy enough combinatorial exercise:

We may find such a sequence $\{r_j\}$ only for the "main example" above, i.e. for the partition

$$(\pi_n) \qquad n = n_1 + n_2 \quad \text{with} \quad n_1 = n - 1 \quad \text{and} \quad n_2 = 1$$

where r_1 is taken to be at the top of the cuspidal range for GL_{n-1}, i.e. $r_1 - 1 = t_{n-1}$, $r_2 - 1$ is (necessarily) taken to be 0, and r is taken to be at the bottom of the cuspidal range for GL_n, i.e., $r - 1 = b_n$. This choice does work (because $t_{n-1} + 1 = b_n$).

For each $u \in W(\mathbb{Q})$, we then have the homomorphism induced by Φ_u:

$$\varphi_u : \mathscr{H}_{t_{n-1}+1}(Y_{n-1,u}; A) \otimes \mathscr{H}_1(Y_1; A) \to \mathscr{H}_{b_n+1}(Y_n; A).$$

Letting $1 \in \mathscr{H}_1(Y_1; A) = \mathscr{H}_1(\mathbb{R}; A) \cong A$ denote the fundamental class, and putting $\psi_u(x) = \varphi_u(x \otimes 1)$ we get a homomorphism,

$$\psi_u : \mathscr{H}_{t_{n-1}+1}(Y_{n-1,u}; A) \to \mathscr{H}_{b_n+1}(Y_n; A).$$

Fixing α an element in the Borel-Moore homology group

$$\alpha \in \mathcal{H}_{t_{n-1}}(SY_{n-1}; A) \cong \mathcal{H}_{t_{n-1}+1}(Y_{n-1}; A),$$

and β an element in the cohomology group

$$\beta \in H^{b_n}(SY_n; A) \cong H^{b_n+1}(Y_n; A)$$

we define the *topological relative modular symbol*

$$\{\alpha, \beta; u\} := \beta \cap \psi_u \circ p_u^*(\alpha) \in A.$$

2. A generalized local Birch lemma

Let F be a non-archimedean local field, and $\psi : F \to \mathbb{C}^\times$ an additive character with kernel given by the ring of integers $\mathcal{O}_F$. In this section π denotes a prime element of $\mathcal{O}_F$. Let $U_n(F) \subset GL_n(F)$ denote the subgroup of upper-triangular unipotent matrices, i.e., matrices $u = (u_{ij})$ such that $u_{ij} = 0$ for $i > j$ and $u_{ii} = 1$. If $u \in U_n(F)$, for $i = 1, \dots, n-1$, let u_i denote the i-th off-diagonal entry of u, i.e., $u_i = u_{i,i+1}$. Put

$$\psi(u) := \prod_{1 \leqq i \leqq n-1} \psi(u_i).$$

By a *Whittaker function* (relative to ψ) on $GL_n(F)$ we mean a function w such that

a) $w(ug) = \psi(u) \cdot w(g)$ for $u \in U_n(F)$, $g \in GL_n(F)$,

b) there is an open compact subgroup $K \subseteqq GL_n(\mathcal{O}_F)$ such that $w(gk) = w(g)$ for $k \in K$, $g \in GL_n(F)$, i.e., w is right invariant under K.

Let $\chi : \mathcal{O}_F^\times \to \mathbb{C}^\times$ be a finite character with conductor ideal generated by a suitable integer $f \in \mathcal{O}_F$; we extend χ to $\mathcal{O}_F$ by defining its value to be zero on non-units. Denote by φ the diagonal matrix $\mathrm{diag}(f^{-1}, f^{-2}, \dots, f^{-n}) \in GL_n(F)$, and denote conjugation by φ by the superscript φ, so that if $g = (g_{ij})$, then $g^\varphi = \varphi g \varphi^{-1} = (f^{j-i} \cdot g_{ij})$. Note that $U_n(\mathcal{O}_F)^\varphi \subset U_n(\mathcal{O}_F)$.

For $m \leqq n$, we imbed $GL_m(F)$ in $GL_n(F)$ in the usual manner by

$$g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}$$

and we shall often identify $GL_m(F)$ with its image in $GL_n(F)$ and $U_m(F)$ with $U_n(F) \cap GL_m(F)$. Further let $I = I_n$ denote the Iwahori subgroup of $GL_n(\mathcal{O}_F)$, i.e. those matrices in $GL_n(\mathcal{O}_F)$ which are upper-triangular modulo π.

Definition. Let w be a Whittaker function right-invariant under the Iwahori subgroup I_n. For $c \in GL_{n-1}$ such that $\det(c) \in \mathcal{O}_F$, set

$$w_\chi(c) := \chi(\det(c)) \cdot \sum_u \prod_{i=1}^{n-1} \chi(u_i)^i \cdot w(c \cdot u^{(\varphi^{-1})}),$$

where the summation over u is taken over a representative system for $U_n(\mathcal{O}_F)$ modulo $U_n(\mathcal{O}_F)^{\varphi}$. Put $w_\chi(c) = 0$ if $\det(c)$ is not in $\mathcal{O}_F$.

For the rest of this section let $K \subseteqq GL_{n-1}(\mathcal{O}_F)$ denote the congruence subgroup defined by the conditions:

$$k = (k_{ij}) \in GL_{n-1}(\mathcal{O}_F) \text{ is in } K \text{ if and only if } k_{ij} \equiv 0 \bmod f^{1+i-j} \text{ for all } i > j.$$

In the special case $f = \pi\mathcal{O}_F$ this is just $K = I_{n-1} \cap I_{n-1}^{\varphi^{-1}}$.

The generalized, local, Birch lemma. *Let $\chi : \mathcal{O}_F^\times \to \mathbb{C}^\times$ be a character such that the powers $\chi, \chi^2, \ldots, \chi^{n-1}$ are all primitive, nontrivial characters of conductor equal to f. For any Whittaker function w right-invariant under the Iwahori subgroup I_n, the function w_χ enjoys these properties*:

a) $w_\chi(uc) = \psi(u) \cdot w_\chi(c)$ *for* $u \in U_{n-1}(F)$, $c \in GL_{n-1}(F)$.

b) $w_\chi(ck) = w_\chi(c)$ *for* $k \in K$, $c \in GL_{n-1}(F)$.

c) *The support of* w_χ *is contained in* $U_{n-1}(F) \cdot K$.

Proof. Assertion a) is evident. We now show b), i.e., that $w_\chi(c)$ depends only on the left-coset of $c \bmod K$. To see this, we need only show b) for k running through a system of generators of K. Specifically, we choose the system consisting of:

(i) diagonal matrices with entries in $\mathcal{O}_F^\times$,

(ii) elementary matrices $E_{\alpha\beta}(a)$ for all $\alpha < \beta \leqq n-1$, where we allow a to run through the elements of $\mathcal{O}_F$,

(iii) elementary matrices $E_{\alpha\beta}(a)$ for all $\beta < \alpha \leqq n-1$, where we allow a to run through the elements of the ideal $f^{1+\alpha-\beta}\mathcal{O}_F$.

Recall that an *elementary matrix* $E_{\alpha\beta}(a)$ is a matrix whose ij-entry is given by $\delta_{ij} + a \cdot \delta_{i\alpha} \cdot \delta_{j\beta}$, where δ_{ij} is Kronecker's delta-function.

Case (i). If k is of type (i), then $k \in I_n$ and k commutes with φ and therefore

$$w_\chi(ck) = \chi\big(\det(ck)\big) \cdot \sum_u \prod_i \chi(u_i)^i \cdot w\big(c(kuk^{-1})^{\varphi^{-1}}\big)$$

and by the change of variables $u \mapsto \tilde{u} := kuk^{-1}$, where $\tilde{u}_i = k_i \cdot u_i \cdot k_{i+1}^{-1}$, we find

$$w_\chi(ck) = \chi\big(\det(c)\big) \cdot \sum_{\tilde{u}} \prod_i w(c\varphi^{-1}\tilde{u}\varphi) = w_\chi(c).$$

Case (ii). In this case we have $k \cdot u^{\varphi^{-1}} = \varphi^{-1}(k' \cdot u)\varphi$ where we set

$$k' := \varphi k \varphi^{-1} = E_{\alpha\beta}(f^{\beta-\alpha} \cdot a).$$

As u runs through a representative system of $U_n(\mathcal{O}_F) \bmod U_n(\mathcal{O}_F)^\varphi$, so does $\tilde{u} := k' \cdot u$, and moreover we have $\tilde{u}_i \equiv u_i \bmod f$. Again we get invariance:

$$w_\chi(ck) = w_\chi(c)$$

for all $a \in \mathcal{O}_F$.

Case (iii). Here we put $k' := \varphi \cdot k \cdot \varphi^{-1}$ hence we have $k' = E_{\alpha\beta}(a')$ where $a' \equiv 0 \bmod f$. For each $u \in U_n(\mathcal{O}_F)$ we shall construct a matrix $v \in U_n(\mathcal{O}_F)$ with the property that

a) $v \cdot k' \cdot u \in I_n^\varphi$,

b) $(v^{-1})_i \equiv u_i \bmod f$ for $i = 1, \dots, n-1$.

By the right invariance of w under I_n the equality $w_\chi(ck) = w_\chi(c)$ will then follow.

Note that it is sufficient to show instead of a) the weaker statement

a$'$) $v \cdot k' \cdot u \in GL_n(\mathcal{O}_F)^\varphi$,

since $vk'u \in GL_n(\mathcal{O}_F)$ and $GL_n(\mathcal{O}_F) \cap GL_n(\mathcal{O}_F)^\varphi \subset I_n^\varphi$ implies a) assuming a$'$).

We shall define the matrix $v = (v_{ij})$ inductively (in the parameter r) as follows.

The r-th step of the induction. For integers $r \geqq 0$ we shall find a choice of elements $v_{ij} = v_{ij}(r)$ satisfying:

$$(H_{ij}(r)) \qquad \sum_s v_{is} \cdot u_{sj} + a \cdot v_{i\alpha} \cdot u_{\beta j} \in f^{j-i} \cdot \mathcal{O}_F$$

if $j - i \leqq r$, and

$$\sum_s v_{is} \cdot u_{sj} + a \cdot v_{i\alpha} \cdot u_{\beta j} \in f^r \cdot \mathcal{O}_F$$

if $j - i \geqq r$.

$r = 1$. We wish to define $v_{ij}(1)$, which we will refer to as simply v_{ij} in this paragraph. Set $v_{ij} = 0$ for $i > j$, $v_{ij} = 1$ for $i = j$, and for $i < j \leqq n$ define elements $v_{ij} \in \mathcal{O}_F$, by a second rising induction on $j - i$, so that they satisfy the equation

$$\sum_{s=i}^{j} v_{is} \cdot u_{sj} = 0.$$

Noting that for any choice of elements $v_{ij} \in \mathcal{O}_F$, we have $a \cdot v_{i\alpha} \cdot u_{\beta j} \in f \cdot \mathcal{O}_F$ we see that $(H_{ij}(1))$ will then automatically be satisfied.

Passage from r to r + 1. We suppose given a matrix $\big(v_{ij}(r)\big)$ satisfying $(H_{ij}(r))$ and seek a matrix $\big(v_{ij}(r+1)\big)$ satisfying $(H_{ij}(r+1))$. Take $v_{ij}(r+1) = v_{ij}(r)$ for $j - i \leqq r$. Now suppose that $j - i > r \geqq 1$. Define elements $v_{ij}(r+1) \in \mathcal{O}_F$, by a second rising induction on $j - i$, so that they satisfy the congruences $v_{ij}(r+1) \equiv v_{ij}(r) \bmod f^r$, and

$$\sum_{s=i}^{j} v_{is}(r+1) \cdot u_{sj} + a \cdot v_{i\alpha}(r) \cdot u_{\beta j} \equiv 0 \bmod f^{r+1} \tag{1}$$

if $\alpha > j$, and

$$\sum_{s=i}^{j} v_{is}(r+1) \cdot u_{sj} + a \cdot v_{i\alpha}(r+1) \cdot u_{\beta j} \equiv 0 \bmod f^{r+1} \tag{2}$$

if $\alpha \leqq j$.

Noting that, in either equation, $v_{ij}(r+1)$ occurs with a coefficient which is a unit in $\mathcal{O}_F^\times$, we may solve, inductively for the $v_{ij}(r+1)$'s to obtain the unique system satisfying (1), (2).

To check that the system of $v_{ij}(r+1)$'s satisfies $(H_{ij}(r+1))$ we need only show that if $j - i \geqq r + 1$, then

$$\sum_s v_{is}(r+1) \cdot u_{sj} + a \cdot v_{i\alpha}(r+1) \cdot u_{\beta j} \in f^{r+1} \cdot \mathcal{O}_F.$$

But (1) and (2) above, together with the realization that

$$a \cdot v_{i\alpha}(r+1) \equiv a \cdot v_{i\alpha}(r) \bmod f^{r+1} \cdot \mathcal{O}_F$$

(recall that $a \in f \cdot \mathcal{O}_F$) gives this immediately. Finally, the matrix $v = (v_{ij})$ for $v_{ij} = v_{ij}(r)$ with any $r \geqq n - 1$ has the properties we seek, concluding the proof of b).

As for c), since $U_{n-1}(F)$ is a closed subgroup, and K is a compact open subgroup of $GL_{n-1}(F)$, it follows that the set $U_{n-1}(F) \cdot K$ is open and closed in $GL_{n-1}(F)$. Let $B_m^-(F)$ denote the Borel subgroup of lower-triangular matrices in $GL_m(F)$. To check that the support of w_χ lies in the set $U_{n-1}(F) \cdot K$ it suffices to show that the support of the restriction of w_χ to the open dense cell, $U_{n-1}(F) \cdot B_{n-1}^-(F)$, of the Bruhat decomposition of GL_{n-1} is contained in the set $U_{n-1}(F) \cdot \big(K \cap B_{n-1}^-(F)\big)$. For this consider the assertion:

$(S(m))$ The support of the restriction of w_χ to the open dense cell, $U_m(F) \cdot B_m^-(F)$, of the Bruhat decomposition of $GL_m(F)$ is contained in the set $U_m(F) \cdot \big(K \cap B_m^-(F)\big)$.

The assertion $(S(1))$ is evident. So suppose we have established $(S(m-1))$ for an integer m in the range $2 \leqq m \leqq n - 1$; we must show $(S(m))$. For this we need two lemmas.

Lemma 1. *Let m be an integer $<n$ and let $c \in B_m^-(F)$ be in the support of w_χ. Then the entries c_{mi} for $i < m$ are in $f^{1+m-i} \cdot \mathcal{O}_F$ and c_{mm} is a unit in $\mathcal{O}_F$.*

Proof. By replacing n by $m+1$, and restricting the Whittaker function w from $GL_n(F)$ to $GL_{m+1}(F)$, one sees that it suffices to prove our lemma in the special case

$m = n-1$. We assume, then, that $m = n-1$. For $u' \in U_n(F)$ let $tu' \in U_{n-1}(F)$ denote the *truncation* of u', i.e., for $i, j \leqq n-1$, put $(tu')_{ij} = u'_{ij}$. For varying u' in $U_n(F)$, we first want to find an upper-triangular matrix $v \in U_n(F)$, given as a function of the entries u'_{in} such that its truncation tv is the identity matrix, and such that $v \cdot c \cdot u'$ only depends on tu'. Since for any choice of the matrix $v \in U_n(F)$,

$$(v \cdot c \cdot u')_{ij} = \sum_{i \leqq r, s \leqq j} v_{ir} c_{rs} u'_{sj},$$

all the entries of $v \cdot c \cdot u'$ except, possibly, those in the last row (i.e., when $j = n$) are independent of the entries u'_{in}, to achieve the condition that $v \cdot c \cdot u'$ depends only upon tu' we need only consider the last row of $v \cdot c \cdot u'$, i.e., $j = n$. For $i < n$, and for any choice of v, we would have

$$(v \cdot c \cdot u')_{in} = \sum_{r \geqq i} \sum_{s \leqq r} v_{ir} c_{rs} u'_{sn} = v_{in} + \sum_{n > r \geqq i} \sum_{s \leqq r} v_{ir} c_{rs} u'_{sn}.$$

Furthermore demanding that tv be the identity matrix, i.e., $v_{ir} = \delta_{ir}$ for $i \leqq r < n$, defining, then, the v_{in} $(v_{nn} = 1)$ by the formula

$$v_{in} := -\sum_{s \leqq i} c_{is} \cdot u'_{sn}$$

we obtain our desired v.

Now put $g := v \cdot c \cdot u'$ where $u' = \varphi^{-1} u \varphi$ with $u \in U_n(\mathcal{O}_F)$. Then we have

$$\begin{aligned} w_\chi(c) &= \chi(\det c) \cdot \sum_u \prod_i \chi(u_i)^i w(v^{-1} \cdot g) \\ &= \chi(\det c) \cdot \sum_{tu} \prod_{i=1}^{n-2} \chi(u_i)^i \cdot w(g) \cdot \left\{ \sum_{u_{in}, i=1,\dots,n-1} \chi(u_{n-1})^{n-1} \cdot \psi(-v_{n-1,n}) \right\} \end{aligned}$$

where we have separated variables, the first summation above being over all choices of tu in $U_{n-1}(\mathcal{O}_F)$ modulo $U_{n-1}(\mathcal{O}_F)^\varphi$, and the second summation (in the curly brackets) being over all choices of u_{in} in $\mathcal{O}_F$ modulo $f^{n-i} \cdot \mathcal{O}_F$ for $0 < i < n$.

If c is in the support of w_χ, we must then have that the curly brackets term be non-zero, hence

i)

$$\sum_{u_{n-1}} \chi^{n-1}(u_{n-1}) \cdot \psi(c_{n-1,n-1} \cdot u_{n-1}/f) \neq 0,$$

which implies that the entry $c_{n-1,n-1}$ is a unit in $\mathcal{O}_F$, since χ^{n-1} is of conductor f, and

ii)

$$\sum_{u_{in}} \psi(c_{n-1,i} \cdot u_{i,n} \cdot f^{i-n}) \neq 0$$

for $1 \leqq i \leqq n-2$, which implies $c_{n-1,i} \equiv 0 \bmod f^{n-i}$, so we proved Lemma 1.

We can now establish the inductive step, i.e., the proof that $(S(m-1))$ implies $(S(m))$.

Lemma 2. *Let m be an integer in the range $1 < m < n$. Let χ be a character such that the powers $\chi, \chi^2, \ldots, \chi^m$ all have conductor equal to f. Suppose $(S(m-1))$. Let $c \in B_m^-(F) \subset GL_n(F)$ be in the support of w_χ. Then $c \in K$.*

Proof. As discussed in the proof of Lemma 1, there is no loss of generality in assuming $m = n-1$, which we do.

Consider the matrix $k = (k_{ij})$ defined as follows. The entries k_{ij} are equal to the Kronecker δ_{ij} unless $i = n-1$, and $j \leqq n-1$, in which case $k_{n-1,j} = -c_{n-1,j}/c_{n-1,n-1}$ if $j < n-1$, and $k_{n-1,n-1} = 1/c_{n-1,n-1}$. Note that by Lemma 1, $c_{n-1,n-1}$ is a unit, and $k \in K$. Part b) gives us that ck is in the support of w_χ if and only if c is. But a straight computation gives us that ck lies in $GL_{n-2}(F) \subset GL_{n-1}(F)$, and consequently, by $(S(m-1)) = (S(n-2))$, we have that $ck \in K$. Therefore so is c.

This concludes the proof of the generalized local Birch lemma.

3. Modular symbols and zeta integrals

3.1. The local zeta integrals at *p*. We specialize the situation of the previous section to the case $F = \mathbb{Q}_p$ and an additive character ψ_p. Let $\mathscr{W}(\psi_p)$ denote the space of all ($\mathbb{C}$-valued) Whittaker functions relative to ψ_p on $GL_n(\mathbb{Q}_p)$ considered as a representation of $GL_n(\mathbb{Q}_p)$ by right-translation. Let π_p be an irreducible admissible representation of $GL_n(\mathbb{Q}_p)$ which is a subrepresentation of $\mathscr{W}(\psi_p)$, i.e., π_p is *generic* (cf. [15], [10]). The unique corresponding subspace $\mathscr{W}(\pi_p, \psi_p) \subset \mathscr{W}(\psi_p)$ is usually called the *Whittaker space* of π_p. Now let σ_p be a generic irreducible representation of $GL_{n-1}(\mathbb{Q}_p)$ and suppose that π_p and σ_p both are unramified. Let w resp. v be Whittaker functions in the Whittaker spaces $\mathscr{W}(\pi_p, \psi_p)$ resp. $\mathscr{W}(\sigma_p, \bar{\psi}_p)$.

Hypothesis. We suppose that w (resp. v) is right invariant under the Iwahori subgroup I_n (resp. I_{n-1}).

Further let χ denote a character of $\mathbb{Q}_p^\times$ such that all powers $\chi, \chi^2, \ldots, \chi^{n-1}$ are nontrivial of conductor equal to f. There is an associated (primitive) Dirichlet character $\tilde{\chi}$ on $\mathbb{Z}_p$ defined by $\tilde{\chi}(x) := \chi(x)$ for $x \in \mathbb{Z}_p^\times$ and $\tilde{\chi}(x) := 0$ for $x \in p\mathbb{Z}_p$. As in the Birch lemma we define a Whittaker function $w_\chi \in \mathscr{W}(\pi_p \otimes \chi, \psi_p)$ by setting

$$w_\chi(g) := \chi\big(\det(g)\big) \cdot \sum_u \prod_{i=1}^{n-1} \tilde{\chi}(u_i^i) \cdot w(gu^{\varphi^{-1}}),$$

where the summation over u is taken over a representative system for $U_n(\mathbb{Z}_p)$ modulo $U_n(\mathbb{Z}_p)^\varphi$. Recall that the local zeta integral $\Psi(w_\chi, v, s)$ is defined by the formula

$$\Psi(w_\chi, v, s) = \int_{U_{n-1}\backslash G_{n-1}} w_\chi(g) v(g) |\det(g)|^{s-\frac{1}{2}} \, dg$$

where $U_{n-1} := U_{n-1}(\mathbb{Q}_p)$, $G_{n-1} = G_{n-1}(\mathbb{Q}_p)$ and $w_\chi(g) := w_\chi\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right)$.

Proposition 3.1. *The local zeta integral $\Psi(w_\chi, v, s)$ is constant in s. Moreover we have*

$$\Psi(w_\chi, v, s) = w(1) \cdot v(1) \cdot \prod_{i=1}^{n-1} G(\chi^i) \cdot \prod_{j=1}^{n-1} \frac{1 - p^{-1}}{1 - p^{-j}},$$

where $G(\chi^i)$ is the usual Gauß sum

$$G(\chi^i) = \sum_{x \bmod f} \tilde{\chi}^i(x) \psi_p\left(\frac{x}{f}\right).$$

Proof. Consider the Iwasawa decomposition $G_{n-1} = U_{n-1} \cdot A \cdot G_{n-1}(\mathbb{Z}_p)$. Recall that v is right-invariant under I_{n-1} and that the restriction of w_χ to G_{n-1} is right-invariant under the congruence subgroup

$$K = \left\{k \in G_{n-1}(\mathbb{Z}_p) \mid k_{ij} \equiv 0(f^{1+i-j}) \text{ for } i > j\right\}$$

by the generalized local Birch lemma. Since that restriction has its support in $U_{n-1} \cdot K$ and $K \subseteqq I_{n-1}$, we get

$$\Psi(w_\chi, v, s) = v(1) \cdot w_\chi(1) \cdot \mathrm{vol}(K).$$

Hence the proposition will follow from

Lemma 3.2. *With $e_n := n(n+1)(n+2)/6 - n$ we have*

a) $w_\chi(1) = w(1) \cdot f^{e_{n-1}} \cdot \prod_{i=1}^{n-1} G(\chi^i)$,

b) $\left(G_{n-1}(\mathbb{Z}_p) : K\right) = f^{e_{n-1}} \cdot \prod_{j=1}^{n-1} \frac{1 - p^{-j}}{1 - p^{-1}}$.

Proof of the lemma. a) We want to compute

$$w_\chi(1) = \sum_u \prod_{i=1}^{n-1} \tilde{\chi}(u_i^i) \cdot w(u^{\varphi^{-1}})$$

where u runs over a representative system for $U_n(\mathbb{Z}_p)/U_n(\mathbb{Z}_p)^\varphi$. Since we have

$$w(u^{\varphi^{-1}}) = w(1) \cdot \psi_p\left(\frac{1}{f} \sum_{i=1}^{n-1} u_i\right),$$

each term in the sum over u only depends on the residue class of u modulo the kernel of the homomorphism

$$pr : U_n(\mathbb{Z}_p) \to (\mathbb{Z}_p/f)^{n-1}, \quad u \mapsto (\ldots, u_i \bmod f, \ldots).$$

Hence we get

$$w_\chi(1) = w(1) \cdot \sum_{u_1 \bmod f} \cdots \sum_{u_{n-1} \bmod f} \prod_{i=1}^{n-1} \tilde{\chi}(u_i)^i \cdot \psi_p\left(\frac{u_i}{f}\right) \cdot \big(\mathrm{Ker}(pr) : U_n(\mathbb{Z}_p)^\varphi\big)$$

$$= w(1) \cdot \prod_{i=1}^{n-1} G(\chi^i) \cdot \big(\mathrm{Ker}(pr) : U_n(\mathbb{Z}_p)^\varphi\big),$$

and it remains to compute the index of $U_n(\mathbb{Z}_p)^\varphi$ in $\mathrm{Ker}(pr)$. This index can easily be read off from the filtration

$$U_n(\mathbb{Z}_p) \supset F^{(1)} \supset F^{(2)} \supset \cdots \supset F^{(n-1)}$$

with $F^{(k)} := \mathrm{Ker}(pr_{(k)})$ for $pr_{(1)} := pr$ and where

$$pr_{(k)} : F^{(n-1)} \to (\mathbb{Z}_p/f^k)^{n-k}, \quad u \mapsto (u_{1,k+1}, u_{2,k+2}, \ldots, u_{n-k,n}) \bmod f^k.$$

Obviously the last member in this filtration is

$$U_n(\mathbb{Z}_p)^\varphi = \{u \in U_n(\mathbb{Z}_p) \,|\, u_{ij} \equiv 0 \bmod f^{j-i} \text{ for } j > i\}.$$

Hence we find the index

$$(F^{(1)} : F^{(n-1)}) = f^{\sum\limits_{k=2}^{n-1} k(n-k)}$$

thus proving a) by the (well known) formula

$$\sum_{k=2}^{n-1} k(n-k) = e_{n-1} \quad (\text{for } n \geqq 3).$$

For $n = 2$ there is nothing to show.

b) We replace $n-1$ by n and prove the appropriate index formula for

$$K := \{k \in G_n(\mathbb{Z}_p) \,|\, k_{ij} \equiv 0(f^{1+i-j}) \text{ for } i > j\}.$$

Obviously K contains the principal congruence groups

$$K(f^m) := \{k \in G_n(\mathbb{Z}_p) \,|\, k \equiv 1 \bmod f^m\}$$

for all $m \geqq n$. Reduction modulo f^m produces the following standard exact sequences

$$0 \to K(f^m) \to G_n(\mathbb{Z}_p) \to G_n(\mathbb{Z}/f^m) \to 0,$$

$$0 \to K(f^m) \to K \to \bar{K} \to 0$$

where we put

$$\bar{K} := \{k \in G_n(\mathbb{Z}/f^m) \,|\, k_{ij} \equiv 0(f^{1+i-j}) \text{ for } i > j\}.$$

Hence the index

$$\bigl(G_n(\mathbb{Z}_p) : K\bigr) = |G_n(\mathbb{Z}/f^m)|/|\bar{K}|$$

can be determined by "counting elements". We have

$$|G_n(\mathbb{Z}/f^m)| = f^{m\cdot n^2} \cdot \prod_{v=0}^{n-1} (1 - p^{v-n})$$

and

$$|\bar{K}| = \varphi(f^m)^n \cdot |U_n(\mathbb{Z}/f^m)| \cdot (f^{m-2})^{n-1} \cdot (f^{m-3})^{n-2} \cdots (f^{m-n})^1,$$

where

$$|U_n(\mathbb{Z}/f^m)| = f^{m\cdot \sum\limits_{i=1}^{n-1} i},$$

so

$$|\bar{K}| = (1 - p^{-1})^n \cdot f^{m\cdot n^2 - \sum\limits_{i=2}^{n} i(n+1-i)}.$$

This completes the proof of b).

3.2. A global Birch Lemma. Let π resp. σ be cuspidal automorphic representations of $G_n(\mathbb{A})$ resp. $G_{n-1}(\mathbb{A})$ both unramified at our fixed prime number p. For a fixed prime number ℓ and each pair of Whittaker functions

$$(w_\ell, v_\ell) \in \mathscr{W}(\pi_\ell, \psi_\ell) \times \mathscr{W}(\sigma_\ell, \bar{\psi}_\ell)$$

the associated zeta integral

$$\psi(w_\ell, v_\ell, s) := \int\limits_{U_{n-1}\backslash G_{n-1}} w_\ell\begin{pmatrix} g & \\ & 1 \end{pmatrix} v_\ell(g)|\det(g)|^{s-\frac{1}{2}}\,dg$$

for $G_{n-1} := G_{n-1}(\mathbb{Q}_\ell)$, $U_{n-1} := U_{n-1}(\mathbb{Q}_\ell)$ converges for $\mathrm{Re}(s)$ sufficiently large. These zeta integrals span a fractional ideal $L(\pi_\ell, \sigma_\ell, s) \cdot \mathbb{C}[\ell^s, \ell^{-s}]$ of the ring $\mathbb{C}[\ell^s, \ell^{-s}]$ in the field $\mathbb{C}(l^s)$ of rational functions in the variable l^s, thus defining the local zeta function $L(\pi_\ell, \sigma_\ell, s)$ uniquely by fixing a polynomial $P(X) \in \mathbb{C}[X]$ such that $P(0) = 1$ and

$$P(\ell^{-s})^{-1} = L(\pi_\ell, \sigma_\ell, s)$$

(cf. [14]). Obviously we have a map on the tensor product $T_\ell := \mathscr{W}(\pi_\ell, \psi_\ell) \otimes \mathscr{W}(\sigma_\ell, \bar{\psi}_\ell)$ given by

$$\Psi : T_\ell \to \mathbb{C}(\ell^s), \quad w_\ell \otimes v_\ell \mapsto \Psi(w_\ell \otimes v_\ell, s) := \psi(w_\ell, v_\ell, s).$$

Moreover if π_ℓ and σ_ℓ both are unramified, the zeta integral for the respective new vectors w_ℓ^0 and v_ℓ^0 represents the L-function

$$L(\pi_\ell, \sigma_\ell, s) = \Psi(w_\ell^0 \otimes v_\ell^0, s).$$

Let $t_\ell^0 := w_\ell^0 \otimes v_\ell^0$. Let S denote the finite set of primes ℓ where π_ℓ or σ_ℓ is ramified. For $\ell \in S$ there is at least some "good tensor" $t_\ell^0 \in T_\ell$ such that we have

$$L(\pi_\ell, \sigma_\ell, s) = \Psi(t_\ell^0, s).$$

We will next consider certain pairs (w, v) of global Whittaker fuctions on $G_n(\mathbb{A})$ and $G_{n-1}(\mathbb{A})$ given as products of local Whittaker functions $w := \prod_\ell w_\ell$, $v := \prod_\ell v_\ell$, where we choose $w_\ell := w_\ell^0, v_\ell := v_\ell^0$ to be the respective new vector for ℓ not contained in $S \cup \{p\}$. For $\ell = p$, as in the previous section we will eventually let w_p and v_p vary among all Whittaker functions which are right invariant under the respective Iwahori subgroup. For $\ell \in S$ we will choose good tensors as previously described.

For any choice of $w_\infty \in \mathscr{W}_0(\pi_\infty, \psi_\infty)$ resp. $v_\infty \in \mathscr{W}_0(\sigma_\infty, \bar{\psi}_\infty)$ (in the notation of [12]) and for arbitrary w_ℓ, v_ℓ for $\ell \in S$ we get global Whittaker functions (w, v) with associated automorphic forms (ϕ, φ). The product of all local zeta integrals then becomes a Rankin-Selberg convolution (cf. [15], II.3.3 or [7], Prop. 6.1)

$$\prod_\ell \psi(w_\ell, v_\ell, s) = \int_{G_{n-1}(\mathbb{Q})\backslash G_{n-1}(\mathbb{A})} \phi\begin{pmatrix} g & \\ & 1 \end{pmatrix} \varphi(g) |\det(g)|^{s-\frac{1}{2}}\, dg$$

for $\mathrm{Re}(s) \gg 0$, admitting an analytic continuation to an entire function in s. This function which only depends on the pure tensor $w \otimes v$ can be linearly extended to the (algebraic) tensor product of global Whittaker spaces

$$T := \mathscr{W}_0(\pi, \psi) \otimes \mathscr{W}_0(\sigma, \bar{\psi})$$

by sending

$$\prod_\ell w_\ell \otimes \prod_\ell v_\ell \mapsto \prod_\ell \Psi(w_\ell \otimes v_\ell, s).$$

In particular we find (up to the infinity factor) the global L-function

$$L(\pi, \sigma, s) := \prod_\ell L(\pi_\ell, \sigma_\ell, s)$$

in the image of this map. For each choice of the pair (w_∞, v_∞) there is an entire function $P(s)$ such that

$$P(s) \cdot L(\pi, \sigma, s) = \Psi(w_\infty, v_\infty, s) \cdot \prod_{\ell \neq \infty} \Psi(t_\ell^0, s).$$

Recall that $L(\pi_\infty, \sigma_\infty, s)$ is defined by the corresponding Weil group representation as in [12].

Note. a) We still can vary (w_∞, v_∞) which will effect $P(s)$.

b) Writing each t_ℓ^0 for $\ell \in S$ as a sum of pure tensors leads to a finite sum of pure (global) tensors in T

$$\sum_j w_j \otimes v_j = (w_\infty \otimes v_\infty) \cdot \prod_{\ell \neq \infty} t_\ell^0.$$

We fix this explicit sum decomposition and, in what follows below, our formulas will depend upon this decomposition. Separating finite and infinite parts we sometimes write $w_j = w_\infty \cdot w_{j,f}$ and similar for v_j. The associated automorphic forms (ϕ_j, φ_j) yield the integral representation

$$P(s) \cdot L(\pi, \sigma, s) = \sum_j \int \phi_j \begin{pmatrix} g & \\ & 1 \end{pmatrix} \varphi_j(g) |\det(g)|^{s-\frac{1}{2}} \, dg.$$

We will in particular consider modified (w_j, v_j)'s and (ϕ_j, φ_j)'s, where at $\ell = p$ the local component (w_p^0, v_p^0) is replaced by an arbitrary pair (w_p, v_p) of Whittaker functions invariant under the respective Iwahori subgroup.

We want to consider χ-twists of π for a finite idele class character $\chi = \prod_\ell \chi_\ell$. For simplicity we assume:

Hypotheses. χ_∞ is trivial, and $\chi, \chi^2, \dots, \chi^{n-1}$ have the same non-trivial conductor f which is a p-power.

The first assumption ensures that $P(s)$ will not change when passing from π to $\pi \otimes \chi$ since $\pi_\infty = (\pi \otimes \chi)_\infty$. By the second assumption we can apply Proposition 3.1 to conclude the

Global Birch Lemma. *For any choice of (w_∞, v_∞) and any (w_p, v_p) right-invariant under the respective Iwahori subgroup the corresponding triples (P, ϕ_j, φ_j) satisfy*

$$w_p(1) \cdot v_p(1) \cdot P(s) \cdot \prod_{i=1}^{n-1} \frac{G(\chi_p^i) \cdot (1 - p^{-1})}{(1 - p^{-i})} \cdot L(\pi \otimes \chi, \sigma, s)$$

$$= \sum_u \prod_{i=1}^{n-1} \tilde{\chi}_p(u_i^i) \cdot \sum_j \int_{G_{n-1}(\mathbb{Q}) \backslash G_{n-1}(\mathbb{A})} \phi_j \left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} u^{\varphi_f^{-1}} \right) \cdot \varphi_j(g) \cdot \chi(\det g) |\det g|^{s-\frac{1}{2}} \, dg,$$

where $u = u_p$ (with $u_\ell = 1$ for all $\ell \neq p$) is taken from a representative system for $U_n(\mathbb{Z}_p)$ modulo $U_n(\mathbb{Z}_p)^{\varphi_f}$ with $\varphi_f := \operatorname{diag}(f^{-1}, f^{-2}, \dots, f^{-n})$ as in the local Birch Lemma.

Proof. We observe that any good tensor t_ℓ^0 for (π_ℓ, σ_ℓ) supplies the good tensor $\chi_\ell(\det) \cdot t_\ell^0$ for $(\pi_\ell \otimes \chi_\ell, \sigma_\ell)$ if $\ell \neq p$. So the respective local L-function is given by

$$L(\pi_\ell \otimes \chi_\ell, \sigma_\ell, s) = \Psi(\chi_\ell(\det) \cdot t_\ell^0, s)$$

At the prime $\ell = p$ where π and σ are unramified but χ is ramified, we have the trivial local L-function $L(\pi_p \otimes \chi_p, \sigma_p, s) = 1$, whereas by Proposition 3.1 the specific Whittaker function

$w_{p,\chi_p} \in \mathscr{W}(\pi_p \otimes \chi_p, \psi_p)$ leads to the zeta integral

$$\Psi(w_{p,\chi_p}, v_p, s) = w_p(1) \cdot v_p(1) \cdot \prod_{i=1}^{n-1} \frac{G(\chi_p^i) \cdot (1 - p^{-1})}{(1 - p^{-i})},$$

which is constant in the variable s. Again we can piece together the local Whittaker functions to form the global Whittaker function

$$w_{j,\chi} := w_\infty \cdot w_{p,\chi_p} \cdot \prod_{\ell \neq p, \infty} \chi_\ell(\det) \cdot w_{j,\ell},$$

which by definition of w_{p,χ_p} as in the previous section easily can be transformed into

$$w_{j,\chi}(g) = \chi\big(\det(g)\big) \cdot \sum_u \prod_{i=1}^{n-1} \tilde{\chi}_p(u_i^i) \cdot w_j(gu^{\varphi_f^{-1}})$$

with the same convention for the summation as in the statement of the lemma. Note that $u = u_p$ has trivial ℓ-component for $\ell \neq p$ and therefore we have

$$w_j(gu^{\varphi_f^{-1}}) = \prod_{\ell \neq p} w_{j,\ell}(g_\ell) \cdot w_p(g_p u_p^{\varphi_f^{-1}}).$$

Via Fourier transformation there is an automorphic form $\phi_{j,\chi}$ attached to $w_{j,\chi}$, which can be expressed by ϕ_j in the form

$$\phi_{j,\chi}(g) = \chi\big(\det(g)\big) \cdot \sum_u \prod_{i=1}^{n-1} \tilde{\chi}_p(u_i^i) \cdot \phi_j(gu^{\varphi_f^{-1}}).$$

Summing up, the Rankin-Selberg convolutions of $\phi_{j,\chi}$ and φ_j obviously become the right hand side of the formula in the lemma. By the general theory this is equal to the product of local zeta integrals above (in the region of convergence), hence the formula in the lemma is valid by analytic continuation.

For later use we want to reformulate the global Birch Lemma such that the integrals do not involve the character χ. Let $C_{1,f}$ denote the inverse image of the idele class group

$$\mathbb{Q}^\times \backslash \mathbb{Q}^\times \cdot \left(\mathbb{R}_{>0} \times \prod_{\ell \neq p, \infty} \mathbb{Z}_\ell^\times \times (1 + f\mathbb{Z}_p) \right) \subset \mathbb{Q}^\times \backslash \mathbb{A}^\times$$

under the determinant map

$$\det : G_{n-1}(\mathbb{Q}) \backslash G_{n-1}(\mathbb{A}) \to \mathbb{Q}^\times \backslash \mathbb{A}^\times.$$

Let a be an arbitrary positive integer. We decompose the domain of integration into finitely many shifts of C_{1,f^a} as follows. Write

$$G_{n-1}(\mathbb{Q}) \backslash G_{n-1}(\mathbb{A}) = \bigcup_x C_{1,f^a} \cdot \operatorname{diag}(x, 1, \ldots, 1)$$

where x runs over a representative system of $(\mathbb{Z}/f^a\mathbb{Z})^\times$ in $\mathbb{Z}_p^\times$. Note that the shift only effects the p-component.

Corollary. *Specializing to* $s = \frac{1}{2}$ *we get for each positive* $a \in \mathbb{Z}$

$$w_p(1) \cdot v_p(1) \cdot P\left(\frac{1}{2}\right) \cdot \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{(1-p^{-i})} \cdot L\left(\pi \otimes \chi, \sigma, \frac{1}{2}\right)$$
$$= (1-p^{-1}) f^a \sum_u \prod_{i=1}^{n-1} \tilde{\chi}_p(u_i^i) \sum_j \int_{C_{1,f^a}} \phi_j\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} u^{\varphi_f^{-1}}\right) \cdot \varphi_j(g)\, dg.$$

Proof. We integrate over each shift of C_{1,f^a} separately. Observe that $\chi(\det)$ is constant on each of these sets. A change of variables shows that these separate integrals are all equal, i.e. do not depend on x, since multiplication by the diagonal matrix $\mathrm{diag}(x, 1, \dots, 1)$ permutes the u's such that u_1 transforms into $x \cdot u_1$ and $u_2, \dots, u_{n-1}$ remain unchanged. This proves the formula.

Our next aim will be to interpret the occurring integrals under certain conditions as terms resulting from integrating cohomology classes against cycles on arithmetic quotients of symmetric spaces (assuming the non-vanishing hypothesis discussed in the introduction).

3.3. Differential forms and cohomology. From now on we assume that our representations π and σ both occur in cohomology with constant coefficients, in particular they have trivial central characters. The global representations π and σ have finite parts π_f and σ_f with respective new vectors w_f, v_f right-invariant under some open compact subgroup $K \subseteqq G_n(\hat{\mathbb{Z}})$ resp. $K' \subseteqq G_{n-1}(\hat{\mathbb{Z}})$ such that the respective image under the determinant map is the full unit group $\hat{\mathbb{Z}}^\times$, i.e.

$$\det(K) = \det(K') = \hat{\mathbb{Z}}^\times.$$

Moreover the canonical embedding

$$j : G_{n-1} \to G_n, \quad g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}$$

sends K' into K since by [13], Théorème (4.1), w_f is even right invariant under $j(G_{n-1}(\hat{\mathbb{Z}}))$, so we may choose K containing $j(K')$.

The advantage of the surjectivity of the determinant map is that we can easily switch from 'adelic quotients' to standard arithmetic quotients and vice versa. Separating finite and infinite parts of adelic elements we write $g = (g_f, g_\infty)$ for

$$g \in GL_n(\mathbb{A}) = GL_n(\mathbb{A}_f) \times GL_n(\mathbb{R}).$$

We put

$$\mathscr{X}_n := G_n(\mathbb{R})/O_n(\mathbb{R}) = G_n^+(\mathbb{R})/SO_n(\mathbb{R}) = \mathbb{R}_{>0} \times \mathscr{X}_n^1$$

with

$$\mathscr{X}_n^1 := SL_n(\mathbb{R})/SO_n(\mathbb{R}) = SL_n^{\pm}(\mathbb{R})/O_n(\mathbb{R}) \subseteqq \mathscr{X}_n$$

and

$$\Gamma := \{\gamma \in G_n^+(\mathbb{Q}) \mid \gamma_f \in K\} \subseteqq SL_n(\mathbb{Z}).$$

Then we have the bijection

$$\Gamma\backslash\mathscr{X}_n \to G_n(\mathbb{Q})\backslash G_n(\mathbb{A})/K \cdot O_n(\mathbb{R}),$$

$$\Gamma g_\infty SO_n(\mathbb{R}) \mapsto G_n(\mathbb{Q}) \cdot (1_f, g_\infty) \cdot K \cdot O_n(\mathbb{R})$$

with $g_\infty \in G_n^+(\mathbb{R})$. The reason is that by strong approximation we have

$$G_n(\mathbb{A}) = G_n(\mathbb{Q}) \cdot G_n^+(\mathbb{R}) \cdot K.$$

The same argument applies to G_{n-1} with a discrete subgroup $\Gamma' \subseteqq SL_{n-1}(\mathbb{Z})$ attached to K'. The embedding $j : G_{n-1} \to G_n$ induces an embedding of symmetric spaces

$$j : \mathscr{X}_{n-1} \to \mathscr{X}_n, \quad g \cdot O_{n-1}(\mathbb{R}) \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix} \cdot O_n(\mathbb{R}).$$

Moreover we can create a whole family of embeddings by composing j with left translation by any element $h \in G_n(\mathbb{R})$. Set

$$j_h : \mathscr{X}_{n-1} \to \mathscr{X}_n, \quad g \cdot O_{n-1}(\mathbb{R}) \mapsto h\begin{pmatrix} g & \\ & 1 \end{pmatrix} O_n(\mathbb{R}).$$

We are in particular interested in those embeddings j_h which define maps of arithmetic quotients.

Remark. For any $h \in G_n(\mathbb{Q})$ let

$$\Gamma_h' := \{\gamma \in \Gamma' \mid j(\gamma) \in h^{-1}\Gamma h\}.$$

Then j_h induces a proper mapping

$$\bar{j}_h : \Gamma_h'\backslash\mathscr{X}_{n-1} \to \Gamma\backslash\mathscr{X}_n, \quad \Gamma_h' g O_{n-1}(\mathbb{R}) \mapsto \Gamma h\begin{pmatrix} g & \\ & 1 \end{pmatrix} O_n(\mathbb{R}).$$

Note that $j(K') \subseteqq K$ implies $\Gamma' \subseteqq \Gamma$ hence we have, taking $h = 1$, the map

$$\bar{j}_1 : \Gamma'\backslash\mathscr{X}_{n-1} \to \Gamma\backslash\mathscr{X}_n.$$

We want to compose the maps j_h with projection p_2 into the second component of $\mathscr{X}_n = \mathbb{R}_{>0} \times \mathscr{X}_n^1$, induced by the map

$$p_2 : G_n(\mathbb{R}) \to SL_n^{\pm}(\mathbb{R}), \quad g \mapsto g \cdot \mathrm{diag}(\ldots, |\det g|^{-1/n}, \ldots).$$

Recall that the passage to quotients only effects the second component, i.e.

$$\Gamma\backslash\mathscr{X}_n = \mathbb{R}_{>0} \times \Gamma\backslash\mathscr{X}_n^1.$$

On arithmetic quotients we have a homotopy equivalence

$$\bar{p}_2 : \Gamma\backslash\mathscr{X}_n \to \Gamma\backslash\mathscr{X}_n^1.$$

Of course the same arguments apply to $n-1$ instead of n.

The construction of modular symbols in the first section gives for each $u \in U_n(\mathbb{Q})$ the proper map

$$J_u : \Gamma'_u\backslash\mathscr{X}_{n-1} \to \Gamma\backslash\mathscr{X}_n^1$$

defined by

$$J_u := \bar{p}_2 \circ \bar{j}_u : \Gamma'_u g O_{n-1} \mapsto \Gamma u \begin{pmatrix} g & \\ & 1 \end{pmatrix} \cdot |\det g|^{-\frac{1}{n}} O_n.$$

One aim of this section is to keep track of the effect of these maps J_u on certain differential forms. We denote by l_u left translation by u and we decompose the map

$$p_2 \circ j_u : G_{n-1}(\mathbb{R}) \to SL_n^{\pm}(\mathbb{R}), \quad g \mapsto p_2\big(u \cdot j(g)\big)$$

further into $p_2 \circ j_u = p_2 \circ l_u \circ j$. Since $\det(u) = 1$, the maps p_2 and l_u commute, hence we have

$$p_2 \circ j_u = l_u \circ p_2 \circ j.$$

We observe that $p_2 \circ j$ is an injective Lie group homomorphism and hence the induced map on invariant 1-forms is surjective . Specifically, letting $*$ denote dual vector space, this induced mapping

$$\delta(p_2 \circ j) : sl_n^* \to gl_{n-1}^*$$

is given by the formula

$$\delta(p_2 \circ j)(\omega)X := \omega\big(d(p_2 \circ j)(X)\big)$$

for a vector field $X \in gl_{n-1}$. Since l_u^* acts trivially on sl_n^* we have

$$\delta(p_2 \circ j) = \delta(p_2 \circ j_u) = \big(d(p_2 \circ j_u)\big)^*.$$

The map $\delta(p_2 \circ j)$ respects the Cartan decompositions

$$sl_n = k_n \oplus \tilde{\wp}_n$$

resp. $gl_{n-1} = k_{n-1} \oplus \wp_{n-1}$, where $k_n = so_n$ denotes the set of skew symmetric $n \times n$ matrices and $\wp_n$ (resp. $\tilde{\wp}_n$) stands for the set of symmetric $n \times n$ matrices (resp. of trace equal to zero). In particular we have

$$\delta(p_2 \circ j)(\tilde{\wp}_n^*) = \wp_{n-1}^*.$$

We can now describe the map of differential forms

$$J_u^* : \Omega^r(\Gamma \backslash \mathscr{X}_n^1) \to \Omega^r(\Gamma_u' \backslash \mathscr{X}_{n-1})$$

in terms of the complex defining the Lie algebra cohomology. Recall that one has canonical isomorphisms

$$\Omega^r(\Gamma_n \backslash \mathscr{X}_n^1) \xrightarrow{\sim} \left(\bigwedge^r \tilde{\wp}_n^* \otimes C^\infty\left(\Gamma_n \backslash SL_n^{\pm}(\mathbb{R})\right)\right)^{O_n(\mathbb{R})}$$

and

$$\Omega^r(\Gamma_{n-1} \backslash \mathscr{X}_{n-1}) \xrightarrow{\sim} \left(\bigwedge^r \wp_{n-1}^* \otimes C^\infty\left(\Gamma_{n-1} \backslash G_{n-1}(\mathbb{R})\right)\right)^{O_{n-1}(\mathbb{R})}$$

for any discrete subgroup Γ_n (resp. Γ_{n-1}) of $SL_n^{\pm}(\mathbb{R})$ (resp. $G_{n-1}(\mathbb{R})$) (cf. [3], §2).

We fix Maurer-Cartan forms $\omega_1, \ldots, \omega_{\tilde{d}_n}$ which form a basis of $\tilde{\wp}_n^*$ such that

$$\omega_i' := \delta(p_2 \circ j)(\omega_i) \quad \text{for } i = 1, \ldots, d_{n-1}$$

is a basis of $\wp_{n-1}^*$ and $\omega_i' = \delta(p_2 \circ j)(\omega_i) = 0$ for $i > d_{n-1}$. For $I \subseteqq \{1, \ldots, \tilde{d}_n\}$ with $I = \{i_1, \ldots, i_r\}$, $|I| = r$ we put $\omega_I := \omega_{i_1} \wedge \cdots \wedge \omega_{i_r}$ resp. $\omega_I' := \omega_{i_1}' \wedge \cdots \wedge \omega_{i_r}'$. One easly finds

Lemma 3.3. *Given a differential r-form*

$$\eta_n = \sum_{|I|=r} \omega_I \phi_I \quad \textit{with } \phi_I \in C^\infty\left(\Gamma \backslash SL_n^{\pm}(\mathbb{R})\right)$$

we have

$$J_u^*(\eta_n) = \sum_{|I|=r} \omega_I' \cdot (\phi_I \circ p_2 \circ j_u)$$

in $\Omega^r(\Gamma_u' \backslash \mathscr{X}_{n-1})$.

Since J_u is proper we also get a map on differential forms with compact support

$$J_u^* : \Omega_c^r(\Gamma \backslash \mathscr{X}_n^1) \to \Omega_c^r(\Gamma_u' \backslash \mathscr{X}_{n-1}),$$

just by replacing C^∞-functions by compactly supported C^∞-functions in our description above. We will later need a version of J_u^* on differential forms with growth conditions.

Recall that a function $\phi \in C^\infty\big(SL_n^\pm(\mathbb{R})\big)$ is of *moderate growth* or *slowly increasing*, if there is a constant C and a positive integer m such that for all $g \in SL_n^\pm(\mathbb{R})$ we have

$$|\phi(g)| \leqq C \cdot \|g\|^m,$$

where $\|g\| := tr({}^t g \cdot g)^{1/2}$. The function ϕ is *fast decreasing*, if for each integer m there is a constant $C = C_m$ such that this inequality holds for all g. A differential form $\eta = \sum\limits_I \omega_I \phi_I$ on $(\Gamma\backslash\mathscr{X}_n^1)$ is by definition of moderate growth (resp. fast decreasing) if the ϕ_I have this property (cf. [1]). Following Borel we denote by $\Omega^{\cdot}_{mg}(\Gamma\backslash\mathscr{X}_n^1)$ (resp. $\Omega^{\cdot}_{fd}(\Gamma\backslash\mathscr{X}_n^1)$) the complex of forms $\eta \in \Omega^{\cdot}(\Gamma\backslash\mathscr{X}_n^1)$ which together with their exterior differentials $d\eta$ are of moderate growth (resp. fast decreasing).

Lemma 3.4. *The forms on*

$$\Gamma_u'\backslash\mathscr{X}_{n-1} = \Gamma_u'\backslash\mathscr{X}_{n-1}^1 \times \mathbb{R}_{>0}$$

which are in the image $J_u^*\big(\Omega_{fd}(\Gamma\backslash\mathscr{X}_n^1)\big)$, can be integrated along the fibre.

We thus get a composed chain map

$$\Omega^r_{fd}(\Gamma\backslash\mathscr{X}_n^1) \to \mathrm{Im}(J_u^*) \to \Omega^{r-1}_{mg}(\Gamma_u'\backslash\mathscr{X}_{n-1}^1),$$

similar to the Poincaré Lemma for forms with compact support.

Proof. Let $\eta = \sum\limits_{|I|=r} \omega_I \phi_I$ be in $\Omega^r_{fd}(\Gamma\backslash\mathscr{X}_n^1)$ and $J_u^*(\eta) = \sum\limits_{|I|=r} \omega_I' \cdot (\phi_I \circ p_2 \circ j_u)$.

Since ϕ_I is fast decreasing we know (cf. also (8) in [15], p. 799) that for each $N > 0$ there is a constant $C_{(N)}$ such that

$$\left|\phi_I\left(u\begin{pmatrix} g & \\ & 1\end{pmatrix} \cdot |\det g|^{-\frac{1}{n}}\right)\right| \leqq C_{(N)} \cdot \min\{|\det g|^{-N}, |\det g|^N\}.$$

Note, that on $SL_n^\pm$ the norm $\|g\| := (\sum\limits_{i,j} g_{ij}^2)^{1/2}$ satisfies: $\|g\| \geqq 1$ and

$$\|x\| \cdot \|y^{-1}\|^{-1} \leqq \|x \cdot y\| \leqq \|x\| \cdot \|y\|.$$

Hence from $|\phi_I(g)| \leqq C_m \|g\|^m$ we specialize to

$$\begin{aligned}\left|\phi_I\left(u\begin{pmatrix} g & \\ & 1\end{pmatrix}|\det g|^{-\frac{1}{n}}\right)\right| &\leqq C_m \cdot \left\|u\begin{pmatrix} g & \\ & 1\end{pmatrix} \cdot |\det g|^{-\frac{1}{n}}\right\|^m \\ &\leqq C_m \cdot \left\|\begin{pmatrix} g & \\ & 1\end{pmatrix}|\det g|^{-\frac{1}{n}}\right\|^m \begin{cases} \|u\|^m & \text{for } m \geqq 0, \\ \|u^{-1}\|^{-m} & \text{for } m < 0.\end{cases}\end{aligned}$$

Put

$$C'_m := C_m \cdot \begin{Bmatrix} \|u\|^m & \text{for } m \geqq 0 \\ \|u^{-1}\|^{-m} & \text{for } m < 0 \end{Bmatrix}.$$

Now for any $m < 0$ we have with $\delta := |\det g|$

$$\left\| \begin{pmatrix} g & \\ & 1 \end{pmatrix} \cdot |\det g|^{-\frac{1}{n}} \right\|^m = \delta^{-\frac{m}{n}} \Big(1 + \sum g_{ij}^2\Big)^{m/2}$$

which, since

$$\sum_{ij} g_{ij}^2 \geqq \delta^{\frac{2}{n-1}},$$

implies for $n \geqq 3$

$$\begin{aligned} \| \cdots \|^m &\leqq \delta^{-\frac{m}{n}} (1 + \delta^{\frac{2}{n-1}})^{m/2} = (\delta^{-\frac{2}{n}} + \delta^{\frac{2}{n-1} - \frac{2}{n}})^{m/2} \\ &\leqq \min\{\delta^{-m/n}, \delta^{\frac{2m}{n(n-1)}}\} \leqq \min\{\delta^{\pm \frac{2m}{n(n-1)}}\}. \end{aligned}$$

Put $N := -2m/n(n-1)$. Then with $C_{(N)} := C'_m$ we get the required inequality.

Let t denote the global parameter of the factor $\mathbb{R}_{>0}$ in $\Gamma'_u \backslash \mathscr{X}_{n-1}$. Integration along the fibre means (cf. for instance [4], p. 38) that for each ω'_I having the invariant differential $\dfrac{dt}{t} =: \omega'_{d_{n-1}}$ as a wedge factor we must consider the integrals

$$\int_0^\infty \phi_I \left(u \begin{pmatrix} \check{g} t & \\ & 1 \end{pmatrix} t^{\frac{1-n}{n}} \right) \frac{dt}{t} =: \check{\phi}_{I,u}(\check{g})$$

for $\check{g} \in SL_{n-1}^{\pm}(\mathbb{R})$. These integrals are absolutely convergent by the estimate above. Moreover the resulting functions $\check{\phi}_{I,u}$ are bounded, hence in particular they are of moderate growth. The same proof as for compact supports shows that integration along the fibre is a chain map lowering the degree of forms by 1, i.e.

$$\pi_* : J_u^* \big(\Omega_{fd}^r (\Gamma \backslash \mathscr{X}_n^1)\big) \to \Omega_{mg}^{r-1} (\Gamma'_u \backslash \mathscr{X}_{n-1}^1).$$

Note, that

$$d\big(\pi_* J_u^*(\eta)\big) = \pi_* J_u^*(d\eta)$$

has coefficient functions of moderate growth, since for $\eta \in \Omega_{fd}^r$ the coefficient functions of $d\eta$ are by definition also fast decreasing. So the proof of the lemma is complete.

As an immediate consequence we get a pairing

$$\begin{aligned} \Omega_{fd}^{b_n}(\Gamma \backslash \mathscr{X}_n^1) \times \Omega_{fd}^{b_{n-1}}(\Gamma'_u \backslash \mathscr{X}_{n-1}^1) &\to \Omega_{fd}^{\tilde{d}_{n-1}}(\Gamma'_u \backslash \mathscr{X}_{n-1}^1), \\ (\eta, \eta') &\mapsto \pi_* J_u^*(\eta) \wedge \eta' \end{aligned}$$

where $b_n + b_{n-1} - 1 = \tilde{d}_{n-1} = \dim \mathscr{X}_{n-1}^1$. By Borel's theorem [1], 5.2, the inclusion $\Omega_C^{\cdot} \to \Omega_{fd}^{\cdot}$ induces isomorphisms in cohomology. In particular each fast decreasing cohomology class can be represented by a form of compact support. So our pairing induces the following pairing on cohomology:

$$\mathcal{B}_u : H_C^{b_n}(\Gamma\backslash\mathcal{X}_n^1) \times H_C^{b_{n-1}}(\Gamma'\backslash\mathcal{X}_{n-1}^1) \to \mathbb{C},$$
$$([\eta],[\eta']) \mapsto \int_{\Gamma_u'\backslash\mathcal{X}_{n-1}^1} \pi_* J_u^*(\eta) \wedge \eta'.$$

Recall that an invariant $\tilde{d}_{n-1}$-form on $\mathcal{X}_{n-1}^1$ uniquely corresponds to an invariant measure μ on SL_{n-1}/SO_{n-1} induced from a Haar measure on SL_{n-1}. So for

$$\eta = \sum_{|I|=r} \omega_I \cdot \phi_I, \quad \eta' = \sum_{|J|=r'} \omega_J' \cdot \varphi_J$$

with $r + r' = \tilde{d}_{n-1} + 1$ we get

$$\pi_* J_u^*(\eta) \wedge \eta' = \sum_{|I|=r} \varepsilon_I \check{\phi}_{I,u} \cdot \varphi_{I'} \cdot \omega_1' \wedge \cdots \wedge \omega_{\tilde{d}_{n-1}}'$$

where $\varepsilon_I = \pm 1$ for $I \,\dot\cup\, I' = \{1, \ldots, d_{n-1}\}$ and $\varepsilon_I := 0$ otherwise. Now integrating over the whole space we get

$$\int_{\Gamma_u'\backslash\mathcal{X}_{n-1}^1} \pi_* J_u^*(\eta) \wedge \eta' = \int_{\Gamma_u'\backslash G_{n-1}(\mathbb{R})} \sum_I \varepsilon_I \phi_I\left(u\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right) \cdot \varphi_{I'}(g)\, dg$$

with a Haar measure dg on $G_{n-1}(\mathbb{R})$ normalized by demanding $\mathrm{vol}\big(O_{n-1}(\mathbb{R})\big) = 1$. Note that we have extended the functions $(\phi_I, \varphi_{I'})$ by demanding trivial action of the respective centers.

Note that these integrals look very much like those that came up in the corollary of the global Birch lemma. To get the precise connection we have to choose η and η' attached to π and σ in the following manner.

Let $a := 2(n-1)$ and $u \in U_n(\mathbb{Z}_p)^{\varphi^{-1}}$. Let

$$K_u' := \{k \in K' \mid uj(k)u^{-1} \in K\}$$

and

$$K'(f^a) := \{k \in K' \mid k \equiv 1 \bmod f^a\}.$$

Then we have for all p where π, σ are unramified,

$$K'(f^a) \subseteqq K_u'.$$

Note that $K'(f^a)$ acts by right translation on C_{1,f^a} and we have an isomorphism

$$\Gamma'(f^a)\backslash G_{n-1}(\mathbb{R}) \to C_{1,f^a}/K'(f^a),$$
$$\Gamma'(f^a)g_\infty \mapsto G_{n-1}(\mathbb{Q})(g_\infty, 1_f)K'(f^a)$$

where $\Gamma'(f^a) := \{\gamma \in \Gamma' \mid \gamma \equiv 1(f^a)\} \subseteq \Gamma'_u$. In the corollary of the global Birch Lemma the integrand is right-invariant under $K'(f^a)$. Hence we can rewrite that integral as

$$\int_{C_{1,f^a}} = \mathrm{vol}\big(K'(f^a)\big) \cdot \sum_j \int_{\Gamma'(f^a)\backslash G_{n-1}(\mathbb{R})} \phi_j\big(j(g_\infty)u^{\varphi^{-1}}\big) \cdot \varphi_j(g_\infty)\, dg_\infty.$$

Note that by the left invariance of $\phi_j(g)$ under $G_n(\mathbb{Q})$ the p-adic factor $u^{\varphi^{-1}} = u_p^{\varphi^{-1}}$ can be transformed into an infinity factor by taking representatives $u = u_f$ such that they belong to some $(u_\infty, u_f) \in U_n(\mathbb{Q})$ with $u_f \in K \cap U_n(\hat{\mathbb{Z}})$. Then we have

$$\phi_j\big(j(g_\infty)u^{\varphi^{-1}}\big) = \phi_j\big((u_\infty^{-1})^{\varphi^{-1}} j(g_\infty)\big),$$

and we are reduced to consider non-adelic integrals of the form

$$I(u_\infty^{-1}, \phi, \varphi) := \int_{\Gamma'(f^a)\backslash G_{n-1}(\mathbb{R})} \phi\left(u_\infty^{\varphi^{-1}} \begin{pmatrix} g_\infty & \\ & 1 \end{pmatrix}\right) \cdot \varphi(g_\infty)\, dg_\infty$$

with $u_\infty \in U_n(\mathbb{Z})$.

Finally we will exploit the fact, that π (resp. σ) occurs in r-dimensional cohomology for $r = b_n$ (resp. b_{n-1}) with multiplicity one. This is a local property at ∞; specifically:

$$H^{b_n}(sl_n, O_n; H_{\pi_\infty}^{(O_n)}) \cong \mathbb{C},$$

where $H_{\pi_\infty}^{(O_n)}$ denotes the space of O_n-finite elements in the representation space H_{π_∞} of π_∞. Note, that π_∞ uniquely corresponds to the irreducible representation of $SL_n^{\pm}(\mathbb{R})$ given by restriction, since π_∞ has trivial central character. As a realization of $H_{\pi_\infty}^{(O_n)}$ we take the space $\mathscr{W}_0(\pi_\infty, \psi_\infty)$ of O_n-finite Whittaker functions. Moreover by Prop. 3.1, p. 52 in [3] we have

$$H^{b_n}(sl_n, O_n; H_{\pi_\infty}^{(O_n)}) \cong \left(\bigwedge^{b_n} \tilde{\wp}_n^* \otimes \mathscr{W}_0(\pi_\infty, \psi_\infty)\right)^{O_n},$$

which up to a scalar uniquely determines a form $\eta_\infty \in \left(\bigwedge^{b_n} \tilde{\wp}_n^* \otimes \mathscr{W}_0(\pi_\infty, \psi_\infty)\right)^{O_n}$ representing a generator of the cohomology. We write

$$\eta_\infty = \sum_{|I|=b_n} \omega_I \cdot w_{I,\infty}(g)$$

with Whittaker functions $w_{I,\infty} \in \mathscr{W}_0(\pi_\infty, \psi_\infty)$. The same procedure applies to σ_∞ instead of π_∞ when we replace n by $n-1$, thus leading to a form of degree b_{n-1}

$$\eta'_\infty = \sum_{|I'|=b_{n-1}} \omega'_{I'} \cdot v_{I',\infty}(g')$$

with Whittaker functions $v_{I',\infty} \in \mathscr{W}_0(\sigma_\infty, \bar{\psi}_\infty)$. Starting from each pair of finite Whittaker functions $(w_{j,f}, v_{j,f})$ as in 3.2 we compose global elements

$$\eta_{j,0} := w_{j,f} \otimes \eta_\infty \in \left(\bigwedge^{b_n} \tilde{\wp}_n^* \otimes \mathscr{W}_0(\pi, \psi)\right)^{O_n}$$

and

$$\eta'_{j,0} := v_{j,f} \otimes \eta'_\infty \in \left(\bigwedge^{b_{n-1}} \tilde{\wp}_{n-1}^* \otimes \mathscr{W}_0(\sigma, \bar{\psi})\right)^{O_{n-1}}$$

which by Fourier transformation yield

$$\eta_j \in \left(\bigwedge^{b_n} \tilde{\wp}_n^* \otimes L_0^2\big(G_n(\mathbb{Q})\backslash G_n(\mathbb{A})/KZ_\infty^0\big)\right)^{O_n}$$

i.e. cohomology classes

$$[\eta_j] \in H^{b_n}_{\mathrm{cusp}}(\Gamma\backslash\mathscr{X}_n^1) \subseteqq H_C^{b_n}(\Gamma\backslash\mathscr{X}_n^1),$$

and similarly

$$[\eta'_j] \in H^{b_{n-1}}_{\mathrm{cusp}}(\Gamma'\backslash\mathscr{X}_{n-1}^1).$$

We have

$$\eta_j = \sum_{|I|=b_n} \omega_I \phi_{j,I}(g)$$

and

$$\eta'_j = \sum_{|I'|=b_{n-1}} \omega'_{I'} \cdot \varphi_{j,I'}(g')$$

where the coefficient functions result as the Fourier transforms of the global Whittaker functions $w_I := w_{I,\infty} \cdot w_{j,f}$ and $v_{I'} := v_{I',\infty} \cdot v_{j,f}$. Evaluating our previously described pairing $\mathscr{B}_u$ we find

$$\begin{aligned}\mathscr{B}_u(\eta_j, \eta'_j) &= \int_{\Gamma'_u\backslash\mathscr{X}_{n-1}^1} \pi_* J_u^*(\eta_j) \wedge \eta'_j \\ &= \sum_I \varepsilon_I \int_{\Gamma'_u\backslash G_{n-1}(\mathbb{R})} \phi_{j,I}\left(u\begin{pmatrix} g & \\ & 1\end{pmatrix}\right) \cdot \varphi_{j,I'}(g)\, dg\end{aligned}$$

with $\phi_{j,I}$ and $\varphi_{j,I'}$ restricted to infinity components. We can easily switch to the adelic setting, and obtain:

$$\mathscr{B}_u(\eta_j, \eta'_j) = \mathrm{vol}(K'_u)^{-1} \cdot \sum_I \varepsilon_I \int_{C_{1,f^a}} \phi_{j,I}\big(j(g)u_f^{-1}\big) \cdot \varphi_{j,I'}(g)\, dg.$$

In the sequel we are mainly interested in this quantity as a function in the variable u. We put $\mathscr{B}(u) := \sum_j \mathscr{B}_u(\eta_j, \eta'_j)$. We can now express our special L-values in terms of this function $\mathscr{B}(u)$. Let $P_I(s)$ denote the entire function attached to the pair $(w_{I,\infty}, v_{I,\infty})$ as in 3.2

such that

$$\Psi(w_{I,\infty} \otimes v_{I,\infty}, s) = P_I(s) \cdot L(\pi_\infty, \sigma_\infty, s),$$

and

$$P_\infty(s) := \sum_I \varepsilon_I \cdot P_I(s).$$

From the corollary of the global Birch Lemma we conclude:

Theorem 3.5. *For all finite idele class characters $\chi = \prod_\ell \chi_\ell$ with the properties*

a) $\chi_\infty = 1$,

b) $\chi, \chi^2, \ldots, \chi^{n-1}$ *have the same non-trivial conductor $f = p$-power,*

we have the formula

$$w_p(1) \cdot v_p(1) \cdot P_\infty(1/2) \cdot \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{(1-p^{-i})} \cdot L\left(\pi \otimes \chi, \sigma, \frac{1}{2}\right)$$
$$= f^{2(n-1)} \cdot \sum_u \prod_{i=1}^{n-1} \tilde{\chi}_p(u_i^i) \cdot \mathrm{vol}(K'_{u^{\varphi^{-1}}}) \cdot \mathscr{B}\big((u^{-1})^{\varphi^{-1}}\big),$$

where $u = u_p$ is taken from a representative system for $U_n(\mathbb{Z}_p)$ modulo $U_n(\mathbb{Z}_p)^\varphi$ as in the local Birch Lemma.

Remarks. 1. We do not know if P_∞ is non-zero at $s = 1/2$ in general. For $n = 3$ it is a fact that $P_\infty(1/2) \neq 0$ (see [18]).

2. We are still free to vary the pair (w_p, v_p) of local Whittaker functions, right-invariant under the respective Iwahori subgroup, which of course causes a variation of the function $\mathscr{B}$. This will eventually allow us to vary $\mathscr{B}$ such that it becomes a p-adic distribution.

3.4. Algebraicity. We recall from the first section that the previously considered pairings $\mathscr{B}_u$ of cohomology spaces are defined purely topologically and moreover with coefficients in an arbitrary ring A. On the other hand the cuspidal cohomology is well-known to carry a $\mathbb{Q}$-structure (cf. Théorème 3.19 in [6]). So this suggests that we try to choose the cuspidal cohomology classes $[\eta_n]$ (and $[\eta_{n-1}]$) subject to good rationality conditions, which then will automatically lead to rationality properties of the *modular symbols* $\mathscr{B}(u)$ and hence of special L-values. We use Clozel's terminology from [loc.cit.]. In particular $\mathbb{Q}(\pi_f)$ resp. $\mathbb{Q}(\sigma_f)$ denotes the field of rationality which by Proposition 3.1 in [loc.cit.] is a field of definition and in our case in fact is a number field by the Drinfeld-Manin argument (see Proposition 3.16 in [loc.cit.]). Let $F := \mathbb{Q}(\pi_f, \sigma_f)$ denote the smallest number field containing $\mathbb{Q}(\pi_f)$ and $\mathbb{Q}(\sigma_f)$. In particular the global (finite) Whittaker spaces $\mathscr{W}(\pi_f, \psi_f)$ and $\mathscr{W}(\sigma_f, \overline{\psi}_f)$ carry an F-structure whose underlying F-vector space we denote by $\mathscr{W}_F(\cdots)$. An immediate consequence of the multiplicity one property of b_n-dimensional cohomology is

Proposition 3.6. *We can normalize the* ∞*-part* $\eta_{n,\infty}$ *(by a non-trivial scalar factor) such that for any Whittaker function* $w_f \in \mathscr{W}_F(\sigma_f, \psi_f)$ *the cohomology class* $[\eta_n]$ *attached to* $\eta_{n,0} = w_f \otimes \eta_{n,\infty}$ *is F-rational, i.e.*

$$[\eta_n] \in H^{b_n}_{\mathrm{cusp}}(\Gamma \backslash \mathscr{X}^1_n, F) \subseteqq H^{b_n}_C(\Gamma \backslash \mathscr{X}^1_n, \bar{\mathbb{Q}}).$$

Corollary 3.7. *There is a choice of good local tensors* t^0_ℓ *of Whittaker functions for all* $\ell \neq p$ *such that for any 'Iwahori fixed' pair*

$$(w_p, v_p) \in \mathscr{W}_F(\pi_p, \psi_p)^{I_n} \times \mathscr{W}_F(\sigma_p, \bar{\psi}_p)^{I_{n-1}}$$

the formula in Theorem 3.5 *holds for the associated pairing* $\mathscr{B}$ *with values* $\mathscr{B}(u)$ *in the number field F.*

4. Hecke operators and distributions

4.1. A parabolic extension of the standard Hecke algebra. For a fixed prime number p let $K := GL_n(\mathbb{Z}_p)$ and $G := GL_n(\mathbb{Q}_p)$. We consider the standard Hecke algebra $\mathscr{H} := \mathscr{H}(K, G)$ attached to the Hecke pair (K, G), i.e. the convolution algebra of $\mathbb{C}$-valued compactly supported functions on G which are bi-invariant under K. Recall that in general a 'Hecke pair' (R, S) is given by any group S with a subgroup $R \subseteqq S$ such that each double coset RsR for $s \in S$ is the union of a finite number of left or right cosets modulo R. Thus the free $\mathbb{C}$-vector space $\mathscr{H}(R, S)$ over the set of all double cosets RsR is embedded into the free $\mathbb{C}$-vector space $\mathscr{R}(R, S)$ over the set of all right cosets sR by sending a double coset $RsR = \biguplus s_j R$ to $\sum_j s_j R$ and moreover under this map $\mathscr{H}(R, S)$ is identified with the R-invariants in $\mathscr{R}(R, S)$ under the action

$$R \times \mathscr{R}(R, S) \to \mathscr{R}(R, S), \quad (r, sR) \mapsto rsR.$$

The vector space $\mathscr{H}(R, S)$ becomes an algebra with the multiplication defined by

$$\left(\sum_j s_j R\right) \cdot \left(\sum_k t_k R\right) := \sum_{i,j} s_j t_k R$$

and is called the Hecke algebra attached to (R, S).

The standard Hecke algebra is well known to be isomorphic under the Satake map $\mathscr{S}$ to the ring of symmetric polynomials in the variables $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$ (see for instance [5], 4.2). The map $\mathscr{S}$ can be normalized by fixing the values of the standard Hecke operators

$$T_v := K \begin{pmatrix} I_{n-v} & 0 \\ 0 & pI_v \end{pmatrix} K \quad (v = 0, \ldots, n)$$

where I_v denotes the $v \times v$-unit matrix. As usual, we put

$$\mathscr{S}(T_v) = p^{\langle v \rangle} \cdot \sigma_v(X_1, \ldots, X_n)$$

where $\langle v \rangle := \frac{1}{2}v(v+1)$ and where σ_v denotes the v-th elementary symmetric function in the variables $X_1, \dots, X_n$.

Let B denote the standard Borel subgroup of upper triangular matrices in $GL_n(\mathbb{Q}_p)$ and $K_B := K \cap B$. Then by [11], Theorem 2 the Hecke algebra $\mathscr{H}_B := \mathscr{H}(B, K_B)$ is a ring extension of $\mathscr{H}$, such that the Hecke polynomial

$$H(X) := \sum_{v=0}^{n} (-1)^v p^{\langle v-1 \rangle} T_v X^{n-v} \in \mathscr{H}[X]$$

decomposes into linear factors over $\mathscr{H}_B$

$$H(X) = \prod_{i=1}^{n} (X - U_i)$$

where

$$U_i := K_B \begin{pmatrix} I_{i-1} & & \\ & p & \\ & & I_{n-i} \end{pmatrix} K_B \in \mathscr{H}_B$$

(note that Gritsenko considers $Q_n(t) = t^n \cdot H(t^{-1})$).

The underlying ring embedding $\varepsilon : \mathscr{H} \hookrightarrow \mathscr{H}_B$ sends an arbitrary element $T = \sum_i a_i \cdot g_i K \in \mathscr{H}$ with upper-triangular matrices g_i (always possible by Iwasawa decomposition) to

$$\varepsilon(T) := \sum_i a_i \cdot g_i K_B.$$

Lemma 4.1. *For $v = 1, \dots, n$ let $V_v := p^{-\langle v-1 \rangle} \cdot U_1 U_2 \cdots U_v \in \mathscr{H}_B$.*

a) *Then in $\mathscr{R}(K_B, B)$ the double coset $K_B \begin{pmatrix} pI_v & 0 \\ 0 & I_{n-v} \end{pmatrix} K_B$ corresponds to*

$$V_v = \sum_A \begin{pmatrix} pI_v & A \\ 0 & I_{n-v} \end{pmatrix} K_B$$

where A runs over a system of representatives of matrices in $M^{v,n-v}(\mathbb{Z}_p)$ modulo p.

b) *Furthermore the Hecke operators V_v commute and the double coset $K_B t K_B$ corresponds to*

$$\prod_{v=1}^{n-1} V_v = \sum_u u t K_B$$

where $t = \mathrm{diag}(p^{n-1}, p^{n-2}, \dots, 1)$ and u runs over a set of representatives for $U_n(\mathbb{Z}_p)/tU_n(\mathbb{Z}_p)t^{-1}$.

Proof. By elementary row and column transformations our double cosets obviously decompose as demanded into right cosets

$$K_B\begin{pmatrix} pI_v & 0 \\ 0 & I_{n-v} \end{pmatrix}K_B = \dot{\bigcup_A}\begin{pmatrix} pI_v & A \\ 0 & I_{n-v} \end{pmatrix}K_B,$$

and for the same reason we have

$$U_i = \dot{\bigcup_{\underline{a}}}\begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & p & a_1, \dots, a_{n-i} \\ 0 & 0 & I_{n-i} \end{pmatrix}K_B$$

where $\underline{a} = (a_1, \dots, a_n)$ runs over a set of representatives of the a_k mod p. Now the equality for V_v follows immediately by induction over v using the identity

$$p^v V_{v+1} = V_v \cdot U_{v+1}.$$

b) By Lemma 2 in [11] the V_v commute since they belong to the commutative subring of $\mathscr{H}_B$ generated as a $\mathbb{C}$-vector space by the elements

$$V_{\underline{i}} := U_1^{i_1} \cdot \ldots \cdot U_n^{i_n}$$

with $\underline{i} = (i_1, \dots, i_n)$ such that $i_1 \geqq i_2 \geqq \cdots \geqq i_n \geqq 0$. Now let

$$t_{(v)} := \operatorname{diag}(p^{v-1}, p^{v-2}, \dots, 1) \in GL_v(\mathbb{Q}_p).$$

Then we get

$$\prod_{v=1}^{n-1} V_v = \sum_{A_1, \dots, A_{n-1}} u(A_1, \dots, A_{n-1}) \cdot t \cdot K_B$$

with

$$u(A_1, \dots, A_{n-1}) := \begin{pmatrix} 1 & A_1 \\ & I_{n-1} \end{pmatrix}\begin{pmatrix} I_2 & t_{(2)}A_2 \\ & I_{n-2} \end{pmatrix}\cdots\begin{pmatrix} I_{n-1} & t_{(n-1)}A_{n-1} \\ & 1 \end{pmatrix}$$

and where A_v runs over a representative system in $M^{v,n-v}(\mathbb{Z}_p)$ mod p. An easy induction shows that these $u(A_1, \dots, A_n)$ form a system of representatives for $U_n(\mathbb{Z}_p)/tU_n(\mathbb{Z}_p)t^{-1}$, which completes the proof of the lemma.

4.2. Eigenfunctions for the modified Hecke algebra. Let $\mathscr{M}$ denote the $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions on $GL_n(\mathbb{Q}_p)$ which are right-invariant under K_B. The extended Hecke algebra $\mathscr{H}_B$ has a natural action on $\mathscr{M}$ given by

$$\mathscr{H}_B \times \mathscr{M} \to \mathscr{M}, \quad \left(\sum_i a_i \cdot g_i K_B, \psi(g)\right) \mapsto \sum_i a_i \cdot \psi(g g_i).$$

We are interested in the action of the commutative subring generated by $V_1, \dots, V_n$. Let $V_0 := K_B I_n K_B$ denote the unit element of $\mathscr{H}_B$.

Proposition 4.2. *Suppose we are given a function $\psi \in \mathscr{M}$ and an $n-1$-tuple of complex numbers $\underline{\lambda} = (\lambda_1, \dots, \lambda_{n-1}) \in \mathbb{C}^{n-1}$ such that the Hecke operators*

$$H(\lambda_\nu) = \prod_{i=1}^{n} (\lambda_\nu - U_i) \in \mathscr{H}_B$$

annihilate ψ, i.e.

$$H(\lambda_\nu)(\psi) = 0 \quad \textit{for } \nu = 1, \dots, n-1.$$

Then

$$\psi_{\underline{\lambda}} := \prod_{i=1}^{n-1} \prod_{\substack{j=1 \\ j \neq i}}^{n} (\lambda_i p^{1-j} V_{j-1} - V_j)\psi$$

is a simultaneous eigenfunction of $V_1, \dots, V_{n-1}$.

Moreover with $\eta_\nu := p^{-\langle \nu-1 \rangle} \prod_{i=1}^{\nu} \lambda_i$ we have

$$V_\nu \psi_{\underline{\lambda}} = \eta_\nu \cdot \psi_{\underline{\lambda}} \quad \textit{for } \nu = 1, \dots, n-1.$$

The proposition will follow from

Lemma 4.3. *For any $\lambda \in \mathbb{C}$ we have the identity in $\mathscr{H}_B$:*

$$\prod_{j=1}^{n} (\lambda p^{1-j} V_{j-1} - V_j) = p^{-\langle n-1 \rangle} \prod_{\nu=0}^{n-1} V_\nu \cdot H(\lambda).$$

Proof of the lemma. Since $V_j = p^{1-j} \cdot V_{j-1} U_j$ we have for each factor

$$\lambda p^{1-j} V_{j-1} - V_j = p^{1-j} \cdot V_{j-1} \cdot (\lambda - U_j),$$

and since any V_ν commutes with this expression we can collect the factors V_ν on the left hand side. Hence we get the claimed formula of the lemma.

Proof of Proposition 4.2. By our assumption on ψ we know by Lemma 4.3 that for each $\nu = 1, \dots, n-1$ we have

$$\prod_{j=1}^{n} (\lambda_\nu p^{1-j} V_{j-1} - V_j)\psi = 0.$$

We proceed by induction over ν. Since $\eta_1 = \lambda_1$ we have

$$(\eta_1 - V_1)\psi_{\underline{\lambda}} = (\lambda_1 - V_1) \cdot \prod_{i=1}^{n-1} \cdot \prod_{j \neq i} (\lambda_i p^{1-j} V_{j-1} - V_j)\psi = 0$$

which is a multiple of $\prod_{i=1}^{n}(\lambda_1 p^{1-j} V_{j-1} - V_j)\psi = 0$. This settles the case $v = 1$. Now we suppose

$$(\eta_k - V_k)\psi_{\underline{\lambda}} = 0 \quad \text{for } k = 1, \dots, v-1.$$

We decompose η_v as $\eta_v = \eta_{v-1} \cdot \lambda_v p^{1-v}$ and get

$$(\eta_v - V_v)\psi_{\underline{\lambda}} = (\lambda_v p^{1-v} V_{v-1} - V_v)\psi_{\underline{\lambda}}$$

by the induction hypothesis. As in the case $v = 1$ this must be zero and the proof is complete.

Remark 4.4. Suppose the λ_i are algebraic numbers which with respect to a fixed embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$ have absolute values $|\lambda_i|_p = p^{1-i}$. Then all eigenvalues η_v are p-adic units.

4.3. Distributions of the unipotent group into function spaces. We consider the unipotent group $U = U_n$ of upper triangular matrices in GL_n. Conjugating by the diagonal matrix $t := \operatorname{diag}(p^{n-1}, p^{n-2}, \dots, 1)$ defines a mapping $\beta := \operatorname{inn}_t|_{U(\mathbb{Z}_p)}$ where

$$\operatorname{inn}_t : U(\mathbb{Q}_p) \to U(\mathbb{Q}_p), \quad u \mapsto tut^{-1},$$

and iterating β gives us a filtration of $U^{(0)} := U(\mathbb{Z}_p)$ by setting

$$U^{(k)} := \beta^k(U^{(0)}) = \beta(U^{(k-1)}),$$

which obviously extends to a filtration of $U(\mathbb{Q}_p)$ by admitting negative exponents k.

Any function μ on the set of cosets $uU^{(k)}$ into an additive group A is called an (A-valued)-*distribution* if μ satisfies the distribution relation

$$\text{(DR)} \qquad \mu(uU^{(k)}) = \sum_{v} \mu\big(u\beta^k(v)U^{(k+1)}\big)$$

where v runs over a representative system of $U^{(0)}/U^{(1)}$.

Proposition 4.5. *Let $\psi \in \mathscr{M}$ be an eigenfunction for the modified Hecke operator $V := V_1 \dots V_{n-1}$ with eigenvalue equal to* 1. *Then*

$$\mu(uU^{(k)}) := \psi(gut^k)$$

is a distribution with values in the $\mathbb{C}$-vector space $A = \mathbb{C}^{GL_n(\mathbb{Q}_p)}$ of $\mathbb{C}$-valued functions on $GL_n(\mathbb{Q}_p)$.

Note that μ is well defined, since replacing u by another representative $u' = ut^k vt^{-k}$ with $v \in U^{(0)}$ does not affect the function ψ by virtue of its K_B right-invariance.

Proof. Recall from Lemma 4.1 that the Hecke operator V is given by the double coset $K_B t K_B$ and moreover that we have

$$V = \sum_v vtK_B$$

with v running over representatives of $U^{(0)}/U^{(1)}$. So we get

$$\psi(g) = V\psi(g) = \sum_v \psi(gvt),$$

hence

$$\begin{aligned}\mu(uU^{(k)}) &= \sum_v \psi(gut^k vt)\\ &= \sum_v \psi\big(gu\beta^k(v)t^{k+1}\big)\\ &= \sum_v \mu\big(u\beta^k(v)U^{(k+1)}\big)\end{aligned}$$

i.e. μ is a distribution.

We can weaken the condition on the eigenvalue and obviously still get a distribution.

Corollary 4.6. *Let $\psi_{\underline{\lambda}} \in \mathscr{M}$ be an eigenfunction of $V_1, \dots, V_{n-1}$ as in Proposition* 4.2 *with corresponding eigenvalues $\eta_1, \dots, \eta_{n-1}$ (and suppose $\kappa_{\underline{\lambda}} := \prod_{v=1}^{n-1} \eta_v$ the eigenvalue of V does not vanish). Then*

$$\mu_{\underline{\lambda}}(uU^{(k)}) := \kappa_{\underline{\lambda}}^{-k} \cdot \psi_{\underline{\lambda}}(gut^k)$$

is a distribution.

4.4. Periods and $\mathbb{C}$-valued distributions. We return to the global situation in the previous section where we have expressed special L-values in terms of certain adelic integrals (see Corollary of the global Birch Lemma). Locally at p, the automorphic forms ϕ, φ involved in these integrals can and will both be chosen to be fixed under the respective maximal compact subgroup and hence to be eigenforms under the respective local standard Hecke algebra. Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_{n-1}) \in \mathbb{C}^{n-1}$ respectively $\underline{\alpha} = (\alpha_1, \dots, \alpha_{n-2}) \in \mathbb{C}^{n-2}$ denote fixed choices of roots of the respective Hecke polynomial attached to (π_p, σ_p). Let $\phi_{\underline{\lambda}}$ respective $\varphi_{\underline{\alpha}}$ denote the associated eigenform under the action of the respective modified Hecke algebra as in Proposition 4.2. In this section we want to study the '*periods*'

$$P(u, f) := \int_{C_{1,f}} \phi_{\underline{\lambda}}\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} t^{-r} u t^r\right) \cdot \varphi_{\underline{\alpha}}(g)\, dg$$

for $u \in U^{(0)} = U_n(\mathbb{Z}_p)$, $f = p^r$ and $t = \operatorname{diag}(p^{n-1}, p^{n-2}, \dots, 1)$. Note that these integrals (when $(\phi_{\underline{\lambda}}, \varphi_{\underline{\alpha}})$ is replaced by (ϕ, φ)) are precisely those occurring in the formula for the special L-value. We will see in the following section that this modification of the pair (ϕ, φ)

only changes the L-value by a certain non vanishing constant in the case we are interested in.

Lemma 4.7. *The period $P(u,f)$ is constant on the double coset $U_{n-1}^{(r)}uU_n^{(r)}$.*

Proof. Recall that $\phi_{\underline{\lambda}}$ and $\varphi_{\underline{\alpha}}$ are right-invariant under the respective K_B's and in particular under $U_n^{(0)}$ respective $U_{n-1}^{(0)}$. Thus multiplying u from the right by an element of the form $t^r v t^{-r} \in U_n^{(r)}$ with $v \in U_n^{(0)}$ does not affect the value of the integrand. Similarly a factor $t^r v t^{-r} \in U_{n-1}^{(r)}$ with $v \in U_{n-1}^{(0)}$ on the left hand side of u can be transformed into the right hand factor v in the argument g of $\varphi_{\underline{\alpha}}$ by the invariance property of the Haar measure dg and again by right-invariance $\varphi_{\underline{\alpha}}(gv) = \varphi_{\underline{\alpha}}(g)$ the integral becomes independent of v. So the lemma follows.

Lemma 4.8. *For any diagonal matrix $\varepsilon \in GL_{n-1}(\mathbb{Z}_p)$ with $\det(\varepsilon) \equiv 1$ modulo f we have*

$$P(\varepsilon u \varepsilon^{-1}, f) = P(u,f).$$

Proof. Again this follows by the right-invariance of $\phi_{\underline{\lambda}}$ and $\varphi_{\underline{\alpha}}$ under the respective K_B's since ε commutes with t and because the change of variables $g \mapsto g\varepsilon$ does not effect the integral.

Proposition 4.9. *Let $\kappa_{\underline{\lambda}}$ resp. $\kappa_{\underline{\alpha}}$ denote the eigenvalue of $\phi_{\underline{\lambda}}$ resp. $\varphi_{\underline{\alpha}}$ under the respective operator V in the modified Hecke algebra. Then we have the following relation of periods*

$$P(u,f) = (\kappa_{\underline{\lambda}}\kappa_{\underline{\alpha}})^{-1} \cdot \sum_{v,w,x} P\big(\varepsilon_x \beta^r(w^{-1}) u \beta^r(v) \varepsilon_x^{-1}, fp\big)$$

with $\varepsilon_x := \operatorname{diag}(1+xf, 1, \dots, 1)$ and where we sum over representatives v of $U_n^{(0)}/U_n^{(1)}$, w of $U_{n-1}^{(0)}/U_{n-1}^{(1)}$ and x mod p.

Proof. By Corollary 4.6 we have the distributions

$$\mu_{\underline{\lambda}}(uU_n^{(k)})(g) = \kappa_{\underline{\lambda}}^{-k} \cdot \phi_{\underline{\lambda}}(gut^k) \quad \text{on } U_n,$$

$$\mu_{\underline{\alpha}}(uU_{n-1}^{(k)})(g) = \kappa_{\underline{\alpha}}^{-k} \cdot \phi_{\underline{\alpha}}(gut^k) \quad \text{on } U_{n-1}.$$

Expressing the periods in terms of these distributions we get

$$\begin{aligned} P(u,f) &= \int_{C_{1,f}} \phi_{\underline{\lambda}}\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} ut^r\right) \cdot \varphi_{\underline{\alpha}}(gt^r)\, dg \\ &= (\kappa_{\underline{\lambda}}\kappa_{\underline{\alpha}})^r \int_{C_{1,f}} \mu_{\underline{\lambda}}(uU_n^{(r)})\begin{pmatrix} g & \\ & 1 \end{pmatrix} \cdot \mu_{\underline{\alpha}}(U_{n-1}^{(r)})(g)\, dg \end{aligned}$$

which (by the additivity property (DR) (cf. 4.3) of the distributions $\mu_{\underline{\lambda}}$ and $\mu_{\underline{\alpha}}$) transforms into

$$P(u,f) = (\kappa_{\underline{\lambda}}\kappa_{\underline{\alpha}})^r \int_{C_{1,f}} \sum_{v,w} \mu_{\underline{\lambda}}\big(u\beta^r(v)U_n^{(r+1)}\big)\begin{pmatrix} g & \\ & 1 \end{pmatrix} \cdot \mu_{\underline{\alpha}}\big(\beta^r(w)U_{n-1}^{(r+1)}\big)(g)\,dg$$

$$= (\kappa_{\underline{\lambda}}\kappa_{\underline{\alpha}})^{-1} \sum_{v,w} \int_{C_{1,f}} \phi_{\underline{\lambda}}\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} u\beta^r(v)t^{(r+1)}\right) \cdot \varphi_{\underline{\alpha}}\big(g\beta^r(w)t^{(r+1)}\big)\,dg.$$

Decomposing the domain of integration as

$$C_{1,f} = \dot{\bigcup_{x(\mathrm{mod}\,p)}} C_{1,fp} \cdot \varepsilon_x$$

and after the change of variables $g \mapsto g\beta^r(w^{-1})$ we get the period relation as claimed.

Note that the period relation in Proposition 4.9 is different from the distribution relation which one might have expected to hold in view of Corollary 4.6. We will in fact attach a $\mathbb{C}$-valued distribution to our periods. But this distribution is rather adjusted to the canonical filtration of $U_n(\mathbb{Z}_p)$ given by the kernel $K(r)$ of the reduction maps modulo p^r than to the filtration by the subgroups $U_n^{(r)}$. Obviously the reduction map modulo p^r

$$\beta_r : U_n(\mathbb{Z}_p) \to U_n(\mathbb{Z}/p^r\mathbb{Z})$$

is surjective and factors through

$$\bar{\beta}_r : U_n^{(0)}/U_n^{(r)} \to U_n(\mathbb{Z}/p^r\mathbb{Z}).$$

Summing over fibres we attach to each $\tilde{u} \in U_n(\mathbb{Z}/p^r\mathbb{Z})$ the '*smoothed period*'

$$\tilde{P}(\tilde{u}, r) := \sum_u P(u, p^r)$$

summing over all u from a representative system for $U_n^{(0)}/U_n^{(r)}$ such that $\beta_r(u) = \tilde{u}$. For $u \in U_n^{(0)}$ we put

$$\hat{P}(u,r) := \tilde{P}\big(\beta_r(u), r\big),$$

such that $\hat{P}(u,r)$ as a function in u is constant on cosets $uK(r)$.

Theorem 4.10. *Let* $\hat{\kappa} := p^{1+\sum_{i=1}^{n-2} i(n-i-1)} \cdot (\kappa_{\underline{\lambda}}\kappa_{\underline{\alpha}})^{-1}$. *Then*

$$\hat{\mu}\big(uK(r)\big) := \hat{\kappa}^r \cdot \hat{P}(u,r)$$

defines a distribution $\hat{\mu} := \hat{\mu}_{\underline{\lambda},\underline{\alpha}}$. *In particular* $\hat{\mu}$ *satisfies the distribution relation*

$$\hat{\mu}\big(uK(r)\big) = \sum_{\substack{v\,(\mathrm{mod}\,K(r+1))\\ \beta_r(v)=\beta_r(u)}} \hat{\mu}\big(vK(r+1)\big).$$

Proof. We exploit the period relations as described in Proposition 4.9. For $u \in U_n^{(0)}$ let $\tilde{u} = \tilde{u}_r := \beta_r(u)$ denote the image under β_r. The proof of the distribution relation is based on the following commutative diagram involving canonical projections

$$\begin{array}{ccc} U^{(0)}/U^{(r+1)} & \xrightarrow{\bar{\beta}_{r+1}} & U(\mathbb{Z}/p^{r+1}\mathbb{Z}) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\ U^{(0)}/U^{(r)} & \xrightarrow{\bar{\beta}_r} & U(\mathbb{Z}/p^r\mathbb{Z}) \end{array}$$

when passing from cosets with respect to our group to cosets with respect to a group containing the first one. By definition and Proposition 4.9 we get

$$\begin{aligned} \hat{\mu}\big(uK(r)\big) &= \hat{\kappa}^r \cdot \sum_{\substack{u' \bmod U^{(r)} \\ \tilde{u}' = \tilde{u}}} P(u', p^r) \\ &= \hat{\kappa}^r (\kappa_{\underline{\lambda}} \kappa_{\underline{\alpha}})^{-1} \cdot \sum_{u', v, w, x} P\big(\varepsilon_x \beta^r(w^{-1}) u' \beta^r(v) \varepsilon_x^{-1}, p^{r+1}\big) \end{aligned}$$

where $u'\beta^r(v)$ runs over a representative system of the fibre $\alpha^{-1}\big(\bar{\beta}_r^{-1}(\tilde{u})\big) = \bar{\beta}_{r+1}^{-1}\big(\gamma^{-1}(\tilde{u})\big)$. Multiplying by $\beta^r(w^{-1})$ and conjugating by ε_x only permutes the fibre, hence we find

$$\hat{\mu}\big(uK(r)\big) = \hat{\kappa}^r (\kappa_{\underline{\lambda}} \kappa_{\underline{\alpha}})^{-1} \cdot p \cdot (U_{n-1}^{(0)} : U_{n-1}^{(1)}) \cdot \sum_{\substack{v \bmod U^{(r+1)} \\ \tilde{v} \in \gamma^{-1}(\tilde{u})}} P(v, p^{r+1}).$$

Recall from section 4.1 that we have

$$(U_{n-1}^{(0)} : U_{n-1}^{(1)}) = p^{\sum_{i=1}^{n-2} i(n-1-i)},$$

which implies

$$\hat{\mu}\big(uK(r)\big) = \hat{\kappa}^{r+1} \cdot \sum_{\substack{\tilde{v} \bmod p^{r+1} \\ \gamma(\tilde{v}) = \tilde{u}}} \hat{P}(v, r+1),$$

and this completes the proof.

The distribution that we have constructed in Theorem 4.10 is a distribution on a manifold of dimension $\dim(U_n) = \dfrac{n(n-1)}{2}$. For our application to special L-values we are in particular interested in a certain push-down $\tilde{\mu}$ to a one-variable distribution.

Corollary 4.11. *Let $\tilde{\kappa} := \hat{\kappa} \cdot p^{n-2}$. Then we get a distribution $\tilde{\mu} = \tilde{\mu}_{\underline{\lambda}, \underline{\alpha}}$ on $\mathbb{Z}_p$ by*

$$\tilde{\mu}(i + p^r\mathbb{Z}_p) := \tilde{\kappa}^r \cdot \sum_u \hat{P}(u, r)$$

where $u = (u_{jk})$ *runs over a representative system of the* $\tilde{u} \in U(\mathbb{Z}/p^r\mathbb{Z})$ *with* $u_{1,2} = i$ *and* $u_{j,j+1} = 1$ *for* $j = 2, \dots, n-1$.

Proof. By the distribution property we get

$$\tilde{\mu}(i + p^r\mathbb{Z}_p) = p^{(n-2)r} \sum_u \hat{\mu}\big(uK(r)\big)$$
$$= p^{(n-2)r} \sum_u \sum_{\substack{v \bmod p^{r+1} \\ v \equiv u\,(p^r)}} \hat{\mu}\big(vK(r+1)\big).$$

Note that the v's run over all members of a representative system of $U^{(0)}/K(r+1)$ with entries $v_{j,j+1}$ in the first off-diagonal of the form $v_{1,2} = i + a_1 p^r$, $v_{j,j+1} = 1 + a_j p^r$ for $j = 2, \dots, n-1$, and where $a_1, \dots, a_{n-1}$ run through congruence classes mod p. Conjugating v by a suitable diagonal matrix $\varepsilon \in GL_n(\mathbb{Z}_p)$ with $\det(\varepsilon) = 1$ brings the off-diagonal of $v' := \varepsilon v \varepsilon^{-1}$ into the form

$$v'_{1,2} = i + ap^r, \quad v_{j,j+1} = 1 \quad \text{for } j = 2, \dots, n-1.$$

Since by Lemma 4.8 this conjugation does not affect the periods we have

$$\hat{\mu}\big(vK(r+1)\big) = \hat{\mu}\big(v'K(r+1)\big)$$

and hence we find

$$\hat{\mu}(i + p^r\mathbb{Z}_p) = p^{(n-2)(r+1)} \sum_{a \bmod p} \sum_{v'} \hat{\mu}\big(v'K(r+1)\big)$$

where v' runs over a system of representatives of $U^{(0)}/K(r+1)$ such that $v'_{1,2} = i + ap^r$, $v_{j,j+1} = 1$ for $j = 2, \dots, n-1$. In terms of $\tilde{\mu}$ this says

$$\tilde{\mu}(i + p^r\mathbb{Z}_p) = \sum_{a \bmod p} \tilde{\mu}(i + ap^r + p^{r+1}\mathbb{Z}_p),$$

hence $\tilde{\mu}$ is a distribution on $\mathbb{Z}_p$.

4.5. Integrating characters. Let $w^0_{\underline{\lambda}}$ resp. $v^0_{\underline{\alpha}}$ denote the local Whittaker function attached to the respective new-vector of π_p resp. σ_p as in Proposition 4.2.

Proposition 4.12. *The value of* $w^0_{\underline{\lambda}}$ *at the unit matrix* 1_n *is given by*

$$w^0_{\underline{\lambda}}(1_n) = (-1)^{\frac{(n-1)(n-2)}{2}} p^{A_n} \prod_{t=1}^{n-1} \lambda_t^{1+(n-1)(n-t)} \cdot \prod_{i=2}^{n-1} \prod_{j<i} (\lambda_j - \lambda_i)$$

where we put $A_n := (n-2) \sum_{v=1}^{n} \frac{v(1-v)}{2} + \frac{n(1-n)}{2}$.

The same formula applies to $v^0_{\underline{\alpha}}$ as well.

Proof. First we write

$$w^0_{\underline{\lambda}} = \prod_{i=1}^{n-1} \prod_{\substack{j=1 \\ j \neq i}}^{n} (\lambda_i p^{1-j} V_{j-1} - V_j)\, w^0$$

as a sum of monomials in the V_i's. Let $N_i := \{1, \ldots, n\} \backslash \{i\}$. Then we have

$$\begin{aligned} w^0_{\underline{\lambda}} &= \prod_{i=1}^{n-1} \left(\sum_{J_i \subset N_i} \lambda_i^{|J_i|} (-1)^{n-1-|J_i|} \cdot \prod_{k \in J_i} (p^{1-k} V_{k-1}) \cdot \prod_{t \in N_i \backslash J_i} V_t \right) w^0 \\ &= (-1)^{n-1} \sum_{J_1 \subset N_1} \cdots \sum_{J_{n-1} \subset N_{n-1}} \prod_{i=1}^{n-1} (-\lambda_i)^{|J_i|} \cdot \left(\prod_{k \in J_i} p^{1-k} V_{k-1} \cdot \prod_{t \in N_i \backslash J_i} V_t \right) w^0. \end{aligned}$$

We first deal with each separate term and consider for an arbitrary monomial

$$\underline{V}^{\underline{r}} := \prod_{v=1}^{n} V_v^{r_v},$$

the value $(\underline{V}^{\underline{r}} w^0)(1_n)$ at the unit matrix 1_n. In particular we are interested in this value for $\underline{r} = \underline{r}(\underline{J})$ for each fixed $n-1$-tuple of sets $\underline{J} = (J_1, \ldots, J_{n-1})$ where

$$\underline{V}^{\underline{r}(\underline{J})} := \prod_{i=1}^{n-1} \left(\prod_{k \in J_i} V_{k-1} \cdot \prod_{t \in N_i \backslash J_i} V_t \right).$$

Lemma 4.13. *With the diagonal matrix*

$$d(\underline{r}) := \prod_{v=1}^{n} \begin{pmatrix} p 1_v & \\ & 1_{n-v} \end{pmatrix}^{r_v}$$

we have

$$(\underline{V}^{\underline{r}} w^0)(1_n) = \prod_{v=1}^{n} p^{v(n-v) r_v} \cdot w^0\big(d(\underline{r})\big).$$

Proof of Lemma 4.13. Recall that the action of V_v on a K_B-right-invariant Whittaker function w is given by

$$(V_v w)(g) = \sum_A w\left(g \begin{pmatrix} p 1_v & A \\ & 1_{n-v} \end{pmatrix} \right)$$

where A runs over a representative system of matrices in $M^{v, n-v}(\mathbb{Z}_p)$ modulo p. This easily generalizes to powers of V_v where we get

$$(V_v^r w)(g) = \sum_A w\left(g \begin{pmatrix} p^r 1_v & A \\ & 1_{n-v} \end{pmatrix} \right)$$

where A runs over a representative system modulo p^r. Hence we have

$$(\underline{V}^{\underline{r}}w)(g) = \sum_{A_1} \cdots \sum_{A_{n-1}} w\left(g\begin{pmatrix} p^{r_1} & A_1 \\ & 1_{n-1}\end{pmatrix} \cdots \begin{pmatrix} p^{r_{n-1}}1_{n-1} & A_{n-1} \\ & 1\end{pmatrix} p^{r_n}1_n\right)$$

where each A_ν runs over a representative system in $M^{\nu,n-\nu}(\mathbb{Z}_p)$ modulo p^{r_ν}. An easy calculation shows that for fixed $A_1, \dots, A_{n-1}$ there is a unipotent matrix $B \in U_n(\mathbb{Z}_p)$ such that

$$\begin{pmatrix} p^{r_1} & A_1 \\ & 1_{n-1}\end{pmatrix}\begin{pmatrix} p^{r_2}1_2 & A_2 \\ & 1_{n-2}\end{pmatrix} \cdots p^{r_n}1_n = B \cdot d(\underline{r}).$$

Since our Whittaker functions are left-invariant under $U_n(\mathbb{Z}_p)$ we find

$$(\underline{V}^{\underline{r}}w)(1_n) = w\left(d(\underline{r}) \cdot \prod_\nu |\{A_\nu' \mathrm{s}\}|\right)$$

which proves the lemma.

By Shintani's explicit formula for class-1 Whittaker functions (in Shintani's terminology) [19] we have

$$w^0\big(d(\underline{r})\big) = p^{\sum_{i=1}^{n}(i-n)s_i}\,\frac{\det(\lambda_i^{s_j+n-j})}{\det(\lambda_i^{n-j})}$$

where $\operatorname{diag}(p^{s_1}, \dots, p^{s_n}) = d(\underline{r})$, i.e. $s_j = \sum_{\nu=j}^{n} r_\nu$ and $i, j = 1, \dots, n$ in the determinants. So the formula in the lemma becomes

$$(\underline{V}^{\underline{r}}w^0)(1_n) = p^{\frac{1}{2}\sum_{\nu=1}^{n}\nu(1-\nu)r_\nu} \cdot \frac{\det(\lambda_i^{s_j+n-j})}{\det(\lambda_i^{n-j})},$$

taking into account that

$$\sum_{i=1}^{n}(i-n)s_i + i(n-i)r_i = \sum_i \frac{i(1-i)}{2} r_i.$$

In terms of the characteristic function χ_M of subsets $M \subset N_i$ we can describe the special $\underline{r} = \underline{r}(\underline{J})$ by

$$r_\nu = \sum_{i=1}^{n-1} \chi_{J_i}(\nu+1) + \chi_{N_i \setminus J_i}(\nu)$$

and therefore

$$\begin{aligned} s_j &= \sum_{\nu=j}^{n}\sum_{i=1}^{n-1}\big(\chi_{J_i}(\nu+1) + \chi_{N_i\setminus J_i}(\nu)\big) \\ &= \sum_{i=1}^{n-1}\left(\chi_{N_i\setminus J_i}(j) + \sum_{\nu=j+1}^{n}\chi_{N_i}(\nu)\right). \end{aligned}$$

Since $\sum_{i=1}^{n-1} \chi_{N_i}(v)$ is equal to the number of i's different from v, we have for $j < n$

$$\sum_{i=1}^{n-1} \sum_{v=j+1}^{n} \chi_{N_i}(v) = (n-1) + (n-2)(n-j-1)$$
$$= 1 + (n-2)(n-j)$$

hence in terms of the Kronecker δ

$$s_j = 1 - \delta_{n,j} + (n-2)(n-j) + \sum_{i=1}^{n-1} \chi_{N_i \setminus J_i}(j).$$

By Leibnitz' determinant formula we get

Lemma 4.14. *For $\underline{r} = \underline{r}(\underline{J})$ we have*

$$(\underline{V}^{\underline{r}} w^0)(1_n) = \frac{p^{\frac{1}{2}\sum_{v=1}^{n} v(1-v)r_v}}{\det(\lambda_i^{n-j})} \cdot \sum_{w \in W} \operatorname{sgn}(w) \cdot \prod_{j=1}^{n-1} \lambda_{w_j}^{1+(n-1)(n-j)+\sum_{t=1}^{n-1} \chi_{N_t \setminus J_t}(j)}.$$

Before we plug this into our formula for $w_{\underline{\lambda}}^0(1_n)$ we determine the p-power factor of each summand for fixed $\underline{J}$.

Lemma 4.15. *We have for $\underline{r}(\underline{J}) = (r_1, \ldots, r_n)$*

$$\frac{1}{2}\sum_{v=1}^{n} v(1-v)r_v + \sum_{i=1}^{n-1} \sum_{k \in J_i} 1 - k = A_n.$$

In particular, the left-hand-side of the above formula is an integer which only depends on n and is independent of $\underline{J}$.

Proof of Lemma 4.15. We have expressed the components r_v of $\underline{r}$ by the formula

$$r_v = \sum_{i=1}^{n-1} \chi_{J_i}(v+1) + \chi_{N_i \setminus J_i}(v).$$

So we get

$$\sum_{v=1}^{n} v(1-v)r_v = \sum_{i=1}^{n-1} \sum_{v=1}^{n} \big(\chi_{J_i}(v+1) + \chi_{N_i \setminus J_i}(v)\big) v(1-v)$$
$$= \sum_{i=1}^{n-1} \left(\sum_{v=1}^{n} \chi_{J_i}(v) \cdot (v-1)(2-v) + \sum_{v=1}^{n} \chi_{N_i \setminus J_i}(v) \cdot v(1-v) \right)$$

which by the identity $\chi_{J_i} + \chi_{N_i \setminus J_i} = \chi_{N_i}$ becomes equal to

$$\sum_{i=1}^{n-1}\left(\sum_{v \neq i} v(1-v) + 2\sum_{v\in J_i}(v-1)\right)$$
$$= (n-1)\sum_{v=1}^{n} v(1-v) - \sum_{i=1}^{n-1} i(1-i) + 2\sum_{i=1}^{n-1}\sum_{v\in J_i}(v-1),$$

hence our lemma follows.

We now can complete the proof of Proposition 4.12. By Lemma 4.14 and Lemma 4.15 we have

$$w^0_{\underline{\lambda}}(1_n) = \frac{(-1)^{n-1}p^{A_n}}{\det(\lambda_i^{n-j})} \cdot \sum_{\underline{J}} \prod_{i=1}^{n-1} (-\lambda_i)^{|J_i|}$$
$$\times \sum_{w\in W} \operatorname{sgn}(w) \cdot \prod_{j=1}^{n-1} \lambda_{w_j}^{1+(n-1)(n-j)+\sum_{t=1}^{n-1}\chi_{N_t\setminus J_t}(j)}$$
$$= \frac{(-1)^{n-1}p^{A_n}}{\det(\lambda_i^{n-j})} \cdot \sum_{w\in W} \operatorname{sgn}(w) \cdot \prod_{j=1}^{n-1} \lambda_{w_j}^{(n-1)(n-j)+1}$$
$$\times \sum_{\underline{J}} \prod_{i=1}^{n-1} (-\lambda_i)^{|J_i|} \cdot \prod_{j=1}^{n}\prod_{t=1}^{n-1} \lambda_{w_j}^{\chi_{N_t\setminus J_t}(j)}.$$

For fixed w the sum over the $\underline{J}$'s becomes equal to

$$\prod_{i=1}^{n-1}\prod_{\substack{j=1\\ j\neq i}}^{n} (\lambda_{w_j} - \lambda_i),$$

hence vanishes for any nontrivial permutation $w \neq \mathrm{id}$. Thus our formula simplifies to

$$w^0_{\underline{\lambda}}(1_n) = \frac{(-1)^{n-1}p^{A_n}}{\det(\lambda_i^{n-j})} \cdot \prod_{i=1}^{n-1}\prod_{\substack{j=1\\ j\neq i}}^{n} (\lambda_j - \lambda_i) \cdot \prod_{v=1}^{n-1} \lambda_v^{(n-1)(n-v)+1}$$

which by the well known formula for the Vandermonde determinant

$$\det(\lambda_i^{n-j}) = \prod_{i<j} (\lambda_i - \lambda_j)$$

eventually yields what we want. So the proof of Proposition 4.12 is complete.

Let $\chi = \prod_{\ell} \chi_\ell$ be an idele class character of conductor $f = p^r$ which satisfies the hypothesis in the global Birch Lemma in section 3.2. Recall that we denote by $\tilde{\chi}$ the Dirichlet character on $\mathbb{Z}_p$ attached to the p-component χ_p of χ (cf. 4.1). Recall that there is always a good pair (w_∞, v_∞) such that the associated $P(s)$ satisfies $P(1/2) \neq 0$. For each pair of automorphic forms (ϕ_j, φ_j) occuring in the global Birch Lemma let $P_j(u, f)$ denote the period attached to $(\phi_{j,\underline{\lambda}}, \varphi_{j,\underline{\alpha}})$ as in section 4.4. By Corollary 4.11 we have an associated distribution $\tilde{\mu}_j = \tilde{\mu}_{j,\underline{\lambda},\underline{\alpha}}$ for each j.

Theorem 4.16. *There is a choice of the Whittaker functions at infinity* (w_∞, v_∞) *such that integrating any* χ_p *of conductor* $f = p^r$ *against the distribution* $\tilde{\mu} = \sum_j \tilde{\mu}_{j,\underline{\lambda},\underline{\alpha}}$ *we get*

$$\int_{\mathbb{Z}_p^\times} \chi_p \, d\tilde{\mu} = \delta(\underline{\lambda}, \underline{\alpha}) \cdot (\tilde{\kappa} p^{1-n})^r \cdot \prod_{i=1}^{n-1} G(\chi_p^i) \cdot L\left(\pi \otimes \chi, \sigma, \frac{1}{2}\right)$$

where $\delta(\underline{\lambda}, \underline{\alpha}) = w_{\underline{\lambda}}^0(1_n) \cdot v_{\underline{\alpha}}^0(1_{n-1})$ *is explicitly given by Proposition* 4.12, *and where* $G(\chi_p^i)$ *is the Gauss sum*; *cf.* 3.1.

Remark. If we multiply our distribution with the constant $\Omega := P_\infty(1/2)$ in Theorem 3.5 then the renormalized distribution $\mu := \Omega \cdot \tilde{\mu}$ has its values in the number field $F = \mathbb{Q}(\pi, \sigma)$.

Proof of Theorem 4.16. The key to our p-adic integral lies in the global Birch Lemma in section 3.2 and its corollary, where we are still free to choose (w_∞, v_∞) and (w_p, v_p) fixed under the respective Iwahori subgroup. We fix the latter pair to be $(w_p, v_p) = (w_{\underline{\lambda}}^0, v_{\underline{\alpha}}^0)$, which in particular replaces the pairs of automorphic forms (ϕ_j, φ_j) in the global Birch Lemma by $(\phi_{j,\underline{\lambda}}, \varphi_{j,\underline{\alpha}})$ and allows us to rewrite the formula for the special L-value in terms of the periods $P_j(u, f)$ as

$$\begin{aligned}\delta(\underline{\lambda}, \underline{\alpha}) \cdot P\left(\frac{1}{2}\right) \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{(1-p^{-i})} \cdot L\left(\pi \otimes \chi, \sigma, \frac{1}{2}\right) \\ = (1-p^{-1}) f \sum_u \prod_{i=1}^{n-1} \tilde{\chi}_p(u_i^i) \cdot \sum_j P_j(u, f).\end{aligned}$$

Furthermore we choose (w_∞, v_∞) such that

$$P\left(\frac{1}{2}\right) \cdot \prod_{i=1}^{n-1} (1-p^{-i})^{-1} = 1$$

which simplifies our formula to become

$$\begin{aligned}(1-p^{-1})^{2-n} \cdot \sum_u \prod_{i=1}^{n-1} \tilde{\chi}_p(u_i^i) \cdot \sum_j P_j(u, f) \\ = \delta(\underline{\lambda}, \underline{\alpha}) p^{-r} \cdot \prod_{i=1}^{n-1} G(\chi_p^i) \cdot L\left(\pi \otimes \chi, \sigma, \frac{1}{2}\right).\end{aligned}$$

It remains to relate the left hand side of this equality with the p-adic integral of the theorem. By definition we have (cf. Corollary 4.11)

$$\int \chi_p \, d\tilde{\mu}_j = \sum_x \chi_p(x) \tilde{\kappa}^r \sum_u \hat{P}_j(u, r)$$

summing over all $x \bmod f$ not divisible by p and u taken from a representative system in $U_n^{(0)}$ of the $\tilde{u} \in U_n(\mathbb{Z}/f\mathbb{Z})$ with off-diagonal elements $u_{1,2} = x$ and $u_{j,j+1} = 1$ for $j = 2, \ldots, n-1$. Recall that we defined

$$\hat{P}_j(u, r) := \sum_v P_j(v, p^r)$$

summing over a representative system in $U_n^{(0)}$ of the cosets in $U_n^{(0)}/U_n^{(r)}$ which have the same image as u under the canonical map to $U_n(\mathbb{Z}/f\mathbb{Z})$. Hence we have

$$\int \chi_p \, d\tilde{\mu}_j = \tilde{\kappa}^r \cdot \sum_x \chi_p(x) \cdot \sum_v P_j(v, p^r)$$

with v running through a representative system in $U_n^{(0)}$ of all cosets in $U_n^{(0)}/U_n^{(r)}$ with off-diagonal entries $v_{1,2} \equiv x \bmod f$ and $v_{j,j+1} \equiv 1 \bmod f$ for $j = 2, \ldots, n-1$.

The group of diagonal matrices $\varepsilon \in GL_{n-1}(\mathbb{Z}_p)$ with $\det(\varepsilon) \equiv 1 \bmod f$ acts by conjugation on those cosets in $U_n^{(0)}/U_n^{(r)}$ with all elements $u_{j,j+1}$ in the off-diagonal being p-adic units. Moreover in each orbit there is a representative v with unique $x \bmod f$ as above so that by the period relation in Lemma 4.8 we get

$$\int \chi_p \, d\tilde{\mu}_j = \tilde{\kappa}^r \cdot \left((p-1)p^{r-1}\right)^{2-n} \cdot \sum_u \tilde{\chi}_p\left(\prod_{i=1}^{n-1} u_i^i\right) \cdot P_j(u, f)$$

where now u runs over a full representative system of $U_n^{(0)}/U_n^{(r)}$. Here we exploit the fact that, by definition, $\tilde{\chi}_p$ vanishes on non-units. So the theorem is proven.

Note added in proof. The expected boundedness of the distribution in the ordinary case recently has been proven by the third author. See forthcoming publication.

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Harvard University, Department of Mathematics, Science Center 325, One Oxford Street, Cambridge, Massachusetts 02138
e-mail: kazhdan@math.harvard.edu
e-mail: mazur@math.harvard.edu

Universität (TH) Karlsruhe, Mathematisches Institut II, Englerstraße 2, D-76128 Karlsruhe
e-mail: cs@ma2pccs.mathematik.uni-karlsruhe.de

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