Curves Over Discrete Valuation Rings

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CURVES OVER DISCRETE VALUATION RINGS.

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Introduction. The purpose of this paper is to lay the foundations for the study of geometrical questions on regular two-dimensional preschemes. We will pay particular attention to the case when the two-dimensional prescheme is given together with a flat morphism to the spectrum of a Dedekind domain A, usually, but not always, local. We will be motivated throughout by the problems of determining the existence and uniqueness of minimal models over A, and of classifying the fibers of minimal models when they do exist. We note that, in particular, A may be the ring of integers in an algebraic number field, or a localization of such a ring.

In the first section we develop intersection theory. This proceeds along the usual lines, with the only complications being caused by the fact that the residue fields may not be algebraically closed. We then show that a complete non-singular curve over a Dedekind domain is projective, a result which is necessary for much of the later applications. In particular we use it next to derive a "first difference form" of the Riemann-Roch theorem for surfaces, which makes sense in our situation.

In the second section, we discuss the theorems leading up to the existence of minimal models. First we show that any proper birational morphism between two regular two-dimensional preschemes can be factored into a product of locally quadratic transformations. Then we give a generalization of Castelnuovo's criterion for the existence of exceptional curves on a surface. Then we prove the existence and uniqueness of minimal models over a discrete valuation ring, in the case when the genus of the generic fiber is greater than zero. (Otherwise, we have an analogue of a ruled surface, and the result is not true.)

As must be evident already, this paper is written in the language of Grothendieck, and his multi-volume work "Eléments de Géometrie Algébrique" (referred to as E.G.A.) is a basic reference. For the conveniences of the author and the reader, all rings and preschemes are assumed to be noetherian.

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John Tate. An earlier version of the material in this paper formed part of the author's doctoral dissertation at Harvard University.

I. 1. Intersection theory. Our aim in this section is to develop intersection theory for regular two-dimensional preschemes. We will begin with a series of definitions and elementary results on invertible sheaves, and gradually specializes our hypotheses to the case which will interest us most. First, we need to know something about algebraic curves.

Definition 1.1. Let k be a field, X a prescheme of finite type over k. X is said to be an *algebraic curve over* k if for every closed point x of X, $\dim O_{X,x=1}$. This is equivalent to the statement that the residue class fields of the local rings of the generic points of the irreducible components of X have transcendence degree 1 over k. (E.G.A. II, Proposition 7.4.1.) X is said to be complete if $f\colon X \to \operatorname{Spec} k$ is proper.

We now recall some elementary facts about divisors and invertible sheaves on a regular prescheme. A prescheme X is said to be *regular* if all its local rings are regular. It is *normal* if it is integral and all its local rings are integrally closed. A closed subscheme D of X is a *prime divisor of* X if it is integral and $\dim 0_{X,x=1}$, where x is the generic point of D. The free abelian group generated by the prime divisors of X is called the *group of divisors* of X. If X is normal, a prime divisor D determines a discrete valuation v_D of the field $K(X)$ of rational functions on X. If f is a non-zero element of $K(X)$, the *divisor associated with* f, denoted (f) is the sum $\Sigma_D v_D(f)$ where D runs over all prime divisors of X. (This sum is finite, since all our preschemes are noetherian.) A divisor is *principal* if it is of the form (f) for some f. If $D_1 - D_2$ is principal, we say D_1 is *linearly equivalent* to D_2 $(D_1 \equiv D_2)$. The principal divisors form a subgroup of the group of divisors of X, and the quotient group is called the *divisor class group* of X, and denoted by $C(X)$. If X is a connected regular prescheme, then X is normal, so $C(X)$ makes sense. In this case, we have a map from $C(X)$ to *Pic* (X), the group of invertible sheaves on X, defined by sending a prime divisor D to the inverse of the (necessarily invertible) sheaf of ideals defining D, and this map is an isomorphism. We use here of course the fact that a regular local ring is a unique factorization domain, in order to show that any prime divisor is defined by an invertible sheaf of ideals. If D is any divisor, we denote the image in Pic (X) of the class of D by $\mathcal{L}(D)$.

Now let X be a complete connected regular curve over a field k. Let $\mathcal{L}$ be an invertible sheaf on X, and $D = \Sigma n_P P$ a divisor on X corresponding to $\mathcal{L}$.

We define the *degree* of $\mathcal{L}$ with respect to k (denoted $\deg_k(\mathcal{L})$) to be $\Sigma n_P[\kappa(P):k]$. This is independent of the choice of D, since X is complete. If we only assume that X is complete and integral, let $\bar{X}$ be the normalization of X, π the canonical map from $\bar{X}$ to X. We define the *degree* of $\mathcal{L}$ with respect to k to be $\deg_k(\pi^*\mathcal{L})$.

Let X be a complete connected curve over k, $\mathcal{L}$ an invertible sheaf on X. Let $h^0(\mathcal{L}) = \dim_k H^0(X, \mathcal{L})$, $h^1(\mathcal{L}) = \dim_k H^1(X, \mathcal{L})$, $\pi(X) = \dim H^1(X, O_X)$. If X is regular, we have the Riemann-Roch Theorem:

THEOREM 1.2. $h^0(\mathcal{L}) - h^1(\mathcal{L}) = 1 - \pi(X) + \deg_k(\mathcal{L})$. It is easy to see that this result still holds if we only assume that X is integral, and that $\dim_k H^0(X, O_X) = 1$.

We now are ready to discuss intersection theory.

Definition 1.3. Let X be a prescheme, F a closed subscheme of X. F is a positive *Cartier divisor* on X if the sheaf of ideals defining F is invertible.

Definition 1.4. Let X be a prescheme. Let E be a closed subscheme of X, given together with a morphism f making E into a complete integral algebraic curve over a field k. Let F be a positive Cartier divisor on X, defined by a sheaf of ideals I. Then the *intersection number of E and F with respect to k* (written $i_k(E,F)$) is defined to be $\deg_k(i^*I^{-1})$, where i is the closed immersion of E into X.

We now are obliged to show that this intersection number is symmetric if $i_k(E,F)$ and $i_k(F,E)$ are both defined. We first need a routine algebraic lemma.

LEMMA 1.5. *Let A be the local ring of a point on an integral algebraic curve over a field k. Let f be in the maximal ideal of A. Let $\bar{A}$ be the integral closure of k. Then* $\dim_k(A/fA) = \dim_k(\bar{A}/f\bar{A})$, *where $\bar{A}$ is the integral closure of A.*

Proof. $\bar{A}$ is an A-module of finite type and rank one. Hence $\bar{A}/A$ is a torsion A-module of finite type, so $\dim_k(\bar{A}/A)$ is finite, since A is one-dimensional. It follows that $\dim_k(\bar{A}/Af) = \dim_k(\bar{A}/\bar{A}f) + \dim_k(\bar{A}f/Af) = \dim_k(\bar{A}/\bar{A}f) + \dim_k(\bar{A}/A)$. Also

$$\dim_k(\bar{A}/Af) = \dim_k(\bar{A}/A) + \dim_k(A/Af).$$

So

$$\dim_k(\bar{A}/\bar{A}f) = \dim_k(A/Af).$$

PROPOSITION 1.6. *Let X be a prescheme, k a field, E and F complete integral algebraic curves over k. Let i, j be closed immersions of E, F into X. Let $E \cap F = E \underset{X}{\times} F$. Assume that the following diagram commutes:*

$$\begin{array}{ccccc} & & E \cap F & & \\ & \swarrow & & \searrow & \\ E & & & & F \\ & \searrow & & \swarrow & \\ & & k & & \end{array}$$

where the maps are all the obvious ones. Assume further that both E and F are Cartier divisors on X. Then $i_k(E,F) = i_k(F,E)$.

Proof. We may assume $E \neq F$. Then $|E \cap F|$ is a finite set of closed points on X, say $P_1, \cdots, P_n$. Since E and F are Cartier divisors, we may assume that e_i and f_i are local equations for F at P_i. It is sufficient for us to show that we have the obviously symmetric formula

$$i_k(E,F) = \sum_{i=1}^{n} \dim_k(O_i/(e_i, f_i))$$

where O_i is the local ring of X at P_i.

Let I be the sheaf of ideals defining F, so we have the exact sequence $0 \to I \to O_X \to O_F \to 0$. Tensoring with O_E, we obtain

$$0 \to I \otimes O_E \to O_E \to O_F \otimes O_E \to 0.$$

(Since $E \neq F$, and E and F are irreducible, a local equation for F is not zero on E. Hence $I \otimes O_E \to O_E$ is injective.) Similarly we see that the sequence

$$0 \to I \otimes \bar{O}_E \to \bar{O}_E \to O_F \otimes \bar{O}_E \to 0$$

is exact, where $\bar{O}_E$ is the sheaf of local rings of the integral closure $\bar{E}$ of E. So we have $i_k(E,F) = \deg_k((I \otimes O_E)^{-1}) =$ (by definition) $\deg_k((I \otimes \bar{O}_E)^{-1})$

$$= \dim_k(\bar{O}_E/I \otimes \bar{O}_E) = \sum_{i=1}^{n} \dim_k(\bar{O}_{E,i}/f_i\bar{O}_{E,i}) = \text{(by Lemma 1.5)}$$

$$\sum_{i=1}^{n} \dim_k(O_{E,i}/f_i\, O_{E,i}) = \sum_{i=1}^{n} \dim_k(O_i/(e_i, f_i)).$$

We will now apply and extend this definition in some special cases:

1. Let E be a closed subscheme of a prescheme X such that
   a) $H^0(E, O_E)$ is a field k.
   b) E is a complete integral curve over k.
   c) E is a Cartier divisor on X.

Then we define the self-intersection number of E (written $E^{(2)}$) to be $i_k(E, E)$.

2. *Definition* 1.7. Let A be a Dedekind domain, $Y = \operatorname{Spec} A$. A *curve* over Y is a prescheme X together with a morphism $f: X \to Y$ such that

a) f is flat and of finite type.

b) the fibers of f, which are preschemes over fields, are algebraic curves.

c) X is connected.

If f is proper, we say X is *complete*. If X is a regular prescheme, we say X is a *regular curve*. Note that the fiber of X over the closed point of Y (the *special fiber*) may be singular.

Definition 1.8. Let A be a discrete valuation ring, and let X, Y be as in Definition 1.7. Assume that X is regular and complete. Let E and F be prime divisors of X such that E is contained in the special fiber. Let k be the residue field of A. Define $(E \cdot F)$ to be $i_k(E, F)$. By linearity this extends to a definition of $(E \cdot F)$ if E is any divisor on X with support in the special fiber and F is any divisor on X.

It is clear that the following properties hold:

1) If E is in the closed fiber, $(E \cdot F_1 + F_2) = (E \cdot F_1) + (E \cdot F_2)$.

2) If E_1, E_2 are in the closed fiber, $(E_1 + E_2 \cdot F) = (E_1 \cdot F) + (E_2 \cdot F)$.

3) If E and F are in the closed fiber, $(E \cdot F) = (F \cdot E)$.

4) If E is a prime divisor in the closed fiber, $(E \cdot F) = \deg_k(\mathcal{L}(F) \otimes O_E)$. (This follows from the additivity of $\deg_k$.)

5) If E is in the closed fiber and F is principal, $(E \cdot F) = 0$.

2. Projective embeddings. In this section we will show that a regular curve which is proper over a Dedekind domain A is projective over A. We start by reviewing some lemmas about ample sheaves.

Proposition 2.1. *Let Y be a locally noetherian prescheme, $f: X \to Y$ a proper morphism, $\mathcal{L}$ an invertible sheaf on X, y a point of Y, X_y the fiber of f over y, g the projection of X_y to X. If $g^*\mathcal{L}$ is ample on X_y, then there exists an open neighborhood U of y in Y such that $\mathcal{L} \mid f^{-1}(U)$ is ample for the restriction of f to $f^{-1}(U)$.*

Proof. This is E.G.A. III, Theorem (4.7.1).

Proposition 2.2. *Let X be a prescheme, $f: X \to Y$ a morphism. Let g be the closed immersion $g: X_{red} \to X$. Let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is ample over Y if and only if $g^*\mathcal{L}$ is ample over Y.*

Proof. This is E.G.A. II, Corollary (4.6.16).

PROPOSITION 2.3. *Let X and X' be Y-preschemes. Let $g\colon X' \to X$ be a finite surjective Y-morphism. Let X be proper over Y. Let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is ample over Y if and only if $g^*\mathcal{L}$ is ample over Y.*

Proof. This is an immediate corollary of E.G.A. III, Proposition 2.6.2.

PROPOSITION 2.4. *Let $f\colon X \to Y$ be a morphism, Y an affine scheme, and $\mathcal{L}$ an invertible sheaf on X. Then $\mathcal{L}$ is ample if and only if it is ample over Y.*

Proof. This is E.G.A. II, Corollary 4.6.6.

PROPOSITION 2.5. *Let $f\colon X \to Y$ be a morphism, $\mathcal{L}$ an invertible sheaf on Y, $\{U_\alpha\}$ an open cover of Y. Then $\mathcal{L}$ is ample over Y if and only if $\mathcal{L} \mid f^{-1}(U_\alpha)$ is ample relative to U_α for all α.*

Proof. This is E.G.A. II, Corollary 4.6.4.

COROLLARY 2.6. *Let $f\colon X \to Y$ be a proper morphism, Y a noetherian prescheme, $\mathcal{L}$ an invertible sheaf on X. Then $\mathcal{L}$ is ample over Y if and only if its restriction to each fiber is ample.*

Proof. This follows immediately from Propositions 2.1 and 2.5.

We note that in the above situation, some multiple of $\mathcal{L}$ will be very ample over Y, and hence f will be a propjective morphism.

PROPOSITION 2.7. *Let $f\colon X \to Y$ be a proper morphism of noetherian preschemes. Then the set V of points y of Y such that the fiber $f^{-1}(y)$ is smooth over* $\operatorname{Spec}\kappa(y)$ *is an open subset of Y.*

Prof. By E.G.A. IV, Corollary 6.8.7, the set U of points $x \in X$ such that f is smooth at x is an open subset of X. Let $y \in V$. Then $f^{-1}(y) \subset U$, so $f(X - f^{-1}(y)) \supset f(X - U)$ which is a closed set which clearly contains all points y in Y such that $f^{-1}(y)$ is not smooth over $\kappa(y)$. Hence $y \in Y - f(X - U)$, which is an open subset of Y contained in V. So V is open.

THEOREM 2.8. *Let A be a Dedekind domain, X a connected, regular prescheme, proper and flat over* $\operatorname{Spec} A$, *such that the fibers of X over* $\operatorname{Spec} A$ *are algebraic curves. Then X is projective over A.*

Proof. We first prove the theorem under the assumption that A is local. Let $F_1, \cdots, F_n$ be the reduced components of the special fiber F of X. Let

G be the disjoint sum of the normalizations $\bar{F}_i$ of the F_i's. Let g be the induced map from G to F and let $\mathscr{L}$ be an invertible sheaf on F. Then Propositions 2.2 and 2.3 imply that if $g^*\mathscr{L}$ is ample on G then $\mathscr{L}$ is ample on F.

An invertible sheaf $\mathscr{L}$ is ample on G if and only if the associated divisor class is of positive degree on each $\bar{F}_i$. (This is well-known if the ground field is algebraically closed, and the general case may be easily deduced from this by applying Propositions 2.3 and 2.4.) So our problem reduces to finding a positive divisor D on X, none of whose components lie in the special fiber, which meets every component of the special fiber.

Let P be a point on one component of the special fiber, and let O_P be the local ring at P on X. Let t be a uniformizing parameter for the local ring A. Then it is easy to construct a prime ideal in O_P which does not contain t. This corresponds to a prime divisor D_P on X which passes through P. Taking a point P on each component of the special fiber, and taking the sum of the D_P's, we obtain a divisor D which corresponds to an invertible sheaf $\mathscr{L}$ (since X is regular) which is ample on the special fiber. So $\mathscr{L}$ is ample on X by Proposition 2.1, and hence X is projective over $\operatorname{Spec} A$.

We now pass to the general case. Let Q be any point in $\operatorname{Spec} A$. Then the above argument shows that there exists a divisor D on X such that the corresponding invertible sheaf $\mathscr{L}$ is ample on the fiber over Q. Again by Proposition 2.1, there exists an open set U of $\operatorname{Spec} A$ such that $\mathscr{L}$ is ample on $f^{-1}(U)$. Since A is one-dimensional, the complement of U is a finite set of points $P_1 \cdots P_n$. By the argument used in proving the local case, there exist positive divisors $D_1, \cdots, D_n$, which have no components in any fiber, such that the invertible sheaf $\mathscr{L}_i$ which corresponds to D_i is ample on the fibre over P_i. If we now let $D' = D + \sum_{i=1}^{n} D_i$, then the invertible sheaf corresponding to D' is ample on each fiber, so is ample on X by Corollary 2.6. So X is projective over Y.

3. On the additivity formula for Euler characteristics. We begin with some motivation. Let F be a non-singular surface over an algebraically closed field k. Let D be a divisor on F, and let K be the canonical class on F. Then the Riemann-Roch theorem for surfaces asserts that

$$\chi(\mathscr{L}(D)) = \chi(O_F) + \tfrac{1}{2} D \cdot (D - K).$$

In this form, the formula does not admit a generalization to our situation, but we can generalize the formula obtained from this by "taking first differences" as follows:

Applying the above formula to $D = -E_1$ and $D = -E_1 - E_2$ successively, where E_1, E_2 are positive divisors, and subtracting, we obtain

$$\chi(O_{E_1+E_2}) - \chi(O_{E_2}) = \chi(O_{E_1}) - (E_1 \cdot E_2), \text{ or}$$
$$\chi(O_{E_1+E_2}) = \chi(O_{E_2}) + \chi(O_{E_1}) - (E_1 \cdot E_2).$$

This formula can now be generalized to curves over discrete valuation rings.

We remark that although this "first difference form" of the Riemann-Roch theorem becomes much easier to prove in the geometric case, we are still forced to use a fairly involved argument for our proof.

Definition 3.1. Let A be a discrete valuation ring, (X, f) a complete regular curve over $Y = \operatorname{Spec} A$. Let F be a coherent sheaf on X with support in the closed fibre. Since f is proper, $H^0(X, F)$ and $H^1(X, F)$ are A-modules of finite type, and since the support of F is contained in the closed fiber, they are annihilated by a sufficiently high power of the maximal ideal of A. Hence they are A-modules of finite length. We define the *Euler characteristic* $\chi(F)$ by

$$\chi(F) = \operatorname{length} H^0(X, F) - \operatorname{length} H^1(X, F).$$

Let E be a positive divisor on X with support in the closed fiber. Then we define the Euler characteristic $\chi(E)$ to be $\chi(O_E)$. We are now in a position to state our theorem:

THEOREM 3.2. *Let ϕ be the map from the semi-group of positive divisors with support in the closed fiber to the integers defined by* $\phi(E) = 2\chi(E) + (E \cdot E)$. *Then ϕ is a homomorphism, i.e.,*

$$\chi(E_1 + E_2) = \chi(E_1) + \chi(E_2) - (E_1 \cdot E_2).$$

We will show in a subsequent paper that $\phi(E)$ actually equals $-(E \cdot K)$, where K is the relative canonical class of the morphism f.

Proof. By Theorem 2.8, X is projective over Y. Let $\mathcal{L}$ be a sheaf on X which is very ample with respect to Y. If F is a sheaf of O_X-modules, we define $F(n)$ to be $F \otimes \mathcal{L}^{\otimes n}$. Let E_1 and E_2 be positive divisors with support in the closed fiber. We want to prove $\phi(E_1 + E_2) = \phi(E_1) + \phi(E_2)$ and it is clearly sufficient to prove this for E_2 irreducible. Let I be the sheaf of ideals corresponding to E_1 and J the sheaf of ideals corresponding to $E_1 + E_2$. Let H be a divisor such that $\mathcal{L}(H) \cong \mathcal{L}$. We know of course that J is included in I. Let

$$P(n) = 2\chi(I/J(n)) + (E_1 \cdot E_1) + 2(E_1 \cdot E_2) + 2n(H \cdot E_1) - \phi(E_1).$$

We claim:

1) P is a polynomial in n, for all n.

2) $P(n)=0$ for n large.

We first show how 1) and 2) imply the theorem. Certainly 1) and 2) imply $P(n)=0$ for all n. So let $n=0$. Then we have

$$2\chi(I/J)+(E_1\cdot E_1)+2(E_1\cdot E_2)-\phi(E_1)=0.$$

From the exact sequence $0\to I/J\to O_X/J\to O_X/I\to 0$, we have $\chi(E_1+E_2)=\chi(I/J)+\chi(E_2)$. It immediately follows that $\phi(E_1+E_2)=\phi(E_1)+\phi(E_2)$.

So it suffices to prove 1) and 2). To prove 1) we only have to show that $\chi(I/J(n))$ is a polynomial in n. But I/J is a sheaf with support in the closed fiber. It has a composition series whose factors are all sheaves on a prescheme proper over a field. Then $\chi(I/J(n))$ is the Hilbert polynomial of I/J, so we are done.

In order to prove 2) we first prove

LEMMA 3.3. *Let E_1 be a positive divisor on X with support in the closed fiber. Let E_2 be a positive divisor having no common components with E_1. Let I be the sheaf of ideals corresponding to E_2, and J the sheaf of ideals corresponding to E_1+E_2. Then $\chi(I/J)=\chi(E_1)-(E_1\cdot E_2)$.*

Proof. Let K be the sheaf of ideals corresponding to E_1. We have the exact sequence $0\to(I+K)/K\to O_X/K\to O_X/I+K\to 0$. By looking locally, it is immediate that $\chi(O_X/(I+K)=\dim_k H^0(X,O_X/(I+K))=(E_1\cdot E_2)$. On the other hand, $(I+K)/K\cong I(K\cap I)\cong I/KI=I/J$, since X is regular and E_1 and E_2 have no common components. The result follows by taking Euler characteristics of the above exact sequence.

Now to complete the proof of 2) we need only show that, given E_2 irreducible with support in the closed fiber and E_1, for large n there exists a positive divisor D_n such that $D_n-(E_1+nH)$ is a principal divisor and D_n does not have E_2 as a component. To see this let I_n be the sheaf of ideals corresponding to D_n and J_n the sheaf of ideals corresponding to E_2+D_n. Then $I_n/J_n=I(n)/J(n)\cong I/J(n)$ and Lemma 3.3 says that $\chi(I_n/J_n)=\chi(E_2)-(D_n\cdot E_2)$. So

$$P(n)=2\chi(E_2)-2(E_1+nH\cdot E_2)+(E_2\cdot E_2)+2(E_2\cdot E_1)-2n(E_2\cdot H)-\phi(E_2)$$
$$=2\chi(E_2)+(E_2\cdot E_2)-\phi(E_2)=0.$$

So we are reduced to finding a D_n with the desired properties. We know (E.G.A., Chapter II, Corollary 4.6.12) that the invertible sheaf corresponding to $E_1 + nH$ is very ample for large n, and thus defines an imbedding of X into some projective space over A. We now have only to prove the following lemma:

LEMMA 3.5. *Let Z be a closed irreducible subscheme of projective n-space P over $Y = \operatorname{Spec} A$, A a discrete valuation ring. Assume further that $|Z|$ is contained in the closed fiber. Let $O_P(1)$ be the canonical sheaf. Then there exists a divisor D in the equivalence class determined by $O_P(1)$ such that $|D| \not\supset |Z|$.*

Proof. Since $|Z|$ is contained in the closed fiber, it is sufficient to show that there is a hyperplane section of projective space over a field k not containing any given non-empty closed subscheme. But the hyperplane sections correspond to linear forms in the graded ring $k[X_0, \cdots, X_n]$. But any homogeneous ideal which contains every linear form must be the irrelevant ideal.

II. A. The factorization theorem. In this section we prove the analogue for general regular surfaces of the factorization theorem for non-singular surfaces over an algegraically closed field. We repeat:

Definition 1.1. A *surface* X is an integral prescheme such that

a) For every point x on X, $\dim O_{X,x} \leqq 2$.

b) There exists at least one point x on X such that $\dim O_{X,x} = 2$.

In fact, b) will never be used, but it assures us that our propositions are not vacuous.

Definition 1.2. Let X be a regular surface. Let P be a closed point of X such that $\dim O_{X,P} = 2$. Let I be the ideal of O_X which defines the closed subscheme $(P, \operatorname{Spec} \kappa(P))$. Let $X' = \operatorname{Proj}(\sum_{n \geqq 0} I^n)$, and let $\pi \colon X' \to X$ be the natural morphism. Then we say that π is the *locally quadratic transformation* of X with center P, and that X' is the *locally quadratic transform* of X with center P (X' is the "X-scheme obtained by blowing up $(P, \operatorname{Spec} \kappa(P))$" in E.G.A., II, Definition 8.1.3).

Proposition 1.3. Let X, X', π be as above. Then π is projective, birational, surjective, and induces an isomorphism of $X' - \pi^{-1}(|P|)$ onto $X' - P$.

Proof. This is contained in the discussion following Definition 8.1.3 and in Proposition 8.1.4 of E.G.A., II.

PROPOSITION 1.4. *Let X', X, π be as above. Let x and y be generators for the maximal ideal of $O_{X,P}$. Let U be an affine open subset of X containing P, such that x and y generate the maximal ideal of P in $\Gamma(U, O_U) = A$. Let X'' be the open subset of X equal to $\pi^{-1}(U)$. Then X'' is covered by two affine open sets $U_1 = \operatorname{Spec} A[x/y]$ and $U_2 = \operatorname{Spec} A[y/x]$.*

Proof. This follows directly from the definition of $\operatorname{Proj}(\Sigma I^n)$.

COROLLARY 1.5. *Let X', X, π be as above. Then X' is a regular surface.*

Proof. In view of Propositions 1.3 and 1.4, it is sufficient to prove that if $\operatorname{Spec} A$ is a regular surface, and x and y are a minimal set of generators for the maximal ideal of a closed point of A, then $\operatorname{Spec} A[x/y]$ is a regular surface. Let $B = A[x/y]$, $\mathfrak{Y}$ be a prime ideal of B, and $C = B_{\mathfrak{Y}}$. If y is not in $\mathfrak{Y}$, then C is isomorphic to $A_{\mathfrak{Y} \cap A}$, and so is regular. It is easy to see that B/yB is isomorphic to a polynomial ring in one variable over $k = A/(x, y)$. Hence, if y is in $\mathfrak{Y}$, C is an integral domain with the property that there exeists an element y in C such that C/yC is a regular local ring of dimension $\leqq 1$. Therefore C is a regular local ring of dimension $\leqq 2$.

Definition 1.6. Let A be a regular two-dimensional local ring with maximal ideal $m = (x, y)$. Let v be a valuation of the quotient field K of A having center m in A. Assume $v(y) \geqq v(x)$. Let $B = A[y/x]$. Let $J = B \cap M_v$ where $M_v = \{x \in K : v(x) > 0\} \cup \{0\}$. Let $A' = B_J$, $m' = JA'$. Then A' is the *first quadratic transform of A along v.* Assume that the n-th quadratic transform A_n of A along v has been defined and has dimension two. Then we define the *$(n+1)$-st quadratic transform of A along v* to be the first quadratic transform of A_n along v. If B is the n-th quadratic transform of A along v for some n, we say that B is a *quadratic transform of A along v.* If there exists a v such that B is a qadratic transform of A along v, we say that B is a *quadratic transform of A.*

THEOREM 1.7. (Zariski-Abhyankar). a) *Let A be a two-dimensional regular local ring and let B be a two-dimensional regular local ring with the same quotient field as A and which dominates A. Then B is a quadratic transform of A.*

b) *Let $A_0 < A_1 < A_2 \cdots$ be strictly ascending sequence of two-dimensional regular local domains with a common quotient field K. Let $B = \bigcup_{i=0} A_i$.*

Assume that A_{i+1} is a quadratic transform of A_i for $i = 0, 1, 2, \cdots$. Then B is the valuation ring of a valuation v of K which dominates each A_i.

Proof. These are Theorem 3 and Lemma 12 of [1].

We now recall the following definitions and results from the usual source (E.G.A., I, Sections 6 and 7):

Definition 1.8. Let X and Y be two preschemes over a base prescheme S. Let U and V be two open subsets of X and $f\colon U \to Y$ and $g\colon V \to Y$ be two S-morphisms. We say that f is *equivalent* to g if f and g agree on a dense open subset of $U \cap V$.

Definition 1.9. An *S-rational map* from X to Y is an equivalence class of morphisms from open dense subsets of X to Y, with the above equivalence relation.

For convenience, we shall use the words "rational map" rather than "rational morphism." Thus, if X and Y are integral preschemes, a birational *map* from X to Y is a rational map which induces an isomorphism on the fields of functions, while a birational *morphism* is a birational map which is everywhere defined, i.e. a morphism.

Definition 1.10. Let X and Y be preschemes, f a rational map from X to Y. We say that f is *defined at a point x of X* if there exists a dense open set U of X containing x and a morphism mapping U to Y belonging to the equivalence class of f. The set U_0 of all points x in X where f is defined is called the *domain of definition of f*.

PROPOSITION 1.11. *Let X and Y be two S-preschemes. Assume that X is reduced and Y is separated over S. Let f be an S-rational map from X to Y, U_0 its domain of definition. Then there exists a unique S-morphism $g\colon U_0 \to Y$ belonging to the equivalence class of f.*

Proof. This is Proposition 7.2.2 of E.G.A., I.

PROPOSITION 1.12. *Let X and Y be two S-preschemes, Y of finite type over S. Let x in X and y in Y be over the same point s in S.*

(i) *If two S-morphisms $f = (\psi, \theta)$, $f' = (\psi', '\theta)$ of X in Y are such that $\psi(x) = \psi'(x) = y$, and the O_S-homomorphisms θ^* and θ'^* of O_y in O_x are identical, f and f' coincide in an open neighborhood of x.*

(ii) *For every local O_S-homomorphism $\phi\colon O_y \to O_x$, there exists an open neighborhood U of x in X and an S-morphism $f = (\psi, \theta)$ of U in Y such that $\psi(x) = y$ and $\theta^* = \phi$.*

Proof. This is Proposition 6. 5. 1 of E.G.A., I.

COROLLARY 1.13. *Let X and Y be two integral S-preschemes, Y separated over S and f an S-birational map from X to Y. We identify $K(X)$ with $K(Y)$ by means of f. Then f is defined at a point x of X iff there exists a point y in Y such that $O_y \subseteq O_x$ and $m_y \subseteq m_x$.*

Proof. It is clear that if f is defined at x, then such a y exists, namely $y = g(x)$, where g is in the class of f. To go the other way, let ϕ be the inclusion of O_y in O_x. By Proposition 1.12(ii), there exists an open neighborhood U of x and an S-morphism $\psi\colon U \to Y$ such that ψ induces ϕ. Let g be some map in the class of f. Let z be the generic point of X. Then it is clear that $g(z) = \psi(z)$ and that ψ and g induce the same map of $O_{f(z)}$ to O_z. By Proposition 1.12(i), g and ψ agree in some open neighborhood of z, i.e. since X is irreducible, ψ is in the class of f. So f is defined at x.

PROPOSITION 1.14. *Let X and Y be two S-preschemes. Assume that Y is normal, and X proper over S. Let f be an S-rational map from Y to X, and let Y' be the closed set of points of Y where f is not defined. Then* $\operatorname{codim}(Y') \geqq 2$, *that is,* $\dim O_{Y,y} \geqq 2$ *for every y in Y'.*

Proof. This is an immediate corollary of 7.3.5, E.G.A., II.

THEOREM 1.15. (The Factorization Theorem). *Let F and F' be regular surfaces. Let $f\colon F' \to F$ be a proper birational morphism. Then F is isomorphic to a prescheme obtained from F' by a finite number of successive locally quadratic transformations.*

Proof. We first prove the following key lemma:

LEMMA 1.16. *Let F, G, H be three regular surfaces and P a point of H such that* $\dim O_{H,P} = 2$. *Assume*

1) *There exists a proper birational morphism $f\colon F \to H$.*
2) *f is not biregular at P.*
3) *$g\colon G \to H$ is the locally quadratic transformation of H with center P.*

Then there exists a proper birational morphism $f'\colon F \to G$ such that $g \circ f' = f$.

Proof of Lemma. Let f'' be the H-rational map of F into G determined by f and g. It is clear that f'' induces an isomorphism of $K(G)$ to $K(F)$. Let R be a point of F. Since G is separated over H, Corollary 1.13 applies, and if we can show that there exists a point Q of G such that O_Q is dominated by O_R, f'' will be defined at R.

1) Assume that $f(R) \neq P$. Then we may take $Q = g^{-1}f(R)$.

2) If $f(R) = P$, then either $\dim O_{F,R} = 1$ or $\dim O_{F,R} = 2$. If $\dim O_{F,R} = 1$, then $O_{F,R}$ is a discrete valuation ring and by the valuative criterion of properness, it must dominate some point of H. (Corollary 7.3.10, E.G.A., II). If $\dim O_{F,R} = 2$ then Theorem 1.7 applies, and $O_{F,R}$ is an n-th quadratic transform of $O_{H,P}$ along some valuation v. Since f is proper, it is separated, and it therefore easily follows that $n \geqq 1$. But then by the definition of G, the first quadratic transform of $O_{H,P}$ along v is the local ring $O_{G,Q}$ of a point Q of F and $O_{F,R}$ dominates $O_{G,Q}$. Hence f'' is defined at every point P of F. By Proposition 1.11, there exists an H-morphism $f'\colon F \to G$, which induces f'', and it is clear that $g \circ f' = f$. Since f is proper and g is separated, f' is proper, and f' is clearly birational. So we have completed the proof of the lemma.

In order to complete the proof, it is sufficient to prove the following lemma:

LEMMA 1.17. *Let F, G be two regular surfaces and f a proper birational morphism from F to G. We define a sequence of regular surfaces G_n and rational maps $g_n\colon G_n \to F$ as follows: Let $G_0 = G$ and g_0 be the rational map induced by f. Assume that g_{n-1} is not defined at some point P_{n-1} of G_{n-1}. Let G_n be the locally quadratic transform of G_{n-1} with center P_{n-1} and g_n the rational map induced by g_{n-1}. Then this sequence is necessarily finite, i.e. for some n, g_n is a morphism.*

Proof. Assume that an infinite sequence existed. By Proposition 1.14 the set of points of G_n where g_n is not defined is finite, since G is a normal surface. It is easy to see then that there must exist a subsequence Q_n of the P_n, such that for all $n \geqq 0$, the local ring O_{n+1} of Q_{n+1} is a quadratic transform of O_n. By Theorem 1.7 $\bigcup_{n=0} O_n$ is a valuation ring B of the common field of functions K of F and the G_n. By the valuative criterion of properness, B dominates the local ring A of some point R on F. Since A is the localization of a ring of finite type over O_0, there exists an n such that O_n dominates A. By Corollary 1.13, g_n is defined at Q_n, which is a contradiction.

It follows immediately from Lemmas 1.16 and 1.17 that F is isomorphic to a surface obtained from G by a finite number of locally quadratic transformations.

B. Necessary lemmas on birational transformations. In this section we prove some basic lemmas about proper birational morphisms of surfaces which will be necessary for the proof of Castelnuovo's criterion for exceptional curves. In particular, we will show that if a curve E is exceptional, then it has arithmetic genus zero and self-intersection minus-one. We will proceed as far as possible without using the factorization theorem, but its use will eventually become necessary. Throughout this section, let X and Y be regular surfaces, and $f\colon X \to Y$ a proper, birational morphism.

PROPOSITION 2.1. *Let $\mathcal{L}$ be an invertible sheaf of ideals on Y, corresponding to a prime divisor C. So we have $0 \to \mathcal{L} \to O_Y \to O_C \to 0$. Then the sequence $0 \to (f^*\mathcal{L}) \to O_X \to f^*(O_C) \to 0$ is exact, and if g is a local equation for C at a point y in Y, then g is also a local equation for the subscheme C' of O_X defined by $f^*\mathcal{L}$ at any point x of X such that $f(x) = y$.*

Proof. The question is clearly local, so let x be in X such that $f(x) = y$. By definition of a local equation, we have $0 \to O_y \xrightarrow{g} O_y \to O_{c,y} \to 0$. Tensoring with O_x, we obtain $O_x \xrightarrow{g} O_x \to O_{c',x} \to 0$, and since O_x is a domain and $g \neq 0$, we are done.

Definition 2.2. Let C be a prime divisor on Y. Let $\mathcal{L}$ be the invertible sheaf of ideals corresponding to C. We define the *total transform* $f^{-1}(C)$ of C to be the divisor associated to $f^*\mathcal{L}$. We extend the definition of total transform to arbitrary divisors on Y by linearity.

Remark. It is clear from Proposition 2.1 that this definition agrees in the geometric case with the usual one, as defined for instance in Zariski [4], p. 70. It is also clear that if $C_1 \equiv C_2$, $f^{-1}(C_1) \equiv f^{-1}(C_2)$.

Definition 2.3. Let C be a prime divisor on Y. By Proposition 1.14, the rational map f^{-1} is defined and an isomorphism on the generic point y of C. We define the proper transform $f^{-1}[C]$ to be the prime divisor $\overline{f^{-1}(y)}$, where the bar denotes closure in the Zariski topology of X, and we take the induced reduced structure. We extend the definition of proper transform to arbitrary divisors by linearity.

Assume from now on that f is an isomorphism outside of a prime divisor E and that $f(E)$ is a point P of Y such that $\dim O_P = 2$. We say that E is an *exceptional curve* on X. Then we have

PROPOSITION 2.4. *Let D be a divisor on Y. Let v be the discrete valuation defined by E, and g a local equation for D at P. Then $f^{-1}(D) = f^{-1}[D] + v(g)E$.*

Proof. It is clear that $f^{-1}[D] = f^{-1}[D] + mE$, and that $m = v(g)$ follows from Proposition 2.1.

PROPOSITION 2.5. *Total transform preserves intersection multiplicities. More precisely, let D be a divisor on Y, and C a divisor on Y such that for every component F of C, $H^0(F, O_F)$ is a field and F is a complete algebraic curve over $H^0(F, O_F)$. Then $f^{-1}(C)$ has the same property, and $i_k(D, C) = i_k(f^{-1}(D), f^{-1}(C))$, where k is any field of finite index in all the fields $H^0(F, O_F)$. (For instance, if X and Y are surfaces over a ground field k', or if $X$and Y are curves over a discrete vaduation ring A with residue field k', we may take $k = k'$. Or, if C is prime, we may take $k = H^0(C, O_C)$.)*

Proof. The first statement follows immediately from the fact that f is proper. For the second, we may assume that D does not contain P by replacing it with an equivalent divisor, if necessary. But then the result follows from the definition of i_k and Proposition 2.1.

PROPOSITION 2.6. *Let D be a divisor on Y. Let $k = \kappa(P)$. Then $i_k(f^{-1}(D), E) = 0$.*

Proof. We may assume that D does not contain P, so that $f^{-1}(D)$ neither meets E nor has E as a component.

PROPOSITION 2.7. *Let k be the residue field of the point P, and D a prime divisor on Y which has a regular point at P. Then $i_k(f^{-1}[D], E) = 1$.*

Proof. We now use the factorization theorem for the first time. Since E is irreducible, the map f must be a locally quadratic transformation with center P. Let D' be the prime divisor of $f^{-1}[D]$. Since D has a regular point at P, the induced map from D' to D must be an isomorphism. In particular, D' can meet only one point Q of E. Let g be a local equation for D at P. Since D is singular at P, we may take (g, y) to be a system of regular parameters for P. Let $A = O_{Y,P}$. Since the local rings of E on X are just $\operatorname{Spec} A[g/y, y] \cup \operatorname{Spec} A[y/g, g]$, we see that the local ring of the point Q is $A[g/y, y]_m$, where $m = (g/y, y)$. It is clear that y is a local equation for E, and hence by Proposition 2.4 that g/y is a local equation for D'. By the definition of intersection multiplicity, $i_k(D', E) = [\kappa(Q) : k]$, but it is also clear that $O_{X,Q}/m_{X,Q} = \kappa(Q) = k$, so we are done.

PROPOSITION 2.8. *Let E be an exceptional curve on X. Then $E^{(2)} = -1$.*

Proof. It is clear that there exists a D having a simple point at P. Then by Proposition 2.6, $i_k(f^{-1}(D), E) = 0$. But $f^{-1}(D) = f^{-1}[D] + E$, so

$i_k(E,E) = i_k(f^{-1}[D],E) = -1$ by Proposition 2.7. Since $k = H^0(E,O_E)$, $E^{(2)} = -1$.

PROPOSITION 2.9. *Let E be an exceptional curve on X. Then $H^1(E,O_E) = 0$.*

Proof. It is immediate that E is a complete curve over $\kappa(P)$, which is covered by two neighborhoods isomorphic to the affine line over $\kappa(P)$, and hence must be the projective line over $\kappa(P)$. It follows that $H^1(E,O_E) = 0$.

PROPOSITION 2.10. *Let C' be a complete integral algebraic curve on X over $k' = H^0(C',O_{C'})$. Assume that $C' \neq E$, so that $f(C') = C$, a prime divisor on Y. Let $H^0(C,O_C) = k$. Then*

a) $C^{(2)} = i_k(C,C) \geqq i_k(C',C') = [k':k]i_{k'}(C',C') = [k':k]C'^{(2)}$, *with equality if and only if C does not contain P.*

b) *If C does contain P, then $C^{(2)} \geqq [k':k](C'^{(2)}+1)$, with equality if and only if C has a regular point at P, in which case $k = k'$.*

Proof. We have $f^{-1}(C) = C' + mE$, where $m \geqq 0$, and $m = 0$ if and only if C does not contain P. Hence $C^2 = i_k(C,C) = i_k(C'+mE, C'+mE) = i_k(C'+mE, C') = i_k(C',C') + mi_k(E,C')$. Since $i_k(E,C') \geqq 0$, with equality if and only if C does not meet P, we have proved a).

By looking at $i_k(E,C')$ as the degree of an invertible sheaf on C', we see that $i_k(E,C')$ is divisible by $[k':k]$. Hence we have the inequality in b), and the equality holds if and only if $m = 1$ and $i_k(E,C') = [k':k]$. C has a regular point at P exactly when the local equation g of C is not in the square of the maximal ideal at P, which is equivalent to saying that $m = 1$. By Proposition 2.7, if C has a regular point at P, $i_k(E,C') = 1$. So we have proved b).

C. Castelnuovo's Criterion. This section is devoted to proving Castelnuovo's Criterion for curves over discrete valuation rings. Presumably it holds for an arbitrary surface, and so we do as much as we can in this generality, but we need an ample sheaf to obtain the contraction map, and so make use of the previous results on projective embeddings of curves over valuation rings. The proof of the existence of a map is then a modification of the proof given in [2] for the case of surfaces over algebraically closed fields, and the proof that the image point is regular, which is valid for an arbitrary surface, is an interesting application of Grothendieck's theorem on holomorphic functions. We start with some lemmas necessary to establish the existence of a map.

LEMMA 3.1. *Let X be a regular surface. Let E be a prime divisor of X such that $k=H^0(E,O_E)$ is a field and E is isomorphic to the projective line over k. Assume also that $E^{(2)}<0$. Let I be the ideal defining E in X. Let Z_n be the closed subscheme of X defined by I^n. Then $H^1(Z_n)=0$, and the natural map from $H^0(Z_n)$ to $H^0(Z_{n-1})$ is surjective. (We abbreviate $H^i(Y,O_Y)$ by $H^i(Y)$.)*

Proof. We prove our result by induction on n. For $n=1$ we know $H^1(E,O_E)=0$. We have the exact sequence

$$0\to I^{n-1}/I^n\to O_X/I^n\to O_X/I^{n-1}\to 0.$$

By the exact sequence of cohomology and the induction hypothesis, it is sufficient to show that $H^1(O_E,I^{n-1}/I^n)=0$. By Definition I.1.4. the degree of I^{n-1}/I^n is equal to $-(n-1)E^{(2)}\geqq 0$ if $n\geqq 1$. Since E is isomorphic to the projective line over k, the result follows.

LEMMA 3.2. *Let Z be any Z_n as above. Then* $\operatorname{Pic}(Z)=H^1(Z,O_Z{}^*)$ *is isomorphic to the integers.*

Proof. By the above lemma, it is clear that $H^0(Z,O_Z)\to H^0(E,O_E)$ is surjective. It follows readily that we also have a surjection from $H^0(Z,O_Z{}^*)$ $\to H^0(E,O_E{}^*)$. Hence we have exact sequences

$$0\to H^1(N)\to H^1(O_Z)\to H^1(O_E)\to 0$$

and

$$0\to H^1(M)\to H^1(O_Z{}^*)\to H^1(O_E{}^*)\to 0,$$

where N and M are the kernels of the maps from O_Z to O_E and from $O_Z{}^*$ to $O_E{}^*$ respectively. To complete the proof, it is only necessary to recall the following lemma of M. Artin [2], p. 486, and to observe that the proof given in [2] of this lemma is valid without changing a word:

LEMMA 3.3. *There exist chains of subgroups*

$$H^1(N)=N_0\supseteq N_1\supseteq\cdots\supseteq N_m=0,$$
$$H^1(M)=M_0\supseteq M_1\supseteq\cdots\supseteq M_m=0$$

such that N_{j-1}/N_j is isomorphic to M_{j-1}/M_j for $j=1,\cdots,n$.

LEMMA 3.4. *Let X be a regular surface. Let E be a prime divisor of X such that $k=H^0(E,O_E)$ is a field, E is a complete algebraic curve over k, $H^1(E,O_E)=0$ and $E^{(2)}=-1$. Then E is isomorphic to the projective line over k.*

Proof. By definition of intersection multiplicity, $E^{(2)} = -1$ implies that E has an invertible sheaf F of degree 1. By the Riemann-Roch theorem on E, we conclude that $\dim_k H^0(F) \geqq 1$. Hence there exists a non-trivial map of $O_E \to F$, which must be an injection. Tensoring with F^{-1}, we obtain $0 \to F^{-1} \to O_E \to Q \to 0$. Again by the Riemann-Roch theorem, we obtain $\dim_k(Q) = 1$. Hence the support of Q is a single point P on E, $\kappa(P) = k$, and $Q = \kappa(P)$. So we see that P is a non-singular point of E, rational over k. But the existence of such a point implies that k is algebraically closed in the field of functions $K(E)$ of E, (x in $K(E)$, integral over $k \Rightarrow x$ integral over $O_P \Rightarrow x \in O_P \Rightarrow x \in O_P/mO_P = k$).

Now we claim that E is non-singular. Let $O = O_E$ and $\bar{O} =$ the integral closure of O in $K(E)$, considered as a sheaf on E. So we have $0 \to O \to \bar{O} \to \bar{O}/O \to 0$. Taking homology, we have

$$0 \to H^0(O) \to H^0(\bar{O}) \to H^0(\bar{O}/O) \to H^1(O) = 0.$$

Since $k = H^0(O)$ is algebraically closed in $K(E)$, we have $H^0(O) \xrightarrow{\sim} H^0(\bar{O})$ so $H^0(\bar{O}/O) = 0$. Since $\bar{O}/O$ is concentrated on a finite number of closed points, we have $\bar{O}/O = 0$, i.e. $O = \bar{O}$, and E is non-singular.

But now, since E is a non-singular complete curve over k with a rational point and the genus of E is zero, it is well-known that E is isomorphic to the projective line over k.

LEMMA 3.5. *Let X be a regular surface. Let E be a prime divisor of X such that $k = H^0(E, O_E)$ is a field, E is a complete algebraic curve over k, $E^{(2)} = -1$ and $H^1(E, O_E) = 0$. Let I be the ideal defining E in X. Let $Z_n = \operatorname{Spec} O_X/I^n$. Then the dimension over k of the kernel of the map from $H^0(Z_2)$ to $H^0(E)$ is equal to 2.*

Proof. It is clear that this kernel is just $H^0(E, I/I^2)$. Since E is the projective line over k, and I/I^2 is an invertible sheaf of degree 1 on E, it follows that $(\dim_k(H^0(E, I/I^2))) = 2$, by using the Riemann-Roch Theorem for instance.

LEMMA 3.6. *Let X, E, k, I be as above. Then $\sum_{n \geqq 0} H^0(E, I^n/I^{n+1})$ is a $Z_n = \underline{\operatorname{Spec}} O_X/I^n$. Then the dimension over k of the kernel of the map from polynomial ring over k in two variables, T_1 and T_2 where T_1 and T_2 are generators of $H^0(E, I/I^2)$ over k.*

Proof. I/I^2 is a sheaf of degree 1 on the projective line E, and therefore isomorphic to $O_E(I)$.

LEMMA 3.7. *Let $f\colon X \to Y$ be a morphism of preschemes. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two invertible sheaves on X which are very ample for f, then $\mathcal{L}_1 \otimes \mathcal{L}_2$ is very ample for f.*

Proof. E.G.A., II, Corollary 4.4.9.

LEMMA 3.8. *Let $f\colon X \to Y$ be a morphism of preschemes, and assume that Y is affine. Let $\mathcal{L}$ be an invertible sheaf on X. Let r be the rational map defined by $\mathcal{L}$. Then a sufficient condition for r to be everywhere defined is that for every x in X, there exists an element of $\Gamma(X, \mathcal{L})$ which is not zero at x, i.e. not in $m_x \mathcal{L}_x$.*

Proof. E.G.A., II, Corollary 3.7.4. (See E.G.A., II, Sections 3 and 4 for the definition and fundamental properties of the rational map associated with an invertible sheaf.)

THEOREM 3.9. (Castelnuovo's Criterion). *Let X be a complete regular curve over $Y = \operatorname{Spec} A$, A a discrete valuation ring with residue field k. Let E be a prime divisor of X such that* a) *E is contained in the closed fiber,* b) $H^1(E, O_E) = 0$, c) $E^{(2)} = -1$. *Then there exists a complete regular curve X' over Y, together with a proper birational morphism $\pi\colon X \to X'$ such that π is an isomorphism outside of E and $\pi(E)$ is a point of P of X'.*

Proof. We first observe that $H^0(E, O_E)$ is a field k', so by Lemma 3.4, E is isomorphic to the projective line over k'. By the results of Section I.1 we know that $f\colon X \to Y$ is a projective morphism, so let $\mathcal{L} = O_X(1)$ be an invertible sheaf on X which is very ample over Y. By Lemma 3.7, $\mathcal{L}(n) = \mathcal{L}^{\otimes(n+1)}$ is very ample for all $n > 0$, and by Serre's theorem, for large n we have $H^1(X, \mathcal{L}(n)) = 0$. Therefore we may assume that $H^1(X, \mathcal{L}) = 0$.

Let i be the immersion of E into X. Let $r = \text{degree}_{k'}(i^*\mathcal{L})$. Let $\mathcal{M} = \mathcal{L}(E)^{\otimes r}$, where $\mathcal{L}(E)$ is the invertible sheaf corresponding to E. Let $Z =$ the divisor rE. Since $E^{(2)} = -1$, we have $(\mathcal{L} \otimes \mathcal{M} \cdot E) = 0$, and therefore $(\mathcal{L} \otimes \mathcal{M} \cdot Z) = 0$. Since Z satisfies the hypotheses of Lemma 3.2, we see that the Picard group of Z is isomorphic to the integers. Letting j be the natural injection of Z into X, we see that $j^*(\mathcal{L} \otimes \mathcal{M}) \cong O_Z$. If we tensor the exact sequences of sheaves

$$0 \to \mathcal{M}^{-1} \to O_X \to O_Z \to 0$$

with $\mathcal{L} \otimes \mathcal{M}$, we obtain

$$0 \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{M} \to j^*(\mathcal{L} \otimes \mathcal{M}) = O_Z \to 0.$$

Since $H^1(X, \mathcal{L}) = 0$, we have a surjection from $H^0(X, \mathcal{L} \otimes \mathcal{M})$ to $H^0(Z, O_Z)$.

I now claim that the rational map defined by the invertible sheaf $\mathcal{L} \otimes \mathcal{M}$ is regular, i.e. defined everywhere. Since $\mathcal{L} \otimes \mathcal{M}$ is isomorphic to $\mathcal{L}$ outside of $|Z|$, it is clear that the rational map is defined and in fact biregular outside of $|Z|$. We may take an element in $H^0(X, \mathcal{L} \otimes \mathcal{M})$ whose image is the unit element of $H^0(Z, O_Z)$ and hence is not zero at any point of $|Z|$. So, by Lemma 3.8, the rational map is defined on $|Z|$, as well. So the sheaf $\mathcal{L} \otimes \mathcal{M}$ defines a map $\pi\colon X \to X^*$, where X^* is contained in some projective space over Y, π is an isomorphism outside of $|Z|$, and $\mathcal{L} \otimes \mathcal{M} = \pi^*(O_{X^*}(1))$.

Since $(\mathcal{L} \otimes \mathcal{M} \cdot Z) = 0$, it follows that the image of $|Z|$ cannot be a curve Z^* in $\pi(X) = X^*$. (If it were, the inverse image of $O_{X^*}(1)$ on Z would be ample, and hence of positive degree.) Since $|Z|$ is connected, it follows that π maps $|Z|$ onto a point P'.

Since X is normal, the map π factors through the normalization X' of X^*, and hence X' contains only one point P lying above P'.

To complete the proof, it suffices to demonstrate the following theorem:

THEOREM 3.10. *Let X be a regular surface. Let Y be a normal integral surface and $f\colon X \to Y$ a proper birational morphism. Assume that there exists a prime divisor E of X such that the image of E is a point P of Y and f is an isomorphism outside of E. Assume that $k = H^0(E, O_E)$ is a field, and that E is a complete algebraic curve over k. Assume further that $E^{(2)} = -1$, and $H^1(E, O_E) = 0$. Then $O_{Y,P}$ is a regular local ring, i.e. Y is a regular surface.*

Proof. We first recall a corollary to Grothendieck's theorem on holomorphic functions:

THEOREM 3.11. *Let Y be a prescheme, $f\colon X \to Y$ a proper morphism, F a coherent O_X-module. Then for every y in Y, $f_*(F)$ is an O_Y-module of finite type, and we have a canonical topological isomorphism*

$$((f_*F)_y)^\wedge \xrightarrow{\sim} \varprojlim_n \Gamma(f^{-1}(y), F/m_y{}^nF).$$

(We recall that our preschemes are always assumed to be noetherian.)

Proof. This is Corollary 4.2.4 of E.G.A., III.

We next observe that given any ideal I on X such that $\mathrm{Supp}(O_X/I) = \mathrm{Supp}(O_X/m_y) = f^{-1}(y)$, then $\varprojlim_n \Gamma(f^{-1}(y), F/I^nF)$ is canonically isomorphic to $\varprojlim_n \Gamma(f^{-1}(y), F/m_y{}^nF)$, since both are isomorphic to $\varprojlim \Gamma(f^{-1}(y), F/JF)$, where the limit is taken over the directed set of all ideals J with $\operatorname{Supp}(O_X/J) = f^{-1}(y)$ partially ordered by inclusion. In particular we take I to be the ideal defining E, and we take $F = O_X$, and we obtain

$$(f_*O_X)_y \longrightarrow \varprojlim_n \Gamma(f^{-1}(y), O_X/I^nO_X).$$

Since Y is normal, it is easily seen by the valuative criterion of properness that $O_P = \cap O_{X,x}$ where the intersection is taken over all x in E. Hence $f_*O_X = O_Y$, and we obtain

$$\hat{O}_P \xrightarrow{\sim} \varprojlim_n \Gamma(f^{-1}(y), O_X/I^nO_X).$$

We wish to show that O_P is a regular local ring. Let $A_n = \Gamma(f^{-1}(y), O_X/I^nO_X)$, and let $A = \varprojlim A_n \cdot = \hat{O}_P$. We have shown (Lemmas 3.1 and 3.4) that the natural maps of A_n to A_{n-1} are surjective for all n. Hence the natural map of A onto A_n is surjective for all n. Let I be the kernel of the map of A onto A_n. Since $A_1 = H^0(E, O_E) = k$ is a field, we know that I_1 is the maximal ideal of A. It is clear that $I_1{}^n \subseteq I_n$, and we wish to show equality. In fact, it will be sufficient, as we shall see, to show that $I_1{}^2 = I_2$.

We shall show first by induction on m that $I_1{}^2 \equiv I_2 (\operatorname{mod} I_m)$. Since modulo I_{n+1}, $I_1 = \Gamma(X, I/I^{n+1})$ and $I_n = \Gamma(X, I^n/I^{n+1})$, it follows from Lemma 3.6 that $I_1{}^n \equiv I_n (\operatorname{mod} I_{n+1})$. So in particular, $I_1{}^2 \equiv I_2 (\operatorname{mod} I_3)$. In general, we have the exact sequence $0 \to I_m/I_{m+1} \to I_2/I_{m+1} \to I_2/I_m \to 0$. By the induction hypothesis it is sufficient to show that the image of $I_1{}^2$ contains I_m/I_{m+1}. But in fact the image of $I_1{}^m$ is equal to I_m/I_{m+1}, so we are done.

To complete the proof that $I_1{}^2 = I_2$, it is sufficient to show that the topology defined by the ideals I_n agrees with the topology defined by the ideals $I_1{}^n$. But by the theorem on holomorphic functions, the map

$$\varprojlim A/I_1{}^n \xrightarrow{\sim} \varprojlim A/I_n$$

is a topological isomorphism.

Since we know that A is a 2-dimensional local ring, to show that A is regular it is sufficient to show that $\dim_k I_1/I_1{}^2 = \dim_k I_1/I_2 = 2$. But this is just Lemma 3.5.

D. Relative minimal models. In this section we prove a theorem for curves over discrete valuation rings which is the analogue of Zariski's theorem on the existence of minimal models for surfaces. However, this theorem is by no means as deep as Zariski's; under our hypotheses, in fact, it is a relatively straightforward corollary of Castelnuovo's criterion.

Recall that a prime divisor E on a regular surface X is said to be *exceptional* if there exists a proper birational morphism $\pi: X \to Y$, such that π is an isomorphism outside of E and $\pi(E)$ is a point P of Y.

We first wish to draw the following corollary of Grothendieck's theorem:

PROPOSITION 4.1. *Let A be a discrete valuation ring, $S = \operatorname{Spec} A$, X a normal integral prescheme and $f: X \to S$ a proper, flat morphism. Assume that $K(S)$ is algebraically closed in $K(X)$. (It follows from the hypothesis that f is surjective.) Then the fibers of f are connected.*

Proof. From Grothendieck's connectedness theorem (E.G.A., III, Corollary 4.3.2), we know that it suffices to show that $f_*(O_X) = A$. Since X is flat over A, $f_*(O_X)$ is a torsion-free A-module and it suffices to show that the zero-dimensional cohomology group of the generic fiber is equal to $K(S)$. Since the generic fiber is normal and integral, however, its zero-dimensional cohomology group is equal to the algebraic closure of $K(S)$ in $K(X)$, so we are done.

PROPOSITION 4.2. *Let X and Y be regular surfaces proper over a base prescheme S, and assume that f is an S-birational map from X to Y. Then there exists a regular S-prescheme X' proper over X and S-birational to X, such that the induced S-rational map from X' to Y is defined on all of X', i.e. comes from an S-morphism.*

Proof. We will construct X' by applying to X a sequence of locally quadratic transformations. We know by Proposition 1.14 that f is defined except at a finite number of closed points. We define a sequence of S-preschemes X_n as follows: $X_0 = X$. If the rational map f_n induced by f on X_n is defined everywhere, we stop. If not, we pick a point P_n on X_n where f_n is not defined and we define X_{n+1} to be the locally quadratic transform of X_n with center P. It is easy to see that if this process does not terminate, we may assume that there exists a sequence of points P_n such that

a) P_n is on X_n,

b) P_{n+1} is a first quadratic transform of P_n,

c) f_n is not defined at P_n.

We know, by Theorem 1.7, that $\bigcup_{n=1}^{\infty} O_{P_n}$ is the valuation ring R_v of some valuation v of $K(X)=K(Y)$. It is clear that R_v dominates the local ring of P_0 on $X_0=X$, and hence the local ring of some point of S. By the valuative criterion for a proper morphism, it follows that R_v dominates the local ring $O_{Y,Q}$ of some point Q on Y. Since $O_{Y,Q}$ is the localization of an algebra of finite type over S, it follows that $O_{Y,Q}$ is dominated by some O_{P_n}, and hence, by Corollary 1.13, f_n is defined at P_n, which is a contradiction.

Remark. It is clear that the map $f_n\colon X_n \to Y$ is proper and S-birational.

PROPOSITION 4.3. *Let A be a discrete valuation ring, and $S=\operatorname{Spec} A$. Let X be a regular complete curve over S. Assume $K(S)$ is algebraically closed in $K(X)$. (It is clear that X is a regular surface, in the sense of Definition* 1.1). *Let W be the generic fiber of X and assume that $H^1(W,O_W)\neq 0$. Let P be a point of X (necessarily closed) such that* $\dim O_{X,P}=2$. *Let X' be the locally quadratic transform of X with center P. Let $f\colon X'\to X$ and $E=f^{-1}(P)$. Suppose that C' is an exceptional curve in the closed fiber of X'. Then either $C'=E$ or $f(C')$ is an exceptional curve in X, not containing P.*

Proof. If $C\neq E$, then $f(C')$ is a curve C in X which is birational to C'. We know, by Castelnuovo's criterion, that $H^1(C',O_{C'})=0$ and $C'^{(2)}=-1$. If P is not in C, then C is isomorphic to C' and $C^{(2)}=C'^{(2)}$, so the result is clear. So we may assume that C contains P. By Proposition 2.10 a), it follows that $C^{(2)}\geqq 0$. Since C is contained in the closed fiber which is connected by Proposition 4.1, this implies that the closed fiber is a multiple of C, and $C^{(2)}=0$. By Proposition 2.10 b), it follows that C has a simple point at P, and hence C is isomorphic to C', and $H^1(C,O_C)=0$. Let F be the closed fiber of X. Let I be the ideal defining C in X. From the exact sequence $0\to I^n/I^{n+1}\to O_X/I^{n+1}\to O_X/I^n\to 0$, it follows easily by induction that $H^1(X,O_X/I^n)=0$ for all n. In particular, $H^1(F,O_F)=0$. Let t be a uniformizing parameter for the maximal ideal of A. We have the exact sequence

$$0\to O_X\xrightarrow{\;t\;}O_X\to O_F\to 0,$$

from which we obtain

$$H^1(X,O_X)\xrightarrow{\;t\;}H^1(X,O_X)\to H^1(F,O_F)=0.$$

Since X is proper over A, $H^1(X,O_X)$ is a finitely generated A-module, and hence is zero by Nakayama's Lemma. Since $H^1(W,O_W)=H^1(X,O_X)\otimes_A K$, where K is the quotient field of A, $H^1(W,O_W)=0$, which is a contradiction.

THEOREM 4.4. *Let A be a Dedekind domain, $S = \operatorname{Spec} A$. Let X be a regular complete curve over S. Assume $K(S)$ is algebraically closed in $K(X)$. Let W be the generic fiber of X and assume again that $H^1(W, O_W) \neq 0$. Then there exists a regular complete curve Y over S and a proper S-morphism $\pi\colon X \to Y$ such that*

a) *π is birational.*

b) *Given any X' such that X' is a regular complete curve over S and such that there exists an S-rational map $X' \to X$ which is a birational isomorphism, then the induced rational map from X' to Y is a proper birational morphism.*

Proof. Let $X_K = X \underset{\operatorname{Spec} A}{\times} \operatorname{Spec} K(S)$. Let $\overline{K(S)}$ denote the algebraic closure of $K(S)$. Since $K(S)$ is algebraically closed in $K(X)$, it follows that $K(X) \underset{K(S)}{\otimes} \overline{K(S)}$ has the property that every zero-divisor is nilpotent. (See [5], Theorem 40, p. 197 and Corollary 2, Theorem 38, p. 195). Therefore X_K is geometrically irreducible. By E.G.A., IV, 9.7.8, the set of points of S where the corresponding fibers of X over S are geometrically reducible is contained in a closed set, hence a finite set, since A is a Dedekind domain. Since geometrically irreducible implies irreducible, there are only finitely many reducible fibers of X over S, so only a finite number of fibers may contain exceptional curves. Also, it is clear that if the fiber of X at a point of $\operatorname{Spec} S$ is irreducible, and we contract one of the exceptional curves of X, the fiber remains irreducible. Hence it is clear by the factorization theorem that there exists a regular complete curve Y over S and a proper S-morphism $\pi\colon X \to Y$ such that

a) π is birational,

b) Y contains no exceptional prime divisors.

We say that Y is a minimal model under X. Let X' be given. Let Y' be a minimal model under X'. We claim that Y' is isomorphic to Y, which will complete the proof. By Proposition 4.2, there exists a complete regular curve Y'' over Y, with proper S-birational maps $\pi_1\colon Y'' \to Y$ and $\pi_2\colon Y'' \to Y'$. By the factorization theorem there exist $Y = Y_0, Y_1, \cdots, Y_m = Y''$ and $Y' = Y_0', Y_1', \cdots, Y_n = Y''$ such that Y_i (resp. Y_j') is a locally quadratic transform of Y_{i-1} (resp. Y_{j-1}') for $i = 1, \cdots, m, j = 1, \cdots, n$. We may choose Y'' such that $m + n$ is minimal, and we may assume that $m > 0$. Consider the map $f_m\colon Y'' \to Y_{m-1}$. Let E be the exceptional curve on Y''.

Since Y' has no exceptional curves, the image of E is not exceptional in Y'. Let E be contained in the fibre over the point x of S. Let B be the local ring A_x and $T = \operatorname{Spec} B$. Let $Z' = Y' \times_S T$ and $Z_i' = Y_i' \times_S T$. By Proposition 4.3, there exists an r such that the image of E in Z_r' is the exceptional curve of the map from Z_r' to Z_{r-1}'. Hence the image of E in Y_r' must be the exceptional curve of the map from Y_r' to Y_{r-1}'. Also, the image of E on any of the surfaces $Y_r', Y_{r+1}', \cdots, Y_n'$ does not contain the image point P_j' in Y_j' of the exceptional curve E_{j+1}. Hence we could have rearranged the locally quadratic transformations so that E would be the exceptional curve of f_n'. Hence Y_{m-1} would be isomorphic to Y_{n-1}', contradicting the minimality of $m + n$.

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