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# Maps between Jacobians of Shimura curves and congruence kernels

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## 1 Introduction

Let  $p \ge 5$  be a prime, and  $\wp$  a place of  $\overline{\mathbf{Q}}$  above it. We denote by  $G_{\mathbf{Q}}$  the absolute Galois group  $\operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ . For a finite field k of characteristic p, an absolutely irreducible representation  $\rho : G_{\mathbf{Q}} \to \mathbf{GL}_2(k)$  is said to arise from a newform of weight k and level a positive integer M if  $\rho$  is isomorphic to the reduction mod  $\wp$  of the  $\wp$ -adic representation associated to a newform of weight k and level M. Note that then it is known that  $\rho$  also arises from a newform of weight 2 and level dividing  $Mp^2$ . Thus we assume without loss of generality that  $\rho$  arises from a newform of weight 2 and level a positive integer N.

The question of "raising" the level of  $\rho$ , i.e., determining when a  $\rho$  arising from a newform of weight 2 and level N as above also arises from a newform of weight 2 and level N', where N divides N', has been widely studied by numerous mathematicians (cf. [10,8,5,2]).

In the first paper that considers this kind of question [10], as well as in most of the papers dealing with similar questions (e.g., [2]), the cases considered correspond to N'/N being coprime to p. Some discussion for the case where N'/N is divisible by p can be found in [8] and [5]. The objective of this paper is to study further the question of level raising in the case where p divides N'/N.

Let  $q \ge 5$  be a prime different from p and, for any positive integer M, let  $\Gamma_0(M)$  and  $\Gamma_1(M)$  denote the classical modular subgroups in  $SL_2(\mathbb{Z})$ . We prove in this paper the following theorem:

**Theorem 1** If  $\rho : G_{\mathbf{Q}} \to \mathbf{GL}_2(k)$  is an absolutely irreducible representation, for k a finite field of characteristic p, that arises from a newform in  $S_2(\Gamma_0(N) \cap \Gamma_1(q))$  of level divisible by N, for a squarefree positive integer N coprime to pq, then

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it also arises from a newform in  $S_2(\Gamma_0(Np^r) \cap \Gamma_1(q))$  of level divisible by  $Np^r$ , for all integers  $r \ge 3$ .

In most of the works studying the problem of level raising, the approach adopted has essentially followed that in Ribet's original method in [10]. (See also [2].) Basically, one has to study in detail the degeneracy maps and the main step is to determine the kernel of a natural degeneracy map between some Jacobian varieties. In the case of [10], the kernel of the degeneracy map  $J_0(M)^2 \rightarrow J_0(Mp)$ , where *M* is not divisible by *p*, is determined completely with the help of a result of Ihara and, in a certain technical sense, shown to be irrelevant (or *Eisenstein* in the terminology below). When *M* is divisible by *p*, this kernel in general contains an abelian variety and was studied in [5]: there it was proved that the group of connected components of the kernel is irrelevant.

The approach we have taken in this paper to obtain the result in Theorem 1 is again similar. However, in our present case, the Jacobians used are no longer those coming from elliptic modular curves, but from Shimura curves. We prove an analog of Theorem 1 of [5]. In order to state this technical result in a precise manner, we first recall some definitions.

Let  $\mathcal{B}$  be an indefinite quaternion algebra over  $\mathbf{Q}$  that is ramified at an even number of primes  $l_1, \ldots, l_n$ . Then  $\mathcal{B}$  has reduced discriminant  $D = \prod_i l_i$ . Let  $N \ge 1$  be an integer that is relatively prime to D. Suppose that s is a prime dividing N.

The algebra  $\mathcal{B}_s \stackrel{\text{def}}{=} \mathcal{B} \otimes \mathbf{Q}_s$  is isomorphic to the matrix algebra  $M_2(\mathbf{Q}_s)$ . Let  $\mathcal{O}_{s,m} = \begin{pmatrix} \mathbf{Z}_s & \mathbf{Z}_s \\ s^m \mathbf{Z}_s & \mathbf{Z}_s \end{pmatrix}$  be the canonical Eichler order of level  $s^m \mathbf{Z}_s$  in  $M_2(\mathbf{Q}_s)$ . We define the groups  $\Gamma_0(s^m, \mathbf{Z}_s)$  and  $\Gamma_1(s^m, \mathbf{Z}_s)$  to be:

$$\Gamma_0(s^m, \mathbf{Z}_s) = \mathbf{SL}_2(\mathbf{Z}_s) \cap \mathcal{O}_{s,m}$$
  
$$\Gamma_1(s^m, \mathbf{Z}_s) = \{ x \in \Gamma_0(s^m, \mathbf{Z}_s) \mid x \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod s^m \mathcal{O}_{s,0} \}.$$

Let  $\mathcal{O}(N)$  be a **Z**-Eichler order in  $\mathcal{B}$  of level N. (We will simply use  $\mathcal{O}$  for the **Z**-Eichler order of level 1.) Let  $s^m$  be the exact power of s dividing N. For such an s, let  $i_s : \mathbf{Q} \hookrightarrow \mathbf{Q}_s$  be the canonical embedding. This embedding can be extended to an embedding, also denoted by  $i_s$ , of  $\mathcal{B}$  in  $M_2(\mathbf{Q}_s)$ , such that  $i_s(\mathcal{O}(N))$  is the canonical Eichler order  $\mathcal{O}_{s,m}$  of level  $s^m \mathbf{Z}_s$  ([15], pp. 108–109). The inverse image of  $\Gamma_0(s^m, \mathbf{Z}_s)$  under  $i_s$  is  $\mathcal{O}^1(N)$ , the set of elements of  $\mathcal{O}(N)$ of (reduced) norm 1. Similarly, we denote by  $\mathcal{O}^1$  the set of elements of  $\mathcal{O}$  of (reduced) norm 1. Let  $q \ge 5$  be an auxiliary prime not dividing ND. We define the congruence subgroups  $\Gamma_0(N, D)$  and  $\Gamma_1(N, D)$  as follows:

$$\Gamma_0(N, D) = \{ x \in \mathcal{O}^1(N) | i_s(x) \in \Gamma_0(s^m, \mathbf{Z}_s) \\
\text{for all } s | N \text{ and } i_q(x) \in \Gamma_1(q, \mathbf{Z}_q) \} \\
\Gamma_1(N, D) = \{ x \in \mathcal{O}^1(N) | i_s(x) \in \Gamma_1(s^m, \mathbf{Z}_s) \text{ for all } s | N \}.$$
(1)

Note that the definition of  $\Gamma_0(N, D)$  is a twisted form of the usual one. The reason for this deviation from the usual convention is to ensure that the congruence subgroups considered throughout this paper are torsion-free.

Since  $\mathcal{B}$  is unramified at the infinite place of  $\mathbf{Q}$ , it follows that  $\mathcal{B} \otimes_{\mathbf{Q}} \mathbf{R}$  is isomorphic to  $M_2(\mathbf{R})$ . Upon fixing such an isomorphism, we obtain an embedding  $i_{\infty} : \mathcal{O}^1(N) \hookrightarrow \mathbf{SL}_2(\mathbf{R})$ . The action of  $\mathbf{SL}_2(\mathbf{R})$  on the Poincaré upper half plane  $\mathcal{H} = \{\tau \in \mathbf{C} | \operatorname{Im}(\tau) > 0\}$ 

$$\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) \mapsto \frac{a\tau + b}{c\tau + d}$$

implies that the groups  $\Gamma_i(N, D)$  (i = 0, 1) also act on  $\mathcal{H}$ . When D > 1, the quotient  $\{\pm 1\}\Gamma_i(N, D)\setminus\mathcal{H}$  (i = 0, 1) has a natural structure of compact connected Riemann surface. It is called a *Shimura curve*, which we shall denote by  $Sh_i(N, D)$ . (When D = 1, the quaternion algebra  $\mathcal{B}$  is the matrix algebra  $M_2(\mathbf{Q})$ . The quotients  $\{\pm 1\}\Gamma_i(N, 1)\setminus\mathcal{H}$  (i = 0, 1) are known to be non-compact. Their compactifications are the classical modular curves  $X_i(N)$  (i = 0, 1) described in the introduction of [9].) We consider Shimura's canonical model over  $\mathbf{Q}$  for these curves.

In the rest of this introduction, for the sake of clarity of notation, we restrict our discussion to the case where i = 1. We note that the same results are available for i = 0.

Let  $J_1(N, D)$  denote the Jacobian of  $Sh_1(N, D)$ . Let M be a positive divisor of N. For every divisor d of N/M, there is a holomorphic degeneracy map  $\alpha_d : Sh_1(N, D) \to Sh_1(M, D)$  which is deduced from the map  $z \mapsto dz$  on  $\mathcal{H}$ by passing to the quotients. By Picard functoriality, we obtain a morphism of abelian varieties  $\alpha_d^* : J_1(M, D) \to J_1(N, D)$  for each d dividing N/M. If  $\tau(x)$ denotes the number of distinct positive divisors of a positive integer x, then let  $\alpha$  be the map

$$lpha := \prod_{d \mid N/M} lpha_d^* : J_1(M, D)^{\tau(N/M)} o J_1(N, D).$$

Fix a prime  $p \ge 5$  with (p, NDq) = 1. Consider the Jacobians  $J_1(Np^r, D)$ . They have an action for each prime *r* by the well-known Hecke operators  $T_r$ , induced by Picard functoriality, and that is **Q** rational (Sect. 3 of [11] and Chapter 3 of [14]). We use these operators only for *r* coprime to NDpq and denote the commutative (**Z**-)algebra generated by these operators  $T_r$  in the endomorphism ring of  $J_1(Np^r, D)$  by  $\mathbf{T}_{Np^r,D}^1$ . A maximal ideal **m** of  $\mathbf{T}_{Np^r,D}^1$  is "Eisenstein" if and only if the action of  $G_{\mathbf{Q}}$  on the semisimplification of the  $(\mathbf{T}_{Np^r,D}^1/\mathbf{m})[G_{\mathbf{Q}}]$ module  $J_1(Np^r, D)[\mathbf{m}]$ , the intersection of the kernels of the elements of **m** acting on  $J_1(Np^r, D)$ , factors through its maximal abelian quotient. By a Hecke module being Eisenstein, we mean that all the maximal ideals in the support of the module are Eisenstein (this definition of Eisenstein is the analog in our present context of the one used in [5]: see Remark 1 after Theorem 1 of *loc. cit.*).

We prove the following geometric result:

**Theorem 2** Let  $N \ge 4$  be an integer coprime to Dq and let  $p \ge 5$  be a prime not dividing NDq. Then, for  $r \ge 1$ , the group of connected components of the kernel of the degeneracy map  $\alpha : J_1(Np^r, D)^2 \rightarrow J_1(Np^{r+1}, D)$  is Eisenstein. Further, we have ker $(\alpha)^0 = \{(\alpha_p^*(x), -\alpha_1^*(x)) \mid x \in J_1(Np^{r-1}, D)\}.$ 

*Remark.* We remind the reader here that an exact analog of Theorem 2 remains true if  $J_1$  is replaced by  $J_0$ .

Using modular symbols,  $\ker(\alpha) / \ker(\alpha)^0$  can be proved to be Eisenstein when D = 1. In Theorem 1 of [5] the proof is written for  $J_0$ , but the same proof works for the  $J_1$  situation. For this reason, henceforth we assume D > 1 throughout this paper.

Modular symbols will not yield Theorem 2 above for D > 1, for the simple but compelling reason that for D > 1 the corresponding Shimura curves have no cusps!

As we have proven in Theorem 2 that  $\ker(\alpha)/\ker(\alpha)^0$  is Eisenstein in the setting of Shimura curves, where  $\alpha$  is applied to  $J_0$  instead of  $J_1$ , we can use the methods of [2] and [5] to raise the *p*-levels mod *p* of newforms corresponding to non-Eisenstein maximal ideals while preserving levels at other primes. Though a more exhaustive result about producing congruences in the "(p, p) case" (see [5]), while preserving prime-to-*p* levels, can be deduced from Theorem 2 and results in [10], [2] and [5], we content ourselves with what was stated in Theorem 1. We sketch how the proof of Theorem 1 follows, by techniques that are now standard, from Theorem 2.

*Proof of Theorem 1.* This follows in the case when the number of prime divisors of N is even from Theorem 2, with  $\alpha$  applied to  $J_0$  instead of  $J_1$ , after using the Jacquet-Langlands (JL) correspondence (cf. Sect. 5 of [2] for instance) to switch to an indefinite quaternion algebra ramified at exactly the places dividing N, and using methods of [5] (see proof of Theorem 2 of [5]). When the number of prime divisors of N is odd the corollary follows from the easy analog of Theorem 2 for definite quaternion algebras (see Sect. 2 of [2]), together with the JL correspondence and the methods of [5].

The rest of the paper is devoted to proving Theorem 2.

#### 2 The congruence kernel

Let  $\Gamma$  be a congruence subgroup of  $\mathcal{O}^1$ . Let  $\hat{\Gamma} := \lim_{\leftarrow} \Gamma/\mathcal{N}$ , where  $\mathcal{N}$  runs through all the normal subgroups of finite index in  $\Gamma$ . This is known as the profinite completion of  $\Gamma$ . Let  $\overline{\Gamma} := \lim_{\leftarrow} \Gamma/\mathcal{N}$ , where  $\mathcal{N}$  runs through all the congruence subgroups of  $\mathcal{O}^1$  and contained normally in  $\Gamma$ . Note that such an  $\mathcal{N}$  is automatically of finite index in  $\Gamma$ . The group  $\overline{\Gamma}$  is known as the congruence completion of  $\Gamma$ .

The congruence kernel  $C_{\Gamma}$  is by definition the kernel of the natural map  $\hat{\Gamma} \to \overline{\Gamma}$ . (See [12] for a survey on congruence kernels.) We have the following exact sequence:

$$0 \longrightarrow C_{\Gamma} \longrightarrow \hat{\Gamma} \longrightarrow \overline{\Gamma} \longrightarrow 0.$$
<sup>(2)</sup>

**Proposition 3** If  $\Gamma \leq \Gamma'$  are congruence subgroups in  $\mathcal{O}^1$ , then the natural map  $C_{\Gamma} \to C_{\Gamma'}$  is an isomorphism.

*Proof.* We begin by remarking that by considering the intersection of all the conjugates of  $\Gamma$  in  $\mathcal{O}^1$ , we get a congruence subgroup  $\Gamma''$  that is normal in  $\mathcal{O}^1$  contained in  $\Gamma'$ . Then if we prove that the natural maps  $C_{\Gamma''} \to C_{\Gamma}$  and  $C_{\Gamma''} \to C_{\Gamma'}$  are isomorphisms, as the latter map factors through the former map, it follows that  $C_{\Gamma} \to C_{\Gamma'}$  is an isomorphism. Hence it is enough to prove the statement with  $\Gamma$  normal in  $\mathcal{O}^1$ : we assume this to be the case. Thus we have a surjective homomorphism  $\overline{\Gamma'} \to \Gamma'/\Gamma$  as  $\Gamma$  is a congruence subgroup.

Consider the commutative diagram

where the vertical maps are all induced by the inclusion of  $\Gamma$  in  $\Gamma'$ . The maps  $\hat{\iota}$  and  $\bar{\iota}$  are injective. Therefore, by the Snake Lemma, it follows that  $\iota$  is injective and there is an exact sequence

$$0 \longrightarrow C_{\Gamma'}/C_{\Gamma} \longrightarrow \hat{\Gamma'}/\hat{\Gamma} \longrightarrow \overline{\Gamma'}/\overline{\Gamma} \longrightarrow 0.$$

In fact, the map  $\hat{\Gamma}'/\hat{\Gamma} \longrightarrow \overline{\Gamma'}/\overline{\Gamma}$  is an isomorphism since both are isomorphic to  $\Gamma'/\Gamma$ . The isomorphism  $\hat{\Gamma}'/\hat{\Gamma} \simeq \Gamma'/\Gamma$  follows from the fact that  $\Gamma$  is a *finite index* subgroup of  $\Gamma'$ , while the isomorphism  $\overline{\Gamma'}/\overline{\Gamma} \simeq \Gamma'/\Gamma$  is a consequence of the fact that  $\Gamma$  is a *congruence* subgroup of  $\Gamma'$ . Thus, it follows that  $C_{\Gamma} \to C_{\Gamma'}$ is actually an isomorphism.

As the congruence kernel  $C_{\Gamma}$  does not depend on the choice of the congruence subgroup  $\Gamma$ , we will henceforth use C to denote it.

We write down the following exact sequence, for  $\Gamma$  a congruence subgroup of  $\mathcal{O}^1$ , which follows from (2), for later use:

$$H^{1}(\overline{\Gamma}, \mathbf{Z}/\ell\mathbf{Z}) \longrightarrow H^{1}(\widehat{\Gamma}, \mathbf{Z}/\ell\mathbf{Z}) \longrightarrow H^{1}(C, \mathbf{Z}/\ell\mathbf{Z})^{\widehat{\Gamma}} \longrightarrow H^{2}(\overline{\Gamma}, \mathbf{Z}/\ell\mathbf{Z}).$$
(3)

#### **3** A group theory principle

We will only consider torsion-free congruence subgroups  $\Gamma$  such that the corresponding Riemann surface  $\Gamma \setminus \mathcal{H}$  has a model over **Q**, and denote its canonical model over **Q** by  $X_{\Gamma}$ .

We recall comparison isomorphisms between group, sheaf and étale cohomology groups that we need below, and the definition of Eisenstein maximal ideals of Hecke algebras acting on these cohomology groups in each case. We use [3] and [4] as references. Though these references work with classical modular curves (i.e., the case of D = 1), the results that we quote (and their proofs) are valid in the present context of Shimura curves associated to congruence subgroups of indefinite, non-split quaternion algebras over **Q**. Also note that unlike in the references we do not have to work with parabolic cohomology as in the case of D > 1, the congruence subgroups  $\Gamma$  are co-compact.

For a prime  $\ell$  we consider the cohomology of  $\Gamma$ , denoted  $H^i(\Gamma, \mathbb{Z}/\ell\mathbb{Z})$ , the sheaf cohomology group of  $X_{\Gamma}$ , denoted  $H^i(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z})$ , and the étale cohomology group of  $X_{\Gamma}$ , denoted  $H^i_{et}(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z})$ , for integers *i* (these groups vanish for  $i \neq 0, 1, 2$ ). Let  $J_{\Gamma}$  be the Jacobian of  $X_{\Gamma}$ , and we denote by  $J_{\Gamma}[\ell]$  its  $\ell$ -torsion. Then we have canonical isomorphisms of abelian groups:

$$H^{i}(\Gamma, \mathbf{Z}/\ell\mathbf{Z}) \simeq H^{i}(X_{\Gamma}, \mathbf{Z}/\ell\mathbf{Z}),$$
 (A)

$$H^{i}(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z}) \simeq H^{i}_{et}(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z}),$$
 (B)

$$H^1(\Gamma, \mathbf{Z}/\ell\mathbf{Z}) \simeq J_{\Gamma}[\ell].$$
 (C)

For (*A*), where the hypothesis that  $\Gamma$  is torsion-free is necessary, and (*B*), we refer to Sect. 1 of [3], while (*C*), where again the hypothesis that  $\Gamma$  is torsion-free gets used, is deduced from (*A*) and Sect. 6 (page 253) of [4]. In Sect. 3 of [3], for  $H^i(\Gamma, \mathbb{Z}/\ell\mathbb{Z})$ ,  $H^i(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z})$  and  $H^i_{et}(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z})$ , and Sect. 3 of [11], for  $J_{\Gamma}[\ell]$ , endomorphisms by Hecke operators  $T_r$ , with r a prime, are defined on all the  $\mathbb{Z}/\ell\mathbb{Z}$ -vector spaces above. We will consider the action of all but finitely many  $T_r$ 's: for instance we will ignore the action of  $T_r$ 's for places r at which either  $\mathcal{B}$  is ramified, or at which there is a congruence condition on  $\Gamma$ . The Hecke action on  $H^i_{et}(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z})$  and  $J_{\Gamma}[\ell]$ , that carry an action of  $G_Q$ , is Q-rational.

We consider the  $\mathbb{Z}/\ell\mathbb{Z}$ -algebra generated by the  $T_r$ 's in the ring of endomorphisms in each case, and by abuse of notation denote it by  $\mathbb{T}$  in each case. The isomorphisms (*A*) and (*B*) are equivariant for the action of  $T_r$  (see Sect. 1 of [3]), as also is (*C*) (see (6.9a) of [4]: there the Hecke action on  $J_{\Gamma}$  is defined using Albanese functoriality, while we have defined it using Picard functoriality (Sect. 3 of [11]), and thus the Hecke equivariance of (*C*) follows from [4] as the operator  $T_r^*$  of [4] is our  $T_r$ ).

We define a maximal ideal **m** of the Hecke algebra **T** acting on  $H_{et}^i(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z})$  to be *Eisenstein*, if the action of  $G_{\mathbb{Q}}$  on the semisimplification of the  $(\mathbf{T}/\mathbf{m})[G_{\mathbb{Q}}]$ -module  $H_{et}^i(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z})[\mathbf{m}]$  factors through the maximal abelian quotient of  $G_{\mathbb{Q}}$ . We define subquotients of the Hecke modules  $H_{et}^i(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z})$  to be *Eisenstein*, if and only if the maximal ideals in their support are *Eisenstein*.

Using the canonical Hecke-equivariant isomorphisms (A), (B) and (C), we define a maximal ideal **m** for Hecke algebras acting on  $H^i(\Gamma, \mathbb{Z}/\ell\mathbb{Z})$ ,  $H^i(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z})$  and  $J_{\Gamma}[\ell]$  to be *Eisenstein* if the corresponding maximal ideal (via the isomorphisms above) of the Hecke algebra acting on  $H^i_{et}(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z})$  in the first two cases, and  $H^1_{et}(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z})$  in the last, is Eisenstein. We have a corresponding definition of subquotients of these Hecke modules being Eisenstein. This is consistent with the definition of the introduction because of the Hecke equivariance of (A), (B) and (C), and as we have an isomorphism of  $G_{\mathbf{O}}$ -modules

$$H^{1}_{et}(X_{\Gamma}, \mathbf{Z}/\ell\mathbf{Z}) = \operatorname{Hom}(J_{\Gamma}[\ell], \mu_{\ell}), \qquad (D)$$

where  $\mu_{\ell}$  is the Galois module of  $\ell$ th roots of unity.

(As a technical aside, also note that, although using the going-up theorem we can in a natural way consider maximal ideals "associated to" m in Hecke algebras generated by different sets of almost all  $T_r$ 's (see the discussion in Sect. 7 of [11]), because of the Cebotarev density and Brauer-Nesbitt theorems, and congruence relations, this definition of *Eisenstein* is not sensitive to which set of *almost all* Hecke operators  $T_r$  we elect to consider the action of.)

Let *A* and *B* be torsion-free congruence subgroups of  $\mathcal{O}^1$ , and denote by *E* the group  $\langle A, B \rangle$  generated by *A* and *B* in  $\mathcal{O}^1$ . Note that the restriction maps from any of the cohomology groups  $H^i(A, \mathbb{Z}/\ell\mathbb{Z})$ ,  $H^i(B, \mathbb{Z}/\ell\mathbb{Z})$  or  $H^i(E, \mathbb{Z}/\ell\mathbb{Z})$  to  $H^i(A \cap B, \mathbb{Z}/\ell\mathbb{Z})$  are equivariant with respect to the Hecke operators  $T_r$  for almost all primes *r* (for instance all primes at which there are no congruences conditions on  $A \cap B$ ). Thus we can pull back a maximal ideal of the Hecke algebra generated by such  $T_r$ 's acting on the cohomology groups (of any degree) associated to  $A \cap B$ , to the cohomology groups (of the same degree) associated to A, B and E via these restriction maps.

For  $\Gamma \leq \Gamma'$  torsion-free congruence subgroups of  $\mathcal{O}^1$ , with  $\Gamma$  normal in  $\Gamma'$ , and  $\ell$  a prime, we have the long exact sequence in cohomology:

$$0 \to H^{1}(\Gamma'/\Gamma, \mathbf{Z}/\ell\mathbf{Z}) \to H^{1}(\Gamma', \mathbf{Z}/\ell\mathbf{Z}) \to H^{1}(\Gamma, \mathbf{Z}/\ell\mathbf{Z})^{\Gamma/\Gamma'} \to H^{2}(\Gamma'/\Gamma, \mathbf{Z}/\ell\mathbf{Z}) \to H^{2}(\Gamma', \mathbf{Z}/\ell\mathbf{Z})$$
(4)

(see Proposition 2 of [16] for instance). We consider the standard action of the Hecke operators  $T_r$ 's (for almost all primes r) on the groups  $H^i(\Gamma'/\Gamma, \mathbb{Z}/\ell\mathbb{Z})$  defined analogously as in Sect. 3 of [3], which makes the long exact sequence (4) Hecke equivariant. Then we define  $H^1(\Gamma'/\Gamma, \mathbb{Z}/\ell\mathbb{Z})$  to be Eisenstein if its image in  $H^1(\Gamma', \mathbb{Z}/\ell\mathbb{Z})$  is Eisenstein, and  $H^2(\Gamma'/\Gamma, \mathbb{Z}/\ell\mathbb{Z})$  to be Eisenstein if its image in  $H^2(\Gamma', \mathbb{Z}/\ell\mathbb{Z})$  and the cokernel of the map  $H^1(\Gamma', \mathbb{Z}/\ell\mathbb{Z}) \longrightarrow H^1(\Gamma, \mathbb{Z}/\ell\mathbb{Z})^{\Gamma/\Gamma'}$  as in (4), are both Eisenstein.

**Lemma 4** For  $\Gamma \leq \Gamma'$  torsion-free congruence subgroups of  $\mathcal{O}^1$ , with  $\Gamma$  normal in  $\Gamma'$ , and  $\ell$  a prime, the groups  $H^i(\Gamma'/\Gamma, \mathbb{Z}/\ell\mathbb{Z})$ , for i = 1, 2, are Eisenstein.

*Proof.* The proof follows directly from the considerations in Proposition 4 of Sect. 3 of [2]. Namely, by the long exact sequence (4) and the Hecke equivariant isomorphism (*A*) between  $H^1(\Gamma, \mathbb{Z}/\ell\mathbb{Z})$  and  $H^1(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z})$  (note that  $\Gamma$  is torsion-free), it is enough to prove that the kernel and cokernel of the map

$$H^1(X_{\Gamma'}, \mathbb{Z}/\ell\mathbb{Z}) \to H^1(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z})^{\Gamma'/\Gamma}$$

are Eisenstein, and that  $H^2(X_{\Gamma'}, \mathbb{Z}/\ell\mathbb{Z})$  is Eisenstein. By the Hochschild-Serre spectral sequence

$$H^{i}(\Gamma'/\Gamma, H^{j}(X_{\Gamma}, \mathbb{Z}/\ell\mathbb{Z})) \Rightarrow H^{i+j}(X_{\Gamma'}, \mathbb{Z}/\ell\mathbb{Z}),$$

this will follow if we prove that  $H^i(\Gamma'/\Gamma, H^j(X_{\Gamma}, \mathbf{Z}/\ell\mathbf{Z}))$  is Eisenstein for i = 0, 1, 2 and j = 0, 2, and that  $H^2(\Gamma', \mathbf{Z}/\ell\mathbf{Z})$  is Eisenstein. We claim that  $H^0(X_{\Gamma}, \mathbf{Z}/\ell\mathbf{Z})$  and  $H^2(X_{\Gamma}, \mathbf{Z}/\ell\mathbf{Z})$  (and analogously  $H^2(X_{\Gamma'}, \mathbf{Z}/\ell\mathbf{Z})$ ) are Eisenstein as  $\Gamma$  (resp.,  $\Gamma'$ ) is a congruence subgroup. From Sect. 3 of [2] it follows that the action of  $G_{\mathbf{Q}}$  on the corresponding étale cohomology groups  $H^0_{et}(X_{\Gamma}, \mathbf{Z}/\ell\mathbf{Z})$  and  $H^2_{et}(X_{\Gamma}, \mathbf{Z}/\ell\mathbf{Z})$  (and  $H^2_{et}(X_{\Gamma'}, \mathbf{Z}/\ell\mathbf{Z})$ ) factors through its maximal abelian quotient, and then the Hecke equivariance of (*B*) proves our claim. This proves the lemma.

We come now to the group theoretic principle in the title of this section, which is the analog in the present setting of the group theoretic principle of Sect. 1 of [6], and that is the key to the proof of Theorem 2 in the next section.

**Proposition 5** Let A, B be torsion-free congruence subgroups of  $\mathcal{O}^1$  and let  $E = \langle A, B \rangle$  be the group generated by A and B in  $\mathcal{O}^1$ . For  $\mathfrak{m}$  a non-Eisenstein maximal ideal of residue characteristic  $\ell$  of the Hecke algebra acting on  $H^1(A \cap B, \mathbb{Z}/\ell\mathbb{Z})$ , the following sequence is exact:

$$H^{1}(E, \mathbf{Z}/\ell\mathbf{Z})_{\mathsf{m}} \to H^{1}(A, \mathbf{Z}/\ell\mathbf{Z})_{\mathsf{m}} \oplus H^{1}(B, \mathbf{Z}/\ell\mathbf{Z})_{\mathsf{m}} \to H^{1}(A \cap B, \mathbf{Z}/\ell\mathbf{Z})_{\mathsf{m}},$$
(5)

where the first map is the restriction while the second is the difference of the restriction maps, and the subscript m denotes localisation at the maximal ideal m.

*Proof.* We claim that, to prove the proposition, it is enough to prove that, for any congruence subgroup  $\Gamma$  and a non-Eisenstein maximal ideal m as above, we have the isomorphism

$$H^{1}(\Gamma, \mathbf{Z}/\ell\mathbf{Z})_{\mathsf{m}} \simeq H^{1}(C, \mathbf{Z}/\ell\mathbf{Z})_{\mathsf{m}}^{\tilde{\Gamma}}, \tag{6}$$

where *C* is the congruence kernel defined at the end of Sect. 2. Since  $H^1(\Gamma, \mathbf{Z}/\ell \mathbf{Z}) = H^1(\hat{\Gamma}, \mathbf{Z}/\ell \mathbf{Z})$ , the isomorphism (6) is an immediate consequence of (3) and Lemma 4.

To justify the claim, first note that (6) implies that the restriction map  $H^1(\Gamma', \mathbb{Z}/\ell\mathbb{Z})_m \to H^1(\Gamma, \mathbb{Z}/\ell\mathbb{Z})_m$  is injective for any congruence subgroups  $\Gamma \leq \Gamma'$  of  $\mathcal{O}^1$ . Now if  $res(h_1) = res(h_2) = g$ , say, with  $h_1 \in H^1(A, \mathbb{Z}/\ell\mathbb{Z})_m$ ,  $h_2 \in H^1(B, \mathbb{Z}/\ell\mathbb{Z})_m$  and  $g \in H^1(A \cap B, \mathbb{Z}/\ell\mathbb{Z})_m$ , then under the isomorphism (6), applied with  $\Gamma = A \cap B$ , g corresponds to an element  $\tilde{g}$  in  $H^1(C, \mathbb{Z}/\ell\mathbb{Z})_m^{\widehat{A}\cap B}$ . But as  $g = res(h_i)$  for i = 1, 2, invoking (6) again we deduce that  $\tilde{g}$  is in fact in  $H^1(C, \mathbb{Z}/\ell\mathbb{Z})_m^{\widehat{E}}$ , as E by definition is generated by A and B, and E is dense in  $\hat{E}$ . Another use of (6), this time with  $\Gamma = E$ , proves the claim. This completes the proof of the proposition.

*Remark.* In [6], the proposition was proved for torsion-free congruence subgroups A and B of  $SL_2(\mathbb{Z})$ . In the case of  $SL_2(\mathbb{Z})$ , the exact sequence (5) is an easy consequence of the exact sequence

$$\begin{aligned} H^1_c(\langle A, B \rangle, \mathbf{Z}/\ell \mathbf{Z})_{\mathsf{m}} &\to H^1_c(A, \mathbf{Z}/\ell \mathbf{Z})_{\mathsf{m}} \oplus H^1_c(B, \mathbf{Z}/\ell \mathbf{Z})_{\mathsf{m}} \\ &\to H^1_c(A \cap B, \mathbf{Z}/\ell \mathbf{Z})_{\mathsf{m}}, \end{aligned}$$

which in turn is a direct consequence of modular symbols. Here, m is a non-Eisenstein maximal ideal and the subscript c stands for compactly supported cohomology, and by compactly supported group cohomology, we mean the compactly supported cohomology of the corresponding affine curve. In the present work we have replaced the use of modular symbols by the use of the congruence kernel.

## 4 Proof of Theorem 2

We start by translating what is to be proved into statements about group cohomology, and remind the reader that we assume D > 1. We prefer to work with finite coefficients, and we reduce to working with such by the following reasoning. Recall from the introduction that we are studying the degeneracy map  $\alpha : J_1(Np^r, D)^2 \rightarrow J_1(Np^{r+1}, D)$  of Theorem 2, where  $J_1(Np^r, D)$ and  $J_1(Np^{r+1}, D)$  are the Jacobians of the modular curves  $Sh_1(Np^r, D)$  and  $Sh_1(Np^{r+1}, D)$ , and we are denoting the connected component of ker( $\alpha$ ), the kernel of  $\alpha$ , by ker( $\alpha$ )<sup>0</sup>. Let  $x \in ker(\alpha)/ker(\alpha)^0$  be an element of order  $\ell$ , for some prime  $\ell$ . Thus  $\ell x \in \ker(\alpha)^0$ , and as  $\ker(\alpha)^0$  is an abelian variety and hence  $\ell$ -divisible, there is a  $y \in \ker(\alpha)^0$  such that  $\ell x = \ell y$ . We deduce that there exists x' in  $\ker(\alpha)$  that maps to x and such that  $\ell x' = 0$ . Thus we see that to prove the assertion pertaining to the group of connected components of Theorem 2, it is sufficient to compute the kernel,  $\ker(\alpha_\ell)$ , of the map  $\alpha_\ell : J_1(Np^r, D)[\ell]^2 \rightarrow J_1(Np^{r+1}, D)[\ell]$  induced by  $\alpha$  on the  $\ell$ -torsion of the Jacobians, and show that the quotient  $K_\ell := \ker(\alpha_\ell)/\{(\alpha_p^*(x), -\alpha_1^*(x)) : x \in J_1(Np^{r-1}, D)[\ell]\}$  is Eisenstein, for every prime  $\ell$ .

We interpret the degeneracy map in the group cohomology setting. Let  $A = \Gamma_1(Np^r, D)$ . Let  $\pi$  be the matrix  $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$  and let  $f_1, f_2$  be elements of  $H^1(A, \mathbb{Z}/\ell\mathbb{Z})$ . By abuse, we again denote by  $\alpha$  the degeneracy map on the group cohomology;  $\alpha((f_1, f_2))$  is the element f of  $H^1(\Gamma_1(Np^{r+1}, D), \mathbb{Z}/\ell\mathbb{Z})$  given by  $f(x) = f_1(x) + f_2(\pi x \pi^{-1})$ , for  $x \in \Gamma_1(Np^{r+1}, D)$ . We can also write  $\alpha((f_1, f_2))$  as the sum  $\alpha_1^* f_1 + \alpha_p^* f_2$ , where  $\alpha_1^*$  is the restriction map and  $\alpha_p^*$  is conjugation by  $\pi$ . Note that  $\alpha_1^*$  and  $\alpha_p^*$  are equivariant with respect to the action of the Hecke operators  $T_r$  that we consider; namely, r coprime to NDpq, where q is the auxiliary prime that occurs in (1) in the introduction (see also paragraph before the statement of Theorem 2 in the introduction). It is the kernel of  $\alpha$  that we determine.

Let  $\alpha((f_1, f_2)) = 0$ . Let  $B = \pi \Gamma_1(Np^r, D)\pi^{-1}$  and

 $E := \langle A, B \rangle.$ 

It is easy to see that  $E = \Gamma_1(Np^{r-1}, D)$ . Let m be any non-Eisenstein maximal ideal of the Hecke algebra (generated by the  $T_r$ 's for r coprime to NDpq as before) acting on  $H^1(E, \mathbb{Z}/\ell\mathbb{Z})$ . We may also regard m, by pull back under restriction maps which are equivariant for the  $T_r$ 's that we consider, as a maximal ideal of the Hecke algebras acting on  $H^1(A, \mathbb{Z}/\ell\mathbb{Z})$ ,  $H^1(B, \mathbb{Z}/\ell\mathbb{Z})$  and  $H^1(A \cap B, \mathbb{Z}/\ell\mathbb{Z})$ . Define elements  $h_1 \in H^1(A, \mathbb{Z}/\ell\mathbb{Z})$ ,  $H^1(B, \mathbb{Z}/\ell\mathbb{Z})$  and  $H^1(A \cap B, \mathbb{Z}/\ell\mathbb{Z})$ . Define elements  $h_1 \in H^1(A, \mathbb{Z}/\ell\mathbb{Z})_m$  and  $h_2 \in H^1(B, \mathbb{Z}/\ell\mathbb{Z})_m$  by  $h_1(x) :=$  $-f_2(x)$  and  $h_2(\pi x \pi^{-1}) := f_1(x)$  where  $f_i$  (i = 1, 2) are now considered as elements in  $H^1(A, \mathbb{Z}/\ell\mathbb{Z})_m$ . The subscript m, as in Proposition 4 of Sect. 3, stands for localisation at the maximal ideal m. As  $\alpha((f_1, f_2)) = 0$ , we conclude that the restrictions of  $h_1$  and  $h_2$  to  $H^1(A \cap B, \mathbb{Z}/\ell\mathbb{Z})_m$  coincide. Applying Proposition 5, it follows that there is a  $g \in H^1(E, \mathbb{Z}/\ell\mathbb{Z})_m$  such that its restriction to  $H^1(A, \mathbb{Z}/\ell\mathbb{Z})_m$  and  $H^1(B, \mathbb{Z}/\ell\mathbb{Z})_m$  is  $h_1$  and  $h_2$  respectively. This together with the isomorphism (C) of the previous section, and its Hecke equivariance, proves that  $K_\ell$  is Eisenstein for every prime  $\ell$ , and we deduce that the group of connected components of the degeneracy map of the theorem is Eisenstein.

After noting the identification of the cotangent space of Jacobians  $J_1(Np^r, D)$ with the space  $S_2(\Gamma_1(Np^r, D), \mathbb{C})$  of cusp forms of weight 2, and the fact that the connected component of the kernel of  $\alpha$  of Theorem 2 certainly contains  $\{(\alpha_p^*(x), -\alpha_1^*(x)) \mid x \in J_1(Np^{r-1}, D)\}$ , to prove the description in Theorem 2 of ker( $\alpha$ )<sup>0</sup> to be correct, it suffices to show that if  $f, g \in S_2(\Gamma_1(Np^r, D), \mathbf{C})$ satisfy f(z) + g(pz) = 0 for  $z \in \mathcal{H}$  then  $f(z) = f_1(pz)$  and  $g(z) = -f_1(z)$ for some  $f_1 \in S_2(\Gamma_1(Np^{r-1}, D), \mathbf{C})$ . This follows from the analog of the results of [1] for cusp forms for congruence subgroups of quaternion algebras that can be proved directly using the adelic description of modular forms. We denote by *G* the algebraic group over  $\mathbf{Q}$  whose  $\mathbf{Q}$ -valued points is the  $\mathcal{B}$  of the introduction. Then the automorphic function on  $G(\mathbf{A}_{\mathbf{Q}})$  corresponding to *g* is fixed under the action of the open compact subgroups of  $G(\mathbf{A}_{\mathbf{Q}})$  corresponding to the congruence subgroups *A* and *B*, and hence is fixed by the open compact subgroup corresponding to *E*. From this the description of the connected component of ker( $\alpha$ ) of the theorem follows, thus finishing the proof of Theorem 2.

# **5** Concluding remarks

We conclude the paper with some remarks about possible refinements of Theorem 2.

- The methods of this paper do not directly seem to yield more precise results about the group of connected components that were proven to be Eisenstein above. We would make the guess that the kernel is connected for the  $J_1$  situation, while it is the image of the Shimura subgroup in the  $J_0$  situation. (In fact, it is easy to see that the kernel of  $\alpha : J_0(Np^r, D)^2 \rightarrow J_0(Np^{r+1}, D)$  contains  $\{(x, -x) \mid x \in \Sigma(Np^r, D)\}$ , where  $\Sigma(Np^r, D)$  denotes the Shimura subgroup of  $J_0(Np^r, D)$ .) Indeed, in the elliptic  $J_0$  case, in many cases when the kernel is known to be finite, the kernel has been proven to be the image of the Shimura subgroup (e.g., [10,8]).
- Our guess on the precise description of the kernel in our present case may follow from a more careful study of the exact sequence (3), with Γ replaced by A, B, A ∩ B and E, and more detailed information about congruence subgroups of SL<sub>2</sub>(Z<sub>p</sub>) (note that the congruence completions of A, B and E are identical away from p).
- We have the following information about the closure Γ of the group Γ<sub>1</sub>(Np<sup>r</sup>, D) (r ≥ 1) in SL<sub>2</sub>(Z<sub>p</sub>) ([13]). The dimension over Z/pZ of the cohomology groups H<sup>i</sup>(Γ, Z/pZ) for i = 0, 1, 2, 3 is 1,2,2,1 respectively. Notice that these dimensions are independent of r ≥ 1 which may be useful in proving a more refined version of Theorem 2.

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