TORIC MODULAR FORMS AND NONVANISHING OF L-FUNCTIONS

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ABSTRACT. In a previous paper [1], we defined the space of toric forms $\mathscr{T}(l)$, and showed that it is a finitely generated subring of the holomorphic modular forms of integral weight on the congruence group $\Gamma_1(l)$. In this article we prove the following theorem: modulo Eisenstein series, the weight two toric forms coincide exactly with the vector space generated by all cusp eigenforms f such that $L(f,1) \neq 0$. The proof uses work of Merel, and involves an explicit computation of the intersection pairing on Manin symbols.

1. INTRODUCTION

1.1. Let $l > 1$ be an integer, let $\mathscr{M}(l)$ be the space of weight two modular forms on the congruence group $\Gamma_1(l) \subset SL_2(\mathbb{Z})$, and let $\mathscr{S}(l)$ be the subspace of cusp forms. Let $f \in \mathscr{S}(l)$ be an eigenform for the Hecke operators T_p, where p is coprime to l, and let $L(f,s)$ be the associated L-function. Then the order of vanishing of $L(f,s)$ at $s = 1$ is called the *analytic rank* of f. This terminology comes from the Birch and Swinnerton-Dyer conjecture, which asserts that the analytic rank times $\dim A_f$ is the same as rank of the group $A_f(\mathbb{Q})$, where A_f is the abelian variety associated to f by the Eichler-Shimura construction [6, 14].

1.2. In this paper we present an elementary construction of the subspace of $\mathscr{S}(l)$ spanned by forms of analytic rank zero. Our main result (Theorem 4.11) is that, modulo Eisenstein series, this space is isomorphic as a Hecke module to the space $\mathscr{T}_2(l)$ of weight two *toric modular forms* of level l. These modular forms were constructed and studied in [1], where we presented explicit generators of $\mathscr{T}(l)$ and described their q-expansions at infinity. The construction of $\mathscr{T}(l)$ and its relevant properties are summarized in Theorem 3.3. For the remainder of this introduction, we describe the proof of Theorem 4.11.

1.3. First, in §2 we recall results about the *Manin symbols*. We discuss various homology groups associated to the modular curve in terms of modular symbols, and describe the intersection pairing. We define the space of *plus* (respectively *minus*) symbols M_+ (resp. M_-), and their *cuspidal* subspaces S_+ and S_-, and we describe the intersection pairing in terms of Manin symbols (Proposition 2.11). We finish this section by recalling Merel's description of the Hecke action on Manin symbols (Theorem 2.13).

Next, in §3 we recall results about toric modular forms from [1], specialized to the case of weight two. Using the toric forms, we define a map $\mu\colon M_- \to \mathscr{M}(l)/\mathscr{E}(l)$, where $\mathscr{E}(l) \subset \mathscr{M}(l)$ is the subspace of Eisenstein series. The main result (Theorem 3.16) is that μ is Hecke-equivariant. We also describe the composition of μ with the *Fricke involution* W_l.

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Finally, in §4 we define a map $\rho\colon \mathscr{S}(l) \to \mathscr{S}(l)$ whose image is spanned by Hecke eigenforms of analytic rank zero. Then we put all these maps together to get a sequence

$$\mathscr{S}(l) \xrightarrow{\int} M_+^* \xrightarrow{\pi} M_- \xrightarrow{W_l \circ \mu} \mathscr{M}(l)/\mathscr{E}(l) \xrightarrow{\sim} \mathscr{S}(l),$$

where the final map is the Hecke-equivariant isomorphism between $\mathscr{M}(l)/\mathscr{E}(l)$ and $\mathscr{S}(l)$. Then in Theorem 4.8 we show that this sequence equals ρ, from which we obtain Theorem 4.11.

The Eisenstein series $s_{a/l}$ have appeared in the literature before. They were originally studied by Hecke [4, 7] and have recently appeared in the work of Kato on Euler systems [5, 12]. Moreover, it might be possible to derive Theorem 4.11 using the formulas of [12, §4], which were obtained using the Rankin-Selberg method.

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2. Manin symbols

2.1. Let $l > 1$ be an integer, and let $\Gamma_1(l) \subset SL_2(\mathbb{Z})$ be the subgroup of matrices congruent to $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ mod l. Let $\mathfrak{H}$ be the upper halfplane, and let $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$ be the usual partial compactification obtained by adjoining cusps. Let $X_1(l) = \Gamma_1(l)\backslash\mathfrak{H}^*$ be the modular curve with cusps $\partial X_1(l)$, and let $Y_1(l) := X_1(l) \smallsetminus \partial X_1(l)$.

Let M be the relative homology $H_1(X_1(l), \partial X_1(l); \mathbb{C})$, and let $S \subset M$ be the subspace $H_1(X_1(l), \mathbb{C})$. The intersection product induces a perfect pairing of complex vector spaces

$$H_1(X_1(l), \partial X_1(l); \mathbb{C}) \times H_1(Y_1(l), \mathbb{C}) \longrightarrow \mathbb{C},$$

which allows us to identify $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ with $H_1(Y_1(l), \mathbb{C})$.

2.2. Manin's theory [8] gives a concrete description of M as follows. For any α, β in $\mathbb{P}^1(\mathbb{Q})$, let $\{\alpha, \beta\} \in M$ be the class of the image of a continuous path in $\mathfrak{H}^*$ from α to β. Then the $\mathbb{C}$-linear map

$$\begin{aligned} \mathbb{C}[\pm\Gamma_1(l)\backslash SL_2(\mathbb{Z})] &\longrightarrow M \\ \pm\Gamma_1(l)g &\longmapsto \{g0, g\infty\} \end{aligned}$$

is well defined and surjective. If we denote the basis element corresponding to the coset x by $[x]$, then the kernel of this map is generated by elements of the form $[x]+[x\sigma]$ and $[x]+[x\tau]+[x\tau^2]$. Here x runs through $\pm\Gamma_1(l)\backslash SL_2(\mathbb{Z})$, and σ, τ are elements of $SL_2(\mathbb{Z})$ (of order 4 and 3) that stabilize i and $\rho = \mathrm{e}^{2\pi\mathrm{i}/3}$ respectively, and satisfy $\sigma 0 = \infty$ and $\tau\infty = 0$.

By duality one can describe M^* as the subspace of $\mathbb{C}[\pm\Gamma_1(l)\backslash SL_2(\mathbb{Z})]$ generated by elements $\sum_x \lambda_x[x]$ satisfying $\lambda_x + \lambda_{x\sigma} = 0$ and $\lambda_x + \lambda_{x\tau} + \lambda_{x\tau^2} = 0$. One can realize this description geometrically as follows. Consider the geodesic path in the upper half-plane from i to ρ. For $g \in SL_2(\mathbb{Z})$, the image in $Y_1(l)$ of its translates by g depends only on the coset $\pm\Gamma_1(l)g$. Denote by $c_{\Gamma_1(l)g}$ the arc in $Y_1(l)$ associated to this image. One can show easily that $\sum_x \lambda_x c_x$ is a closed cycle if and only if $\lambda_x + \lambda_{x\sigma} = 0$ and $\lambda_x + \lambda_{x\tau} + \lambda_{x\tau^2} = 0$. In that case, $\sum_x \lambda_x c_x$ belongs to $M^* = H_1(Y_1(l), \mathbb{C})$. Using easy considerations on fundamental domains, one shows that the intersection pairing of the image in M of $[y]$ and of $\sum_x \lambda_x c_x$ is equal to λ_y. For details, we refer to [10].

2.3. We will now calculate explicitly the natural map $\pi\colon M^* \to M$, which is the composition of canonical topological maps

$$M^* = H_1(Y_1(l), \mathbb{C}) \to H_1(X_1(l), \mathbb{C}) \to H_1(X_1(l), \partial X_1(l); \mathbb{C}) = M.$$

Proposition 2.4. *Let $x \mapsto g_x$ be a section of the surjective map $SL_2(\mathbb{Z}) \to \pm\Gamma_1(l)\backslash SL_2(\mathbb{Z})$. Let $\sum_x \lambda_x c_x \in M^*$. The map π is given by the following formulas:*

$$\pi(\sum_x \lambda_x c_x) = \frac{1}{6}\sum_x (\lambda_{x\tau} - \lambda_{x\tau^2})\{g_x 0, g_x\infty\} = \frac{1}{6}\sum_x (\lambda_{x\sigma\tau\sigma} - \lambda_{x\tau^2})\{g_x 0, g_x\infty\},$$

where the sums are taken over $x \in \pm\Gamma_1(l)\backslash SL_2(\mathbb{Z})$.

Proof. This is a simple computation. One has

$$\pi(\sum_x \lambda_x c_x) = \sum_x \lambda_x\{g_x\mathrm{i}, g_x\rho\} = \sum_x \lambda_x\{g_x\mathrm{i}, g_x\infty\} - \sum_x \lambda_x\{g_x\rho, g_x\infty\}.$$

Here we are abusing the notation $\{\alpha, \beta\}$, which now also denotes the arc in $X_1(l)$ that is the image of an arc in $\mathfrak{H}^*$.

We use the fact that $g_{x\sigma}$ and $g_x\sigma$ lie in the same coset of $\pm\Gamma_1(l)$ (and similarly for τ) to rewrite the right hand side of the above expression as

$$\frac{1}{2}\sum_x(\lambda_x\{g_x\mathrm{i}, g_x\infty\}+\lambda_{x\sigma}\{g_x\sigma\mathrm{i}, g_x\sigma\infty\})-\frac{1}{3}\sum_x(\lambda_x\{g_x\rho, g_x\infty\}+\lambda_{x\tau}\{g_x\tau\rho, g_x\tau\infty\}+\lambda_{x\tau^2}\{g_x\tau^2\rho, g_x\tau^2\infty\}).$$

Using the relations $\sigma\mathrm{i} = \mathrm{i}$, $\tau\rho = \rho$, $\lambda_x + \lambda_{x\sigma} = 0$ and $\lambda_x + \lambda_{x\tau} + \lambda_{x\tau^2} = 0$, we have

$$\frac{1}{2}\sum_x(\lambda_x\{g_x\infty, g_x\infty\}+\lambda_{x\sigma}\{g_x\infty, g_x\sigma\infty\})-\frac{1}{3}\sum_x(\lambda_x\{g_x\infty, g_x\infty\}+\lambda_{x\tau}\{g_x\infty, g_x\tau\infty\}+\lambda_{x\tau^2}\{g_x\infty, g_x\tau^2\infty\}).$$

Using the relations $\tau\infty = 0$ and $\{g_x\infty, g_x\infty\} = 0$ and reindexing the last set of terms of the sum, we obtain

$$\frac{1}{2}\sum_x \lambda_x\{g_x 0, g_x\infty\} - \frac{1}{3}\sum_x \lambda_{x\tau}\{g_x\infty, g_x 0\} - \frac{1}{3}\sum_x \lambda_x\{g_x\tau\infty, g_x\infty\}.$$

Using again the relations $\tau\infty = 0$, $\{g_x\infty, g_x 0\} = -\{g_x 0, g_x\infty\}$, and $\lambda_x + \lambda_{x\tau} + \lambda_{x\tau^2} = 0$, we arrive at the first equality of the statement. To get the second equality, one changes the index to $x\sigma$ and uses the relation $\lambda_x + \lambda_{x\sigma} = 0$. ☐

Remark 2.5. Proposition 2.4 remains valid if $\Gamma_1(l)$ is replaced by any finite-index subgroup of $SL_2(\mathbb{Z})$.

2.6. The elements of $\Gamma_1(l)\backslash SL_2(\mathbb{Z})$ can be represented by pairs (u, v), where $u, v \in \mathbb{Z}/l\mathbb{Z}$ and *g.c.d.*$(u, v, l) = 1$. The discussion above shows that M can be described as the $\mathbb{C}$-vector space generated by the symbols (u, v) modulo the relations

1. $(u, v) + (-v, u) = 0$.
2. $(u, v) + (v, -u - v) + (-u - v, v) = 0$.

Pairs (u, v) are called *Manin symbols*. Two subspaces of M will play an important role in what follows. Let $\iota\colon M \to M$ be the involution that takes $(u, v) \mapsto (-u, v)$.

Definition 2.7. The space of *plus symbols* $M_+ \subset M$ is the subspace consisting of symbols x satisfying $\iota(x) = x$. Similarly, the space of *minus symbols* $M_- \subset M$ is the subspace consisting of symbols x satisfying $\iota(x) = -x$.

We have *symmetrization maps* $(\ ,\ )_\pm \colon M \to M_\pm$ given by $(u,v)_\pm := ((u,v) \pm (-u,v))/2$. We also introduce the corresponding spaces of cuspidal symbols $S_\pm \subseteq M_\pm$. The spaces $S_\pm$ can each be seen as the dual of the space of cusp forms as follows. Let $\mathscr{M}(l)$ be the $\mathbb{C}$-vector space of weight two holomorphic modular forms on $\Gamma_1(l)$, and let $\mathscr{S}(l) \subset \mathscr{M}(l)$ be the subspace of cusp forms. Let $(u,v) \in M$, and let the cusps corresponding to u,v be α,β respectively. The pair α,β induces a geodesic on $X_1(l)$; hence given any $f \in \mathscr{S}(l)$, we can form the integral

$$\int_\alpha^\beta f(z)\,dz \in \mathbb{C},$$

which converges since f is a cusp form. In this way we identify Manin symbols with functionals on cusp forms, and likewise cusp forms with elements of the dual space M^*. We obtain a pairing

$$\begin{aligned} M \times \mathscr{S}(l) &\longrightarrow \mathbb{C}, \\ ((u,v),f) &\longmapsto \langle f,(u,v)\rangle. \end{aligned}$$

In general this pairing is degenerate, although we have the following result:

Proposition 2.8. [9, Proposition 8] *The pairings*

$$S_\pm \times \mathscr{S}(l) \to \mathbb{C}$$

are nondegenerate.

Remark 2.9. The involution $(u,v) \mapsto (-u,v)$ on Manin symbols is induced from the action of the map $\tau \mapsto -\bar{\tau}$, $\tau \in \mathfrak{H}$ on geodesics.

2.10. Let (u,v) be a Manin symbol. For any $\varphi \in M^*$, we define φ on "degenerate" symbols (u,v) with $\mathbb{Z}u + \mathbb{Z}v \neq \mathbb{Z}/l\mathbb{Z}$ by setting $\varphi(u,v) = 0$. This convention is somewhat artificial but turns out to be quite useful.

We now rewrite the map π of Proposition 2.4 on $(M_+)^*$ in a form that will be useful later.

Proposition 2.11. *The image of M_+^* under π is S_-. For any element of $\varphi \in (M_+)^*$, we have*

$$\pi(\varphi) = \frac{1}{12} \sum_{a,b=0}^{l-1} \varphi((a,a-b)_+ - (a,a+b)_+)(a,b)_-.$$

In addition, $\pi(M_+^*) = S_-$.

Proof. Because ι comes from the orientation-reversing automorphism of $X_1(l)$, it anticommutes with π. The surjectivity of the map $M^* = H_1(Y_1(l),\mathbb{C}) \to H_1(X_1(l),\mathbb{C})$ then implies $\pi(M_+^*) = S_-$. The second part of the statement follows from the second equality in Proposition 2.4 and the definitions of the symmetrization maps. The coefficient is changed to $\frac{1}{12}$ because the sum is now over $\Gamma_1(l)\backslash SL_2(\mathbb{Z})$ instead of $\pm\Gamma_1(l)\backslash SL_2(\mathbb{Z})$. $\square$

2.12. To conclude this section, we present Merel's description of the Hecke action on the Manin symbols. Let $n \geq 1$ be an integer, and let T_n be the associated Hecke operator (cf. [7]). We denote the action of T_n on a modular form f by $f\,\big|\,T_n$.

Theorem 2.13. [9, Theorem 2 and Proposition 10] *The operator T_n acts on any Manin symbol (u,v) via*

$$T_n(u,v) = \sum_{\substack{a>b\geq 0 \\ d>c\geq 0 \\ ad-bc=n}} (au+cv, bu+dv). \tag{1}$$

If n is not coprime to l, then we omit the terms for which g.c.d.$(l, au+cv, bu+dv) > 1$. This action is compatible with the pairing between cusp forms and Manin symbols

$$\langle f \,|\, T_n, (u,v)\rangle = \langle f, T_n(u,v)\rangle.$$

It is also easy to show that this Hecke action is compatible with the symmetrization maps:

Proposition 2.14.

$$T_n((u,v)_\pm) = (T_n(u,v))_\pm.$$

Proof. This follows from switching a with d and b with c in (1). ☐

3. Toric forms of weight two

3.1. Let us briefly review the contents of [1]. For every integer $l > 1$, we defined a certain Hecke-stable subring of the ring of modular forms for $\Gamma_1(l)$, called the subring of *toric forms* $\mathscr{T}(l)$. In the present paper we are only concerned with weight two toric forms, which greatly simplifies the combinatorial data needed to encode toric varieties.

Let $N = \mathbb{Z}^2$ be a lattice of rank two, where lattice simply means a free abelian group. A (compact) toric variety of dimension two is defined uniquely by a collection of k rational rays from the origin, such that the angle between any two consecutive rays is less than π. We denote the minimum nonzero lattice points on these rays by d_i, $i = 0, \dots, k$, where i increases counterclockwise, and where $d_0 = d_k$.

To every such collection one associates a *fan* Σ, which is a collection of $2k+1$ cones in $N_{\mathbb{Q}}$. This fan contains k two-dimensional cones

$$\{\mathbb{Q}_{\geq 0} d_i + \mathbb{Q}_{\geq 0} d_{i+1} \mid i = 0, \dots, k-1\},$$

k one-dimensional cones

$$\{\mathbb{Q}_{\geq 0} d_i \mid i = 0, \dots, k-1\},$$

and one zero-dimensional cone $\{0\}$. The corresponding toric variety is smooth if and only if (d_i, d_{i+1}) is a basis of $\mathbb{Z}^2$ for every i.

To define a toric form we need an additional piece of data, namely a *degree function* with respect to Σ. This is a piecewise-linear function $\deg\colon N \to \mathbb{Q}$ that is linear on the cones of Σ. Every such function is uniquely determined by the values $\alpha_i = \deg(d_i)$.

Definition 3.2. [1] Suppose $\deg\colon N \to \mathbb{Q}$ is a degree function with respect to the fan Σ, and that $\alpha_i \notin \mathbb{Z}$ for all i. Then the *toric form* associated to $(N, \deg)$ is the function $f_{N,\deg}\colon \mathfrak{H} \to \mathbb{C}$ defined by

$$f_{N,\deg}(q) := \sum_{m\in M}\Big(\sum_{C\in\Sigma}(-1)^{\operatorname{codim} C}\,\text{a.c.}\Big(\sum_{n\in C} q^{m\cdot n}\,\mathrm{e}^{2\pi \mathrm{i} \deg(n)}\Big)\Big).$$

Here $M = \operatorname{Hom}(M, \mathbb{Z})$ is the dual of N, $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ where $\tau \in \mathfrak{H}$, and a.c. means analytic continuation of a sum from its region of convergence to all q and m.

It turns out that $f_{N,\deg}$ does not change if Σ is subdivided, and thus does not depend on Σ. This is why Σ is omitted from the notation. A toric form is, by definition, any linear combination of $f_{N,\deg}$. We will now state in one theorem most of the results of [1], specialized to the case of weight two.

Theorem 3.3. [1] *Suppose that* $\deg(N) \subseteq \frac{1}{l}\mathbb{Z}$, *and that* $\alpha_i \notin \mathbb{Z}$ *for all* i. *Then* $f_{N,\deg}$ *is a holomorphic modular form of weight two with respect to* $\Gamma_1(l)$. *If* $l \geq 5$ *then it is a linear combination of pairwise products of the forms* $s_{a/l}$, $a = 1, \ldots, l-1$, *where*

$$s_{a/l}(\tau) = \frac{1}{2\pi \mathrm{i}} \partial_z (\log \vartheta)(\frac{a}{l}, \tau).$$

Here $\vartheta(z,\tau)$ *is the standard theta function* [2, Chapter 5]. *If* $l < 5$, *then the span* $\mathscr{T}_2(l)$ *of all toric forms of level* l *and weight two coincides with the space of all modular forms of weight two; in particular, it consists only of Eisenstein series. The space* $\mathscr{T}_2(l)$ *is stable under the action of Hecke operators and the Fricke involution. The space of all weight two toric forms of all levels is stable under Atkin-Lehner liftings* $f(\tau) \mapsto f(n\tau)$.

Let p be a prime not dividing l. We will need an explicit formula for the action of the Hecke operator T_p on $f_{N,\deg}$. This follows immediately from a formula in [1, Theorem 5.3] specialized to the case of weight two.

Proposition 3.4. *Let* $f_{N,\deg}$ *be a toric form of weight two. Then*

$$f_{N,\deg} \big| T_p = \sum_S f_{S, p \deg},$$

where the sum is taken over all superlattices $S \supseteq N$ *with* $[S:N] = p$.

3.5. It is not hard to write down the explicit q-expansions of $s_{a/l}$.

Proposition 3.6. *Denote* $w = \exp(2\pi \mathrm{i}/l)$. *Then*

$$s_{a/l}(q) = \frac{w^a + 1}{2(w^a - 1)} - \sum_d q^d \sum_{k|d} (w^{ka} - w^{-ka}).$$

Proof. It is easy to compute the logarithmic derivative of ϑ using the Jacobi triple product formula [2, Chapter 5, Theorem 6]. Details are left to the reader. □

In [1] we introduced weight two modular forms $s^{(2)}_{a/l}$ given by

$$s^{(2)}_{a/l}(q) = -\frac{1}{4\pi^2} \left(\frac{\partial^2}{\partial z^2}\right)_{z=0} \log \left(\frac{z\vartheta(z + a/l)\vartheta'(0)}{\vartheta(z)\vartheta(a/l)} \right).$$

These are also toric forms, and the following relations allow one to express them as linear combinations of products $s_{a/l}s_{b/l}$ for $l \geq 5$.

Proposition 3.7. *If* $a + b + c = 0 \bmod l$ *and* $a, b, c \neq 0 \bmod l$ *then*

$$s_{a/l}s_{b/l} + s_{b/l}s_{c/l} + s_{c/l}s_{a/l} = -\frac{1}{2}\left((s_{a/l})^2 + (s_{b/l})^2 + (s_{c/l})^2 + s^{(2)}_{a/l} + s^{(2)}_{b/l} + s^{(2)}_{c/l} \right).$$

Proof. This is a consequence of a more general formula of [1]. □

Proposition 3.8. *The modular forms* $(s_{a/l})^2$ *and* $s^{(2)}_{a/l}$ *are Eisenstein series for all* a.

Proof. As in Proposition 3.6, we get

$$(2\pi \mathrm{i})^{-1}\partial_z \log\vartheta(z,\tau) = \frac{\mathrm{e}^{2\pi \mathrm{i} z}+1}{2(\mathrm{e}^{2\pi \mathrm{i} z}-1)} - \sum_{d>0} q^d \sum_{k|d} (\mathrm{e}^{2\pi \mathrm{i} k z} - \mathrm{e}^{-2\pi \mathrm{i} k z}).$$

Differentiating it again with respect to z and plugging in $z = a/l$, we get

$$(2\pi \mathrm{i})^{-2}\left(\frac{\partial^2}{\partial z^2}\right)\log\vartheta(\frac{a}{l},\tau) = \frac{\mathrm{e}^{2\pi \mathrm{i} a/l}}{(\mathrm{e}^{2\pi \mathrm{i} a/l}-1)^2} - \sum_{d>0} q^d \sum_{k|d} k(\mathrm{e}^{2\pi \mathrm{i} k a/l} + \mathrm{e}^{-2\pi \mathrm{i} k a/l}).$$

Notice now that

$$\begin{aligned} s^{(2)}_{a/l}(\tau) &= (2\pi \mathrm{i})^{-2}\left(\frac{\partial^2}{\partial z^2}\right)\log\vartheta(\frac{a}{l},\tau) - (2\pi \mathrm{i})^{-2}\left(\frac{\partial^2}{\partial z^2}\right)_{z=0}\log\left(\frac{\vartheta(z,\tau)}{z\partial_z\vartheta(0,\tau)}\right) \\ &= (2\pi \mathrm{i})^{-2}\left(\frac{\partial^2}{\partial z^2}\right)\log\vartheta(\frac{a}{l},\tau) - \frac{1}{12} + 2\sum_{d>0} q^d \sum_{k|d} k \\ &= \frac{\mathrm{e}^{2\pi \mathrm{i} a/l}}{(\mathrm{e}^{2\pi \mathrm{i} a/l}-1)^2} - \frac{1}{12} - \sum_{d>0} q^d \sum_{k|d} k(\mathrm{e}^{2\pi \mathrm{i} k a/l} + \mathrm{e}^{-2\pi \mathrm{i} k a/l} - 2), \end{aligned}$$

which is an Eisenstein series. Indeed, in the notation of [13, Chapter VII], it is

$$\frac{l^2}{4\pi^2}\left(G_{l,2,\binom{0}{a}} - G_{l,2,\binom{0}{0}}\right).$$

To obtain a nice formula for $s^2_{a/l}$, recall that ϑ satisfies the heat equation

$$\vartheta_{zz} = (4\pi \mathrm{i})\vartheta_\tau,$$

which implies

$$(2\pi \mathrm{i})^{-2}\left(\frac{\vartheta_{zz}}{\vartheta}\right)_z = (\pi \mathrm{i})^{-1}\left(\frac{\vartheta_z}{\vartheta}\right)_\tau = -4\pi \mathrm{i}\sum_{d>0} dq^d \sum_{k|d}(\mathrm{e}^{2\pi \mathrm{i} k z} - \mathrm{e}^{-2\pi \mathrm{i} k z}).$$

We can integrate it with respect to z while keeping in mind that

$$(2\pi \mathrm{i})^{-2}\lim_{z\to 0}\frac{\vartheta_{zz}(z,\tau)}{\vartheta(z,\tau)} = (2\pi \mathrm{i})^{-2}\frac{\vartheta_{zzz}(0,\tau)}{\vartheta_z(0,\tau)} = \frac{1}{4} - 6\sum_{d>0} q^d \sum_{k|d}\frac{d}{k}$$

to obtain

$$s^{(2)}_{a/l} + s^2_{a/l} = (2\pi \mathrm{i})^{-2}(\frac{\vartheta_{zz}(a/l,\tau)}{\vartheta(a/l,\tau)} - \frac{\vartheta_{zzz}(0,\tau)}{3\vartheta_z(0,\tau)}) = \frac{1}{6} - 2\sum_{d>0} q^d \sum_{k|d}\frac{d}{k}(\mathrm{e}^{2\pi \mathrm{i} ak/l} + \mathrm{e}^{-2\pi \mathrm{i} ak/l}),$$

which is again an Eisenstein series. In the notation of [13, Chapter VII] it is equal to

$$\frac{l}{2\pi^2}\sum_{a_1,a_2=0}^{l-1}\mathrm{e}^{2\pi \mathrm{i} a a_1/l} G_{l,2,\binom{a_1}{a_2}}.$$

□

Proposition 3.9. *For every even function $\chi\colon \mathbb{Z}/l\mathbb{Z} \to \mathbb{C}$ the series*

$$\sum_{d>0} q^d \sum_{k|d} k\,\chi(k) \qquad \text{and} \qquad \sum_{d>0} q^d \sum_{k|d}\frac{d}{k}\,\chi(k)$$

lie in the linear span of 1, toric Eisenstein series and the standard Eisenstein series E_2.

Proof. Notice that the functions χ_a defined by $k \mapsto e^{2\pi i k a}$ span the space of all functions from $\mathbb{Z}/l\mathbb{Z}$ to $\mathbb{C}$, so the functions $\chi_a + \chi_{-a}$ span the space of all even functions from $\mathbb{Z}/l\mathbb{Z}$ to $\mathbb{C}$. Then use explicit formulas for $s^{(2)}_{a/l}$ and $s^{(2)}_{a/l} + s^2_{a/l}$ from the proof of the above proposition. □

3.10. Recall that $\mathscr{M}(l)$ is the space of all holomorphic modular forms for $\Gamma_1(l)$ of weight two, and denote by $\mathscr{E}(l)$ the subspace of Eisenstein series. Proposition 3.8 allows us to define a map from the space of Manin symbols to toric modular forms modulo Eisenstein series.

Definition 3.11. Let M_- be the space of minus Manin symbols (Definition 2.7). We define a map $\mu\colon M_- \to \mathscr{M}(l)/\mathscr{E}(l)$ by
$$\mu((a,b)_-) = s_{a/l}s_{b/l} \bmod \mathscr{E}(l)$$
if $a, b \neq 0 \bmod l$, and by 0 otherwise.

Remark 3.12. The map μ is well-defined. Indeed, it is clear that $(a,b)_- + (-a,b)_-$ and $(a,b)_- - (b,a)_-$ map to zero, even before we mod out by $\mathscr{E}(l)$. It remains to show that $\mu((a,b)_- + (b,c)_- + (c,a)_-) = 0$ for $a+b+c = 0 \bmod l$. This follows from Propositions 3.7 and 3.8 if a, b, c are non-zero, and from Proposition 3.8 if one of them is zero.

3.13. We will now show that map μ commutes with the action of the Hecke operators T_p for primes p not dividing l. To do this, we recall the description of toric forms in terms of cohomology rings of smooth toric varieties [1].

Let deg be a degree function with respect to Σ in N. Let $\widehat{\Sigma}$ be a subdivision of Σ such that the corresponding toric variety $\widehat{X}$ is smooth. Denote by d_i the minimum nonzero lattice points of the one-dimensional cones of $\widehat{\Sigma}$. Since $\widehat{\Sigma}$ is a subdivision of Σ, it is possible that some $\alpha_i = \deg(d_i)$ are now integral. To circumvent this difficulty, in [1] we introduced a generic degree function $\deg_1$, linear on all cones of the original fan Σ and such that the values $\beta_i = \deg_1(d_i)$ are non-zero.

Proposition 3.14. ([1]) *In the above notation, we have*
$$f_{N,\deg}(q) = \lim_{\varepsilon\to 0} \int_{\widehat{X}} \prod_i \frac{(D_i/2\pi i)\vartheta(D_i/2\pi i - \deg(d_i) - \varepsilon \deg_1(d_i), \tau)\vartheta'(0,\tau)}{\vartheta(D_i/2\pi i, \tau)\vartheta(-\deg(d_i) - \varepsilon\deg_1(d_i), \tau)} \tag{2}$$
where D_i is the cohomology class of the toric divisor corresponding to the one-dimensional cone $\mathbb{Q}_{\geq 0} d_i$ (cf. [3]) and the integral means pairing with the fundamental class of $\widehat{X}$.

If we work modulo Eisenstein series, the above formula greatly simplifies.

Proposition 3.15. *Let $d_1, \dots, d_k$ be generators of one-dimensional cones of $\widehat{\Sigma}$. We denote $d_0 = d_k$. Then*
$$f_{N,\deg} = \sum_{\substack{0\leq i\leq k-1\\ \alpha_i,\alpha_{i+1}\notin\mathbb{Z}}} s_{\alpha_i}s_{\alpha_{i+1}} \bmod \mathscr{E}(l).$$

Proof. Let us first explain what happens in the case where all α_i are non-integral. We are integrating over $\widehat{X}$ the product of cohomology elements
$$(1 - s_{\alpha_i}D_i + r_i D_i^2),$$
where r_i is easily seen to be an Eisenstein series due to Proposition 3.8. The intersection number $\int_{\widehat{X}} D_i D_j$ for $i \neq j$ equals 1 if d_i and d_j come from adjacent cones and equals 0 otherwise. This finishes the argument.

When some of the α_i are zero the argument is a bit more complicated and involves the expansion of the right hand side of (2) in powers of ε. Then up to Eisenstein series one ends up with the integral of the product of $(1 - s_{\alpha_i}D_i)$ over i for which $\alpha_i \notin \mathbb{Z}$. Details are left to the reader. □

Theorem 3.16. *The map $\mu\colon M_- \to \mathscr{M}(l)/\mathscr{E}(l)$ defined above is invariant under the action of the Hecke operators T_p for primes p coprime to l.*

Proof. We will identify N with the lattice of integer row vectors with two components. Consider the fan Σ that has $d_1 = (1,0)$, $d_2 = (0,1)$, $d_3 = (-1,0)$, and $d_4 = (0,-1)$ as generators of its one-dimensional faces, and a degree function on Σ defined by $\deg(1,0) = \deg(-1,0) = m/l$ and $\deg(0,1) = \deg(0,-1) = n/l$. Proposition 3.15 shows that

$$f_{N,\deg} = 4 s_{m/l} s_{n/l} \bmod \mathscr{E}(l).$$

Let us calculate the action of the Hecke operator T_p on this form modulo $\mathscr{E}(l)$. By Proposition 3.4 we have

$$f_{N,\deg}\big| T_p = \sum_{N\subset S\subset \frac{1}{p}N} f_{S,p\deg}.$$

Let us investigate the contribution of each S. To get into the setup of Proposition 3.15, we need a subdivision $\widehat{\Sigma}_S$ of the fan Σ so that the consecutive d_i form a basis. There is a standard way of doing so. For each quadrant (i.e. for each two-dimensional cone of Σ), consider the set A of all non-zero points of S in that quadrant. The boundary of the convex hull of A consists of two half-lines and some segments (Figure 1). We ignore the half lines and add to the list of d_i all points in S that lie on the rest of the boundary. It is easy to show that this choice of d_i guarantees that the new toric variety is smooth. Indeed, if d_i and d_{i+1} did not form a basis of S then there would exist a point in S lying in the convex hull of 0, d_i and d_{i+1} by Pick's Theorem [3, page 113].

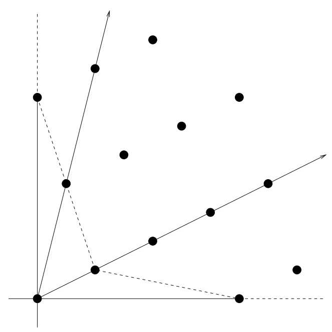

FIGURE 1.

Because of the symmetry, it is enough to consider the first quadrant. Notice that if $d_i = (\frac{a}{p}, \frac{c}{p})$ and $d_{i+1} = (\frac{b}{p}, \frac{d}{p})$, then $ad - bc = p$, with $a > b \geq 0$ and $d > c \geq 0$. Conversely, every pair of $\{(\frac{a}{p}, \frac{c}{p}), (\frac{b}{p}, \frac{d}{p})\}$ with (a,b,c,d) as above generates some superlattice S of coindex p. Moreover, this pair will form a segment of the boundary of the convex hull of all non-zero points of S in the first quadrant. Indeed, if any other non-zero point $(x,y) \in S$ with $x, y \geq 0$ did lie below the line through these two points, then the area of the triangle with vertices (x,y), $(\frac{a}{p}, \frac{c}{p})$ and $(\frac{b}{p}, \frac{d}{p})$ would be positive but less than $\frac{1}{p}$.

The values of $p\deg$ on the points $(\frac{a}{p}, \frac{c}{p})$ and $(\frac{b}{p}, \frac{d}{p})$ are $\frac{am+cn}{l}$ and $\frac{bm+dn}{l}$ respectively, hence by Proposition 3.15 we have

$$f_{N,\deg}\big| T_p = 4 \sum_{\substack{ad-bc=p, a>b\geq 0, d>c\geq 0,\\ (am+cn)/l\notin\mathbb{Z}, (bm+dn)/l\notin\mathbb{Z}}} s_{(am+cn)/l}\, s_{(bm+dn)/l} \bmod \mathscr{E}(l).$$

Therefore, from the definition of μ, Theorem 2.13 and Proposition 2.14 we conclude that

$$\mu(T_p(m,n)_-) = \mu((m,n)_-)\,\big|\,T_p.$$

□

Proposition 3.17. *The image of M_- under μ coincides with the image of S_- under μ.*

Proof. This follows from Theorem 3.16 and the fact that M is isomorphic as a Hecke-module to $S_- \oplus S_+ \oplus \mathscr{E}(l)$ [9, Section 4.2]. □

3.18. Finally, we explicitly describe the composition of μ and the Fricke involution.

Proposition 3.19. *The composition of μ and the Fricke involution W_l is given by*

$$W_l \circ \mu((m,n)_-) = \widetilde{s}_{m/l}(q)\widetilde{s}_{n/l}(q) \bmod \mathscr{E}(l)$$

where $\widetilde{s}_{0/l} = 0$ and

$$\widetilde{s}_{a/l}(q) = (\frac{a}{l} - \frac{1}{2}) - \sum_{d \geq 1} q^d \sum_{k\,|\,d} (\delta_k^{a \bmod l} - \delta_k^{-a \bmod l})$$

for $a = 1, \ldots, l-1$.

Proof. All we need to do is calculate the Fricke involute of $s_{a/l}$. This was accomplished in [1], up to the constant term. To compute the constant term, we remark that these forms have already appeared in the literature as the *Hecke-Eisenstein forms* [7, Chapter 15]. □

4. Forms of analytic rank zero

4.1. In this subsection we define a linear map $\rho\colon \mathscr{S}(l) \to \mathscr{S}(l)$ whose image is spanned by Hecke eigenforms of analytic rank zero.

Definition 4.2. Let $f \in \mathscr{S}(l)$ be a cusp form. We define a linear map ρ by

$$\rho(f) = \sum_{n=1}^{\infty} \left(\int_0^{i\infty} (f\,|\,T_n)(s)ds \right) q^n.$$

Proposition 4.3. *The form $\rho(f)$ is a cusp form with nebentypus equal to that of f.*

Proof. This follows from [9, Theorem 6], since $\rho(f)$ is associated to the linear map on the Hecke algebra that maps T to $\int_0^{i\infty} (f\,|\,T)(s)\,ds$. □

Remark 4.4. It is easy to see that for a Hecke eigenform f that is a newform

$$\rho(f) = L(f,1)f.$$

In particular, the image of the space of newforms is the span of the new Hecke eigenforms that have analytic rank zero.

Proposition 4.5. *The image of ρ is contained in the span of all lifts of all new Hecke eigenforms of analytic rank zero for all levels k, $k|l$.*

Proof. It is clear that if f is a lift of an eigenform g of analytic rank one or more, then $\rho(f)=0$. Indeed, for every n the form $f\big|T_n$ is a linear combination of various lifts of g, which implies $L(f\big|T_n,1)=0$. It remains to show that if f is a lift of g then $\rho(f)$ is a linear combination of lifts of g. This follows from the commutation relation $\rho(f)\big|T_p=\rho(f\big|T_p)$ for all prime p coprime to l. Indeed, from the definition of T_p [7],

$$\begin{aligned}\rho(f)\big|T_p &= \sum_{m>0} L(f\big|T_{mp},1)q^m + p\epsilon_p(\rho(f))\sum_{m>0} L(f\big|T_m,1)q^{mp}\\ &= \sum_{m>0,(m,p)=1} L(f\big|T_{mp},1)q^m + \sum_{m>0,(m,p)=p} L(f\big|(T_{mp}+p\epsilon_p(f)T_{m/p}),1)q^m\\ &= \sum_{m>0} L(f\big|T_pT_m)q^m = \rho(f\big|T_p).\end{aligned}$$

□

Remark 4.6. Even though we suspect that the image of ρ coincides with the above span, we only need the inclusion proved in the above proposition.

4.7. We will now prove a key result that relates the map ρ to the map $W_l\circ\mu$ constructed in the previous section.

Theorem 4.8. *The composition*

$$\mathscr{S}(l)\xrightarrow{\int} M_+^*\xrightarrow{\pi} M_-\xrightarrow{W_l\circ\mu}\mathscr{M}(l)/\mathscr{E}(l)\xrightarrow{\sim}\mathscr{S}(l), \tag{3}$$

of the map induced by integration between cusps, the map π from §2.3, the map $W_l\circ\mu$, and the Hecke-equivariant isomorphism between $\mathscr{M}(l)/\mathscr{E}(l)$ and $\mathscr{S}(l)$, equals ρ.

Proof. Let f be a cusp form, and let $\varphi\colon M_+\to\mathbb{C}$ be the corresponding element of M_+^*. Then by Theorem 2.13 and Proposition 2.14,

$$\begin{aligned}\rho(f)(q) &= \sum_{n>0} q^n\langle f\big|T_n,(1,0)\rangle = \sum_{n>0} q^n\langle f,T_n(1,0)\rangle\\ &\qquad = -\sum_{n>0} q^n\varphi(T_n(0,1)) = -\sum_{n>0} q^n \sum_{\substack{ad-bc=n,\\ a>b\geq 0,d>c\geq 0}}\varphi((c,d)_+).\end{aligned}$$

Notice that we are using our convention that $\varphi(c,d)=0$ for $g.c.d.(c,d,l)>1$. On the other hand, by Theorem 2.11 the composition of all the maps in (3) except for the last one yields an element of $\mathscr{M}(l)/\mathscr{E}(l)$ given by

$$\rho_1(f)(q)=\frac{1}{12}\sum_{a,b=0}^{l-1}(\varphi((a,a-b)_+)-\varphi((a,a+b)_+))\widetilde{s}_{a/l}(q)\widetilde{s}_{b/l}(q),$$

where we again apply our convention. Also, we can formally use the same expression for $\widetilde{s}_{0/l}$ as for the rest of $\widetilde{s}_{a/l}$ because the coefficient at $\widetilde{s}_{0/l}\widetilde{s}_{b/l}$ is zero.

In what follows, it will be convenient for us to ignore the constant terms in all our expressions. Indeed, all our functions are modular forms, so any constant term can always be restored. We will denote all these constant terms by C. Using

$$\widetilde{s}_{a/l}\widetilde{s}_{b/l}=C+\sum_{n>0}q^n\Big(\sum_{k|n,k>0}(\frac{1}{2}-\frac{a}{l})(\delta_k^{b\bmod l}-\delta_k^{-b\bmod l})+$$

$$\sum_{k|n,k>0}(\frac{1}{2}-\frac{b}{l})(\delta_k^{a\bmod l}-\delta_k^{-a\bmod l})+\sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0}}(\delta_{k_1}^{a\bmod l}-\delta_{k_1}^{-a\bmod l})(\delta_{k_2}^{b\bmod l}-\delta_{k_2}^{-b\bmod l})\Big)$$

and symmetry properties of $\varphi((a,a-b)_+)-\varphi(a,a+b)_+)$, we get

$$\rho_1(f)(q)=C+\frac{1}{3}\sum_{n>0}q^n\Big(\sum_{k|n,k>0}\sum_{a=0}^{l-1}(\frac{1}{2}-\frac{a}{l})(\varphi((a,k-a)_+)-\varphi((a,k+a)_+))$$
$$+\sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0}}(\varphi((k_1,k_1-k_2)_+)-\varphi((k_1,k_1+k_2)_+))\Big).$$

Let us now simplify the second part of this expression to make it look more like $\rho(f)$. We split it into four sums

$$\sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0\\ k_1\geq k_2}}\varphi((k_1,k_1-k_2)_+)+\sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0\\ k_1<k_2}}\varphi((k_1,k_1-k_2)_+)$$
$$-\sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0\\ m_1>m_2}}\varphi((k_1,k_1+k_2)_+)-\sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0\\ m_1\leq m_2}}\varphi((k_1,k_1+k_2)_+) \tag{4}$$

and deal with each sum separately. We will give a detailed calculation for one of the sums and will indicate how to manipulate the other three.

Lemma 4.9.

$$\sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0\\ m_1>m_2}}\varphi((k_1,k_1+k_2)_+)=\sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0\\ m_1>m_2}}\varphi((k_1,k_2)_+)+\sum_{\substack{ad-bc=n,\\ a>b>0,d>c>0}}\varphi((c,d)_+)$$

Proof of the lemma. We first of all rewrite $\varphi((k_1,k_1+k_2)_+)$ as $\varphi((k_1,k_2)_+)+\varphi((k_2,k_1+k_2)_+)$. Then for the second term we make the change of variables $(a,b,c,d)=(m_1,m_1-m_2,k_2,k_1+k_2)$. □

We perform similar but easier manipulations for the remaining three sums in (4). For the first sum we make the change of variables $(a,b,c,d)=(m_1+m_2,m_2,k_1-k_2,k_1)$. For the second sum we make the change of variables $(m_1',m_2',k_1',k_2')=(m_1+m_2,m_2,k_1,k_2-k_1)$, so that it cancels the first sum in Lemma 4.9. For the fourth sum in (4) we make the change of variables $(a,b,c,d)=(m_2,m_2-m_1,k_1,k_1+k_2)$. After some straightforward calculations we get

$$\sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0}}(\varphi((k_1,k_1-k_2)_+)-\varphi((k_1,k_1+k_2)_+))=-3\sum_{\substack{ad-bc=n,\\ a>b\geq 0,d>c\geq 0}}\varphi((c,d)_+)$$
$$-\sum_{0<k|n}(\frac{2n}{k}+1)\varphi((k,0)_+)-2\sum_{k_1|n,k_1>k_2>0}\varphi((k_1,k_1-k_2)_+).$$

As a result,

$$\rho_1(f)(q)-\rho(f)(q)=C+\frac{1}{3}\sum_{n>0}q^n\sum_{k|n,k>0}\Big(\sum_{a=0}^{l-1}(\frac{1}{2}-\frac{a}{l})(\varphi((k,a-k)_+)-\varphi((k,a+k)_+))$$
$$-(\frac{2n}{k}-1)\varphi((k,0)_+)-2\sum_{0\leq m\leq k}\varphi((k,m)_+)\Big).$$

It is possible to further simplify this equation to obtain

$$\rho_1(f)(q)-\rho(f)(q)=C-\frac{2}{3}\sum_{n>0}q^n\sum_{k|n,k>0}\frac{n}{k}\varphi((k,0)_+)-\frac{2}{3l}\sum_{n>0}q^n\sum_{k|n,k>0}k\Big(\sum_{b=0}^{l-1}\varphi((k,b)_+)\Big),$$

which is an Eisenstein series. Indeed, it can be easily written (up to a constant) as a linear combination of $s^{(2)}_{a/l}$, $s^2_{a/l}$ and the (non-modular) $SL_2(\mathbb{Z})$-Eisenstein series E_2 by Proposition 3.9. It remains to observe that the coefficient of E_2 must be zero, because of the transformation properties of E_2 under $\Gamma_1(l)$. □

4.10. We are now ready to prove our main result.

Theorem 4.11. *For each integer l the space spanned by pairwise products of $s_{a/l}$ for $a=1,\dots,l-1$ is the direct sum of the span of all Hecke eigenforms of analytic rank zero and some subspace of the space of Eisenstein series.*

Proof. We will prove this theorem by induction on l. For small levels there is nothing to prove, because there are simply no cusp forms.

Fix l and assume that the statement of the theorem is true for all smaller levels. In particular, this implies that lifts of all forms of smaller levels are contained in the span of toric forms, because the space of toric forms is stable under liftings, see Theorem 3.3. Every new Hecke eigenform f of analytic rank zero is contained in the image of ρ, see Remark 4.4. By Theorem 4.8, f is contained in the image of $W_l\circ\mu$, and so is a toric form up to Eisenstein series. Because the space of toric forms is Hecke stable, this implies that f is toric. This proves that the space of toric forms contains the span of all Hecke eigenforms of analytic rank zero.

To prove the opposite inclusion, notice that by the induction assumption it is enough to consider $s_{a/l}s_{b/l}$ with $g.c.d.(a,b,l)=1$. By Proposition 3.17, there is an element $x\in S_-$ such that $\mu(x)=s_{a/l}s_{b/l}$. We use here that $g.c.d.(a,b,l)=1$, because otherwise the symbol $(a,b)_-$ is not defined. By the definition of π in §2.3 there is an element $\varphi\in S_+^*$ such that $\pi(\varphi)=x$. Moreover, we can find a cusp form f which induces the linear map φ on S_+. Then Theorem 4.8 shows that $s_{a/l}s_{b/l}$ is proportional to $W_l\circ\rho(f)$ up to an Eisenstein series. By Proposition 4.5 $s_{a/l}s_{b/l}$ lies in the span of Hecke eigenforms of rank zero and Eisenstein series, which finishes the proof. □

Remark 4.12. It is easy to see that as a corollary we get similar results for any given nebentypus. In particular, modulo Eisenstein series, the span of the forms

$$\sum_{k\in(\mathbb{Z}/l\mathbb{Z})^*}s_{ka}s_{kb},\quad a,b=1,\dots,l-1$$

is the span of all Hecke eigenforms of analytic rank zero for $\Gamma_0(l)$.

Remark 4.13. In general it is not clear which Eisenstein series are toric forms. In particular at level $l=25$ the space of toric Eisenstein series has codimension one.

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