TORIC MODULAR FORMS AND NONVANISHING OF L-FUNCTIONS

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Abstract. In a previous paper [1], we defined the space of toric forms $\mathscr{T}(l)$, and showed that it is a finitely generated subring of the holomorphic modular forms of integral weight on the congruence group $\Gamma_1(l)$. In this article we prove the following theorem: modulo Eisenstein series, the weight two toric forms coincide exactly with the vector space generated by all cusp eigenforms f such that $L(f,1) \neq 0$. The proof uses work of Merel, and involves an explicit computation of the intersection pairing on Manin symbols.

1. Introduction

1.1. Let $l > 1$ be an integer, let $\mathscr{M}(l)$ be the space of weight two modular forms on the congruence group $\Gamma_1(l) \subset SL_2(\mathbb{Z})$, and let $\mathscr{S}(l)$ be the subspace of cusp forms. Let $f \in \mathscr{S}(l)$ be an eigenform for the Hecke operators T_p, where p is coprime to l, and let $L(f,s)$ be the associated L-function. Then the order of vanishing of $L(f,s)$ at $s=1$ is called the *analytic rank* of f. This terminology comes from the Birch and Swinnerton-Dyer conjecture, which asserts that the analytic rank is the same as rank of the group of rational points $A(\mathbb{Q})$, where A is the abelian variety associated to f by the Eichler-Shimura construction (cf. [4, 9]).

1.2. In this paper we present an elementary construction of the subspace of $\mathscr{S}(l)$ spanned by forms of analytic rank zero. Our main result (Theorem 5.11) is that, modulo Eisenstein series, this space is isomorphic as a Hecke module to the space $\mathscr{T}_2(l)$ of weight two *toric modular forms* of level l. These modular forms were constructed and studied in [1], where we presented explicit generators of $\mathscr{T}(l)$ and described their q-expansions at infinity. The construction of $\mathscr{T}(l)$, and its properties we shall need, are summarized in Theorem 4.3. For the remainder of this introduction, we describe the proof of Theorem 5.11.

1.3. First, in §2 we recall results about the *Manin symbols.* We define the *plus* (respectively *minus*) symbols M_+ (resp. M_-), and their *cuspidal* subspaces S_+ and S_-. We recall the nondegerate integration pairing between cusp forms and the cuspidal Manin symbols. We finish this section by recalling Merel's description of the Hecke action on Manin symbols (Theorem 2.10).

Next, in §3 we study the intersection pairing on Manin symbols. First we show how the intersection pairing can be computed as a sum a local contributions, Then we define a map $\pi\colon M_+^* \to M_-$, where $M_+^* = \operatorname{Hom}_{\mathbb{C}}(M_+, \mathbb{C})$, and show that π is induced by the nondegenerate intersection pairing between S_+ and S_- (Theorem 3.13).

Next, in §4 we recall results about toric modular forms from [1], specialized to the case of weight two. Using the toric forms, we define a map $\mu\colon M_- \to \mathscr{M}(l)/\mathscr{E}(l)$, where $\mathscr{E}(l) \subset \mathscr{M}(l)$ is the subspace of Eisenstein series. The main result (Theorem 4.15) is that μ is Hecke-equivariant. We also describe the composition of μ with the *Fricke involution* W_l.

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Finally, in §5 we define a map $\rho\colon \mathscr{S}(l) \to \mathscr{S}(l)$ whose image is spanned by Hecke eigenforms of analytic rank zero. Then we put all these maps together to get a sequence

$$\mathscr{S}(l) \xrightarrow{\int} M_+^* \xrightarrow{\pi} M_- \xrightarrow{W_l \circ \mu} \mathscr{M}(l)/\mathscr{E}(l) \xrightarrow{\sim} \mathscr{S}(l),$$

where the final map is the Hecke-equivariant isomorphism between $\mathscr{M}(l)/\mathscr{E}(l)$ and $\mathscr{S}(l)$. Then in Theorem 5.8 we show that this sequence equals 12ρ, from which we obtain Theorem 5.11.

1.4. **Acknowledgements.** We thank Dorian Goldfeld for encouraging this work, and Siman Wong for some interesting discussions. After this paper was submitted, Loïc Merel has informed us that [7] contains an intersection formula for $\Gamma_0(n)$ analogous to our formula of §3.

2. Manin symbols

This section closely follows [6], to which the reader is referred for more details.

2.1. Let $l > 1$ be an integer, and let $E_l \subset (\mathbb{Z}/l\mathbb{Z})^2$ be the subset of pairs (u, v) such that $\mathbb{Z}u + \mathbb{Z}v = \mathbb{Z}/l\mathbb{Z}$. The space of *Manin symbols* is the $\mathbb{C}$-vector space generated by the symbols $(u, v) \in E_l$ modulo the relations:

1. $(u, v) + (-v, u) = 0$.
2. $(u, v) + (v, -u - v) + (-u - v, v) = 0$.

Two subspaces of M will play an important role in what follows. Let $\iota\colon M \to M$ be the involution that takes $(u, v) \mapsto (-u, v)$.

Definition 2.2. The space of *plus symbols* $M_+ \subset M$ is the subspace consisting of symbols x satisfying $\iota(x) = x$. Similarly, the space of *minus symbols* $M_- \subset M$ is the subspace consisting of symbols x satisfying $\iota(x) = -x$.

We have *symmetrization maps* $(\ ,\ )_\pm\colon M \to M_\pm$ given by $(u, v)_\pm := ((u, v) \pm (-u, v))/2$.

2.3. Let $\mathfrak{H}$ be the upper halfplane, and let $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$ be the usual compactification obtained by adjoining cusps. Consider the spaces

$$X_1(l) := \Gamma_1(l)\backslash\mathfrak{H}^*, \quad \text{and} \quad \partial X_1(l) := \Gamma_1(l)\backslash(\mathfrak{H}^* \smallsetminus \mathfrak{H})$$

where $\Gamma_1(l) \subset SL_2(\mathbb{Z})$ is the congruence subgroup of matrices that are upper unitriangular modulo l. The space $\partial X_1(l)$ is a finite set of points, called the *cusps*. One can show that the set of cusps is in bijection with the set of symbols

$$\left\{\pm\frac{1}{0}\right\} \cup \left\{\pm\frac{r}{s} \;\middle|\; s \in \mathbb{Z}/l\mathbb{Z}, r \in (\mathbb{Z}/(s,l)\mathbb{Z})^\times\right\},$$

where (s, l) denotes the greatest common divisor of s and l.

2.4. Now let (u, v) be a Manin symbol, and let $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ be the dual space to the Manin symbols. For any $\varphi \in M^*$, we define φ on "degenerate" symbols (u, v) with $\mathbb{Z}u + \mathbb{Z}v \neq \mathbb{Z}/l\mathbb{Z}$ by setting $\varphi(u, v) = 0$. This convention is somewhat artificial but turns out to be quite useful.

Choose cusps $\alpha = a/u$ and $\beta = b/v$ such that $av - bu = 1$. The cusps α and β allow us to define a collection of maps $\{D_\gamma \mid \gamma \in \partial X_1(l)\} \subset M^*$ as follows. For any cusp γ and any $(u, v) \in M$, we set

$$D_\gamma(u, v) = \delta_\gamma^\beta - \delta_\gamma^\alpha,$$

where α, β are the cusps constructed above, and the delta function δ_γ^α takes the value 1 or 0 depending on whether $\alpha = \gamma$ or not. One can show that this map is well-defined.

Definition 2.5. The space of *cuspidal Manin symbols* $S \subset M$ is the intersection of the kernels of the maps D_γ, as γ ranges over all the cusps. We let $S_\pm := S \cap M_\pm$, and call these spaces the *cuspidal plus/minus Manin symbols.*

2.6. The spaces $S_\pm$ can each be seen as the dual of the space of cusp forms as follows. Let $\mathscr{M}(l)$ be the $\mathbb{C}$-vector space of weight two holomorphic modular forms on $\Gamma_1(l)$, and let $\mathscr{S}(l) \subset \mathscr{M}(l)$ be the subspace of cusp forms. Let $(u, v) \in M$, and let the cusps corresponding to u, v be α, β respectively. The pair α, β induces a geodesic on $X_1(l)$; hence given any $f \in \mathscr{S}(l)$, we can form the integral

$$\int_\alpha^\beta f(z)\, dz \in \mathbb{C},$$

which converges since f is a cusp form. In this way we identify Manin symbols with functionals on cusp forms, and likewise cusp forms with elements of the dual space M^*. We obtain a pairing

$$\begin{aligned} M \times \mathscr{S}(l) &\longrightarrow \mathbb{C}, \\ ((u, v), f) &\longmapsto \langle f, (u, v)\rangle. \end{aligned}$$

In general this pairing is degenerate, although we have the following result:

Proposition 2.7. [6, Proposition 8] *The pairings*

$$S_\pm \times \mathscr{S}(l) \to \mathbb{C}$$

are nondegenerate.

Remark 2.8. The involution $(u, v) \mapsto (-u, v)$ on Manin symbols is induced from the action of the map $\tau \mapsto -\bar{\tau}, \tau \in \mathfrak{H}$ on geodesics.

2.9. To conclude this section, we present Merel's description of the Hecke action on the Manin symbols. Let $n \geq 1$ be an integer, and let T_n be the associated Hecke operator (cf [5]). We denote the action of T_n on a modular form f by $f \big| T_n$.

Theorem 2.10. [6, Theorem 2 and Proposition 10] *The operator T_n acts on any Manin symbol (u, v) via*

$$T_n(u, v) = \sum_{\substack{a > b \geq 0 \\ d > c \geq 0 \\ ad - bc = n}} (au + cv, bu + dv). \tag{1}$$

If n is not coprime to l, then we omit the terms for which $g.c.d.(l, au + cv, bu + dv) > 1$. This action is compatible with the pairing between cusp forms and Manin symbols

$$\langle f \big| T_n, (u, v)\rangle = \langle f, T_n(u, v)\rangle.$$

It is also easy to show that this Hecke action is compatible with the symmetrization maps:

Proposition 2.11.

$$T_n((u, v)_\pm) = (T_n(u, v))_\pm.$$

Proof. This follows from switching a with d and b with c in (1). ∎

3. The intersection pairing

One can identify the cuspidal Manin symbols S with the homology group $H_1(X_1(l);\mathbb{C})$. This implies that there is a nondegenerate pairing

$$\langle\ ,\ \rangle\colon S \times S \longrightarrow \mathbb{C}$$

induced by intersection of cycles. We use the same notation for the intersection pairing and the pairing induced by integration, which should not cause any confusion. It is easy to see that S_+ and S_- are Lagrangian subspaces for the intersection pairing, that is the intersection of two plus or two minus symbols is zero, and the intersection on $S_+ \times S_-$ is non-degenerate.

In this section we describe the intersection pairing on the Manin symbols by explicitly computing intersection numbers through summing local contributions. We also define a map $\pi\colon M_+^* \to M_-$ that will play an important role in later sections. The main result (Theorem 3.13) is that π is, in a certain sense, induced by the above intersection pairing on S.

3.1. A *simple closed loop* is a class in $H_1(X_1(l);\mathbb{C})$ represented by a simple closed curve on $X_1(l)$ that is a union of geodesics corresponding to Manin symbols. Such classes clearly form a spanning set for $S \cong H_1(X_1(l);\mathbb{C})$.

Let $\xi, \xi' \in S$ be two closed loops. A computation shows that if the end of the geodesic corresponding to a Manin symbol (a,b) is the beginning of the geodesic corresponding to (c,d), then $b = \pm c$. Hence we can write each in the form

$$\sum_{i=0}^{R-1} (c_i, c_{i+1}), \quad c_i \in \mathbb{Z}/l\mathbb{Z}.$$

Let $\widetilde{\xi}, \widetilde{\xi}' \subset X_1(l)$ be the two piecewise geodesic arcs corresponding to ξ and ξ'. We call each arc corresponding to an individual symbol (c_i, c_{i+1}) a *piece* of $\widetilde{\xi}$.

Lemma 3.2. *On $X_1(l)$, the intersection of $\widetilde{\xi}$ and $\widetilde{\xi}'$ consists of cusps, pieces, and no other points.*

Proof. Let (u,v) be a Manin symbol, and let $\widetilde{(u,v)}$ be the corresponding piece on $X_1(l)$. This piece can be lifted to a geodesic on $\mathfrak{H}^*$ with endpoints in $\mathfrak{H}^* \smallsetminus \mathfrak{H}$. If $\alpha, \beta \in \mathbb{Q} \cup \{\infty\}$ are the endpoints of this geodesic, then $|\det(\alpha,\beta)| = 1$, where the determinant is of the matrix with columns corresponding to α and β. The $SL_2(\mathbb{Z})$-translates of such geodesics either coincide or have no intersection in $\mathfrak{H}$, which proves the claim. □

3.3. Now we want to write

$$\langle \xi, \xi' \rangle = \sum_{\gamma \in \partial X_1(l)} \langle \xi, \xi' \rangle_\gamma, \tag{2}$$

where $\langle \xi, \xi' \rangle_\gamma$ is a local contribution at each cusp. If $\widetilde{\xi}$ and $\widetilde{\xi}'$ have no common pieces, then at each cusp γ we assign a contribution of ± 1, depending on the orientations of $\widetilde{\xi}$ and $\widetilde{\xi}'$ and the orientation on $X_1(l)$.

On the other hand, if $\widetilde{\xi}$ and $\widetilde{\xi}'$ have common pieces, then we can do the following "averaging" trick. Let C be the set of common pieces of $\widetilde{\xi}$ and $\widetilde{\xi}'$. To each element $\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^{\#C}$, we can associate paths $\widetilde{\xi}_\varepsilon$ and $\widetilde{\xi}'_\varepsilon$ on $X_1(l)$ by perturbing $\widetilde{\xi}$ and $\widetilde{\xi}'$ (see Figure 1). We make the perturbations so that no new intersections are introduced, and so that $\widetilde{\xi}_\varepsilon$ and $\widetilde{\xi}'_\varepsilon$ transversely intersect only at cusps. If we perturb them slightly, then homology classes are preserved, and this leads to the following lemma:

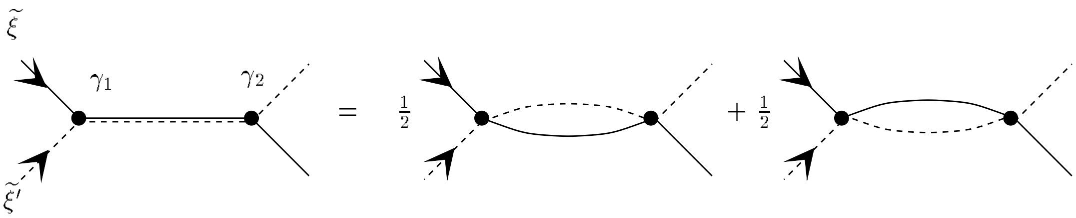

FIGURE 1.

Lemma 3.4.

$$\langle \xi, \xi' \rangle = \frac{1}{2^{\#C}} \sum_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^{\#C}} \langle \xi_\varepsilon, \xi'_\varepsilon \rangle.$$

3.5. Now we can define local intersection numbers by the following rules. Let γ be a cusp.

1. If $\widetilde{\xi}$ and $\widetilde{\xi}'$ meet transversely at γ, then $\langle \xi, \xi' \rangle_\gamma = \pm 1$.
2. If $\widetilde{\xi}$ and $\widetilde{\xi}'$ have common pieces entering and leaving γ, then $\langle \xi, \xi' \rangle_\gamma = 0$.
3. If $\widetilde{\xi}$ and $\widetilde{\xi}'$ have one common piece and two separate pieces at a cusp γ, then $\langle \xi, \xi' \rangle_\gamma = \pm 1/2$.

In these formulas, we choose the sign according to Figure 2. Changing the direction of any curve or switching the labels introduces an extra factor of -1.

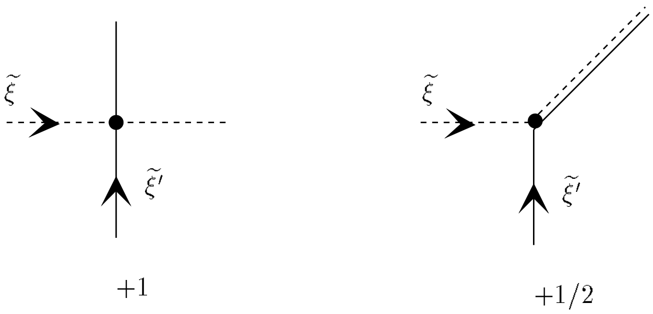

FIGURE 2.

With these choices, using Lemma 3.4 it is easy to show the following:

Proposition 3.6. *With the local contributions above, we have*

$$\langle \xi, \xi' \rangle = \sum_{\gamma \in \partial X_1(l)} \langle \xi, \xi' \rangle_\gamma.$$

Remark 3.7. We extend this local formula to compute the intersection number of two nonsimple closed loops. At each cusp, a nonsimple closed loop splits into a collection of strands, and we sum local contributions over all pairs of strands.

3.8. Let $M^*_\pm$ be the dual vector space to the space $M_\pm$, and let $\varphi \in M^*_+$. We define a map $\pi\colon M^*_+ \to M_-$ by

$$\pi\colon \varphi \longmapsto \sum_{a,b} \varphi((a, a-b)_+ - (a, a+b)_+)(a,b)_- . \tag{3}$$

We claim that π is induced from the intersection form on S in the following sense. The map π induces a map $\pi_S : S_+^* \to S_-$ via the diagram

$$\begin{tikzcd} M_+^* \arrow[r] \arrow[d] & M_- \\ S_+^* \arrow[r] \arrow[ur] & S_- \arrow[u] \end{tikzcd}$$

where the vertical arrows are induced by the inclusions $S_\pm \to M_\pm$. Then in Theorem 3.13, we will prove, given $\xi \in S_+$ and $\varphi \in S_+^*$, that

$$\varphi(\xi) = 12\langle \xi, \pi_S(\varphi)\rangle.$$

Lemma 3.9. *We have the following relation among Manin symbols:*

$$(a, b-a)_+ - (a, b+a)_+ = (b, a-b)_+ - (b, a+b)_+.$$

Proof. By relation 2 in §2.1, we have

$$\begin{aligned}(a, b-a)_+ &= -(b-a, -b)_+ - (-b, a)_+,\\ (a, b+a)_+ &= -(b+a, -b)_+ - (-b, a)_+.\end{aligned}$$

Subtracting the two equations and applying the relations $(u, v) = -(-v, u)$ and $(\pm u, \pm v)_+ = (u, v)_+$ completes the proof. ☐

Lemma 3.10. *For any cusp γ, let $D_\gamma \in M^*$ be the linear map defined in §2.4. Then $\pi(D_\gamma) = 0$.*

Proof. Write γ as r/s, where $s \in \mathbb{Z}/l\mathbb{Z}$ and $r \in (\mathbb{Z}/(s,l)\mathbb{Z})^\times$ as above, and let $r^* \in \mathbb{Z}/(s,l)\mathbb{Z}$ satisfy $rr^* = 1$. By manipulating the summation in the definition of π, we find

$$\pi(D_\gamma) = \sum_{a,b\in\mathbb{Z}/l\mathbb{Z}} \delta_a^{\pm s}\delta_b^{\pm r^*}\big((a, b+a)_- - (a, b-a)_-\big),$$

where the signs in the delta functions are taken independently. Here the convention for the delta function is that $\delta_x^{\pm y} = 1$ when either $x = y$ or $x = -y$ and is otherwise 0. This simplifies further to

$$\sum_{b=\pm r^* \bmod (s,l)} 2\big((s, b+s)_- - (s, b-s)_-\big) = \sum_{b=\pm r^* \bmod (s,l)} 2\big((s, b)_- - (s, b)_-\big) = 0.$$

☐

Lemma 3.11. *Let $\varphi \in M_+^*$. Then $D_\gamma(\pi(\varphi)) = 0$ for any cusp γ.*

Proof. The proof is very similar to that of Lemma 3.10. Again writing $\gamma = r/s$, and manipulating the sum defining $D_\gamma(\pi(\varphi))$, we find

$$D_\gamma(\pi(\varphi)) = \sum_{a,b\in\mathbb{Z}/l\mathbb{Z}} \delta_a^{\pm b}\delta_b^{\pm r^*}\big(\varphi((a, b-a)_+) - \varphi((a, b+a)_+)\big),$$

where r^* is as in the proof of Lemma 3.10. This sum is equivalent to the sum

$$\sum_{b=\pm r^* \bmod (s,l)} 2\big(\varphi((s, b-s)_+) - \varphi((s, b+s)_+)\big),$$

which is seen to be 0 as above. ☐

Proposition 3.12. *The map $\pi\colon M_+^* \to M_-$ induces a map $\pi_S\colon S_+^* \to S_-$.*

Proof. This follows immediately from the Lemmas 3.10 and 3.11. □

Theorem 3.13. *The map π_S is induced by the intersection pairing $\langle\ ,\ \rangle$, in the sense that for any $\xi \in S_+$ and any $\varphi \in S_+^*$, we have*

$$\varphi(\xi) = 12\langle \xi, \pi_S(\varphi)\rangle.$$

In particular, π_S is an isomorphism, since the intersection pairing is nondegenerate.

Proof. We prove the statement by explicitly computing $\langle \xi, \pi_S(\varphi)\rangle$ for any form φ. It suffices to assume that $\xi = (\xi')_+$ where ξ' is a closed loop. The loop ξ' can be written in the form

$$\xi' = \sum_{i=0}^{R-1} (c_i, c_{i+1}), \quad c_i \in \mathbb{Z}/l\mathbb{Z}$$

where $c_0 = c_R$. Because $\pi_S(\varphi) \in S_-$, and $\langle S_-, S_-\rangle = 0$, by Proposition 3.6 and Remark 3.7 we have

$$\langle \xi, \pi_S(\varphi)\rangle = \langle \xi', \pi_S(\varphi)\rangle = \sum_{\gamma \in \partial X_1(l)} \langle \xi', \pi_S(\varphi)\rangle_\gamma,$$

where $\langle\ ,\ \rangle_\gamma$ denotes the local contribution to the intersection pairing at the cusp γ.

Let $\gamma = r/s$, in the notation of §2.3, and consider the consecutive pair of geodesic segments (c_{i-1}, c_i) and (c_i, c_{i+1}), which we denote by $[c_{i-1}, c_i, c_{i+1}]$. If this pair passes through γ, then a simple computation shows that r/s must have the form $-c_{i-1}^{-1}/c_i$ and c_{i+1}^{-1}/c_i, which shows $c_{i-1} = -c_{i+1}$ in $\mathbb{Z}/(c_i, l)\mathbb{Z}$. Hence in the above sum the pair $[c_{i-1}, c_i, c_{i+1}]$ will contribute to the intersection number terms of the form

$$\langle [c_{i-1}, c_i, c_{i+1}], \pi_S(\varphi)\rangle_{-c_{i-1}^{-1}/c_i}.$$

Now we apply the definition of π and compute these local intersections. Notice that we can replace $(a,b)_-$ with (a,b), and so each local contribution equals

$$\langle [c_{i-1}, c_i, c_{i+1}], \sum_{a,b} \varphi((a, b-a)_+ - (a, b+a)_+)(a,b)\rangle_{-c_{i-1}^{-1}/c_i}.$$

We want to know when the symbol (a,b) enters or leaves the cusp $-c_{i-1}^{-1}/c_i$. This happens if $a = \pm c_i$ and $b = \mp c_{i-1} \bmod (c_i, l)$ or if $b = \pm c_i$ and $a = \pm c_{i-1} \bmod (c_i, l)$, where the signs are not taken independently. Hence the contribution becomes

$$\Big\langle [c_{i-1}, c_i, c_{i+1}], \sum_{b=-c_{i-1} \bmod (c_i,l)} 2\varphi((c_i, b-c_i)_+ - (c_i, b+c_i)_+)(c_i, b) + \sum_{a=c_{i-1} \bmod (c_i,l)} 2\varphi((a, c_i-a)_+ - (a, c_i+a)_+)(a, c_i)\Big\rangle_{-c_{i-1}^{-1}/c_i}.$$

We apply Lemma 3.9 to the second sum, and the above expression becomes

$$4\Big\langle [c_{i-1}, c_i, c_{i+1}], \sum_{a=c_{i-1} \bmod (c_i,l)} \varphi((c_i, a-c_i)_+ - (c_i, a+c_i)_+)(a, c_i)\Big\rangle_{-c_{i-1}^{-1}/c_i}.$$

Using the same trick as in the proofs of Lemmas 3.10 and 3.11, we rewrite this as

$$4\Big\langle [c_{i-1}, c_i, c_{i+1}], \sum_{a=c_{i-1} \bmod (c_i,l)} \varphi((c_i, a)_+)((a+c_i, c_i) - (a-c_i, c_i))\Big\rangle_{-c_{i-1}^{-1}/c_i}. \tag{4}$$

Now we determine which a contribute to the intersection (4). If (c_{i-1}, c_i) is a Manin symbol, then the Manin symbols that have right entry c_i and whose geodesics end in the cusp $-c_{i-1}^{-1}/c_i$ have the form $(kc_i + c_{i-1}, c_i)$, where k is an integer. Moreover, it is easy to see that the clockwise order of these geodesics at the cusp corresponds to the ordering on k. This implies that there are *six* values of a that contribute to the local intersection (4): $a = c_{i-1} - c_i$, c_{i-1}, $c_{i-1} + c_i$ and $a = c_{i+1} - c_i$, c_{i+1}, $c_{i+1} - c_i$. We depict the first three in Figure 3. Hence (4) becomes

$$-4\Big(\varphi((c_i, c_{i-1} - c_i)_+)/2 + \varphi((c_i, c_{i-1})_+ + \varphi((c_i, c_{i-1} + c_i)_+)/2 \\ - \varphi((c_i, c_{i+1} - c_i)_+)/2 - \varphi((c_i, c_{i+1})_+) - \varphi((c_i, c_{i+1} + c_i)_+)/2\Big).$$

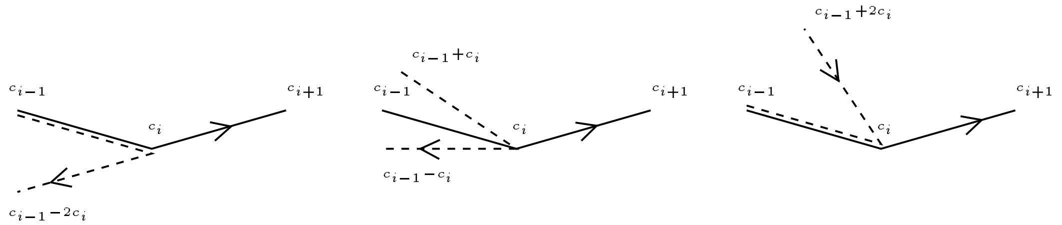

FIGURE 3.

Now we sum this final expression over $i = 0, \ldots, R-1$ to compute the total intersection number. Using the three-term relations

$$(c_i, c_{i-1} - c_i)_+ = -(c_{i-1} - c_i, c_{i-1})_+ - (c_{i-1}, c_i)_+,$$
$$(c_i, c_{i+1} + c_i)_+ = -(c_{i+1} + c_i, c_{i+1})_+ - (c_{i+1}, c_i)_+,$$

we see that most terms in the sum cancel. Altogether, the intersection number is

$$-4\sum_{i=0}^{R-1}\Big(\varphi((c_i, c_{i-1})_+) + \varphi((c_{i+1}, c_i)_+) + \varphi((c_i, c_{i-1})_+)/2 + \varphi((c_{i+1}, c_i)_+)/2\Big) = \\ 12\sum_{i=0}^{R-1}\varphi((c_i, c_{i+1})_+),$$

which completes the proof. □

4. TORIC FORMS OF WEIGHT TWO

4.1. Let us briefly review the contents of the paper [1]. For every integer $l > 1$, we defined a certain Hecke-stable subring of the ring of modular forms for $\Gamma_1(l)$, called the subring of *toric forms* $\mathcal{T}(l)$. In the present paper we are only concerned with weight two toric forms, which greatly simplifies the combinatorial data needed to encode toric varieties.

Let $N = \mathbb{Z}^2$ be a lattice of rank two, where lattice simply means a free abelian group. A (compact) toric variety of dimension two is defined uniquely by a collection of k rational rays from the origin, such that all angles between two consecutive rays are less than π. We denote the minimum nonzero lattice points on these rays by d_i, $i = 0, \ldots, k$, where i increases counterclockwise, and where $d_0 = d_k$.

To every such collection one associates a *fan* Σ, a collection of $2k+1$ cones in $N_{\mathbb{Q}}$. This fan contains k two-dimensional cones

$$\{\mathbb{Q}_{\geq 0} d_i + \mathbb{Q}_{\geq 0} d_{i+1} \mid i = 0, \ldots, k-1\},$$

k one-dimensional cones

$$\{\mathbb{Q}_{\geq 0} d_i \mid i = 0, \dots, k-1\},$$

and one sero-dimensional cone $\{0\}$. The corresponding toric variety is smooth if and only if (d_i, d_{i+1}) is a basis of $\mathbb{Z}^2$ for every i.

To define a toric form we need an additional piece of data, namely a *degree function* with respect to Σ. This is a piecewise-linear function $\deg : N \to \mathbb{Q}$ that is linear on the cones of Σ. Every such function is uniquely determined by the values $\alpha_i = \deg(d_i)$.

Definition 4.2. [1] Suppose $\deg\colon N \to \mathbb{Q}$ is a degree function with respect to the fan Σ, and that $\alpha_i \notin \mathbb{Z}$ for all i. Then the *toric form* associated to $(N, \deg)$ is the function $f_{N,\deg}\colon \mathfrak{H} \to \mathbb{C}$ defined by

$$f_{N,\deg}(q) := \sum_{m \in M} \Big(\sum_{C \in \Sigma} (-1)^{\operatorname{codim} C} \text{a.c.} \Big(\sum_{n \in C} q^{m \cdot n} \mathrm{e}^{2\pi \mathrm{i} \deg(n)} \Big) \Big).$$

Here $M = \operatorname{Hom}(M, \mathbb{Z})$ is the dual of N, and a.c. means analytic continuation of a sum from its region of convergence to all q and m.

It turns out that $f_{N,\deg}$ does not change if Σ is subdivided, and thus does not depend on Σ. This is why Σ is omitted from the notation. A toric form is, by definition, any linear combination of $f_{N,\deg}$. We will now state in one theorem most of the results of [1], specialized to the case of weight two.

Theorem 4.3. [1] *Suppose* $\deg(N) \subseteq \frac{1}{l}\mathbb{Z}$, *and that* $\alpha_i \notin \mathbb{Z}$ *for all* i. *Then* $f_{N,\deg}$ *is a modular form of weight two with respect to* $\Gamma_1(l)$. *If* $l \geq 5$ *then it is a linear combination of products of* $s_{a/l}$, $a = 1, \dots, l-1$, *given by*

$$s_{a/l}(\tau) = \frac{1}{2\pi \mathrm{i}} \partial_z (\log \vartheta)(\frac{a}{l}, \tau).$$

Here $\vartheta(z, \tau)$ *is the standard theta function* [2, Chapter 5]. *If* $l < 5$, *then the span* $\mathscr{T}_2(l)$ *of all toric forms of level* l *and weight two coincides with the space of all modular forms of weight two; in particular, it consists only of Eisenstein series. The space* $\mathscr{T}_2(l)$ *is stable under the action of Hecke operators and the Fricke involution. The space of all weight two toric forms of all levels is stable under Atkin-Lehner liftings* $f(\tau) \to f(n\tau)$.

Let p be a prime not dividing l. We will need an explicit formula for the action of the Hecke operator T_p on $f_{N,\deg}$. This follows immediately from a formula in [1, Theorem 5.3] specialized to the case of weight two.

Proposition 4.4. *Let* $f_{N,\deg}$ *be a toric form of weight two. Then*

$$f_{N,\deg} \big| T_p = \sum_S f_{S, p\deg},$$

where the sum is taken over all suplattices $S \supseteq N$ *with* $[S : N] = p$.

4.5. It is not hard to write down explicit q-expansions of $s_{a/l}$.

Proposition 4.6. *Denote* $w = \exp(2\pi \mathrm{i}/l)$. *Then*

$$s_{a/l}(q) = \frac{w^a + 1}{2(w^a - 1)} - \sum_d q^d \sum_{k|d} (\mathrm{e}^{2\pi \mathrm{i} k a/l} - \mathrm{e}^{-2\pi \mathrm{i} k a/l}).$$

Proof. It is easy to compute the logarithmic derivative of ϑ using the Jacobi triple product formula [2, Chapter 5, Theorem 6]. Details are left to the reader. ∎

In [1] we introduced modular forms $s^{(2)}_{a/l}$ given by

$$s^{(2)}_{a/l}(q) = \frac{1}{(-4\pi^2)}\left(\frac{\partial^2}{\partial z^2}\right)_{z=0} \log\left(\frac{z\vartheta(z+a/l)\vartheta'(0)}{\vartheta(z)\vartheta(a/l)}\right).$$

These are also toric forms, and the following relations allow one to express them as linear combinations of products $s_{a/l}s_{b/l}$ for $l \geq 5$.

Proposition 4.7. *If $a+b+c=0 \bmod l$ and $a,b,c \neq 0 \bmod l$ then*

$$s_{a/l}s_{b/l} + s_{b/l}s_{c/l} + s_{c/l}s_{a/l} = -\frac{1}{2}\left((s_{a/l})^2 + (s_{b/l})^2 + (s_{c/l})^2 + s^{(2)}_{a/l} + s^{(2)}_{b/l} + s^{(2)}_{c/l}\right).$$

Proof. This is a consequence of a more general formula of [1]. ☐

Proposition 4.8. *The modular forms $(s_{a/l})^2$ and $s^{(2)}_{a/l}$ are Eisenstein series for all a.*

Proof. As in Proposition 4.6, we get

$$(2\pi \mathrm{i})^{-1}\partial_z \log\vartheta(z,\tau) = \frac{\mathrm{e}^{2\pi \mathrm{i}z}+1}{2(\mathrm{e}^{2\pi \mathrm{i}z}-1)} - \sum_{d>0} q^d \sum_{k|d}(\mathrm{e}^{2\pi \mathrm{i}kz} - \mathrm{e}^{-2\pi \mathrm{i}kz}).$$

Differentiating it one more time with respect to z and plugging in $z = a/l$, we get

$$(2\pi \mathrm{i})^{-2}\left(\frac{\partial^2}{\partial z^2}\right)\log\vartheta(\frac{a}{l},\tau) = \frac{\mathrm{e}^{2\pi \mathrm{i}a/l}}{(\mathrm{e}^{2\pi \mathrm{i}a/l}-1)^2} - \sum_{d>0} q^d \sum_{k|d} k(\mathrm{e}^{2\pi \mathrm{i}ka/l} + \mathrm{e}^{-2\pi \mathrm{i}ka/l}).$$

Notice now that

$$\begin{aligned} s^{(2)}_{a/l}(\tau) &= (2\pi \mathrm{i})^{-2}\left(\frac{\partial^2}{\partial z^2}\right)\log\vartheta(\frac{a}{l},\tau) - (2\pi \mathrm{i})^{-2}\left(\frac{\partial^2}{\partial z^2}\right)_{z=0}\log\left(\frac{\vartheta(z,\tau)}{z\partial_z\vartheta(0,\tau)}\right) \\ &= (2\pi \mathrm{i})^{-2}\left(\frac{\partial^2}{\partial z^2}\right)\log\vartheta(\frac{a}{l},\tau) - \frac{1}{12} + 2\sum_{d>0} q^d \sum_{k|d} k \\ &= \frac{\mathrm{e}^{2\pi \mathrm{i}a/l}}{(\mathrm{e}^{2\pi \mathrm{i}a/l}-1)^2} - \frac{1}{12} - \sum_{d>0} q^d \sum_{k|d} k(\mathrm{e}^{2\pi \mathrm{i}ka/l} + \mathrm{e}^{-2\pi \mathrm{i}ka/l} - 2), \end{aligned}$$

which is an Eisenstein series. Indeed, in the notation of [8, Chapter VII], it is

$$\frac{l^2}{4\pi^2}\left(G_{l,2,\binom{0}{a}} - G_{l,2,\binom{0}{0}}\right).$$

To obtain a nice formula for $s^2_{a/l}$, recall that ϑ satisfies the heat equation

$$\vartheta_{zz} = (4\pi \mathrm{i})\vartheta_\tau,$$

which implies

$$(2\pi \mathrm{i})^{-2}\left(\frac{\vartheta_{zz}}{\vartheta}\right)_z = (\pi \mathrm{i})^{-1}\left(\frac{\vartheta_z}{\vartheta}\right)_\tau = -4\pi \mathrm{i}\sum_{d>0} dq^d \sum_{k|d}(\mathrm{e}^{2\pi \mathrm{i}kz} - \mathrm{e}^{-2\pi \mathrm{i}kz}).$$

We can integrate it with respect to z while keeping in mind that

$$(2\pi \mathrm{i})^{-2}\lim_{z\to 0}\frac{\vartheta_{zz}(z,\tau)}{\vartheta(z,\tau)} = (2\pi \mathrm{i})^{-2}\frac{\vartheta_{zzz}(0,\tau)}{\vartheta_z(0,\tau)} = \frac{1}{4} - 6\sum_{d>0} q^d \sum_{k|d}\frac{d}{k}$$

to obtain

$$s^{(2)}_{a/l}+s^2_{a/l}=(2\pi\mathrm{i})^{-2}\Big(\frac{\vartheta_{zz}(a/l,\tau)}{\vartheta(a/l,\tau)}-\frac{\vartheta_{zzz}(0,\tau)}{3\vartheta_z(0,\tau)}\Big)=\frac{1}{6}-2\sum_{d>0}q^d\sum_{k|d}\frac{d}{k}(\mathrm{e}^{2\pi\mathrm{i}ak/l}+\mathrm{e}^{-2\pi\mathrm{i}ak/l}),$$

which is again an Eisenstein series. In the notation of [8, Chapter VII] it is equal to

$$\frac{l}{2\pi^2}\sum_{a_1,a_2=0}^{l-1}\mathrm{e}^{2\pi\mathrm{i}aa_1/l}G_{l,2,\binom{a_1}{a_2}}.$$

□

4.9. Recall that $\mathscr{M}(l)$ is the space of all holomorphic modular forms for $\Gamma_1(l)$ of weight two, and denote by $\mathscr{E}(l)$ the subspace of Eisenstein series. Proposition 4.8 allows us to define a map from the space of Manin symbols to toric modular forms modulo Eisenstein series.

Definition 4.10. Let M_- be the space of minus Manin symbols (Definition 2.2). We define a map $\mu\colon M_-\to\mathscr{M}(l)/\mathscr{E}(l)$ by

$$\mu((a,b)_-)=s_{a/l}s_{b/l} \bmod \mathscr{E}(l)$$

if $a,b\neq 0 \bmod l$, and by 0 otherwise.

Remark 4.11. The map μ is well-defined. Indeed, it is clear that $(a,b)_-+(-a,b)_-$ and $(a,b)_--(b,a)_-$ map to zero, even before we mod out by $\mathscr{E}(l)$. It remains to show that $\mu((a,b)_-+(b,c)_-+(c,a)_-)=0$ for $a+b+c=0 \bmod l$. This follows from Propositions 4.7 and 4.8 if a,b,c are non-zero, and from Proposition 4.8 if one of them is zero.

4.12. We will now show that map μ commutes with the action of the Hecke operators T_p for primes p not dividing l. To do this, we recall the description of toric forms in terms of cohomology rings of smooth toric varieties [1].

Let deg be a degree function with respect to Σ in N. Let $\widehat{\Sigma}$ be a subdivision of Σ such that the corresponding toric variety $\widehat{X}$ is smooth. Denote by d_i the minimum lattice points of the one-dimensional cones of $\widehat{\Sigma}$. Since $\widehat{\sigma}$ is a subdivision of Σ, it is possible that some $\alpha_i=\deg(d_i)$ are now integral. To circumvent this difficulty, in [1] we introduced a generic degree function $\deg_1$, linear on all cones of the original fan Σ and such that the values $\beta_i=\deg_1(d_i)$ are non-zero.

Proposition 4.13. ([1]) *In the above notation, we have*

$$f_{N,\deg}(q)=\lim_{\varepsilon\to 0}\int_{\widehat{X}}\prod_i\frac{(D_i/2\pi\mathrm{i})\vartheta(D_i/2\pi\mathrm{i}-\deg(d_i)-\varepsilon\deg_1(d_i),\tau)\vartheta'(0,\tau)}{\vartheta(D_i/2\pi\mathrm{i},\tau)\vartheta(-\deg(d_i)-\varepsilon\deg_1(d_i),\tau)}\tag{5}$$

where D_i is the cohomology class of the toric divisor corresponding to the one-dimensional cone $\mathbb{Q}_{\geq 0}d_i$ (cf. [3]*) and the integral means pairing with the fundamental class of $\widehat{X}$.*

If we work modulo Eisenstein series, the above formula greatly simplifies.

Proposition 4.14. *Let $d_1,\dots,d_k$ be generators of one-dimensional cones of $\widehat{\Sigma}$. We denote $d_0=d_k$. Then*

$$f_{N,\deg}=\sum_{\substack{0\leq i\leq k-1\\ \alpha_i,\alpha_{i+1}\notin\mathbb{Z}}}s_{\alpha_i}s_{\alpha_{i+1}} \bmod \mathscr{E}(l).$$

Proof. Let us first explain what happens in the case where all α_i are non-integer. Then we are integrating over $\widehat{X}$ the product of cohomology elements

$$(1 - s_{\alpha_i} D_i + r_i D_i^2),$$

where r_i is easily seen to be an Eisenstein series due to Proposition 4.8. The intersection number $\int_{\widehat{X}} D_i D_j$ for $i \neq j$ equals 1 if d_i and d_j come from adjacent cones and equals 0 otherwise. This finishes the argument.

When some of the α_i are zero the argument is a bit more complicated and involves the expansion of the right hand side of (5) in powers of ε. Then up to Eisenstein series one ends up with the integral of the product of $(1 - s_{\alpha_i} D_i)$ over i for which $\alpha_i \notin \mathbb{Z}$. Details are left to the reader. □

Theorem 4.15. *The map $\mu : M_- \to \mathscr{M}(l)/\mathscr{E}(l)$ defined above is invariant under the action of the Hecke operators T_p for primes p coprime to l.*

Proof. We will identify N with the lattice of integer row vectors with two components. Consider the fan Σ that has $d_1 = (1,0)$, $d_2 = (0,1)$, $d_3 = (-1,0)$, and $d_4 = (0,-1)$ as generators of its one-dimensional faces, and a degree function on Σ defined by $\deg(1,0) = \deg(-1,0) = m/l$ and $\deg(0,1) = \deg(0,-1) = n/l$. Proposition 4.14 shows that

$$f_{N,\deg} = 4 s_{m/l} s_{n/l} \bmod \mathscr{E}(l).$$

Let us calculate the action of the Hecke operator T_p on this form modulo $\mathscr{E}(l)$. By Proposition 4.4 we have

$$f_{N,\deg} \big| T_p = \sum_{N \subset S \subset \frac{1}{p} N} f_{S, p\deg}.$$

Let us investigate the contribution of each S. To get into the setup of Proposition 4.14, we need a subdivision $\widehat{\Sigma}_S$ of the fan Σ so that the consecutive d_i form a basis. There is a standard way of doing so. For each quadrant (i.e. for each two-dimensional cone of Σ), consider the set A of all non-zero points of S in that quadrant. The boundary of the convex hull of A consists of two half-lines and some segments (Figure 4). We ignore the half lines and add to the list of d_i all points in S that lie on the rest of the boundary. It is easy to show that this choice of d_i guarantees that the new toric variety is smooth. Indeed, if d_i and d_{i+1} did not form a basis of S then there would exist a point in S lying in the convex hull of 0, d_i and d_{i+1} by Pick's Theorem [3, page 113].

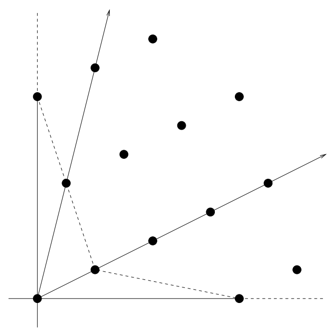

FIGURE 4.

Because of the symmetry, it is enough to consider the first quadrant. Notice that if $d_i = (\frac{a}{p}, \frac{c}{p})$ and $d_{i+1} = (\frac{b}{p}, \frac{d}{p})$, then $ad - bc = p$, with $a > b \geq 0$ and $d > c \geq 0$. Conversely, every pair of $\{(\frac{a}{p},\frac{c}{p}),(\frac{b}{p},\frac{d}{p})\}$ with (a,b,c,d) as above generates some suplattice S of index p. Moreover, this pair will form a segment of the boundary of the convex hull of all non-zero points of S in the first quadrant. Indeed, if any other non-zero point $(x,y)\in S$ with $x,y\geq 0$ did lie below the line through these two points, then the area of the triangle with vertices (x,y), $(\frac{a}{p},\frac{c}{p})$ and $(\frac{b}{p},\frac{d}{p})$ would be positive but less than $\frac{1}{p}$.

The values of $p\deg$ on the points $(\frac{a}{p},\frac{c}{p})$ and $(\frac{b}{p},\frac{d}{p})$ are $\frac{am+cn}{l}$ and $\frac{bm+dn}{l}$ respectively, so by Proposition 4.14 we have

$$f_{N,\deg}\big|T_p = 4\sum_{\substack{ad-bc=p,a>b\geq 0,d>c\geq 0,\\ (am+cn)/l\notin\mathbb{Z},(bm+dn)/l\notin\mathbb{Z}}} s_{(am+cn)/l}\, s_{(bm+dn)/l} \bmod \mathscr{E}(l).$$

Therefore, from the definition of μ and Proposition 2.11 we conclude that

$$\mu(T_p(m,n)_-) = \mu((m,n)_-)\big|T_p.$$

□

Proposition 4.16. *The image of M_- under μ coincides with the image of S_- under μ.*

Proof. This follows from Theorem 4.15 and the fact that M is isomorphic as a Hecke-module to $S_-\oplus S_+\oplus\mathscr{E}(l)$ [6, Section 4.2]. □

4.17. Finally, we will explicitly describe the composition of μ and the Fricke involution.

Proposition 4.18. *The composition of μ and Fricke involution W_l is given by*

$$W_l\circ\mu((m,n)_-) = \widetilde{s}_{m/l}(q)\widetilde{s}_{n/l}(q) \bmod \mathscr{E}(l)$$

where $\widetilde{s}_{0/l}=0$ and

$$\widetilde{s}_{a/l}(q) = (\Big\{\frac{a}{l}\Big\}-\frac{1}{2}) - \sum_{d\geq 1} q^d \sum_{k|d}(\delta_k^{a\bmod l}-\delta_k^{-a\bmod l})$$

for $a\neq 0 \bmod l$. The braces denote fractional part.

Proof. All we need to do is calculate the Fricke involution of $s_{a/l}$. This was accomplished in [1], up to the constant term. To compute the constant term, we remark that these forms have already appeared in the literature as the *Hecke-Eisenstein forms* [5, Chapter 15]. □

5. Forms of analytic rank zero

5.1. In this subsection we define a linear map $\rho\colon\mathscr{S}(l)\to\mathscr{S}(l)$ whose image is spanned by Hecke eigenforms of analytic rank zero.

Definition 5.2. Let $f\in\mathscr{S}(l)$ be a cusp form. We define a linear map ρ by

$$\rho(f) = \sum_{n=1}^{\infty}\left(\int_0^{i\infty}(f\,|\,T_n)(s)ds\right)q^n.$$

Proposition 5.3. *The form $\rho(f)$ is a cusp form with nebentypus equal to that of f.*

Proof. This follows from [6, Theorem 6], since $\rho(f)$ is associated to the linear map on the Hecke algebra that maps T to $\int_0^{i\infty}(f\,|\,T)(s)\,ds$. □

Remark 5.4. It is easy to see that for a Hecke eigenform f that is a newform
$$\rho(f) = L(f,1)f.$$
In particular, the image of the space of newforms is the span of the new Hecke eigenforms that have analytic rank zero.

Proposition 5.5. *The image of ρ is contained in the span of all lifts of all new Hecke eigenforms of analytic rank zero for all levels k, $k|l$.*

Proof. It is clear that if f is a lift of an eigenform g of analytic rank one or more, then $\rho(f) = 0$. Indeed, for every n $f\,\big|\,T_n$ is a linear combination of various lifts of g, which implies $L(f\,\big|\,T_n, 1) = 0$. It remains to show that if f is a lift of g then $\rho(f)$ is a linear combination of lifts of g. This follows from a commutation relation $\rho(f)\,\big|\,T_p = \rho(f\,\big|\,T_p)$ for all prime p coprime to l. Indeed, from the definition of T_p [5],
$$\begin{aligned}\rho(f)\,\big|\,T_p &= \sum_{m>0} L(f\,\big|\,T_{mp},1)q^m + p\epsilon_p(\rho(f))\sum_{m>0} L(f\,\big|\,T_m,1)q^{mp}\\ &= \sum_{m>0,(m,p)=1} L(f\,\big|\,T_{mp},1)q^m + \sum_{m>0,(m,p)=p} L(f\,\big|\,(T_{mp}+p\epsilon_p(f)T_{m/p}),1)q^m\\ &= \sum_{m>0} L(f\,\big|\,T_pT_m)q^m = \rho(f\,\big|\,T_p).\end{aligned}$$
□

Remark 5.6. Even though we suspect that the image of ρ coincides with the above span, we only need the inclusion proved in the above Proposition.

5.7. We will now prove the key result that relates the map ρ to the map $W_l \circ \mu$ constructed in the previous section.

Theorem 5.8. *The composition*
$$\mathscr{S}(l) \longrightarrow M_+^* \longrightarrow M_- \longrightarrow \mathscr{M}(l)/\mathscr{E}(l) \longrightarrow \mathscr{S}(l) \tag{6}$$
of the map induced by integration between cusps, the map π from (3), the map $W_l \circ \mu$, and the Hecke-equivariant isomorphism between $\mathscr{M}(l)/\mathscr{E}(l)$ and $\mathscr{S}(l)$ equals 12ρ.

Proof. Let f be a cusp form and $\varphi : M_+ \to \mathbb{C}$ be the corresponding element of M_+^*. Then. because of Proposition 2.10 and Theorem 2.11,
$$\begin{aligned}\rho(f)(q) = \sum_{n>0} q^n\langle f\,\big|\,T_n,(1,0)\rangle &= \sum_{n>0} q^n\langle f, T_n(1,0)\rangle\\ &= -\sum_{n>0} q^n\varphi(T_n(0,1)) = -\sum_{n>0} q^n \sum_{\substack{ad-bc=n,\\ a>b\geq 0, d>c\geq 0}} \varphi((c,d)_+).\end{aligned}$$

Notice that we are using the convention that $\varphi(c,d) = 0$ for $g.c.d.(c,d,l) > 1$. On the other hand, the composition of all the maps in (6) except for the last one yields an element of $\mathscr{M}(l)/\mathscr{E}(l)$ given by
$$\rho_1(f)(q) = \sum_{a,b=0}^{l-1} (\varphi((a,a-b)_+) - \varphi((a,a+b)_+))\widetilde{s}_{a/l}(q)\widetilde{s}_{b/l}(q),$$

where we again use our convention that $\varphi((a,b)_+) = 0$ if $g.c.d.(a,b,l) > 1$ (§2.4). Also, we can formally use the same expression for $\widetilde{s}_{0/l}$ as for the rest of $\widetilde{s}_{a/l}$ because the coefficient at $\widetilde{s}_{0/l}\widetilde{s}_{b/l}$ is zero.

In what follows, it will be convenient for us to ignore the constant terms in all our expressions. Indeed, all our functions are modular forms, so the constant term can always be restored. We will denote all these constant terms by C. Using

$$\widetilde{s}_{a/l}\widetilde{s}_{b/l} = C + \sum_{n>0} q^n \Big(\sum_{k|n,k>0} (\frac{1}{2} - \Big\{\frac{a}{l}\Big\})(\delta_k^{b \bmod l} - \delta_k^{-b \bmod l}) +$$

$$\sum_{k|n,k>0} (\frac{1}{2} - \Big\{\frac{b}{l}\Big\})(\delta_k^{a \bmod l} - \delta_k^{-a \bmod l}) + \sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0}} (\delta_{k_1}^{a \bmod l} - \delta_{k_1}^{-a \bmod l})(\delta_{k_2}^{b \bmod l} - \delta_{k_2}^{-b \bmod l})\Big)$$

and symmetry properties of $\varphi((a, a-b)_+) - \varphi(a, a+b)_+)$, we get

$$\rho_1(f)(q) = C + 4\sum_{n>0} q^n \Big(\sum_{k|n,k>0} \sum_{a=0}^{l-1} (\frac{1}{2} - \Big\{\frac{a}{l}\Big\})(\varphi((a, k-a)_+) - \varphi((a, k+a)_+))$$

$$+ \sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0}} (\varphi((k_1, k_1-k_2)_+) - \varphi((k_1, k_1+k_2)_+))\Big).$$

Let us now simplify the second part of this expression to make it look more like $\rho(f)$. We split it into four sums

$$\sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0\\ k_1\geq k_2}} \varphi((k_1, k_1-k_2)_+) + \sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0\\ k_1<k_2}} \varphi((k_1, k_1-k_2)_+)$$
$$- \sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0\\ m_1>m_2}} \varphi((k_1, k_1+k_2)_+) - \sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0\\ m_1\leq m_2}} \varphi((k_1, k_1+k_2)_+) \tag{7}$$

and deal with each sum separately. We will give a detailed calculation for one of the sums and will indicate how to manipulate the other three.

Lemma 5.9.

$$\sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0\\ m_1>m_2}} \varphi((k_1, k_1+k_2)_+) = \sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0\\ m_1>m_2}} \varphi((k_1, k_2)_+) + \sum_{\substack{ad-bc=n,\\ a>b>0, d>c>0}} \varphi((c,d)_+)$$

Proof of the lemma. We first of all rewrite $\varphi((k_1, k_1+k_2)_+)$ as $\varphi((k_1,k_2)_+) + \varphi((k_2, k_1+k_2)_+)$. Then for the second term we make the change $(a,b,c,d) = (m_1, m_1-m_2, k_2, k_1+k_2)$. □

We perform similar but easier manipulations for the remaining three sums in (7). For the first sum we make a change of variables $(a,b,c,d) = (m_1+m_2, m_2, k_1-k_2, k_1)$. For the second sum we make the change of variables $(m_1', m_2', k_1', k_2') = (m_1+m_2, m_2, k_1, k_2-k_1)$ so that it cancels the first sum in the Lemma 5.9. For the fourth sum in (7) we make a change of variables $(a,b,c,d) = (m_2, m_2-m_1, k_1, k_1+k_2)$. Then, after some straightforward calculations we get

$$\sum_{\substack{m_1k_1+m_2k_2=n,\\ m_1,k_1,m_2,k_2>0}} (\varphi((k_1, k_1-k_2)_+) - \varphi((k_1, k_1+k_2)_+)) = -3 \sum_{\substack{ad-bc=n,\\ a>b\geq 0, d>c\geq 0}} \varphi((c,d)_+)$$

$$-\sum_{0<k|n}(\frac{2n}{k}+1)\varphi((k,0)_+) - 2\sum_{k_1|n, k_1>k_2>0}\varphi((k_1,k_1-k_2)_+).$$

As a result,

$$\rho_1(f)(q) - 12\rho(f)(q) = C + 4\sum_{n>0}q^n\sum_{k|n,k>0}\Big(\sum_{a=0}^{l-1}(\frac{1}{2}-\Big\{\frac{a}{l}\Big\})(\varphi((k,a-k)_+)-\varphi((k,a+k)_+))$$

$$-(\frac{2n}{k}-1)\varphi((k,0)_+) - 2\sum_{0\leq m\leq k}\varphi((k,m)_+)\Big).$$

Then it is possible to further simplify this equation to show that

$$\rho_1(f)(q) - 12\rho(f)(q) = C - 8\sum_{n>0}q^n\sum_{k|n,k>0}\frac{n}{k}\varphi((k,0)_+) - \frac{8}{l}\sum_{n>0}q^n\sum_{k|n,k>0}k\Big(\sum_{b=0}^{l-1}\varphi((k,b)_+)\Big),$$

which is an Eisenstein series. Indeed, it can be easily written (up to a constant) as a linear combination of $s^{(2)}_{a/l}$, $s^2_{a/l}$ and the (non-modular) $SL_2(\mathbb{Z})$-Eisenstein series E_2. It remains to observe that the coefficient by E_2 has to be zero, because of modular transformation properties of E_2 under $\Gamma_1(l)$. □

5.10. We are now ready to prove our main result.

Theorem 5.11. *For each integer l the space spanned by products of $s_{a/l}$ for $a = 1, \ldots, l-1$ is a direct sum of the span of all Hecke eigenforms of analytic rank zero and some subspace of the space of Eisenstein series.*

Proof. We will prove this theorem by induction on l. For small levels there is nothing to prove, because there are simply no cusp forms.

Fix l and assume that the statement of the theorem is true for all smaller levels. In particular, this implies that lifts of all forms of smaller levels are contained in the span of toric forms, because the space of toric forms is stable under liftings, see Theorem 4.3. Every new Hecke eigenform f of analytic rank zero is contained in the image of ρ, see Remark 5.4. By Theorem 5.8, f is contained in the image of $W_l\circ\mu$, and so is a toric form up to Eisenstein series. Because the space of toric forms is Hecke stable, this implies that f is toric. This proves that the space of toric forms contains the span of all Hecke eigenforms of analytic rank zero.

To prove the opposite inclusion, notice that by the induction assumption it is enough to consider $s_{a/l}s_{b/l}$ with $g.c.d.(a,b,l) = 1$. By Proposition 4.16, there is an element $x\in S_-$ such that $\mu(x) = s_{a/l}s_{b/l}$. We use here that $g.c.d.(a,b,l) = 1$, because otherwise the symbol $(a,b)_-$ is not defined. By Theorem 3.13, there is an element $\varphi\in S_+^*$ such that $\pi(\varphi) = x$. Moreover, we can find a cusp form f which induces the linear map φ on S_+. Then Theorem 5.8 shows that $s_{a/l}s_{b/l}$ is proportional to $W_l\circ\rho(f)$ up to an Eisenstein series. By Proposition 5.5 $s_{a/l}s_{b/l}$ lies in the span of Hecke eigenforms of rank zero and Eisenstein series, which finishes the proof. □

Remark 5.12. It is easy to see that as a corollary we get similar results for any given nebentypus. In particular, modulo Eisenstein series, the span of the forms

$$\sum_{k\in(\mathbb{Z}/l\mathbb{Z})^*}s_{ka}s_{kb},\quad a,b = 1,\ldots,l-1$$

is the span of all Hecke eigenforms of analytic rank zero for $\Gamma_0(l)$.

Remark 5.13. In general it is not clear which Eisenstein series are toric forms. In particular at level $l = 25$ the space of toric Eisenstein series has codimension one.

References

1. L. A. Borisov and P. E. Gunnells, *Toric varieties and modular forms*, submitted, 1999.
2. K. Chandrasekharan, *Elliptic functions*, Springer-Verlag, 1985.
3. W. Fulton, *Introduction to toric varieties*, Princeton University Press, Princeton, NJ, 1993.
4. A. W. Knapp, *Elliptic curves*, Princeton University Press, Princeton, NJ, 1992.
5. S. Lang, *Introduction to modular forms*, Springer-Verlag, 1976.
6. L. Merel, *Universal Fourier expansions of modular forms*, On Artin's conjecture for odd 2-dimensional representations, Springer, Berlin, 1994, pp. 59–94.
7. L. Merel, *The arithmetic of elliptic curves and diophantine equations*, Journal de Théorie des nombres de Bordeaux 11 (1999), 173–200.
8. E. Schoneberg, *Elliptic modular functions*, Springer-Verlag, 1970.
9. J. Tate, *On the conjectures of Birch and Swinnerton-Dyer and a geometric analog*, Séminaire Bourbaki, Vol. 9, Soc. Math. France, Paris, 1995, pp. Exp. No. 306, 415–440.

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