A p-ADIC FAMILY OF KLINGEN-EISENSTEIN SERIES

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Introduction

The p-adic interpolation properties of Fourier coefficients of elliptic Eisenstein series are by now classical. These properties can be considered as the starting point and as an important tool in the theory of p-adic L-functions and p-adic families of modular forms.

In the case of Siegel modular forms there are two types of Eisenstein series. A Siegel-Eisenstein measure which comes from the Siegel-Eisenstein series was recently constructed by A.A. Panchishkin [10].

Another Eisenstein series on the symplectic group are those of Klingen type. The Fourier expansion of these series is of definite interest. One can associate a Klingen-Eisenstein series to an elliptic cusp form f. Then the Fourier coefficients of this series involve special values of certain Dirichlet series connected with f. Namely, the Rankin convolutions of f with theta series associated with positive definite quadratic forms. Though the explicit formulas for these Fourier coefficients [1], [2] are in general considerably complicated, we prove (without making use of these explicit formulas) that, after suitable normalization and regularization, they become p-adically smooth functions.

It becomes natural to consider vector valued Siegel modular forms in this context as well.

The purpose of this paper is to construct a p-adic measure coming from the Klingen-Eisenstein series.

Our main tools are the A.A. Panchishkin's construction of Siegel-Eisenstein measure [10]; the Böcherer-Garrett pull-back formula [1], [3]; H. Hida's theory of p-ordinary Λ-adic forms [4], [5].

The author is grateful to the Minerva Fellowship for the financial support and to the University of Mannheim for its hospitality. It is a pleasure for me to thank Prof. Dr. Böcherer for his patient and detailed explanations connected with the theory of Siegel modular forms. This work was inspired by the talk given by Prof. A.A. Panchishkin on Lundi Arithmétique entitled Familles p-adiques de formes modulaires et de représentations galoisiennes, held at Institut Henri Poincaré. The author is very grateful to Prof. J. Tilouine for the kind invitation to this conference. The influence of the results and ideas of [10] and [7] on this paper is evident, and I am very grateful to Prof. A.A. Panchishkin for transmitting me the manuscripts before publication as well as for helpful discussions.

Math. classification: 11F03, 11F33, 11F46, 11F67, 11F70.
Keywords: formes modulaires, séries d'Eisenstein-Klingen, formes Λ-adiques, familles p-adiques.

The contents of the paper and the main ideas of our construction are as follows.

We introduce necessary notations on vector valued Siegel modular forms and state an appropriate version of the pull-back formula in the first section (Proposition 1 in the text).

The second section is devoted to our main result. In [10], certain subseries of the Siegel-Eisenstein series $G^+_{k,\chi}$ is used for the p-adic construction. Moreover, since one knows the explicit formulae for the Fourier coefficients of Siegel-Eisenstein series (the construction presented in [10] is based on these formulae), it is evident that a statement of such type is not true for $G^+_{k,\chi}$ itself. We show that the subseries in question can be produced from the Siegel-Eisenstein series with the help of a p-adic limit procedure (Proposition 2). Namely, we consider the coefficient-wise p-adically converging sequence $G^+_{k_i,\chi}$ of Siegel-Eisenstein series of increasing weights $k_i = k + (p-1)p^i$. Thus $\lim_{i\to\infty} G^+_{k_i,\chi}$ can be considered as a p-adic Siegel modular form defined à la Serre [11]. We show that the subseries used in the A.A. Panchishkin's construction coincide with this limit. Note that this is known to be a Siegel modular form (on a congruence subgroup). The similarity with the p-adic elliptic Eisenstein series $\frac{1}{2}h(-p) + \sum_{n\geqslant 1}\sum_{d|n}\left(\frac{d}{p}\right) q^n$, where $h(-p)$ is the class number of $\mathbb{Q}(\sqrt{-p})$, (see [11], p. 207, Example), is evident and is not casual. We formulate our main result in the same framework. Consider a Λ-adic modular form f which is a p-ordinary cusp Hecke eigenform. One associates the Klingen-Eisenstein series $F_m^{k,l,\chi}$ to a specialization of f at an arithmetic point (k,χ). One once more considers the sequence of these series $F_m^{k_i,l,\chi}$ with $k_i = k + (p-1)p^i$. The coefficient-wise p-adic limit when $i\to\infty$ once more exists, and, moreover, inherits the p-adic analytic properties of the Siegel-Eisenstein series (Theorem 1 in the text).

Note that the introducing into the play the limits described above, is the main difference between our approach and that of [7]. Another difference is the consideration of the vector valued Siegel modular forms in the same framework.

In the third section we consider the Fourier expansion coefficients of the Siegel-Eisenstein series. The point is that in order to establish the existence of the limit $\lim_{i\to\infty} G^+_{k_i,\chi}$, one should keep control upon all the Fourier expansion coefficients of $G^+_{k,\chi}$. More precisely, one needs certain representation of these numbers as p-adic integrals. These coefficients are numerated by half-integral symmetric positive definite matrices h. The case when $\det(h)$ is divisible by p is not explicitly considered in [10]. However, if the maximal power of p which divides $\det(h)$ is even (as well as if the degree of Eisenstein series is odd), one can almost literary repeat the argument from [10]. We consider the remaining case in section 3 of the paper. We prove the desired property (Proposition 5). As a corollary we get another (it involves other Fourier expansion coefficients then that in [10]) measure associated with Siegel-Eisenstein series Theorem 2 in the text).

1. Siegel modular forms and the pull-back formula

1.1 Siegel-Eisenstein series

Let N be a positive integer. Consider the congruence subgroup of the symplectic group $Sp(n,\mathbb{Z})$ of degree $n>1$:

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} AB \\ CD \end{pmatrix} \in Sp(n,\mathbb{Z}) | C \equiv 0 \bmod N \right\}.$$

It contains the subgroup

$$\Gamma_\infty^{(n)} = \left\{ \begin{pmatrix} AB \\ CD \end{pmatrix} \in Sp(n,\mathbb{Z}) | C = 0 \right\}.$$

Let χ be a primitive Dirichlet character modulo $N > 1$ Put for $Z \in H_n$ and $M = \binom{AB}{CD} \in \Gamma_0^{(n)}(N)$

$$j_{\chi,k}(M,Z) = \overline{\chi}(\det D)\det(CZ+D)^{-k}$$

For a positive integer weight k and degree n we consider the Siegel-Eisenstein series

$$E_n^{k,\chi}(Z) = \sum_{M \in \Gamma_\infty^{(n)} \backslash \Gamma_0^{(n)}(N)} j_{\chi,k}(M,Z).$$

The series converges absolutely and uniformly for $k > n+2$. We also use the involuted series ([10], 1.5):

$$G_n^*(Z;k,\chi) = N^{nk/2}\tilde{\Gamma}(k)L_N(k,\chi)\prod_{1\leqslant i \leqslant [n/2]} L_N(2k-2i,\chi^2)E(-(NZ)^{-1})\det(NZ)^{-k}.$$

Here

$$\tilde{\Gamma}(k) = i^{nk}2^{-n(k+1)}\pi^{-nk}\Gamma_n(k) \times \begin{cases} \Gamma([(k-n/2+1)/2]) & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases}$$

$$\Gamma_n(s) = \pi^{n(n-1)/4}\prod_{0\leqslant j \leqslant n-1} \Gamma(s-j/2)),$$

and the subscript N in Dirichlet L-functions means that the Euler factors over primes dividing N are omitted.

1.2 Vector valued modular forms and differential operators ([1]).

Let ρ be a representation of $GL(n,\mathbb{C})$ with a representation space W. The formula

$$(F|_\rho M)(Z) = \rho((CZ+D)^{-1})f(M\langle Z\rangle)$$

defines the action of $M = \binom{AB}{CD} \in Sp(n,\mathbb{R})$ on the space of C^∞ W-valued functions on the Siegel's upper half plane H_n. Such a function F is called a modular form (of degree n, Dirichlet character χ mod N and level N) if it is holomorphic on H_n and its cusps and satisfies

$$F|_\rho M = \chi(\det D)F$$

for all $M = \binom{AB}{CD} \in \Gamma_0^{(n)}(N)$. When ρ is a representation $\det^k \otimes Sym^l$ of $GL(n,\mathbb{C})$ we write $|_\rho$ as $|_{k,l,n}$. We denote the space of modular forms by $M_{\chi,k,l,n,N}(W)$, and the subspace of cusp forms by $S_{\chi,k,l,n,N}(W)$. Let $W^{(l)}$ denote the l-th symmetric power of a vector space W. We identify $W^{(0)}$ with $\mathbb{C}$. For a row vector $x = (x_1,\dots,x_n)$ consisting of n indeterminants, we put $V = \mathbb{C}x_1 \oplus \dots \oplus \mathbb{C}x_n$. We identify $V^{(l)}$ with $\mathbb{C}[x_1,\dots,x_n]_{(l)}$, where (l) stands for homogeneous polynomials of degree l. The group $GL(n,\mathbb{C})$ acts on $V^{(l)}$ by $(gv)(x) = \det g^k v(xg)$ for $g \in GL(n,\mathbb{C})$, and $v \in V^{(l)}$. We always use this realization and identify C^∞ functions on H_n with values in $V^{(l)}$ with $C^\infty(H_n)[x_1,\dots,x_n]_{(l)}$.

Put

$$B_n = \left\{ \xi \in M_n(\mathbb{R}) | \xi =^t \xi, \xi_{ii}, 2\xi_{ij} \in \mathbb{Z}, \xi \geqslant 0 \right\}.$$

We write the Fourier expansion for $F \in M_{\chi,k,l,n,N}(W)$ as

$$F = \sum_{B_n \ni \xi} a(\xi; F; x_1, \dots, x_n) q^{\xi}.$$

Here $q^{\xi} = \exp(2\pi i(tr(\xi Z)))$. Let $Z = (z_{ij}) \in H_n$. For an integer $l \geqslant 0$ and a function $f \in C^{\infty}(H_n)[x_1, \dots, x_n]_{(l)}$, put

$$Df = \left(\frac{1}{2\pi i} \frac{\partial}{\partial Z} f \right) [x].$$

Here $\frac{\partial}{\partial Z} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial z_{ij}} \right)_{1 \leqslant i,j \leqslant n}$ and $A[x] = xA^t x$. For an integer $l \geqslant 0$ put

$$k^{[l]} = \begin{cases} k(k+1)\dots(k+l-1) & \text{for } l > 0 \\ 1 & \text{for } l = 0 \end{cases}$$

Put $V_1 = \mathbb{C}x_1 \oplus \dots \oplus \mathbb{C}x_m$, $V_2 = \mathbb{C}x_{m+1}$, $n = m+1$. Note that $V_1^{(l)}$ and $V_2^{(l)}$ are subspaces of $V^{(l)} (= \mathbb{C}[x_1, \dots, x_n]_{(l)})$ which are stable under the action of $GL(m, \mathbb{C}) \times \mathbb{C}^*$. For a map $X : A \to C^{\infty}(H_n)[x_1, \dots, x_n]_{(l)}$ (A is an arbitrary set) we denote by $X_{\uparrow} : A \to C^{\infty}(H_n)[x_1, \dots, x_m]_{(l)}$ and $X_{\downarrow} : A \to C^{\infty}(H_n)[x_{m+1}]_{(l)}$ the maps

$$(X_{\uparrow}(a))(x_1, \dots, x_m) = (X(a))(x_1, \dots, x_m, 0)$$

$$(X_{\downarrow}(a))(x_{m+1}) = (X(a))(0, \dots, 0, x_{m+1})$$

Let d^* be the pull-back of the diagonal embedding $d : H_m \times H \to H_{m+1}$. Following Lemma 2.2 of [1] define the operator $L^{(l)}$ by

$$L^{(l)} = \frac{1}{k^{[l]}} d^* \sum_{0 \leqslant 2\nu \leqslant l} \frac{1}{\nu!(l-2\nu)!(2-k-l)^{[\nu]}} (D_{\uparrow} D_{\downarrow})^{\nu} (D - D_{\uparrow} - D_{\downarrow})^{l-2\nu}$$

Let $F \in M_{\chi,k,0,n,N}(\mathbb{C})$. Proposition 2.3 of [1] yields $L^{(l)} F \in M_{\chi,k,l,m,N}(V_1^{(l)}) \otimes M_{\chi,k,l,1,N}(V_2^{(l)})$. One identifies the second factor in this tensor product with the space of elliptic modular forms of weight $k+l$, level N and character χ.

1.3 Klingen type Eisenstein series and a version of the pull-back formula

Let f be an elliptic cusp form of weight $k+l$, level N and character χ. Consider the Klingen-Eisenstein series associated with f:

$$E_m^{k,l,\chi}(f, V_1^{(l)})(Z) = \sum_{\left(\begin{smallmatrix} AB \\ CD \end{smallmatrix}\right) \in P_{m,1}(N) \backslash \Gamma^{(m)}(N)} \overline{\chi}(D) \left(f \circ pr_1^m |_{k,l,m} \begin{pmatrix} AB \\ CD \end{pmatrix} \right) (Z).$$

Here $P_{m,1}$ denotes the parabolic subgroup of $\Gamma^{(m)}(N)$ consisting of all elements whose entries in the last $m+1$ rows and first $m-1$ columns vanish. The projection $pr_1^m : H_m \to H_1$ is defined by $pr_1^m\left(\begin{smallmatrix} * * \\ * w \end{smallmatrix}\right) = w$. One has $E_m^{k,l,\chi}(f, V_1^{(l)}) \in M_{\chi,k,l,m,N}(V_1^{(l)})$. We will also consider the normalized involuted series

$$F_m^{k,l,\chi}(f, V_1^{(l)}) = \\ N^{nk/2}\tilde{\Gamma}(k) L_N(k,\chi) \prod_{1 \leqslant i \leqslant [(m+1)/2]} L_N(2k-2i, \chi^2) E_m^{k,l,\chi}(f, V_1^{(l)})|_{k,l} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

Put

$$\alpha_{k,l} = \begin{cases} \left(-\frac{1}{2\pi i}\right)^l \frac{2(2k-1)^{[l-1]}}{l! k^{[l-1]}} & \text{for } l > 0 \\ 1 & \text{for } l = 0, \end{cases}$$

$$C_{k,l,1} = 2^{3-k-l} i^{k+l} \frac{\pi}{k+l-1},$$

$$\Lambda(f) = L(k,\chi) D_f(2k-2),$$

with $D_f(s) = \sum_{n \geqslant 1} a(n^2) n^{-s}$ denotes the symmetric square of the elliptic cusp form $f = \sum_{n \geqslant 1} a(n) q^n$.

The brackets $\langle\ ,\ \rangle$ will denote the usual Petersson inner product in the space of elliptic modular forms.

The following proposition is a version of the pull-back formulae.

Proposition 1. — *Let f be an (elliptic) cusp Hecke eigenform of weight $k + l$, level N, with Dirichlet character χ. We assume χ to be primitive modulo N. Then*

$$\langle f, L^{(l)} G_{m+1}^* \left(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}; k, \overline{\chi} \right) \rangle = \alpha_{k,l} C_{k,l,1} \Lambda(f) F_m^{k,l,\chi}(f, V_1^{(l)}).$$

The proposition is completely similar to Proposition 4.4 of [1] and Theorem 1.3 of [3]. Though the first assertion mentioned above deals with the full symplectic group, and the second one does not involve differential operators (i.e. treats the case $l = 0$), the necessary changes in the proof are minor. That is why we sketch the argument and refer to [1] and [3] for detailed treatment. We claim that

$$\langle f, L^{(l)} E_{m+1}^{k,\chi} \begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix} \rangle = \alpha_{k,l} C_{k,l,1} \Lambda(f) E_m^{k,l,\chi}(f^\tau, V_1^{(l)})(Z). \tag{1}$$

One begins with the explicit description of the double coset decomposition

$$\Gamma_\infty^{(1+m)} \backslash \Gamma_0^{(1+m)}(N) / \Gamma_0(N)^\uparrow \Gamma_0^{(m)}(N)^\downarrow.$$

This is presented in [3], Theorem 1.1 and Theorem 1.2. The arrows denote the embeddings of the "small" symplectic groups into the "bigger" one (see [3], Part 1). This decomposition allows to subdivide the Eisenstein series $E_{m+1}^{k,\chi}$ into sub-series. The proof of Theorem 1.3 of [3] stays true if one substitutes the automorphy factor by $j_{\chi,k}$. Our claim follows from this argument when $l = 0$. In general when $l > 0$ one needs Lemma 4.2 of [1] to accomplish the proof of (1). We get the formula in Proposition 1 after application the "Fricke-involution" to both sides of (1).

2. Klingen-Eisenstein measure.

2.1 Siegel-Eisenstein measure d'après A.A.Panchishkin

Fix a prime p. We assume for simplicity $p \geqslant 5$ (see [4], remarks after Theorem 7.3.2 for the explanations of this restriction). $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the ring of p-adic integers and its fraction field. We fix an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p \simeq \hat{\overline{\mathbb{Q}}}_p$ of the field of algebraic numbers into the Tate's field (i.e. the completion of the algebraic closure of $\mathbb{Q}_p$). We will tacitly identify algebraic numbers with their images under ι_p. We fix a topological generator u of $(1 + p\mathbb{Z}_p)^*$ (say, $u = p+1$). $\mathcal{O}_p$ denotes the ring of integers in $\mathbb{C}_p$.

The p-adic Lie group $X = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p)$ is the domain of definition for non-Archimedean zeta functions. Since $\mathbb{Z}_p^* = (\mathbb{Z}/p\mathbb{Z})^* \times (1+p\mathbb{Z}_p)^*$, the group X splits into $p-1$ analytical domains.

We identify the elements of finite order from X^{tors} with Dirichlet characters modulo a power of p. We call such a character wild if its tame component $(\mathbb{Z}/p\mathbb{Z})^* \to \mathbb{C}^*$ is trivial.

Note that all integers k can be considered as elements of X. They map $x \in \mathbb{Z}^*$ into x^k.

Denote by ω the Teichmüller character.

The following is a corollary of [10], Theorems 4.3 and 4.4. This serves simultaneously as the template and the motivation for our consideration.

Consider the normalized Siegel modular forms $G^+_{k,\chi}(Z) = \beta(k,\chi,n)G^*(Z;k,\overline{\chi})$, where

$$\beta(k,\chi,n) = \begin{cases} \frac{i^{\mu}\pi^{1/2-k+n/2}}{\Gamma((1-k+(n/2)+\mu)/2)} \frac{C_\chi^{k-(n/2)}}{\tau(\overline{\chi})} & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Here $\mu = 1$ if $(n/2+k)$ is odd and 0 otherwise; $C_\chi > 0$ is the conductor of χ and $\tau(\chi)$ denotes the Gauss sum.

Put $R = \mathcal{O}_p[[q^{C_n}]]$, where C_n is the additive group of positive semidefinite half-integral matrices of size n.

Proposition 2. — *For an integer $i \geqslant 0$ put $k_i = k + (p-1)p^i$. Let $c > 1$ be an integer coprime to p. There exist a (non-zero) measure μ_P^c on $\mathbb{Z}_p^* \times \mathbb{Z}_p^*$ with values in R and a polynomial $H(X,Y) \in \mathcal{O}_p[X,Y]$ such that for all pairs (k,χ) with $k \in \mathbb{Z}$ sufficiently large, $2k > n$, and primitive Dirichlet character χ modulo p^α with $\alpha > 1$ one has:*

$$\int_{\mathbb{Z}_p^* \times \mathbb{Z}_p^*} x^k y^k \chi(y)\, d\mu_P^c = H(c^k - 1, c^k\chi(c) - 1)\, lim_{i\to\infty} G^+_{k_i,\chi\epsilon}(Z).$$

Here ϵ is a (fixed) primitive Dirichlet character with an even conductor T coprime to p.

Remark. One can take

$$H(X,Y) = \begin{cases} 1 - (Y+1)^2 c^n & \text{for } n \text{ even} \\ 1 & \text{for } n \text{ odd.} \end{cases}$$

Proof. Recall [10] that our normalized Siegel-Eisenstein series $G^+_{k,\chi}$ are holomorphic Siegel modular forms with cyclotomic Fourier coefficients:

$$G^+_{k,\chi}(Z) = \beta(k,\chi,n) \sum_{h\in C_n} b^+(h;k,\chi)q^h,$$

with $e_n(Z) = \exp(2\pi i\, tr(Z))$, and

$$b^+(h;k,\chi) = \begin{cases} 2^{-n\kappa}\det(h)^{k-\kappa} L^+_p(k-(n/2), \chi\theta^{n/2}\varepsilon)M(h,\chi,k) & \text{for } n \text{ even,} \\ 2^{-n\kappa}\det(h)^{k-\kappa} M(h,\chi,k) & \text{for } n \text{ odd.} \end{cases} \tag{2}$$

Here $\kappa = (n+1)/2$, the quadratic Dirichlet character ε is associated with the quadratic extension $\mathbb{Q}(\sqrt{\det(h)})$, the character θ is defined by $\theta(a) = \left(\frac{-1}{a}\right)$, the symbol $M(\cdot)$ stays for certain Euler product over the divisors of $\det(h)$, and $L^+_p(\cdot,\cdot)$ is the value of (suitably normalized) Dirichlet L-series:

$$L^+(s,\xi) = (1-p^{-s}\xi(p))L(s,\xi)\frac{2i^\delta\Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi)^s}, \tag{3}$$

where $\delta = 0$ or 1 such that $(-1)^\delta = \xi(-1)$.

We claim that

$$\lim_{i\to\infty} G^+_{k_i,\chi}(Z) = \sum_{\substack{h\in C_n \\ (\det(h),p)=1}} b^+(h;k,\chi)e_n(hZ). \tag{4}$$

Indeed, the quantities $L^+_p(k-(n/2),\chi\varepsilon)M(h,\chi,k)$ and $M(h,\chi,k)$ can be represented as p-adic integrals. This yields our claim. However, the representation in question is proven (see [10], proofs of Theorems 4.3, 4.4) only in the case when the conductor of ε is coprime to p. We put apart the remaining statement to the next section.

Note that one expects the statement of the Proposition to stay true if $T = 1$ (or, more generally, for an odd T). However, since the 2-factor of M is not known in general, we have to introduce this tame level. We refer to loc. cit. for a more detailed discussion and further references.

Notice that the right-hand side of (4) coincide with the series used in the construction of A.A.Panchishkin. Namely, for an integer $c > 1$ coprime to p, one has a measure μ_{E-S} such that

$$\int_{C_n\otimes\mathbb{Z}_p\times\mathbb{Z}_p^*} \det(z)^{k-\kappa}x_p^{k-m/2}\chi(x)\,d\mu_{E-S} =$$
$$H(c^k-1,\chi(c)c^k-1)\sum_{\substack{h\in C_n \\ (\det(h),p)=1}} b^+(h;k,\overline{\chi})q^h.$$

This measure induces our μ_P after the restriction to $C_n\otimes\mathbb{Z}_p^*$ via the map $C_n\otimes\mathbb{Z}_p^* \to \mathbb{Z}_p^*$ which takes a matrix to its determinant. This finishes the proof of Proposition 2.

It follows that there exists a power series $S(X,Y) \in R[[X,Y]]$ such that $S(u^k-1, u^k\chi(u)-1) = H(c^k-1, c^k\chi(c)-1)lim_{i\to\infty}G^+_{k_i,\chi\epsilon}(Z)$. This is just the p-adic Mellin transform of the measure constructed ([9], 4.3).

2.2 Klingen-Eisenstein measure

Let K be a finite extension of $\mathbb{Q}_p$, and $\mathscr{O}$ its ring of integers. Put $\Lambda = \mathscr{O}[[T]]$. Assume now f to be a Λ-adic form ([4], Chapter 7). Let f be a normalized cusp Hecke eigenform with character ϵ. This means that f is a formal power series in $\Lambda[[q]]$ such that its specializations $f(\chi(u)u^k - 1) \in \mathscr{O}[[q]]$ become q-expansions of normalized Hecke eigenforms in $M^{ord}_{k+l}(\chi\epsilon\omega^{-k}, \Gamma_0(M), \mathscr{O}) = M^{ord}_k(\chi\epsilon\omega^{-k}, \Gamma_0(M)) \otimes \mathscr{O}$. We assume f to be primitive of tame conductor T. Thus M is a power of p times T (in fact, the conductor of $\chi\epsilon\omega^{-k}$).

Put for an integer k and χ as above

$$P(k,\chi) = \langle f(\chi(u)u^k - 1), f(\chi(u)u^k - 1)\rangle^{-1} \times$$
$$k^{[l]}\beta(k, \chi\epsilon\omega^{-k}, m+1)\alpha_{k,l}C_{k,l,1}\Lambda(f(\chi(u)u^k - 1))F_m^{k,l,\chi\epsilon\omega^{-k}}(f(\chi(u)u^k - 1), V_1^{(l)}).$$

This is a formal power series in $\overline{\mathbb{Q}}[[q^{B_m}]][x_1, \dots, x_m]_{(l)}$.

Theorem 1. — *For an integer $i \geqslant 0$ put $k_i = k + (p-1)p^i$. There exist a (non-zero) measure μ_K on $\mathbb{Z}_p^* \times \mathbb{Z}_p^*$ with values in $\mathscr{O}_p[[X,Y]][[q^{B_m}]][x_1, \dots, x_m]_{(l)}$ and a polynomial $K(X,Y) \in \mathscr{O}_p[X,Y]$ such that for all pairs (k,χ) with $k \in \mathbb{Z}$ sufficiently large, $2k > n$, and wild primitive Dirichlet character χ modulo p^α with $\alpha > 1$ one has:*

$$\int_{\mathbb{Z}_p^* \times \mathbb{Z}_p^*} x^k y^k \chi(y)\, d\mu_K = K(u^k - 1, u^k\chi(u) - 1) \lim_{i \to 0} P(k_i, \chi)$$

Remark. The existence of the limit in the right becomes clear from the proof.

Proof. Let $\mathscr{E} = \lim_{j\to\infty} T(p)^{j!}$ be the ordinary projector operator. Consider the operator $\tilde{L}^{(l)} = \frac{1}{k^{[l]}} d^*(D - D_\uparrow - D_\downarrow)^l$. Notice that $\mathscr{E}L^{(l)}F = \mathscr{E}\tilde{L}^{(l)}F$ for arbitrary F. Since f is assumed to be p-ordinary, the operator $\mathscr{E}$ does not harm the scalar product with a specialization of f:

$$\begin{aligned}\langle f(\chi(u)u^{k_i} - 1), L^{(l)}G_n^*\left(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}; k_i, \overline{\chi\epsilon}\omega^{k_i}\right)\rangle = \\ \langle f(\chi(u)u^{k_i} - 1), \mathscr{E}\tilde{L}^{(l)}G_n^*\left(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}; k_i, \overline{\chi\epsilon}\omega^{k_i}\right)\rangle.\end{aligned} \tag{5}$$

Combining (5) with Proposition 1, we obtain

$$\begin{aligned} x_n^l P(k_i, \chi) = \langle f(\chi(u)u^{k_i} - 1), f(\chi(u)u^{k_i} - 1)\rangle^{-1} \times \\ \langle f(\chi(u)u^{k_i} - 1), k_i^{[l]}\mathscr{E}\tilde{L}^{(l)}G^+_{k_i, \overline{\chi\epsilon}\omega^{k_i}}\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}\rangle. \end{aligned}$$

Since l is fixed, the differential operator $k_i^{[l]}\tilde{L}^{(l)}$ is interchangeable with the p-adic limits, and does not harm the the smoothness established in Proposition 2. The same is known about the ordinary projector $\mathscr{E}$. Thus the term by term consideration together with Proposition 2 shows then that there exists a power series $\mathscr{G}(X,Y) \in R[[X,Y]] \otimes V_1^{(l)}$ such that

$$\mathscr{G}(u^k - 1, u^k\chi(u) - 1) = H(\chi(u)u^k - 1) \lim_{i\to\infty} k^{[l]}\mathscr{E}\tilde{L}^{(l)}G^+_{k, \overline{\chi\epsilon}\omega^{k_i}}\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}. \tag{6}$$

From the other hand one has

$$x_n^{-l}\mathscr{E}\tilde{L}^{(l)}G^+_{k_i,\chi\epsilon\omega^{-k_i}} \in M^{ord}_{\chi\epsilon\omega^{-k_i},k_i,l,1,N}[[q^{B_m}]][x_1,\dots,x_m]_{(l)}.$$

Let us denote by $\Phi(k_i,\chi,h,s_1,\dots,s_m)$ (here $s_1+\dots+s_m=l$) the coefficient of $q^h x_1^{s_1}\dots x_m^{s_m}$ in this power series. Thus the coefficient of $q^h x_1^{s_1}\dots x_m^{s_m}$ in $P(k_i,\chi)$ equals

$$\langle f(\chi(u)u^{k_i}-1),\Phi(k_i,\chi,h,s_1,\dots,s_m)\rangle/\langle f(\chi(u)u^{k_i}-1),\, f(\chi(u)u^{k_i}-1)\rangle. \tag{7}$$

It follows from (6) that there is a power series $\overline{\mathscr{G}}(X,Y)\in\mathscr{O}_p[[X,Y]]$ such that

$$\overline{\mathscr{G}}(u^k-1,\chi(u)u^k-1)=H(\chi(u)u^k-1)\lim_{i\to\infty}\Phi(k_i,\chi,h,s_1,\dots,s_m) \tag{8}$$

Note that the right hand side is a q-expansion of a p-adic modular form. At the moment, we can not conclude that this is a specialization of a Λ-adic form, because we do not know whether it coincides with a complex-analytic modular form. It follows from the control theorem ([4], Theorem 7.3.1) that there exist Λ-adic forms $\phi_1,\dots,\phi_t\in\mathscr{M}^{ord}(\epsilon,\Lambda)$ such that

$$\Phi(k_i,\chi,h,s_1,\dots,s_m)=\alpha_1(k_i,\chi)\phi_1(u^{k_i}\chi(u)-1)+\dots+\alpha_t(k_i,\chi)\phi_t(u^{k_i}\chi(u)-1). \tag{9}$$

Here the coefficients of the linear combination $\alpha_j(k_i,\chi)$ depend on k_i and χ. Then

$$\begin{aligned}&\langle f(\chi(u)u^{k_i}-1),\Phi(k_i,\chi,h,s_1,\dots,s_m)\rangle=\\&\qquad\sum_{1\leqslant j\leqslant t}\alpha_j(k_i,\chi)\langle f(\chi(u)u^{k_i}-1),\phi_j(\chi(u)u^{k_i}-1)\rangle.\end{aligned} \tag{10}$$

We claim now that the limits $\lim_{i\to\infty}\alpha_j(k_i,\chi)$ exist, and there exist a polynomial $H_1(X)\in\mathscr{O}[X]$ such that

$$H_1(X)\lim_{i\to\infty}\alpha_j(k_j,\chi)\in\Lambda. \tag{11}$$

Moreover, the polynomial $H_1(X)$ does not depend on $h,s_1,\dots,s_m$. Indeed, let $\phi_j=\sum_{n>0}a(j,n)q^n$ with $a(j,n)\in\Lambda$ be the q-expansion of ϕ_j. We can find positive integers $n_1,\dots,n_t$ such that $D=\det(a(j,n_t))\neq 0$. Thus $D\in\Lambda$. The p-adic WeierstraSS preparation theorem yields that there exists a polynomial H_1 such that $H_1/D\in\Lambda$. Note that this polynomial depends only on the character ϵ and on $k \bmod p-1$. Let us now return to (9), and extract from this identity a system of t linear equations. We do this while considering the q-expansion of the right and left sides, and extracting the identities for the coefficients of q^{n_j} for $j=1,\dots,t$. We solve this system of linear equations using the Kramer's method. Our claim about the limits $\lim_{i\to\infty}\alpha_j(k_i,\chi)$ follows now from (8).

Let us now divide (10) by $\langle f(\chi(u)u^{k_i}-1),\, f(\chi(u)u^{k_i}-1)\rangle$, and use the algebraic construction of the scalar product in the space of p-ordinary Λ-adic forms. (The construction and its connection to the usual Petersson inner product is detailly discussed in [4], 7.4 p.222; see also [5], section 1.) Now we are able to interpolate p-adically the series $P(k,\chi)$ coefficient-wise. Namely, the coefficient (7) equals to

$$\sum_{1\leqslant j\leqslant t}\alpha_j(k_i,\chi)\frac{\langle f(\chi(u)u^{k_i}-1),\phi_j(\chi(u)u^{k_i}-1)\rangle}{\langle f(\chi(u)u^{k_i}-1),\, f(\chi(u)u^{k_i}-1)\rangle}.$$

We take the limit when $i \to \infty$ and make use of (11). Note that the ratio of the scalar products interpolates to a meromorphic function. Again by the p-adic Weierstraß preparation theorem it has finitely many poles. The positions of its possible poles depends on nothing but the Λ-adic form f. Thus we can get rid of these poles multiplying by an appropriate polynomial.

So far we get a formal power series $\tilde{P}(X,Y) \in \mathscr{O}_p[[X,Y]][[q^{B_m}]][x_1,\dots,x_m]_{(l)}$ and a polynomial $K(X,Y) \in \mathscr{O}_p[X,Y]$ such that

$$K(u^k-1,u^k\chi(u)-1)\lim_{i\to\infty} P(k,\chi) = \tilde{P}(u^k-1,u^k\chi(u)-1).$$

The series $\tilde{P}$ is the Mellin transform of the measure μ_K required.

3. Another Siegel-Eisenstein measure.

3.1 Twisted Mazur measure.

Let $f : \mathbb{Z} \to \mathbb{C}$ be a periodic function i.e. $f(a+M) = f(a)$. The generalized Bernoulli $B_{k,f}$ numbers are defined by

$$\sum_{k\geqslant 0} \frac{B_{k,f}}{k!}t^k = \sum_{0\leqslant a\leqslant M-1} \frac{f(a)te^{at}}{e^{Mt}-1}.$$

The consideration of the Bernoulli polynomials yields the following p-adic limit representation of Bernoulli numbers ([8], Chapter XIII, Theorem 1.2):

$$B_{k,f} = \lim_{m\to\infty} \frac{1}{Mp^m} \sum_{0\leqslant a\leqslant Mp^m-1} f(a)a^k \tag{12}$$

In order to establish the existence of a measure, we will frequently make use of the following criterion [6], [9].

Proposition 3 (The abstract Kummer congruences). — *Let $f_i \in \mathscr{C}(Y,\mathscr{O}_p)$ be a system of continuous functions on a compact totally disconnected group Y with values in $\mathscr{O}_p$. Assume that $\mathbb{C}_p$-linear span of $\{f_i\}$ is dense in $\mathscr{C}(Y,\mathscr{O}_p)$. Let $\{a_i\}$ be a system of elements $a_i \in \mathscr{O}_p$. Then the existence of an $\mathscr{O}_p$-valued measure μ on Y with the property*

$$\int_Y f_i d\mu = a$$

is equivalent to the following congruences: for an arbitrary choice of elements $b_i \in \mathbb{C}_p$ almost all of which vanish

$$\sum_i b_i f_i(y) \in p^n\mathscr{O}_p \text{ for all } y\in Y \text{ implies } \sum_i b_ia_i \in p^n\mathscr{O}_p.$$

Fix a character ψ modulo d. The numbers $B_{k,\psi f}$ define a distribution on the profinite group $\lim_m \mathbb{Z}/dp^m\mathbb{Z}$. For any function f as above and an integer c we denote by f^c the shifted function $f^c(a) = f(ca)$.

Proposition 4. — *Let $c > 1$ be an integer such that $(c, pd) = 1$. There exist a unique measure μ_ψ^c on $\mathbb{Z}_p^*$ such that for a character χ modulo a power of p and an integer $s \geqslant 0$*

$$\int_{\mathbb{Z}_p^*} x^s \chi(x)\, d\mu_\psi^c = -(1 - c^{s+1}\psi(c)\chi(c))(1 - p^s\psi(p)\chi(p))B_{s+1,\chi\psi}/(s+1). \tag{13}$$

Proof. The proof is fairly standard, and we sketch it. Using (12) and Proposition 3, one shows that, for a fixed $k > 0$, the distributions $E_{k,\psi}^c$ defined by

$$\int_{\mathbb{Z}_p} f\, dE_{k,\psi}^c = B_{k,\psi f} - c^k B_{k,(\psi f)^c}$$

are bounded measures. After that, once more making use of (12), one establishes the connection between these measures for different values of k:

$$\int_{\mathbb{Z}_p} f\, dE_{k,\psi}^c = k \int_{\mathbb{Z}_p} x^{k-1} f\, dE_{1,\psi}^c.$$

Now one defines $\mu_\psi^c = E_{k,\psi}^c$. Since the set of functions $(k, \chi) : x \mapsto x^k\chi(x)$ is an infinite subset in $\mathscr{C}(\mathbb{Z}_p^*, \mathscr{O}_p)$, the p-adic WeierstraSS preparation theorem yields that the measure μ_ψ^c on $\mathbb{Z}_p^*$ is defined by the values (13) uniquely. To accomplish the proof, it stays to calculate the integrals

$$\int_{\mathbb{Z}_p^*} x^s \chi(x)\, d\mu_\psi^c = \int_{Z_p} x^s \chi(s)(1 - \delta(p\mathbb{Z}_p))(x)\, dE_{1,\psi}^c,$$

and to ensure that they coincide with (13). Here $(1 - \delta(p\mathbb{Z}_p))(x) = 0$ if $x \in p\mathbb{Z}_p$ and 0 otherwise. The calculation involves (12).

Let χ be a primitive Dirichlet character modulo p^α with $\alpha > 1$ (this yields, in particular, $\chi(p) = 0$). Assume that ψ is a primitive Dirichlet character modulo pd_0 with $(p, d_0) = 1$. Then the character $\chi\psi$ is primitive. Taking into the account the functional equation for the Dilichlet L-function, one can rewrite (13) as

$$\int_{\mathbb{Z}_p^*} x^s \chi(x)\, d\mu_\psi^c = \\ -(1 - c^{s+1}\psi(c)\chi(c)) \left(\frac{C_{\chi\psi}}{2\pi}\right)^{s+1} \frac{2\Gamma(s+1)\, i^\delta}{\tau(\overline{\chi\psi})} \cos\pi\frac{s+1-\delta}{2} L(s+1, \overline{\chi\psi}). \tag{14}$$

Here δ equals 0 or 1 such that $\chi\psi(-1) = (-1)^\delta$.

In what follows, we need some additional information about the Gauss sum $\tau(\overline{\chi\psi})$ involved.

Lemma 1. — *Let χ be a primitive Dirichlet character modulo p^α with $\alpha > 1$. Let ψ be a primitive Dirichlet character modulo pd_0 with $(p, d_0) = 1$. We write in accordance with the Chinese residue lemma $\psi = \psi_p\psi_{d_0}$, and the characters ψ_p modulo p and ψ_{d_0} modulo d_0 are defined uniquely. Define $l_\chi \in (\mathbb{Z}/p\mathbb{Z})^*$ by $\chi(1 + p^{\alpha-1}) = \exp(2\pi i l_\chi/p)$. Then*

$$\tau(\overline{\chi\psi}) = p^{\alpha-1}\overline{\chi\psi_p}(-d_0)\overline{\psi_{d_0}}(p^\alpha)\overline{\psi_p}(l_\chi)\tau(\overline{\psi})\tau(\overline{\psi_p})/\tau(\chi).$$

In particular, if the character ψ is quadratic, one has

$$\tau(\overline{\chi\psi}) = p^\alpha\overline{\chi}(-d_0)\psi_p(l_\chi d_0)\psi_{d_0}(p^\alpha)\tau(\psi_{d_0})\tau(\chi)^{-1}$$

The proof consists of a computation with Gauss sums using Lemma 3 of [12] (p.87). We omit it.

3.2 Fourier coefficients of Siegel-Eisenstein series as p-adic integrals.

Let us now bound ourselves with the case when the character ψ is quadratic. In this case there exists only one character ψ_p. The number $\psi_p(l_\chi)$ is 1 or -1, and it depends on nothing but the character χ. Combining the assertion of Lemma 1, (3) and (14) we are now able to represent the Fourier expansion coefficients (2) of $G^+_{k,\chi}$ as p-adic integrals. Namely, we have got the following assertion.

PROPOSITION 5. — *Let $h \in C_n$. Assume that $\det(h) = p^t d$ with an odd positive integer $t > 0$ and $(p,d) = 1$. Put, in accordance with the Chinese residue lemma, $\theta^{n/2}\varepsilon = \psi_p\psi_{d_0}$ with ψ_p as above and ψ_{d_0} modulo d_0 such that $(p,d_0) = 1$. Let n be even and $c > 1$ be a positive integer coprime to pd_0. There exist a measure ν_h^c on $\mathbb{Z}_p^*$ which is uniquely defined by the following properties: for all pairs (k,χ) with $k \in \mathbb{Z}$ sufficiently large, $2k > n$, and a primitive Dirichlet character χ modulo p^α with $\alpha > 1$, one has*

$$\int_{Z_p^*} x^{k-n/2}\chi(x)\,d\nu_h^c =$$

$$(1-\chi^2(c)c^{2k-n})b^+(h;k,\overline{\chi})\det(h)^{-k}\psi_{d_0}(p^\alpha)\frac{\psi_p(d_0)}{\tau(\psi_{d_0})}p^{\alpha(k-n/2-1)}\tau(\chi)\chi(-1).$$

The corresponding statement for odd n is clear from [10]. Since the case of even n and $\det(h)$ divisible by an even power of p is also clear from [10], the remaining details in the proof (the validity of (4)) of our Theorem 1 are now fulfilled.

3.3 Another Siegel-Eisenstein measure.

Consider the following subseries of Siegel-Eisenstein series

$$\tilde{G}_{k,\chi}(Z) = p^{-k}\sum_{\substack{h\in C_n\\(\det(h),p^2)=p}} b^+(h;k,\chi)e_n(hZ).$$

This is somehow less natural then the series (4) considered in [10] for the p-adic construction. Nevertheless, it follows from Proposition 5 that one can write

$$\tilde{G}_{k,\chi}(Z) = \lim_{i\to\infty} p^{-k_i}(G_{k_i,\chi} - \lim_{j\to\infty} G_{k_j,\chi}),$$

where $k_i = k + (p-1)p^i$. This last definition is more natural in the framework of our consideration.

THEOREM 2. — *Let $c > 1$ be a positive integer coprime to p. There exist a measure ν_{E-S} on $\mathbb{Z}_p^* \times \mathbb{Z}_p^*$ with values in R and a polynomial $H(X,Y) \in \mathcal{O}_p[X,Y]$ such that for all pairs (k,χ) with $k \in \mathbb{Z}$ sufficiently large, $2k > n$, and primitive Dirichlet character χ modulo p^α with $\alpha > 2$ one has*

$$\int_{\mathbb{Z}_p^*\times\mathbb{Z}_p^*} x^k y^k \chi(y)\,d\nu_{E_S} = H(c^k-1, c^k\chi(c)-1)\tilde{G}_{k,\overline{\chi}}.$$

If n is odd, the proof is absolutely similar to the original proof of Theorem 4.4 in [10]. When n is even, one makes use of the representation of the Fourier coefficients of Siegel-Eisenstein series given in Proposition 5, and goes along the same lines as the proof of Theorem 4.3 of [10]. For these reasons, we omit it.

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