# HILBERT MODULAR FORMS MODULO $p^m$: THE UNRAMIFIED CASE

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ABSTRACT. This paper is about Hilbert modular forms on certain Hilbert modular varieties associated with a totally real field $L$. Let $p$ be unramified in $L$. We reduce to the inert case and consider modular forms modulo $p^m$. We study the ideal of modular forms with $q$-expansion equal to zero modulo $p^m$, find canonical elements in it, and obtain as a corollary the congruences for the values of the zeta function of $L$ at negative integers. Our methods are geometric and have also applications to liftings of Hilbert modular forms and compactification of certain modular varieties.

## 1. INTRODUCTION

1.1. **The contents of this paper.** The subject of this paper is the study of Hilbert modular forms on Hilbert modular varieties and some applications. The modular varieties are those parameterizing abelian varieties of dimension $g$ with a given action of the ring of integers of a totally real field $L$ of degree $g$ over $\mathbb{Q}$ and certain level structures, some indigenous to characteristic $p$. We shall be particularly interested in the case where the domain of the modular form is the modular variety modulo $p^m$. This allows us to study $q$-expansion modulo $p^m$.

The Hilbert modular forms we consider are modular forms in the sense of Katz [11]. Their weights are given by characters of a certain split torus over $\mathcal{O}_L$. Over the complex numbers this just boils down to discussing Hilbert modular forms of possibly non-parallel weight.

We assume a priory that the prime $p$ we are dealing with is non-ramified in $L$. However, one immediately reduces to the case where the prime is inert. This is a well known principle and we refer the reader to [5] to see how this works. *Assume, henceforth, that $p$ is inert.*

Denote the graded ring of Hilbert modular forms of $\mu_N$-level ($(N,p)=1$), defined over $W_m(\mathbb{F})$, by $\oplus_{\chi\in\mathbf{X}}\mathbf{M}(W_m(\mathbb{F}),\chi,\mu_N)$. We refer the reader to Section 1.2 for precise definitions. In brief: $W_m(\mathbb{F})$ is isomorphic to $\mathcal{O}_L/(p^m)$; a level $\mu_N$ means an equivariant embedding of $D_L^{-1}\otimes\mu_N$ into the abelian variety.

The main question we treat is:

*"what can one say on the kernel of the $q$-expansion map on $\oplus_{\chi\in\mathbf{X}}\mathbf{M}(W_m(\mathbb{F}),\chi,\mu_N)$?"*

While in characteristic 0 the kernel is trivial, the situation is different in characteristic $p$. A well-known theorem of P. Swinnerton-Dyer asserts that for $g=1$ and $m=1$, the kernel is generated by $E_{p-1}-1$, where $E_{p-1}$ is an Eisenstein of weight $p-1$ (see (2.21)), and a well-known theorem of P. Deligne asserts that $E_{p-1}$ modulo $p$ is the Hasse invariant.

Our results are a generalization of these theorems for general totally real fields and any $m$. One of the psychological shifts one has to make is to completely abandon the method of obtaining relations from reducing from characteristic zero and to work solely modulo $p^m$. Indeed, the question whether or not $E_{(p-1)p^r}-1$ belongs to this kernel depends, for a given $r$ on the field (and need not hold), and for all $r\gg 0$ is equivalent to Leopoldt's conjecture.

For $m=1$, that is, modulo $p$, our results are a direct and precise analog of the above theorems. The complement of the ordinary locus was studied by F.Oort and the author in [7]. It turns out that it canonically decomposes as a union $\cup_{i=1}^{g}W_{\{i\}}$ (see Section 1.2).

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**Theorem 1.** *(Theorem 2.1) Let $p$ be inert in $L$. There exist Hilbert modular forms $h_1, \dots, h_g$, over $\mathbb{F}$, of weights $\chi_1^p\chi_2^{-1}, \dots, \chi_g^p\chi_1^{-1}$ respectively ($h_i$ being of weight $\chi_i^p\chi_{i+1}^{-1}$), such that*

$$(h_i) = W_{\{i\}}.$$

*(In particular, the divisor of $h_i$ is reduced). The $q$-expansion of $h_i$ at every cusp of $\mathcal{M}^*(\mathbb{F}, \mu_N)$ is 1. Let $h = h_1 \cdots h_g$. Then $h$ is a modular form of weight $\mathrm{Norm}^{p-1}$. It has $q$-expansion equal to 1 at every cusp and its divisor is reduced, equal to the complement of the ordinary locus.*

We remark that $h$ is non-other then the Hasse invariant, i.e., the determinant of the Hasse-Witt matrix, and that if $g > 1$ the $h_i$'s do *not* lift ever to characteristic zero!

We then prove (compare Theorem 2.3)

**Theorem 2.** *Let $p$ be inert in $L$. The kernel of the $q$-expansion map modulo $p$ is the ideal generated by $\{h_1 - 1, \dots, h_g - 1\}$.*

Regarding the situation modulo $p^m$, our results are less complete. Let $I_m$ be the kernel of the $q$-expansion map modulo $p^m$. We are able to identify the quotient $\oplus_{\chi\in\mathbf{X}}\mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)/I_m$ and find some canonical elements in $I_m$ that are a generalization of the $h_i$'s. See Theorem 3.8. After adding level structure one can determine the kernel of the $q$-expansion map modulo $p^m$ completely. See Proposition 3.12.

We provide two applications. One is to construct a normal, and explicit, compactification of Hilbert modular varieties with a "$\Gamma_1(p)$" level structure, i.e., with a $\mu_p$-level. See Theorem 2.8.

The other application is classical. Recall that by a theorem of C.L. Siegel the values of $\zeta_L(1-k)$, for $k \geq 2$ an integer, are rational numbers and are equal to zero if $k$ is odd. From a modern perspective this is quite immediate. One considers the modular form of weight $k$ given by $E_k - E_k^\sigma$ for an automorphism $\sigma$. It turns out that this "rational influence" of the higher coefficients on the constant coefficient can be refined to an "integral influence". This was proved and developed in the case $g = 1$ by J.-P. Serre [16], and in general by P.Deligne and K. Ribet in [4], [15]. In truth, our methods are not that far from Deligne-Ribet's methods [4], [15] (who, in turn, follow ideas of N. Katz [8], [9], [10], [11] and J.-P. Serre [16]), but our approach is more geometric and is based on [7], [5]. The conclusion of the congruences is clearly in "Serre's style".

**Corollary 1.** *(Corollary 3.11) Let $p$ be inert in $L$. Let $k \geq 2$.*

*1. Let $p \neq 2$, then if $k \equiv 0 \pmod{p-1}$*

$$\mathrm{val}_p(\zeta_L(1-k)) \geq -1 - \mathrm{val}_p(k),$$

*and $\zeta_L(1-k)$ is $p$-integral if $k \not\equiv 0 \pmod{p-1}$.*

*2. If $p = 2$, then*

$$\mathrm{val}_2(\zeta_L(1-k)) \geq g - 2 - \mathrm{val}_2(k).$$

**Corollary 2.** *(Corollary 3.15) Let $p$ be inert in $L$. Let $k, k' \geq 2$ and $k \equiv k' \pmod{(p-1)p^m}$.*

*1. If $k \not\equiv 0 \pmod{p-1}$ then*

$$(1 - p^{g(k-1)})\zeta_L(1-k) \equiv (1 - p^{g(k'-1)})\zeta_L(1-k') \pmod{p^{m+1}}.$$

*2. If $k \equiv 0 \pmod{p-1}$ but $p \neq 2$, then*

$$(1 - p^{g(k-1)})\zeta_L(1-k) \equiv (1 - p^{g(k'-1)})\zeta_L(1-k') \pmod{p^{m-1-\mathrm{val}_p(k\cdot k')}}.$$

*3. If $p = 2$, then*

$$(1 - 2^{g(k-1)})\zeta_L(1-k) \equiv (1 - 2^{g(k'-1)})\zeta_L(1-k') \pmod{2^{m+g-2-\mathrm{val}_2(k\cdot k')}}.$$

The derivation of the congruences rests on the following Criterion 3.10:

"Let $\sum_\chi f_\chi \in I_m$. Then there exist $a_\chi$ in some $W_m(\mathbb{F})$-algebra such that $\sum_\chi a_\chi \chi(u) \equiv 0 \pmod{p^m}$ for all $u \in (\mathcal{O}_L/(p^m))^\times$ and $a_{\mathbf{1}} = f_{\mathbf{1}}$."

It is interesting to note that this criterion allows an inverse in some sense. Given such polynomial relations one obtains relations between values of zeta functions, provided certain restrictions are satisfied.

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1.2. **Definitions and Notation.** Let $L$ be a totally real field of degree $g$ over $\mathbb{Q}$. Let $\mathcal{O}_L$ be its ring of integers, $D_L$ the different ideal and $d_L$ the discriminant. Let $\mathfrak{c}$ be a fractional ideal of $L$. Let $p$ a rational prime that is *inert* in $L$. Let $\mathbb{F}$ be a fixed field of $p^g$ elements.

*All schemes in this paper are over* $\mathbb{Z}[d_L^{-1}]$.

• A *HBAS* (Hilbert-Blumenthal abelian scheme) over $S$ is a triple

$$\underline{A} = (A, \iota, \lambda) \tag{1.1}$$

consisting of an abelian scheme $\pi : A \longrightarrow S$, an embedding of rings $\iota : \mathcal{O}_L \hookrightarrow \mathrm{End}_S(A)$, a polarization $\lambda : (M_A, M_A^+) \longrightarrow (\mathfrak{c}, \mathfrak{c}^+)$ identifying the $\mathcal{O}_L$-module $M_A$ of symmetric homomorphisms from $A$ to its dual with $\mathfrak{c}$ such that the cone of polarizations $M_A^+$ is mapped to $\mathfrak{c}^+$. Furthermore, we require that $\mathfrak{t}^*_{A/S}$ be a locally free $\mathcal{O}_L \otimes \mathcal{O}_S$-module of rank 1. In particular, the relative dimension of $A$ is $g$. Here $\mathfrak{t}_{A/S}$ stands for the locally free sheaf of $\mathcal{O}_S$-modules of rank $g$ given by $\mathrm{Lie}(A/S) = s^*\Omega_{A/S}$, where $s : S \longrightarrow A$ is the identity section, and $\mathfrak{t}^*_{A/S}$ stands for its dual. We shall employ this notation for a general group scheme $\pi : G \longrightarrow S$. If $\pi$ is proper then also $\mathfrak{t}^*_{G/S} = \pi_*\Omega_{G/S}$.

By a *non-vanishing differential* on a HBAS $\underline{A}$, we mean an $\mathcal{O}_L \otimes \mathcal{O}_S$ basis to $\mathfrak{t}^*_{A/S}$. Locally Zariski, every HBAS possesses a non-vanishing differential.

• A $\mu_N$*-level structure* on a HBAS is a closed immersion of $S$-group schemes,

$$D_L^{-1} \otimes_{\mathbb{Z}} \mu_N \hookrightarrow A, \tag{1.2}$$

that is equivariant for the $\mathcal{O}_L$-action. Here $\mathcal{O}_L$ acts canonically on $D_L^{-1} \otimes_{\mathbb{Z}} \mu_N$ from the left. If $p|N$ this, of course, implies that $A$ is ordinary at every fibre of characteristic $p$.

• Let $\mathbb{T}$ be the split torus over $W(\mathbb{F})$, associating to a $W(\mathbb{F})$-algebra $R$ the group

$$\mathbb{T}(R) = (\mathcal{O}_L \otimes_{\mathbb{Z}} R)^\times. \tag{1.3}$$

Let $\{\sigma_1, \ldots, \sigma_g\}$ be the embeddings of $L$ into $W(\mathbb{F})$, ordered cyclically with respect to the Frobenius automorphism $\sigma$ of $W(\mathbb{F})$: $\sigma \circ \sigma_i = \sigma_{i+1}$ (the subscripts read $\pmod g$). Once we have fixed a choice of $\sigma_1$, we have then a canonical isomorphism

$$\mathcal{O}_L \otimes_{\mathbb{Z}} W(\mathbb{F}) = \bigoplus_{i=1}^{g} W(\mathbb{F}). \tag{1.4}$$

That gives a canonical isomorphism $\mathbb{T} = {\mathbb{G}_m}^g$, and in particular, a canonical isomorphism

$$\mathbb{T}(R) = \bigoplus_{i=1}^{g} R^\times, \quad R \in W(\mathbb{F}) - \mathrm{Alg.} \ . \tag{1.5}$$

We let $\chi_1, \ldots, \chi_g$ denote the projections of $\mathbb{T}$ on its $g$ components.

• Let $\mathbf{X}$ be the group of characters of $\mathbb{T}$. It is the free abelian group on $\chi_1, \dots, \chi_g$. We write $\mathbf{X}$ multiplicatively:

$$\mathbf{X} = \left\{ \chi_1^{r_1} \cdots \chi_g^{r_g} : r_i \in \mathbb{Z} \right\}. \tag{1.6}$$

It is a principal homogeneous space for the group $\mathbb{Z}[\mathbb{Z}/g\mathbb{Z}]$. We denote by $\mathbf{1}$ the trivial character.

Let $\mathbf{X}(1)$ be the subgroup of $\mathbf{X}$ generated by the elements $\chi_i^p \chi_{i+1}^{-1}$:

$$\mathbf{X}(1) = < \chi_1^p \chi_2^{-1}, \chi_2^p \chi_3^{-1}, \dots, \chi_g^p \chi_1^{-1} > . \tag{1.7}$$

It is the subgroup of $\mathbf{X}$ consisting of all characters trivial on $(\mathcal{O}_L/(p))^\times$ via

$$(\mathcal{O}_L/(p))^\times \hookrightarrow \mathbb{T}(\mathbb{F}) = \overset{g}{\underset{i=1}{\oplus}} \mathbb{F}^\times. \tag{1.8}$$

Similarly, we let $\mathbf{X}(m)$ be the subgroup of $\mathbf{X}$ consisting of all characters trivial on $(\mathcal{O}_L/(p^m))^\times$. See Section 3.2.

• Let $B$ be a $W(\mathbb{F})$-algebra. Let $\chi \in \mathbf{X}$. A *HMF* (Hilbert modular form) over $B$, of weight $\chi$, and $\mu_N$-level is a rule,

$$(\underline{A}, \beta, \omega)_{/R} \mapsto f((\underline{A}, \beta, \omega)_{/R}) \in R, \tag{1.9}$$

associating to a HBAS $\underline{A}$ over a $B$-algebra $R$, endowed with a $\mu_N$-level $\beta$ and a non-vanishing differential $\omega$, an element $f((\underline{A}, \beta, \omega)_{/R})$ of $R$. One requires that $f((\underline{A}, \beta, \omega)_{/R})$ depends only on the $R$-isomorphism class of $(\underline{A}, \beta, \omega)$, commutes with base-change, and satisfies

$$f((\underline{A}, \beta, \alpha^{-1}\omega)_{/R}) = \chi(\alpha) f((\underline{A}, \beta, \omega)_{/R}), \quad \forall \alpha \in (\mathcal{O}_L \otimes R)^\times. \tag{1.10}$$

We let $\mathbf{M}(B, \chi, \mu_N)$ denote the $B$-module of HMFs over $B$, of weight $\chi$ and $\mu_N$-level.

• In [7], a stratification of Hilbert modular varieties in characteristic $p$ was obtained by means of a *type*. In the case where $p$ is inert, the type of $\underline{A}$ is the structure of the Dieudonné module of the $\alpha$-group of $\underline{A}$ as an $\mathcal{O}_L \otimes \mathcal{O}_S$-module. In [5], the reader can find how to define this stratification under less restrictions.

We recall that for every HBAS $\underline{A}$ over a field $k$ containing $\mathbb{F}$ there is associated a type $\tau(\underline{A})$, which is a subset of $\{1, \cdots, g\}$. It simply encodes the structure of the Dieudonné module of the $\alpha$-group of $\underline{A}$, $\alpha(\underline{A})$, as an $\mathcal{O}_L \otimes k$-module. For $k$ a perfect field this $\alpha$-group is $\mathrm{Ker}(F) \cap \mathrm{Ker}(\mathrm{Ver})$. In this case, the Dieudonné module $\mathbb{D}(\alpha(A))$ of $\alpha(\underline{A})$ is a $k$-vector space, of dimension between zero and $g$, on which $\mathcal{O}_L \otimes k$ acts. As $\mathbb{D}(\alpha(A))$ is contained in the Dieudonné module of the kernel of Frobenius, i.e., in the relative cotangent space, we have that $\mathbb{D}(\alpha(A))$ is a sub-sum of $\oplus_{i=1}^g k = \mathcal{O}_L \otimes k$. The type $\tau(A)$ is just defined by the identity

$$\mathbb{D}(\alpha(A)) = \oplus_{i \in \tau(A)} k. \tag{1.11}$$

For every subset $\tau$ of $\{1, \cdots, g\}$, one lets $W_\tau$ be the closed reduced subscheme of the moduli space, universal for the property "the type contains $\tau$". It has codimension $|\tau|$. We have $W_\tau \cap W_\sigma = W_{\tau \cup \sigma}$. For a rigid level structure, $W_\tau$ is regular.

**Lemma 1.1.** *Let $N \geq 4$. The moduli problem of HBAS with $\mu_N$-level over $\mathbb{Z}[d_L^{-1}]$-schemes is rigid.*

*Proof.* Let $\underline{A}$ be a HBAS. Let $D$ be the centralizer of $L$ in $\mathrm{End}(A) \otimes \mathbb{Q}$. It is known that $D$ is either $L$, a CM field such that $D^+ = L$, or a quaternion algebra over $L$ that is ramified everywhere at $\infty$. See [2], Lemma 6.

Let $\mathcal{O}_D = D \cap \mathrm{End}(\underline{A})$. If $\xi \in \mathcal{O}_D$ is an automorphism of $A$ preserving the polarization, then $\xi\xi^* = 1$, where $*$ is the unique positive involution of $D$. Hence, $\xi$ is of finite order. It follows that the field $L(\xi)$ is either $L$, or a CM field whose totally real subfield is $L$, and that $\xi$ is a root of unity of order $n$. The case of $L(\xi) = L$ is just the case of $\xi = \pm 1$ and is easily dispensed with. We assume that $L(\xi) \neq L$. Hence, $[L(\xi) : \mathbb{Q}] = 2g$. Equivalently, $1 < \phi(n)$, $\phi(n) | 2g$ and $L \cap \mathbb{Q}(\xi) = \mathbb{Q}(\xi)^+$.

If $\xi$ preserves a $\mu_N$-level structure, it follows that $N^g|\deg(1-\xi)$. Hence, $n$ is a prime power. Say $n=\ell^r$, $\ell$ a prime. Then $\deg(1-\xi)=\ell^{2g/\phi(n)}$. Since $\phi(n)>1$, this is divisible by a $g$-th power if and only if $\phi(n)=2$. On the other hand, $\phi(n)=\ell^{r-1}(\ell-1)$. This implies $r=1$ and $\ell=3$, or $r=2$ and $\ell=2$. Both imply $N<4$. $\square$

• Let $B$ be a $W(\mathbb{F})$-algebra. We let $\mathcal{M}^c(B,\mu_N)$ denote the base change to $B$ of the ordinary locus of the moduli space of HBAS with $\mu_N$-level compactified at infinity by its canonical minimal compactification. This notation is chosen so that for every $(N_1,N_2)=1$, the map

$$\mathcal{M}^c(B,\mu_{N_1N_2})\longrightarrow\mathcal{M}^c(B,\mu_{N_1}) \tag{1.12}$$

is an étale Galois covering with Galois group canonically isomorphic to $(\mathcal{O}_L/(N_2))^\times$. We elaborate on this:

Let $A$ be a commutative ring with 1. Let $M,M'$ be finitely generated free abelian groups, $N=\mathrm{Hom}(M,\mathbb{Z})$ and $N'=\mathrm{Hom}(M',\mathbb{Z})$. Let $\mathbb{G}_m=\mathrm{Spec}(A[q,q^{-1}])$. We consider the torus

$$\begin{aligned}G(M) &:= \mathrm{Spec}(A[M])\\ &= \mathrm{Spec}(A[x^m:\ m\in M]/(x^0-1,x^mx^{m'}-x^{m+m'}\ \ \forall m,m'\in M)).\end{aligned} \tag{1.13}$$

As a functor on schemes over $A$ we may identify it with the functor $N\otimes\mathbb{G}_{m/A}$, where

$$(N\otimes\mathbb{G}_{m/A})(R):=N\otimes_{\mathbb{Z}}R^\times,\quad R\in A-\mathrm{Alg.}\,. \tag{1.14}$$

One verifies that

$$\mathrm{Lie}(G(M)/A)=N\otimes\mathrm{Lie}(\mathbb{G}_m/A)=N\otimes A\cdot q\frac{\partial}{\partial q}, \tag{1.15}$$

and hence,

$$\mathfrak{t}^*_{G(M)/A}=M\otimes\mathfrak{t}^*_{\mathbb{G}_m/A}=M\otimes A\cdot\frac{dq}{q}. \tag{1.16}$$

See [1], Exposé II. In the last isomorphism we have $m\otimes a\cdot\frac{dq}{q}$ corresponding to $ax^{-m}dx^m$.

Let $\phi:M\longrightarrow M'$ be a homomorphism. It induces a homomorphism of group schemes $\Phi: G(M')\longrightarrow G(M)$, whose effect on functions is $x^m\mapsto x^{\phi(m)}$. The induced map

$$\Phi^*:\mathfrak{t}^*_{G(M)/A}\longrightarrow\mathfrak{t}^*_{G(M')/A} \tag{1.17}$$

is given, innocently enough, by $\frac{dx^m}{x^m}\mapsto\frac{dx^{\phi(m)}}{x^{\phi(m)}}$. Alternately, $m\otimes a\cdot\frac{dq}{q}\mapsto\phi(m)\otimes a\cdot\frac{dq}{q}$.

Consider now the case $M=M'=\mathcal{O}_L$ and $\phi=[\alpha]$, the map of multiplication by an element $\alpha\in\mathcal{O}_L$. That is, we consider the group scheme $D_L^{-1}\otimes\mathbb{G}_m$ over $A$, which is the torus

$$\mathrm{Spec}(A[\mathcal{O}_L])=\mathrm{Spec}(A[x^m:m\in\mathcal{O}_L]/(x^0-1,\ x^mx^{m'}-x^{m+m'}\ \ \forall m,m'\in\mathcal{O}_L)). \tag{1.18}$$

Thus, $[\alpha]$ acts on functions by $x^m\mapsto x^{\alpha m}$. The identification of $\mathfrak{t}^*_{D_L^{-1}\otimes\mathbb{G}_m/A}$ with $\mathcal{O}_L\otimes A\cdot\frac{dq}{q}$ agrees with the action of $\mathcal{O}_L$. In particular, the differential $1\otimes\frac{dq}{q}$ generates $\mathfrak{t}^*_{D_L^{-1}\otimes\mathbb{G}_m/A}$ as an $\mathcal{O}_L\otimes A$-module.

Let $N$ be prime to $p$. Given a HBAS with $\mu_{Np^n}$-level, say $(\underline{A},\beta_N\times\beta_{p^n})$, we define

$$[\alpha](\underline{A},\beta_N\times\beta_{p^n})=(\underline{A},\beta_N\times(\beta_{p^n}\circ[\alpha])). \tag{1.19}$$

We let $(\mathcal{O}_L/(p^n))^\times$ act on functions $f$ on $\mathcal{M}^c(B,\mu_{Np^n})$ by

$$([\alpha]f)(\underline{A},\beta_N\times\beta_{p^n})=f([\alpha](\underline{A},\beta_N\times\beta_{p^n})). \tag{1.20}$$

## 2. Mod $p$

Let $N \geq 4$ and prime to $p$. Let $\mathcal{M}^*(B, \mu_N)$ denote the base change to $B$ of the *whole* moduli space of HBAS with $\mu_N$-level compactified at infinity. For $B$ an $\mathbb{F}$-algebra, we let $W_{\{i\}}$ be the closed reduced subscheme of $\mathcal{M}^*(B, \mu_N)$ where the type contains $i$. See above and [7] for more details.

**Theorem 2.1.** *There exist HMFs $h_1, \dots, h_g$, over $\mathbb{F}$, of weights $\chi_1^p \chi_2^{-1}, \dots, \chi_g^p \chi_1^{-1}$ respectively ($h_i$ being of weight $\chi_i^p \chi_{i+1}^{-1}$), such that*

$$(h_i) = W_{\{i\}}. \tag{2.1}$$

*(In particular, the divisor of $h_i$ is reduced). The q-expansion of $h_i$ at every cusp of $\mathcal{M}^*(\mathbb{F}, \mu_N)$ is 1. Let $h = h_1 \cdots h_g$. Then $h$ is a modular form of weight $\mathrm{Norm}^{p-1}$. It has q-expansion equal to 1 at every cusp and its divisor is reduced, equal to the complement of the ordinary locus.*

We refer the reader to [5] for complete details and discussion of the partial Hasse invariants $h_i$. For completeness, we sketch the proof of the theorem. The following lemma follows immediately from the discussion in [7].

**Lemma 2.2.** *Let $\underline{A}$ be a HBAS over a perfect field $k$ containing $\mathbb{F}$. Assume that $\underline{A}$ is not ordinary. Then the p-divisible group of $\underline{A}$, say $\underline{A}(p)$, is local and a universal display over $\mathrm{Spec}(k[[t_1, \dots, t_g]])$ for its infinitesimal deformations as a HBAS is given by*

$$\Phi = \begin{pmatrix} A + TC & B + TD \\ C & D \end{pmatrix}. \tag{2.2}$$

*Here $A, B, C$ and $D$ are $g \times g$ matrices that are Teichmüller lifts to $W(k[[t_1, \dots, t_g]])$ of the display $\Phi_0 = \left(\begin{smallmatrix} A \pmod p & B \pmod p \\ C \pmod p & D \pmod p \end{smallmatrix}\right)$ of $\underline{A}$, and can be chosen to be of the form*

$$A = \begin{pmatrix} & & & a_1 \\ a_2 & & & \\ & \ddots & & \\ & & a_g & \end{pmatrix}. \tag{2.3}$$

*(Similarly for $B, C, D$). The matrix $T$ is diagonal, with diagonal elements $T_1, \dots, T_g$, where $T_i$ is the Teichmüller lift of $t_i$.*

Let

$$e_1, \dots, e_g \tag{2.4}$$

be the idempotents of $\mathcal{O}_L \otimes \mathbb{F}$. Given $(\underline{A}, \omega)_{/R}$ we get a basis $\{e_1\omega, \dots, e_g\omega\}$ for $\mathfrak{t}^*_{A/R}$. Let $\{\eta_1, \dots, \eta_g\}$ be the basis of $\mathfrak{t}_{A/R}$ dual to that basis. Let $F$ be the Frobenius morphism. It is induced by a choice of prime-to-$p$ $\mathcal{O}_L$-polarization that identifies $\mathfrak{t}_{A/R}$ with $H^1(A, \mathcal{O}_A)$. Put

$$h_i((\underline{A}, \omega)) = F\eta_i / \eta_{i+1}. \tag{2.5}$$

One verifies that indeed $F\eta_i$ is a multiple of $\eta_{i+1}$ and that $h_i$ is a modular form of weight $\chi_i^p \chi_{i+1}^{-1}$. See [5]. Moreover, by the theory of displays, the matrix $A + TC$ modulo $p$ is giving the action of Frobenius on the tangent space of the universal local deformation. One finds that $a_i \pmod p$ is, up to a unit of the base, $h_i(\underline{A}, \omega)$, and that $a_i + T_i c_i \pmod p$ is, up to a unit of the base, $h_i$ of the universal deformation with some choice of a non-vanishing differential on it. On the other hand, one can prove that $a_i = 0$ if and only if $i \in \tau(\underline{A})$. We see that $(h_i) = W_{\{i\}}$. $\square$

Let $R_{Np^n}$ denote the ring of regular functions on the scheme $\mathcal{M}^c(\mathbb{F}, \mu_{Np^n})$.

**Theorem 2.3.** *1. There exists a natural surjective homomorphism*

$$r : \underset{\chi\in\mathbf{X}}{\oplus}\mathbf{M}(\mathbb{F},\chi,\mu_N) \longrightarrow R_{Np}, \tag{2.6}$$

*whose kernel $I$ is precisely the kernel of the $q$-expansion map. It is an ideal graded by $\mathbf{X}/\mathbf{X}(1)$ and*

$$I = (h_i - 1 : i = 1,\dots,g). \tag{2.7}$$

*2. Under the isomorphism provided above, $\underset{\chi\in\mathbf{X}}{\oplus}\mathbf{M}(\mathbb{F},\chi,\mu_N)/I \cong R_{Np}$, we have*

$$\underset{\chi\in\mathbf{X}(1)}{\oplus}\mathbf{M}(\mathbb{F},\chi,\mu_N)/I \cong R_N. \tag{2.8}$$

*Proof.* Let $\pi : (\underline{A}^u,\beta^u) \longrightarrow \mathcal{M}^c(\mathbb{F},\mu_{Np})$ be the universal object. Let

$$\Omega = \mathbf{t}^*_{(\underline{A}^u,\beta^u)\to\mathcal{M}^c(\mathbb{F},\mu_{Np})}. \tag{2.9}$$

Via $\beta^u$ we get an isomorphism

$$\Omega \cong \mathbf{t}^*_{D_L^{-1}\otimes\mu_p\to\mathrm{Spec}(\mathbb{F})} \otimes_{\mathbb{F}} \mathcal{O}_{\mathcal{M}^c(\mathbb{F},\mu_{Np})}. \tag{2.10}$$

Hence $\Omega$ has a canonical generator $\omega_{\mathrm{can}}$: The image of $(1\otimes\frac{dq}{q})\otimes 1$. The idempotents $\{e_1,\dots,e_g\}$ (see (2.4)) give a decomposition

$$\Omega = \overset{g}{\underset{i=1}{\oplus}}\Omega(\chi_i),\quad \omega_{\mathrm{can}} = \overset{g}{\underset{i=1}{\oplus}} a(\chi_i). \tag{2.11}$$

Given any $\chi\in\mathbf{X}$, $\chi = \chi_1^{r_1}\cdots\chi_g^{r_g}$, we put

$$\Omega(\chi) = \overset{g}{\underset{i=1}{\otimes}}\Omega(\chi_i)^{\otimes r_i},\quad a(\chi) = \overset{g}{\underset{i=1}{\otimes}} a(\chi_i)^{\otimes r_i}. \tag{2.12}$$

Clearly $a(\chi)$ is a canonical section of $\Omega(\chi)$ ($\omega_{\mathrm{can}}$ is non-vanishing!).

Let $f\in\mathbf{M}(\mathbb{F},\chi,\mu_N)$. We write $f$ also for the pull-back of $f$ to $\mathcal{M}^c(\mathbb{F},\mu_{Np})$. Let

$$r(f) = f/a(\chi). \tag{2.13}$$

We extend the definition linearly and obtain a ring homomorphism

$$\underset{\chi\in\mathbf{X}}{\oplus}\mathbf{M}(\mathbb{F},\chi,\mu_N) \longrightarrow R_{Np}. \tag{2.14}$$

It can be interpreted as follows. Given $(\underline{A},\beta_N\times\beta_p)_{/R}$, we have

$$r(\sum f_\chi)((\underline{A},\beta_N\times\beta_p)) = \sum f_\chi(\underline{A},\beta_N,(\beta_p^*)^{-1}(1\otimes\frac{dq}{q})). \tag{2.15}$$

From Equation (2.15) we can conclude two facts:

- The map,

$$\underset{\chi\in\mathbf{X}}{\oplus}\mathbf{M}(\mathbb{F},\chi,\mu_N) \longrightarrow R_{Np}, \tag{2.16}$$

is $W(\mathbb{F})^\times$-equivariant, where $\alpha\in W(\mathbb{F})^\times$ acts on $f\in\mathbf{M}(\mathbb{F},\chi,\mu_N)$ by $[\alpha]f = \chi(\alpha)f$. Indeed $r([\alpha]f)(\underline{A},\beta_N\times\beta_p) = \chi(\alpha)r(f)(\underline{A},\beta_N\times\beta_p) = \chi(\alpha)f(\underline{A},\beta_N,(\beta_p^*)^{-1}1\otimes\frac{dq}{q}) = f(\underline{A},\beta_N,\alpha^{-1}\cdot(\beta_p^*)^{-1}1\otimes\frac{dq}{q}) = r(f)(\underline{A},\beta_N\times\beta_p\circ[\alpha]) = [\alpha](r(f))(\underline{A},\beta_N\times\beta_p)$.

- Let $B$ be a $W(\mathbb{F})$-algebra. Let Std be the standard cusp of $\mathcal{M}^c(B,\mu_{Np^n})$. It is the Tate object $D_L^{-1}\otimes\mathbb{G}_m/\underline{q}((\mathfrak{c}D_L)^{-1})$ with its canonical $\mathcal{O}_L$-action and polarization (see [11] for details) with its visible $\mu_{Np^n}$-level structure and non-vanishing differential. Evaluation at that object is a $q$-expansion map.

Taking again $B=\mathbb{F}$ and $n=1$ and employing (2.15), we see that the following diagram commutes:

$$\begin{array}{ccc} \underset{\chi\in\mathbf{X}}{\oplus}\mathbf{M}(\mathbb{F},\chi,\mu_N) & \xrightarrow{r} & R_{Np} \\ q-\text{expansion} & \searrow & \downarrow \\ & & \mathcal{O}_{\mathcal{M}^c(\mathbb{F},\mu_{Np}),\mathrm{Std}} \end{array} \tag{2.17}$$

It follows that $I$ is the kernel of the $q$-expansion map.

Note that since $(W(\mathbb{F})/(p))^\times$ is of order prime to $p$, we have

$$R_{Np}=\underset{\psi\in\mathbf{X}/\mathbf{X}(1)}{\oplus}R_{Np}^{\psi}, \tag{2.18}$$

where $f\in R_{Np}^{\psi}$ if for every $\alpha$, $[\alpha]f=\psi(\alpha)f$.

Given such $f$, choose some lift $\chi$ of $\psi$ to $\mathbf{X}$ and define first a meromorphic modular form $g$ in $\mathbf{M}(\mathbb{F},\chi,\mu_N)$ by

$$g=f\cdot a(\chi). \tag{2.19}$$

In terms of points,

$$g(\underline{A},\beta_N,\omega)=f(\underline{A},\beta_N\times\beta_p)\cdot\psi\left(\frac{(\beta_p^*)^{-1}(1\otimes\frac{dq}{q})}{\omega}\right), \tag{2.20}$$

for any $\mu_p$-level $\beta_p$. This shows that $g$ is indeed of $\mu_N$-level. Clearly, $r(g)=f$ and $g$ has no poles on the ordinary locus. It follows that $g'=g\cdot h^k$ is a holomorphic modular form for $k\gg 0$. Here $h$ is the total Hasse invariant from Theorem 2.1.

Because $I$ is the kernel of the $q$-expansion, it follows that for every $i$, $h_i-1\in I$. In particular:

- $r(h)=1$ and hence $r(g')=f$ and the map $r$ is therefore surjective.
- $(h_1-1,\dots,h_g-1)\subseteq I$.

We next show that $I=(h_1-1,\dots,h_g-1)$. Suppose that $r(\sum_{i=i}^m f_i)=0$. By multiplying by various $h_j-1$ we may assume that $f_i$ is of weight $\psi_i$ and for $i\neq j$ we have $\psi_i\neq\psi_j \pmod{\mathbf{X}(1)}$. But, since the map $r$ is $W(\mathbb{F})^\times$-equivariant, it follows that each $r(f_i)=0$, because they fall into different summands of (2.18). However, on each $\mathbf{M}(\mathbb{F},\chi,\mu_N)$, the $q$-expansion map, hence $r$, is injective. It follows that each $f_i=0$.

To conclude the proof it only remains to prove part 2. But this follows immediately from Equation (2.18) and the fact that $I$ is generated by elements with weights in $\mathbf{X}(1)$. □

*Remark* 2.4. Let $R=\oplus_{\gamma\in\Gamma}R_\gamma$ be a ring graded by an abelian group $\Gamma$. Let $\Gamma_0$ be a subgroup of $\Gamma$. Let $J$ be an ideal generated by elements in $\oplus_{\gamma\in\Gamma_0}R_\gamma$. Then $J$ is an ideal graded by $\Gamma/\Gamma_0$: Let $\delta\in\Gamma$. If a finite sum $\sum_{\gamma\in\Gamma}f_\gamma\in J$, then $\sum_{\gamma\in\delta+\Gamma_0}f_\gamma\in J$.

Although the following corollary will be superseded by Corollary 3.15 below, we include it to demonstrate the principle of deriving congruences between zeta values from modular forms, as well as to set notation.

**Corollary 2.5.** *Let $L$ be a totally real field. Let $p$ be a prime that is unramified in $L$. Let $k\geq 2$.*

1. *If $k\not\equiv 0 \pmod{p-1}$ then $\zeta_L(1-k)$ is $p$-integral.*
2. *If $k\not\equiv 0 \pmod{p-1}$ then $\zeta_L(1-k)\equiv\zeta_L(1-(k+p-1)) \pmod{p}$.*

*Proof.* There exists an Eisenstein series of parallel weight $k$ (i.e., weight $\mathrm{Norm}^k$)

$$E_k=1+2^g\zeta_L(1-k)^{-1}\sum c_{k-1,\alpha}q^\alpha, \tag{2.21}$$

where $\alpha$ runs over a lattice depending on the cusp at which the $q$-expansion is created and the $c_{k-1,\alpha}$ are sums of $(k-1)$-powers of certain rational integers depending on $\alpha$ and the cusp but not on $k$.

More precisely, under appropriate choices, the $q$-expansion on a component of the moduli space has coefficients

$$c_{k-1,\alpha} = \begin{cases} \sigma_{k-1}((\nu)\mathfrak{a}^{-1}D_L) & \nu \in \mathfrak{a}D_L^{-1} \\ 0 & \text{otherwise} \end{cases}, \tag{2.22}$$

where for any integral ideal $\mathfrak{b}$ we let $\sigma_{k-1}(\mathfrak{b}) = \sum_{\mathcal{O}_L \supset \mathfrak{c} \supset \mathfrak{a}} \mathbf{N}(\mathfrak{c})^{k-1}$. See [5] and (3.51). We let

$$E_k^* = 2^{-g}\zeta_L(1-k) \cdot E_k. \tag{2.23}$$

If $2^g\zeta_L(1-k)^{-1}$ is not $p$-integral, then $E_k - 1 \equiv 0 \pmod p$. If $k \not\equiv 0 \pmod{p-1}$ then $\mathrm{Norm}^k \neq \mathbf{1} \pmod{\mathbf{X}(1)}$. This and the fact that $I$ is graded by $\mathbf{X}/\mathbf{X}(1)$, imply that $1 \in I$, which is a contradiction.

Assume that $k \not\equiv 0 \pmod{p-1}$. Because the coefficients $c_{k-1,\alpha} \pmod p$ depend only on $k \pmod{p-1}$, there exists some $\alpha \in \mathbb{Z}_p$ such that

$$E_k^* - E_{k+p-1}^* - \alpha \equiv 0 \pmod p. \tag{2.24}$$

But, using the grading, this implies $\alpha \pmod p$ belong to $I$. That is, $\alpha \equiv 0 \pmod p$. Hence,

$$2^{-g}\zeta_L(1-k) \equiv 2^{-g}\zeta_L(1-(k+p-1)) \pmod p. \quad \square \tag{2.25}$$

**Corollary 2.6.** *Let $H$ be the kernel of the Norm map $(\mathcal{O}_L/(p))^\times \longrightarrow (\mathbb{Z}/(p))^\times$. Let $R_{Np}^{\parallel}$ be the ring of regular functions of the scheme $\mathcal{M}^c(\mathbb{F}, \mu_{Np})/H$. We have isomorphisms*

$$\bigoplus_{k=0}^{\infty} \mathbf{M}(\mathbb{F}, \mathrm{Norm}^{k(p-1)}, \mu_N)/(h-1) \cong R_N, \tag{2.26}$$

$$\bigoplus_{k=0}^{\infty} \mathbf{M}(\mathbb{F}, \mathrm{Norm}^{k}, \mu_N)/(h-1) \cong R_{Np}^{\parallel}. \tag{2.27}$$

*Proof.* Let $\mathbf{X}^{\parallel} \subset \mathbf{X}$ be the characters trivial on $H$. Clearly, $\mathbf{X}^{\parallel} =<\mathrm{Norm}, \mathbf{X}(1)>$. It follows immediately from the theorem that

$$\bigoplus_{\chi \in \mathbf{X}(1)} \mathbf{M}(\mathbb{F}, \chi, \mu_N)/I \cong R_N, \quad \bigoplus_{\chi \in \mathbf{X}^{\parallel}} \mathbf{M}(\mathbb{F}, \chi, \mu_N)/I \cong R_{Np}^{\parallel}. \tag{2.28}$$

Thus, the assertion is that

$$\bigoplus_{\chi \in \mathbf{X}(1)} \mathbf{M}(\mathbb{F}, \chi, \mu_N)/I \cong \bigoplus_{k=0}^{\infty} \mathbf{M}(\mathbb{F}, \mathrm{Norm}^{k(p-1)}, \mu_N)/(h-1), \tag{2.29}$$

and

$$\bigoplus_{\chi \in \mathbf{X}^{\parallel}} \mathbf{M}(\mathbb{F}, \chi, \mu_N)/I \cong \bigoplus_{k=0}^{\infty} \mathbf{M}(\mathbb{F}, \mathrm{Norm}^{k}, \mu_N)/(h-1). \tag{2.30}$$

In both cases the inclusion $\supset$ is clear. Thus, the claim amounts to that for any element $\chi \in \mathbf{X}^{\parallel}$ (resp. $\mathbf{X}(1)$) we may find suitable non-negative $r_i$'s such that $\chi \cdot (\chi_1^p\chi_2^{-1})^{r_1} \cdots (\chi_g^p\chi_1^{-1})^{r_g}$ is a power of Norm. This is clear. $\square$

The modular forms $a(\chi)$ have other interesting applications. We now discuss how they may be used to construct a compactification with nice properties of $T_{1,1} := \mathcal{M}^c(\mathbb{F}, \mu_{Np})$ – the moduli space of HBAS over $\mathbb{F}$-algebras together with $\mu_{Np}$-level.

**Lemma 2.7.** *We have an equality of modular forms on $T_{1,1}$:*

$$a(\chi_i)^{p^g-1} = h_i^{p^{g-1}} h_{i+1}^{p^{g-2}} \cdots h_{i-2}^p h_{i-1}\,. \tag{2.31}$$

*Proof.* Indeed, both sides are modular forms on $T_{1,1}$ of the same weight and the same $q$-expansion, namely, 1. $\square$

Let, therefore,

$$b_i = a(\chi_i)^{p^g-1}, \tag{2.32}$$

be the modular form on $\mathcal{M}^*(\mathbb{F}, \mu_N)$ of weight $\chi_i^{p^g-1}$ and $q$-expansion 1. We fix $i$ and consider the scheme

$$\mathcal{M} = \mathcal{M}^*(\mathbb{F}, \mu_N)[b_i^{1/(p^g-1)}]. \tag{2.33}$$

We explain our notation and terminology:

The map of global sections

$$\Gamma(\mathcal{M}^*(\mathbb{F}, \mu_N), \Omega(\chi_i)) \longrightarrow \Gamma(\mathcal{M}^*(\mathbb{F}, \mu_N), \Omega(\chi_i^{p^g-1})) \tag{2.34}$$

is induced from a morphism of schemes over $\mathcal{M}^*(\mathbb{F}, \mu_N)$

$$\alpha : \Omega(\chi_i) \longrightarrow \Omega(\chi_i^{p^g-1}), \tag{2.35}$$

given locally by taking $(p^g-1)$-powers along the fibre. We define $\mathcal{M} = \mathcal{M}^*(\mathbb{F}, \mu_N)[b_i^{1/(p^g-1)}]$ to be the fibre product with respect to the maps $\alpha$ and $b_i$:

$$\mathcal{M} = \Omega(\chi_i) \underset{\Omega(\chi_i^{p^g-1})}{\times} \mathcal{M}^*(\mathbb{F}, \mu_N). \tag{2.36}$$

Let $p_2 : \mathcal{M} \longrightarrow \mathcal{M}^*(\mathbb{F}, \mu_N)$ be the projection and consider the line bundles $p_2^*\Omega(\chi_i)$ and $p_2^*\Omega(\chi_i^{p^g-1})$ on $\mathcal{M}$. Let $s^u$ be the tautological section

$$s^u : \mathcal{M} \longrightarrow p_2^*\Omega(\chi_i), \tag{2.37}$$

and let $p_2^*b_i$ be the induced section

$$p_2^*b_i : \mathcal{M} \longrightarrow p_2^*\Omega(\chi_i^{p^g-1}). \tag{2.38}$$

The equation

$$(s^u)^{p^g-1} = p_2^*b_i \tag{2.39}$$

holds on $\mathcal{M}$. In fact $\mathcal{M}$ has the following universal property: Given a scheme $f : S \longrightarrow \mathcal{M}^*(\mathbb{F}, \mu_N)$ and $s \in \Gamma(S, f^*\Omega(\chi_i))$ such that $s^{p^g-1} = f^*b_i$, there exists a unique morphism $g : S \longrightarrow \mathcal{M}$ over $\mathcal{M}^*(\mathbb{F}, \mu_N)$ such that $s = g^*s^u$. We leave the verification of this fact to the reader.

One also sees easily that $(\mathcal{O}_L/(p))^\times$, identified with $\mathbb{F}^\times$, acts faithfully on $\mathcal{M}$. The morphism $\mathcal{M} \longrightarrow \mathcal{M}^*(\mathbb{F}, \mu_N)$ is $(\mathcal{O}_L/(p))^\times$-equivariant and exhibits $\mathcal{M}^*(\mathbb{F}, \mu_N)$ as the quotient for this action.

We conclude from Lemma 2.7 and the universal property the existence of an $(\mathcal{O}_L/(p))^\times$-equivariant open immersion

$$T_{1,1} \longrightarrow \mathcal{M}. \tag{2.40}$$

Note the identity

$$a(\chi_i)^p a(\chi_{i+1})^{-1} = h_i. \tag{2.41}$$

We have $a(\chi_{i+1}) = a(\chi_i)^p/h_i$. A priory this is a meromorphic modular form on $\mathcal{M}$. But raising both sides of the equation to the $p^g-1$ power, and using Lemma 2.7, we find it must be holomorphic. It follows that $\mathcal{M}$ does not depend on $i$.

**Theorem 2.8.** *There exists a normal scheme $f : \mathcal{M} \longrightarrow \mathcal{M}^*(\mathbb{F}, \mu_N)$, an open immersion $T_{1,1} \longrightarrow \mathcal{M}$, and a faithful $(\mathcal{O}_L/(p))^\times$ action extending the one on $T_{1,1}$ such that $f$ exhibits $\mathcal{M}^*(\mathbb{F}, \mu_N)$ as the quotient by this action. In particular, $f$ is finite.*

*The scheme $\mathcal{M}$ is the universal scheme for the equation*

$$s^{p^g-1} = h_i^{p^{g-1}} h_{i+1}^{p^{g-2}} \cdots h_{i-2}^{p} h_{i-1}, \tag{2.42}$$

*and is independent of i. The map f is ramified precisely along the complement of the ordinary locus, and is totally ramified there. The singular locus of $\mathcal{M}$ is of pure codimension 2 and is the pre-image of $\cup_{i\neq j} W_{\{i,j\}}$.*

A point of interest in this theorem is that the "reflexive way" to compactify $T_{1,1}$, namely by a Drinfeld $\Gamma_1(p)$-level structure, though having the virtue of being defined over the integers, gives worse results in characteristic $p$. See [13].

## 3. Mod $p^m$

3.1. **Construction of modular forms.** Following Katz [8], we let

$$T_{m,n} = \mathcal{M}^c(W_m(\mathbb{F}), \mu_{Np^n}), \tag{3.1}$$

where $W_m(\mathbb{F})$ is the ring of Witt vectors of length $m$ over $\mathbb{F}$. We retain our convention that $T_{m,0}$ consists only of the ordinary locus (compactified at $\infty$). For every $n$, the morphism $T_{m,n} \longrightarrow T_{m,0}$ is étale Galois with Galois group equal to $(\mathcal{O}_L/(p^n))^\times$. For every $m, n$, the morphism $T_{m,n} \longrightarrow T_{m+1,n}$ is a closed immersion and $T_{m,n} = T_{m+1,n} \otimes W_m(\mathbb{F})$. Thus $T_{m,n}$ is an affine scheme, smooth over $W_m(\mathbb{F})$, for every $m, n$. We let $V_{m,n}$ be its ring of regular functions. Note that $T_{1,1} = \mathcal{M}^c(\mathbb{F}, \mu_{Np})$, that $T_{1,0} = \mathcal{M}^c(\mathbb{F}, \mu_N)$, and thus that $V_{1,1} = R_{Np}$ and $V_{1,0} = R_N$ in the notation of Section 2. The schemes $T_{m,n}$ and the rings $V_{m,n}$ all fit into the following commutative diagrams:

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \downarrow & & \downarrow & & \downarrow & & \\ T_{1,2} & \hookrightarrow & T_{2,2} & \hookrightarrow & T_{3,2} & \hookrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ T_{1,1} & \hookrightarrow & T_{2,1} & \hookrightarrow & T_{3,1} & \hookrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ T_{1,0} & \hookrightarrow & T_{2,0} & \hookrightarrow & T_{3,0} & \hookrightarrow & \dots \end{array} \quad , \quad \begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \uparrow & & \uparrow & & \uparrow & & \\ V_{1,2} & \twoheadleftarrow & V_{2,2} & \twoheadleftarrow & V_{3,2} & \twoheadleftarrow & \dots \\ \uparrow & & \uparrow & & \uparrow & & \\ V_{1,1} & \twoheadleftarrow & V_{2,1} & \twoheadleftarrow & V_{3,1} & \twoheadleftarrow & \dots \\ \uparrow & & \uparrow & & \uparrow & & \\ V_{1,0} & \twoheadleftarrow & V_{2,0} & \twoheadleftarrow & V_{3,0} & \twoheadleftarrow & \dots \end{array} \quad . \tag{3.2}$$

We let

$$T_{m,\infty} = \varprojlim_n T_{m,n}, \quad T_{\infty,\infty} = \varinjlim_m T_{m,\infty}, \tag{3.3}$$

and

$$V_{m,\infty} = \varinjlim_n V_{m,n}, \quad V_{\infty,\infty} = \varprojlim_m V_{m,\infty} \quad . \tag{3.4}$$

**Lemma 3.1.** *1. Fix $1 \le i \le g$. For every $m \le n$ there exist a modular form $a(\chi_i) = a_{m,n}(\chi_i)$ on $T_{m,n}$ of weight $\chi_i$. It has q-expansion equal to* 1 *at the standard cusp* Std.

*2. The $a(\chi_i) = a_{m,n}(\chi_i)$ are compatible in the following sense:*

*a. Under the map $f : T_{m,n+n'} \longrightarrow T_{m,n}$ we have*

$$f^* a_{m,n}(\chi_i) = a_{m,n+n'}(\chi_i). \tag{3.5}$$

*b. Under the map $f : T_{m,n} \longrightarrow T_{m+m',n}$, where $m + m' \le n$, we have*

$$f^* a_{m+m',n}(\chi_i) = a_{m,n}(\chi_i). \tag{3.6}$$

*Proof.* Let $(\underline{A}^u, \beta_N^u \times \beta_{p^n}^u) \longrightarrow T_{m,n}$ be the universal object. Note that

$$\mathfrak{t}^*_{D_L^{-1}\otimes\mu_{p^n}\to W_m(\mathbb{F})} \cong \mathcal{O}_L \otimes \mathfrak{t}^*_{\mu_{p^n}\to W_m(\mathbb{F})}. \tag{3.7}$$

The invariant differentials $\mathfrak{t}^*_{\mu_{p^n}\to W_m(\mathbb{F})}$ are contained in

$$\Omega_{\mu_{p^n}\to W_m(\mathbb{F})} = W_m(\mathbb{F})[q]/(q^{p^n} - 1, p^n q^{p^n - 1}) \cdot dq. \tag{3.8}$$

The differential $\omega = q^{p^n-1}dq$ is invariant and $p^n\omega = 0$. Thus, $m \le n$ if and only if $\mathfrak{t}^*_{D_L^{-1}\otimes\mu_{p^n}\to W_m(\mathbb{F})}$ is a free $\mathcal{O}_L \otimes W_m(\mathbb{F})[q]/(q^{p^n} - 1)$ module of rank 1. Since we assume that $m \le n$, it follows as in the proof of Theorem 2.3 that the relative cotangent space of $(\underline{A}^u, \beta_N^u \times \beta_{p^n}) \longrightarrow T_{m,n}$ is a free $\mathcal{O}_L \otimes \mathcal{O}_{T_{m,n}}$ module of rank 1 with a *canonical* generator $\omega_{\mathrm{can}}$ – "the pull-back of $(1 \otimes \frac{dq}{q}) \otimes 1$".

Let $\{e_1, \dots, e_g\}$ be the idempotents as in (2.4). Let

$$a(\chi_i) = e_i \cdot \omega_{\mathrm{can}}. \tag{3.9}$$

It is a modular form of weight $\chi_i$. The compatibility assertions are easily reduced to the following simple observations:

- The canonical map

$$D_L^{-1} \otimes \mu_{p^n}{}_{/W_m(\mathbb{F})} \hookrightarrow D_L^{-1} \otimes \mu_{p^{n+n'}}{}_{/W_m(\mathbb{F})} \tag{3.10}$$

induces an isomorphism of the relative cotangent spaces.

- The canonical map

$$D_L^{-1} \otimes \mu_{p^n}{}_{/W_{m+m'}(\mathbb{F})} \hookrightarrow D_L^{-1} \otimes \mu_{p^n}{}_{/W_m(\mathbb{F})} \tag{3.11}$$

induces an isomorphism $\mathfrak{t}^*_{D_L^{-1}\otimes\mu_{p^n}\to W_{m+m'}(\mathbb{F})} \otimes_{W_{m+m'}(\mathbb{F})} W_m(\mathbb{F}) \cong \mathfrak{t}^*_{D_L^{-1}\otimes\mu_{p^n}\to W_m(\mathbb{F})}$. □

**Corollary 3.2.** *Let $\chi = \chi_1^{r_1} \cdots \chi_g^{r_g} \in \mathbf{X}$. Define for $m \le n$*

$$a(\chi) = a(\chi_1)^{r_1} \cdots a(\chi_g)^{r_g}. \tag{3.12}$$

*Then the $a(\chi)$ are "independent of $(m, n)$" and define a modular form $a(\chi)$ on $T_{\infty,\infty}$. This modular form is of weight $\chi$ and has $q$-expansion 1 at the standard cusp* Std *of $T_{\infty,\infty}$.* □

The group $(\mathcal{O}_L \otimes \mathbb{Z}_p)^\times$ acts as automorphisms of $T_{m,n}$. This action is given in terms of points by

$$[\alpha](\underline{A}, \beta_N \times \beta_{p^n}) \mapsto (\underline{A}, \beta_N \times (\beta_{p^n} \circ [\alpha])). \tag{3.13}$$

Of course the action factors through $(\mathcal{O}_L/(p^n))^\times$. We let

$$[\alpha] : T_{m,n} \longrightarrow T_{m,n} \tag{3.14}$$

denote the automorphism induced by $\alpha$. The morphism $[\alpha]$ induces an automorphism of modular forms (a diamond operator). This may be seen as follows: The modular forms of weight $\chi$ are sections of $\Omega(\chi)$ (see (2.11), (2.12)). Let $\mathrm{pr} : T_{m,n} \longrightarrow T_{m,0}$ be the natural projection. Then "$\mathrm{pr}^*\Omega(\chi) = \Omega(\chi)$". Indeed, $(\underline{A}^u, \beta_N^u \times \beta_{p^n}^u) \cong (\underline{A}^u, \beta_N^u) \times_{T_{m,0}} T_{m,n}$. But $[\alpha]^*\mathrm{pr}^* = (\mathrm{pr} \circ [\alpha])^* = \mathrm{pr}^*$. Moreover, the formula for the action on a modular form $f$ *is*

$$([\alpha]f)(\underline{A}, \beta_n \times \beta_{p^n}, \omega) = f(\underline{A}, \beta_N \times (\beta_{p^n} \circ [\alpha]), \omega). \tag{3.15}$$

**Lemma 3.3.** *Let $\alpha \in (\mathcal{O}_L/(p^m))^\times$. Let $a(\chi)$ be the modular form on $T_{m,n}$ constructed above. Then*

$$[\alpha]a(\chi) = \chi(\alpha)^{-1}a(\chi). \tag{3.16}$$

*Let $c(\chi)$ be the minimal non-negative integer such that*

$$p^{c(\chi)}(1 - \chi)(t) \equiv 0 \pmod{p^m}, \quad \forall t \in (\mathcal{O}_L/(p^m))^\times. \tag{3.17}$$

*Then $p^{c(\chi)}a(\chi)$ is invariant under $(\mathcal{O}_L/(p^m))^\times$, and in particular, $a(\chi)$ is invariant under $(\mathcal{O}_L/(p^m))^\times$ if and only if $\chi$ is the trivial map $\pmod{p^m}$.*

*Proof.* Let $\chi = \chi_1^{r_1} \cdots \chi_g^{r_g}$. In terms of points, we have

$$a(\chi)(\underline{A}, \beta_n \times \beta_{p^n}, \omega) = \prod_{i=1}^{g} (e_i \cdot (\beta_{p^n}^*)^{-1}(1 \otimes \frac{dq}{q})/e_i \cdot \omega)^{r_i}. \tag{3.18}$$

The assertion follows. □

Let $\mathbf{X}(m)$ be the characters in $\mathbf{X}$ that are trivial on $(\mathcal{O}_L/(p^m))^\times$ under the composition

$$(\mathcal{O}_L/(p^m))^\times \hookrightarrow (\mathcal{O}_L \otimes W_m(\mathbb{F}))^\times = \mathbb{T}(W_m(\mathbb{F})) \xrightarrow{\chi} \mathbb{G}_m(W_m(\mathbb{F})) = W_m(\mathbb{F})^\times. \tag{3.19}$$

We shall discuss $\mathbf{X}(m)$ further below. For now, note that $\mathbf{X}(m+1) \subset \mathbf{X}(m)$, and if $j$ is the minimal non-negative integer such that $\chi \in \mathbf{X}(m)$ then

$$c(\chi) = m - j. \tag{3.20}$$

We say that an element $\chi$ of $\mathbf{X}(m)$ is *p-positive* if in its expression as

$$\chi = (\chi_1^p \chi_2^{-1})^{r_1} \cdots (\chi_g^p \chi_1^{-1})^{r_g}, \tag{3.21}$$

every $r_i \geq 0$.

**Corollary 3.4.** *1. For every $\chi \in \mathbf{X}$ there exists a modular form $p^{c(\chi)}a(\chi)$ on $T_{m,0} = \mathcal{M}^c(W_m(\mathbb{F}), \mu_N)$ of weight $\chi$ ($a(\chi)$ is given by (3.12)). Its q-expansion at every cusp is $p^{c(\chi)}$. In particular, for every $\chi \in \mathbf{X}(m)$, the modular form $a(\chi)$ is a modular form of weight $\chi$ and q-expansion 1 on $T_{m,0}$.*

*2. Let $\chi \in \mathbf{X}(m)$. The modular form $a(\chi)$ extends to the non-ordinary locus, i.e., it is a modular form over $\mathcal{M}^*(W_m(\mathbb{F}), \mu_N)$, if and only if $\chi = (\chi_1^p \chi_2^{-1})^{r_1} \cdots (\chi_g^p \chi_1^{-1})^{r_g}$ is p-positive. Furthermore,*

$$a(\chi) = h_1^{r_1} \cdots h_g^{r_g} \pmod p. \tag{3.22}$$

*Proof.* It follows from Lemma 3.3 that $p^{c(\chi)}a(\chi)$ is a modular form on $T_{m,0}$, of weight $\chi$, and that its $q$-expansion at every cusp is $p^{c(\chi)}$. This is clear if one thinks on a modular form as in (1.9).

Consider $a(\chi) \pmod p$. It has the same weight and $q$-expansion as the r.h.s. of Equation (3.22) and that proves the equation. The divisor of $a(\chi)$ on $T_{m,n}$ intersects the special fibre in the divisor of $h_1^{r_1} \cdots h_g^{r_g}$. But according to Theorem 2.1 we have

$$(h_1^{r_1} \cdots h_g^{r_g}) = r_1 W_{\{1\}} + \cdots + r_g W_{\{g\}}. \tag{3.23}$$

Hence, this divisor is effective if and only if each $r_i \geq 0$. □

3.2. **Digression on X(m).** We consider now more closely the group $\mathbf{X}(m)$. Let us change notation. Let $G = <\sigma>$ be a cyclic group of order $g$. Let $\mathbb{Z}[G]$ be the group ring of $G$ and $\mathbb{Z}_p[G]$ be the group ring of $G$ over $\mathbb{Z}_p$. The group $W(\mathbb{F})^\times$ is a module over $\mathbb{Z}[G]$, where $\sigma$ acts as $\sigma$ - the Frobenius.

• Assume first that $p \neq 2$.
We have

$$W(\mathbb{F})^\times = \mu \times U_1, \tag{3.24}$$

where $\mu$ is the cyclic group of order $p^g - 1$ consisting of the roots of unity in $W(\mathbb{F})$, and $U_m$ are the units congruent to 1 modulo $(p^m)$. Clearly, as a $\mathbb{Z}[G]$ module,

$$\mu \cong \mathbb{Z}[G]/(p^g - 1, p - \sigma) = \mathbb{Z}[G]/(p - \sigma). \tag{3.25}$$

By a theorem of Krasner, [12] Theorem 17, $U_1$ is a free $\mathbb{Z}_p[G]$-module of rank 1. Hence,

$$W_m(\mathbb{F})^\times = \mu \times U_1/U_m \cong \mu \times U_1/U_1^{p^{m-1}} \tag{3.26}$$

and it follows that as a $\mathbb{Z}[G]$-module

$$W_m(\mathbb{F})^\times \cong \mathbb{Z}[G]/(p-\sigma) \oplus \mathbb{Z}[G]/(p^{m-1}) \cong \mathbb{Z}[G]/(p^{m-1}(p-\sigma)). \tag{3.27}$$

Otherwise said:

$$\mathbf{X}(m) \cong <\chi_1^{p^m}\chi_2^{-p^{m-1}}, \ldots, \chi_g^{p^m}\chi_1^{-p^{m-1}}>. \tag{3.28}$$

Note that these are $p$-positive generators.

• Assume now that $p = 2$. We have

$$W(\mathbb{F})^\times = \mu \times U_1 = \mu \times \{\pm 1\} \times U, \tag{3.29}$$

where $\mu$ are the $2^g - 1$ roots of unity and $U$ is a torsion free subgroup of $U_1$.

Assume that $g$ is odd. Then by [12], Theorem 17, we have

$$U \cong \mathbb{Z}_p[G]. \tag{3.30}$$

Thus, for $m = 1$,

$$W_1(\mathbb{F})^\times \cong \mathbb{Z}[G]/(2-\sigma), \tag{3.31}$$

and for $m \geq 2$

$$W_m(\mathbb{F})^\times \cong \mathbb{Z}[G]/(2-\sigma) \oplus \mathbb{Z}[G]/(\sigma, 2) \oplus \mathbb{Z}[G]/(2^{m-2}). \tag{3.32}$$

The group $\mathbf{X}(m)$ is thus the intersection of ideals $(2-\sigma) \cap (\sigma, 2) \cap (2^{m-2})$. We have $(2-\sigma) \subset (\sigma, 2)$, $(2^{m-2}) \subset (\sigma, 2)$ if $m > 2$ and $(2^{m-2}) \supset (\sigma, 2)$ if $m = 2$. Thus,

$$\mathbf{X}(m) = \begin{cases} (2-\sigma) & m = 1 \\ (2^{m-2}(2-\sigma)) & m \geq 2 \end{cases}. \tag{3.33}$$

In any case $\mathbf{X}(m)$ has naturally chosen $p$-positive generators,

$$x, x\sigma, \ldots, x\sigma^{g-1}, \tag{3.34}$$

where $x$ is $2-\sigma$ or $2^{m-2}(2-\sigma)$, depending on the case.

If $g$ is even, the situation is more complicated. The decomposition (3.29) still hold, but $U$ can not always be chosen to be a $G$-module. We allow ourselves simply to remark that in the case $p = 2$ in fact $\mathbf{X}(1) = \mathbf{X}$ is the free abelian group generated by $\chi_1, \ldots, \chi_g$. The notion of positivity is the one obtained by identifying $\mathbf{X}$ with $\mathbb{Z}^g$ by sending $\chi_i$ to the $i$-th standard basis element. The group $\mathbf{X}(m)$ is a sub-lattice and is therefore automatically generated by $p$-positive elements. Without going into the details of its structure, we let

$$\psi_1, \ldots, \psi_s \tag{3.35}$$

($s = s(g)$) be $p$-positive generators for it. For the applications we give, the following observation suffices:

*Remark* 3.5. The character $\mathrm{Norm}^k$ belongs to $\mathbf{X}(m)$ if and only if $2^{e(m)}|k$, where $2^{e(m)}$ is the exponent of the group $(\mathbb{Z}/(2^m))^\times$. I.e., $e(m) = m-1$ for $m = 1, 2$, and $m-2$ for $m > 2$.

3.3. **The $q$-expansion map mod $p^m$.** In this section we study the kernel of the $q$-expansion map on Hilbert modular forms modulo $p^m$ and level prime to $p$. Our results are not complete in the sense that we fail to produce a complete set of generators for the kernel $I_m$ of the $q$-expansion map. However, see Theorem 3.8 and Remark 3.13. We do obtain enough information on $I_m$ to deduce, after introducing a "technical device", the classical congruences and estimates on values of $\zeta_L$ at negative integers. See Corollaries 3.11 and 3.15 below.

We remark that our techniques apply to more general $L$-functions. But the true difficulty now is in the *construction* of Hilbert modular forms with a $q$-expansion whose constant term is the desired special value and whose higher coefficients have integrality and congruence properties. For this, see [4] and [18].

**Definition 3.6.** Let $\chi \in \mathbf{X}$ and consider it as a character $\chi : (\mathcal{O}_L/(p^m))^\times \longrightarrow W_m(\mathbb{F})^\times$. Let

$$V_{m,m}^\chi = \left\{ f \in V_{m,m} : \alpha \cdot f = \chi(\alpha) f,\ \forall \alpha \in (\mathcal{O}_L/(p^m))^\times \right\}. \tag{3.36}$$

Let $V_{m,m}^K$ – the "Kummer part" of $V_{m,m}$ – be given by

$$V_{m,m}^K = \sum_{\chi \in \mathbf{X}/\mathbf{X}(m)} V_{m,m}^\chi. \tag{3.37}$$

*Remark* 3.7. Note that if $m > 1$ the inclusion $V_{m,m}^K \hookrightarrow V_{m,m}$ is always a *strict* inclusion and the sum in (3.37) is never a direct sum.

**Theorem 3.8.** *1. There exists a natural surjective homomorphism of rings*

$$r : \bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N) \longrightarrow V^K_{m,m}. \tag{3.38}$$

*Let $I_m$ be the kernel of $r$. Then $I_m$ is equal to the kernel of the $q$-expansion map.*

*2. Let $I'_m$ be the ideal $I_m \cap \oplus_{\chi \in \mathbf{X}(m)} \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)$. The map $r$ induces an isomorphism*

$$\bigoplus_{\chi \in \mathbf{X}(m)} \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)/I'_m \cong V_{m,0}. \tag{3.39}$$

*3. If $p \neq 2$, the ideal $I_m$ contains the ideal $< a(\chi_1^{p^{m+1}} \chi_2^{-p^m}) - 1, \dots, a(\chi_g^{p^{m+1}} \chi_1^{-p^m}) - 1 >$, and if $p = 2$, it contains $< a(\psi_1) - 1, \dots, a(\psi_s) - 1 >$, (where for $g$ odd we have $s = g$ and generators as in (3.34), and for $g$ even the generators are as in (3.35)).*

*Proof.* The proof follows the same line as the proof of Theorem 2.3. We shall therefore be brief.

The map $r$ is defined as in Theorem 2.3. Namely, if $f \in \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)$, we let $r(f) = f/a(\chi)$. Using Corollary 3.2 we see that $f$ and $r(f)$ have the same $q$-expansion, and since $V_{m,m}$ is irreducible, we conclude that $I_m$ is the kernel of the $q$-expansion map. Certainly Corollary 3.4 implies that if $p \neq 2$,

$$I_m \supseteq < a(\chi_1^{p^{m+1}} \chi_2^{-p^m}) - 1, \dots, a(\chi_g^{p^{m+1}} \chi_1^{-p^m}) - 1 >, \tag{3.40}$$

and if $p = 2$,

$$I_m \supseteq < a(\psi_1) - 1, \dots, a(\psi_s) - 1 > . \tag{3.41}$$

Moreover, one verifies that the map $r$ is $(\mathcal{O}_L \otimes \mathbb{Z}_p)^\times$-equivariant. where $([\alpha]f) = \chi(\alpha)f$ for $f \in \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)$, and $([\alpha]f)(\underline{A}, \beta_N \times \beta_{p^n}) = f(\underline{A}, \beta_N \times (\beta_{p^n} \circ [\alpha]))$ for $f \in V_{m,m}$. This shows that the image of $r$ is contained in $V^K_{m,m}$. On the other, a construction as in Theorem 2.3, shows that $r$ is surjective onto $V^K_{m,m}$.

It remains only to note that the equivariance implies also (3.39). □

*Remark* 3.9. For $m > 1$, it is not true that $I'_m = I_m$. This has to do again with (3.37) not being a direct sum.

The following Criterion follows directly from the methods of the proof of Theorem 3.8. Weak as it seems, it will suffice to derive the classical congruences between values of $\zeta_L$ (and more generally, of suitable $L$-functions).

**Criterion 3.10.** *Let $\sum_\chi f_\chi \in I_m$. Then there exist $a_\chi$ in some $W_m(\mathbb{F})$-algebra such that*

$$\sum_\chi a_\chi \chi(u) \equiv 0 \pmod{p^m}, \ \ \forall u \in (\mathcal{O}_L/(p^m))^\times, \tag{3.42}$$

*and $a_{\mathbf{1}} = f_{\mathbf{1}}$.*

*Proof.* Consider the relation $\sum_\chi r(f_\chi) = 0$. Evaluate it at a point and let the Galois group act. □

**Corollary 3.11.** *Let $k \geq 2$.*

*1. Let $p \neq 2$, then if $k \equiv 0 \pmod{p-1}$*

$$\mathrm{val}_p(\zeta_L(1-k)) \geq -1 - \mathrm{val}_p(k), \tag{3.43}$$

*and $\zeta_L(1-k)$ is $p$-integral if $k \not\equiv 0 \pmod{p-1}$.*

*2. If $p = 2$, then*

$$\mathrm{val}_2(\zeta_L(1-k)) \geq g - 2 - \mathrm{val}_2(k). \tag{3.44}$$

*Proof.* 1. The case $k \not\equiv 0 \pmod{p-1}$ was treated in Corollary 2.5. Hence, assume $k \equiv 0 \pmod{p-1}$. Let $E_k$ be as in (2.21). Let

$$\ell = \max\{-\mathrm{val}_p(2^{-g}\zeta_L(1-k)), 0\}. \tag{3.45}$$

If $\ell = 0$ there is nothing to prove. Assume therefore that $\ell > 0$ and consider the congruence

$$E_k - 1 \equiv 0 \pmod{p^\ell}. \tag{3.46}$$

Then Criterion 3.10 says that for some $a$ in a $W_\ell(\mathbb{F})$ algebra, the polynomial $a \cdot \mathrm{Norm}(x)^k - 1$ is identically zero on $(\mathcal{O}_L/(p^\ell))^\times$ or, equivalently, the polynomial $ax^k - 1$ is identically zero on $(\mathbb{Z}/p^\ell\mathbb{Z})^\times$ – a cyclic group of order $(p-1)p^{\ell-1}$. Taking $x = 1$ we see that $a = 1$. It follows that $p^{\ell-1}$ divides $k$ and, hence, $\mathrm{val}_p(k) \geq \ell - 1 \geq -\mathrm{val}_p(2^{-g}\zeta_L(1-k)) - 1$.

2. When $p = 2$ one argues the same and obtains that $ax^k - 1$ is identically zero on $(\mathbb{Z}/2^\ell\mathbb{Z})^\times$. Analysis of the structure of this group yields the result. □

3.4. **Adding level $p$-structure.** In this section we briefly discuss modular forms of level $\mu_N$ (for $(N,p) = 1$) together with an extra level structure of either the form $\mu_{p^m}$, or the form $\Gamma_0(p)$. The first additional level structure already appeared above as involving the *target* of the $q$-expansion map modulo $p^m$. It would now appear in the level of the modular forms themselves. This would clarify the nature of the ideal $I_m$ of Theorem 3.8.

The second level structure is introduced to derive the precise congruences between, say, values of the zeta function, that are needed to construct the $p$-adic zeta function. The same technique would work for a wide variety of $L$-functions.

**Adding $\mu_{p^m}$ level.** Let us consider the graded ring modular forms on the scheme $T_{m,m}$. We denote it by $\mathbf{M}_m(Np^m) := \oplus_{\chi\in\mathbf{X}}\mathbf{M}(W_m(\mathbb{F}), \chi, \mu_{Np^m})$. The modular forms on $T_{m,0}$, say $\mathbf{M}_m(N)$, embed in $\mathbf{M}_m(Np^m)$ via the canonical projection $T_{m,m} \longrightarrow T_{m,0}$.

**Proposition 3.12.** *Let $I_m(Np^m)$ be the kernel of the $q$-expansion map on $\mathbf{M}_m(Np^m)$. Then*

$$I_m(Np^m) =< a(\chi) - 1 : \chi \in \mathbf{X} >, \tag{3.47}$$

*and*

$$I_m(N) = I_m(Np^m) \cap \mathbf{M}_m(N) \subset I_m(Np^m)^{\mathbf{1}}, \tag{3.48}$$

*where $I_m(Np^m)^{\mathbf{1}}$ stands for the elements of $I_m(Np^m)$ invariant under the Galois group $(\mathcal{O}_L/(p^m))^\times$.*

*Proof.* First, by Corollary 3.2 indeed $a(\chi) - 1$ belongs to $I_m(Np^m)$. Suppose that the $q$-expansion of $\sum_\chi f_\chi$ is zero. Then we may replace an $f_\chi$ by $f_\chi + f_\chi(a_\chi - 1)$. Repeating this as necessary we obtain a modular form $g$ of parallel positive weight whose $q$-expansion is zero. Hence, $g$ is zero. That is $\sum_\chi f_\chi \in< a(\chi) - 1 : \chi \in \mathbf{X} >$. The rest is clear. □

*Remark* 3.13. The proposition above clearly demonstrates the problem of determining $I_m(N)$ explicitly. The elements in $I_m(Np^m)^{\mathbf{1}}$ need not extend to a *holomorphic* modular form on $T_{m,0}$.

**Adding $\Gamma_0(p)$ level.** By a $\Gamma_0(p)$ level structure on a HBAS $\underline{A}$ we mean a subgroup scheme $H \subset A[p]$, $\mathcal{O}_L$-invariant and of order $p^g$. Such a subgroup is automatically isotropic with respect to any $\mathcal{O}_L$-polarization. We refer the reader to [13], [17] and [6] for details. However, it may benefit the exposition to recall some basic facts without proofs.

Let us denote the fine moduli scheme representing HBAS with level $\mu_N$ and level $\Gamma_0(p)$, over $W_m(\mathbb{F})$-algebras, by $S_m$ ($m \leq \infty$). Let us denote by $S_m^0$ the ordinary locus. The scheme $S_1$ has two "horizontal" components, denoted $S_1^F$ and $S_1^V$, that correspond to taking as $H$ the kernel of Frobenius or the kernel of Verschiebung, respectively. The natural morphism

$$\pi : S_1 \longrightarrow \mathcal{M}^*(W_m(\mathbb{F}), \mu_N), \tag{3.49}$$

induces an isomorphism, $S_1^F \longrightarrow \mathcal{M}^*(W_m(\mathbb{F}), \mu_N)$, and a totally inseparable morphism of degree $p^g$, $S_1^V \longrightarrow \mathcal{M}^*(W_m(\mathbb{F}), \mu_N)$. The scheme $S_1$ has many other components parameterized by the type and the geometric fibers of $\pi$ are stratified by projective spaces.

Consider the restriction of the section $\mathcal{M}^*(W_m(\mathbb{F}), \mu_N) \longrightarrow S_1^F$ to $T_{1,0}$, where as above, $T_{1,0}$ stands for the ordinary part of $\mathcal{M}^*(W_m(\mathbb{F}), \mu_N)$. Let $S_m^{F,0}$ be the open subscheme of $S_m$ consisting of ordinary HBAS $\underline{A}$ with $H$ being the connected part $A[p]$. We have the following commutative diagram in which the vertical arrows are isomorphisms:

$$\begin{array}{ccccccc} S_1^F & \hookleftarrow & S_1^{F,0} & \hookrightarrow & S_m^{F,0} & \longleftarrow & T_{m,1} \\ \downarrow & & \downarrow & & \downarrow & \swarrow & \\ \mathcal{M}^*(W_m(\mathbb{F}), \mu_N) & \hookleftarrow & T_{1,0} & \hookrightarrow & T_{m,0} & & \end{array}. \tag{3.50}$$

Consider the modular form

$$E_k^* = 2^{-g}\zeta_L(1-k) + \sum_{\nu \in \mathcal{O}_L{}^+} \left( \sum_{\mathfrak{c}|(\nu)} \mathrm{Norm}(\mathfrak{c})^{k-1} \right) e^{2\pi i \mathrm{Tr}(\nu z)}. \tag{3.51}$$

It is a modular form of weight $\mathrm{Norm}^k$ on $\mathrm{SL}_2(\mathcal{O}_L \oplus D_L)$ - a component of $\mathcal{M}^*(\mathbb{C}, \mu_N)$. The coefficient of $e^{2\pi i \mathrm{Tr}(\nu z)}$ can also be written as $\sigma_{k-1}((\nu))$, where for every integral ideal $\mathfrak{b}$ we let

$$\sigma_{k-1}(\mathfrak{b}) = \sum_{\mathcal{O}_L \supseteq \mathfrak{c} | \mathfrak{b}} \mathrm{Norm}(\mathfrak{c})^{k-1}. \tag{3.52}$$

The function $\sigma_{k-1}$ is multiplicative:

$$\sigma_{k-1}(\mathfrak{b}\mathfrak{c}) = \sigma_{k-1}(\mathfrak{b})\sigma_{k-1}(\mathfrak{c}), \quad (\mathfrak{b}, \mathfrak{c}) = 1. \tag{3.53}$$

It follows that for every prime ideal $\mathfrak{q}$, an ideal $\mathfrak{b} \subset \mathcal{O}_L$ prime to $\mathfrak{q}$, and any $r \geq 0$, we have

$$\sigma_{k-1}(\mathfrak{q}^{r+1}\mathfrak{b}) - q^{f(k-1)}\sigma_{k-1}(\mathfrak{q}^r\mathfrak{b}) = \sigma_{k-1}(\mathfrak{b}), \tag{3.54}$$

where $q$ is the rational prime below $\mathfrak{q}$ and $f = f(\mathfrak{q}/q)$.

Retaining our assumption that $p$ is inert in $L$, let us put

$$\sigma_{k-1,p}(p^r\mathfrak{b}) = \sigma_{k-1,p}(\mathfrak{b}), \quad (p, \mathfrak{b}) = 1. \tag{3.55}$$

We obtain then the expansion

$$E_k^\dagger(\tau_1, \ldots, \tau_g) \stackrel{def}{=} E_k^*(\tau_1, \ldots, \tau_g) - p^{g(k-1)}E_k^*(p\tau_1, \ldots, p\tau_g) \tag{3.56}$$

$$= (1 - p^{g(k-1)})2^{-g}\zeta_L(1-k) + \sum_{\nu \in \mathcal{O}_L{}^+} \sigma_{k-1,p}((\nu))e^{2\pi i \mathrm{Tr}(\nu z)}. \tag{3.57}$$

The point important to us is that all the coefficients (except the constant one) are $k-1$ powers of natural numbers that are prime to $p$. Hence, the following holds:

Let $k, k' \geq 2$ and $k \equiv k' \pmod{(p-1)p^m}$. Let

$$\ell = \max\{-\mathrm{val_p}(2^{-g}\zeta_L(1-k)), -\mathrm{val_p}(2^{-g}\zeta_L(1-k')), 0\}, \tag{3.58}$$

and put

$$r = \max\{\mathrm{val_p}(k), \mathrm{val_p}(k')\}, \quad r' = \min\{\mathrm{val_p}(k), \mathrm{val_p}(k')\}. \tag{3.59}$$

Note the following points: (i) If $p \neq 2$ then $0 \leq \ell \leq 1 + r$; (ii) If $p = 2$ then $0 \leq \ell \leq r + 2$; (iii) If $k \not\equiv 0 \pmod{p-1}$ then $\ell = 0$. They all follow from Corollary 3.11.

We may further assume, w.l.o.g., that if $p = 2$ then $\mathrm{val}_2(k) \leq \mathrm{val}_2(k')$ and that $k$ and $k'$ are even. Let

$$i = \begin{cases} 1 & p \neq 2 \\ 2 & p = 2 \end{cases}. \tag{3.60}$$

There exists an $\alpha \in \mathbb{Z}_p$ such that

$$p^\ell E_k^\dagger - p^\ell E_{k'}^\dagger - \alpha \equiv 0 \pmod{p^{m+i+\ell}}. \tag{3.61}$$

(The congruence meaning a congruence of $q$-expansions). In fact

$$\alpha = p^\ell \left((1-p^{g(k-1)})2^{-g}\zeta_L(1-k) - (1-p^{g(k'-1)})2^{-g}\zeta_L(1-k')\right). \tag{3.62}$$

Now, the point is that $p^\ell E_k^\dagger, p^\ell E_{k'}^\dagger$ and $\alpha$ are modular forms over $\mathbb{C}$ of level $\Gamma_0(p)$ having integral $q$-expansion, hence are modular forms on $S_{m+i+\ell}$, hence on $S^{F,0}_{m+i+\ell}$. Therefore, $p^\ell E_k^\dagger, p^\ell E_{k'}^\dagger$ and $\alpha$ are meromorphic modular forms on $T_{m+i+\ell,0}$ with poles supported on the complement of the ordinary locus (The poles coming from the singularities of $S_m$). Criterion 3.10 holds also for meromorphic modular forms and we obtain that there exist $a, b$ such that

$$ap^\ell x^k - bp^\ell x^{k'} - \alpha \equiv 0, \ \forall x \in (\mathbb{Z}/(p^{m+i+\ell}))^\times. \tag{3.63}$$

Since, for every $x \in \mathbb{Z}/(p^{m+i}))^\times$ we have $x^k = x^{k'} \pmod{p^{m+i}}$, we deduce that there exists a $c$ in a $W_{m+i}$-algebra such that $cx^k - \alpha \equiv 0 \pmod{p^{m+i}}$ for every $x$ in $(\mathbb{Z}/(p^{m+i}))^\times$. Taking $x = 1$ we see that the following holds

$$\alpha(x^k - 1) \equiv 0 \pmod{p^{m+i}}, \ \forall x \in (\mathbb{Z}/(p^{m+i}))^\times. \tag{3.64}$$

*Remark* 3.14. The reader notices that we "lose" information by going from (3.63) to (3.64). We remark that the congruences obtained are "good enough" for the purposes of $p$-adic interpolation.

We separate cases:
(i) $k \not\equiv 0 \pmod{p-1}$. Then $\ell = 0$, and one gets that $\alpha \equiv 0 \pmod{p^{m+1}}$.
(ii) $k \equiv 0 \pmod{p-1}$ but $p \neq 2$. We observe that

$$\mathrm{val}_p(k) + 1 = \min\{\mathrm{val_p(x^k - 1) : x \in \mathbb{Z}, p \not| x}\}. \tag{3.65}$$

We therefore obtain that $\mathrm{val}_p(\alpha) \geq m + 1 - (r+1) = m - r$.
(iii) $k \equiv 0 \pmod{p-1}$ and $p = 2$. (We still assume that $k$ is even, since $k$ odd implies that $k'$ is odd and we get $\zeta_L(1-k) = \zeta_L(1-k') = 0$). Observe:

$$\mathrm{val}_2(k) + 2 = \min\{\mathrm{val_2(x^k - 1) : x \in \mathbb{Z}, 2 \not| x}\}. \tag{3.66}$$

Therefore, $\alpha \equiv 0 \pmod{p^{m+i-(\mathrm{val}_2(k)+i)}}$.

We sum up the above discussion in

**Corollary 3.15.** *Let $k, k' \geq 2$ and $k \equiv k' \pmod{(p-1)p^m}$.*
*1. If $k \not\equiv 0 \pmod{p-1}$ then*

$$(1-p^{g(k-1)})\zeta_L(1-k) \equiv (1-p^{g(k'-1)})\zeta_L(1-k') \pmod{p^{m+1}}. \tag{3.67}$$

*2. If $k \equiv 0 \pmod{p-1}$ but $p \neq 2$, then*

$$(1-p^{g(k-1)})\zeta_L(1-k) \equiv (1-p^{g(k'-1)})\zeta_L(1-k') \pmod{p^{m-1-\mathrm{val}_p(k\cdot k')}}. \tag{3.68}$$

*3. If $p = 2$ then*

$$(1-2^{g(k-1)})\zeta_L(1-k) \equiv (1-2^{g(k'-1)})\zeta_L(1-k') \pmod{2^{m+g-2-\mathrm{val}_2(k\cdot k')}}. \tag{3.69}$$

## 4. Liftings

**Proposition 4.1.** *Any modular form $f \in \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)$ can be lifted to $T_{\infty,\infty}$.*

*Proof.* Clearly the regular function $f/a(\chi) \in V_{m,m} \subset V_{m,\infty}$ can be lifted to $V_{\infty,\infty}$. Indeed, $V_{m,\infty} = V_{\infty,\infty} \otimes W_m(\mathbb{F})$. On the other hand, by Corollary 3.2, $a(\chi)$ itself lifts to $T_{\infty,\infty}$. □

A much more subtle question is that of lifting a modular form $f \in \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)$ to a modular form in $\mathbf{M}(W(\mathbb{F}), \chi, \mu_N)$. For example, take $m = 1$. The modular forms $h_i$ do not lift, because any non-cusp form of *finite* level must have parallel weight. Or, any modular form of finite level must have non-negative weights. This does not contradict Proposition 4.1. The level there is *infinite*. The following theorem says, heuristically, that the $h_i$'s are the prototype of modular forms that can not be lifted. The geometric explanation for this phenomenon is that the line bundle $\Omega(\chi)$, for $\chi$ not a multiple of Norm, does not extend to a line bundle over the minimal compactification, though it does extend to a line bundle over any smooth toroidal compactification.

**Theorem 4.2.** *Let $B$ be any $W(\mathbb{F})$-algebra and let $B_m = B \otimes W_m(\mathbb{F})$. The map*

$$\bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(B, \chi, \mu_N) \longrightarrow \bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(B_1, \chi, \mu_N)/I_1, \tag{4.1}$$

*is surjective. The map*

$$\bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(B, \chi, \mu_N)^{\mathrm{cusp}} \longrightarrow \bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(B_m, \chi, \mu_N)^{\mathrm{cusp}}/I_m, \tag{4.2}$$

*is surjective.*

*Proof.*

**Lemma 4.3.** ( [14], Proposition 6.11.) *If $f \in \mathbf{M}(B_1, \chi, \mu_N)$ has some $q$-expansion in which the constant term is non-zero then $\chi \in \mathbf{X}(1)$.*

Thus, if $f$ is not a cusp form then for a suitable $g \in I_1$ we have that $f + g$ is of weight $\mathrm{Norm}^k$ for some $k > 0$, which we may take as large as needed.

Let us put $T^S = \mathcal{M}^*(W(\mathbb{F}), \mu_N)$ – the moduli space of HBAS over $W(\mathbb{F})$-algebras with $\mu_N$-level with its Satake compactification. Recall the notation (2.12). It is well know that $\Omega(\mathrm{Norm})$ extends to $T^S$ and that $\Omega(\mathrm{Norm})$ is an ample line bundle (our level is rigid). It follows that for $k$ large enough every section of $\Omega(\mathrm{Norm}^k)$ can be lifted. We may therefore restrict our attention to cusp forms.

Let $D \hookrightarrow T^S$ be the cusps and $T^0 = T^S - D$. Let $T^{\mathrm{tor}}$ be a smooth toroidal compactification. We have a commutative diagram

$$\begin{array}{ccc} T^0 & \longrightarrow & T^{\mathrm{tor}} \\ & \searrow & \downarrow b \\ & & T^S \end{array} \tag{4.3}$$

The map $b$ is proper and the other two maps are open immersions. Let $D^{\mathrm{tor}}$ be the pre-image of $D$.

**Lemma 4.4.** *There exists a quasi-coherent sheaf $\mathcal{S}(\chi)$ on $T^S$ whose global sections are cusp forms of weight $\chi$.*

The theorem follows immediately from the lemma. For $k$ large enough all the higher cohomology of $\mathcal{S}(\chi) \otimes \Omega(\mathrm{Norm}^k)$ vanishes and there are thus no obstructions to liftings. It remains to prove the lemma:

There exists a semi-abelian variety with real multiplication

$$(\mathcal{A}, \beta_N) \xrightarrow{\pi} T^{\mathrm{tor}}. \tag{4.4}$$

Let $\Omega = \mathbf{t}^*_{(\mathcal{A},\beta_N)\to T^{\rm tor}}$ and define $\Omega(\chi)$ as usual (on $T^0$ this agrees with our previous definition). Let $\mathcal{I}$ be the ideal sheaf defining $D^{\rm tor}$. Let

$$\mathcal{S}(\chi) = \pi_*(\Omega(\chi)\otimes\mathcal{I}). \tag{4.5}$$

The sheaf $\mathcal{S}(\chi)$ is quasi-coherent sheaf on $T^S$. We need only show that its global sections are cusp forms. The map from $\Gamma(T^S,\mathcal{S}(\chi)) = \Gamma(T^{\rm tor},\Omega(\chi)\otimes\mathcal{I})$ to $\Gamma(T^0,\Omega(\chi))\subset \mathbf{M}(W(\mathbb{F}),\chi,\mu_N)$, given by restriction, is clearly injective. It has image contained in the cusp forms. Indeed, if $f\in\Gamma(T^S,\mathcal{S}(\chi))$ and $\tilde f$ its image, then the $q$-expansion of $\tilde f$ is non-other then $f$ viewed as an element of the structure sheaf of the completion of $T^{\rm tor}$ along $\mathcal{I}$. For this one needs to choose a particular trivialization of $\Omega(\chi)$ in a neighbourhood of the component of $D^{\rm tor}$ under consideration. See [3], Main Theorem.

Conversely, a cusp form $\tilde f$, viewed as a section of $\Gamma(T^0,\Omega(\chi))$, or $\Gamma(T^0,\mathcal{S}(\chi))$ extends to an a priory meromorphic section $f$ of $\Gamma(T^S,\mathcal{S}(\chi))$, whose expression as an element of the structure sheaf of the completion of $T^{\rm tor}$ along $\mathcal{I}$ has zero constant coefficient. That just means that locally around $D^{\rm tor}$ it belongs to $\mathcal{I}$. See loc. cit. (x). $\square$

*Remark* 4.5. The point of Theorem 4.2 is that it says that every HMF modulo $p$, say $f$, can lifted to characteristic zero, in the sense that its *q-expansion* can be lifted. I.e., though often one can not lift the modular form $f$ itself, there *does* exist a modular form $g$ of characteristic zero and weight equal to the weight of $f$ modulo $\mathbf{X}(1)$, whose $q$-expansion is equal to the $q$-expansion of $f$ modulo $p$.

Practically the same proof gives the following:

Let $f$ be a modular form over $W_m(\mathbb{F})$ whose constant coefficient in one $q$-expansion is a unit. Then $f$ has weight in $\mathbf{X}(m)$ and its $q$-expansion lifts to a $q$-expansion of a HMF over $W(\mathbb{F})$ of the same level and weight in $\mathbf{X}(m)$. A similar statement holds for cusp forms.

In fact the method of the proof allows one to control the difference between the weights of $f$ and the "lift" if one has an effective bound on $k$ such that $H^1(T^S,\mathcal{S}(\chi)\otimes\Omega(\mathrm{Norm}^k)) = 0$.

## 5. Tabulation of some zeta values

*Remark* 5.1. The calculations were done using PARI and are subject to the following reservations: (i) My lack of expertise in such calculations. (ii) The validity of a factor being a prime. In particular, almost surely, those numerators which are not decomposed at all are composite. (iii) However, the factorization of the denominator is always into true primes.

Field: $L=\mathbb{Q}$.

| $k$ | $\zeta_{\mathbb{Q}}(1-k)$ | | $k$ | $\zeta_{\mathbb{Q}}(1-k)$ |
|---|---|---|---|---|
| 2 | $\frac{-1}{2^2\cdot 3}$ | | 20 | $\frac{283\cdot 617}{2^3\cdot 3\cdot 5^2\cdot 11}$ |
| 4 | $\frac{1}{2^3\cdot 3\cdot 5}$ | | 22 | $\frac{-131\cdot 593}{2^2\cdot 3\cdot 23}$ |
| 6 | $\frac{-1}{2^2\cdot 3^2\cdot 7}$ | | 24 | $\frac{103\cdot 2294797}{2^4\cdot 3^2\cdot 5\cdot 7\cdot 13}$ |
| 8 | $\frac{1}{2^4\cdot 3\cdot 5}$ | | 26 | $\frac{-657931}{2^2\cdot 3}$ |
| 10 | $\frac{-1}{2^2\cdot 3\cdot 11}$ | | 28 | $\frac{9349\cdot 362903}{2^3\cdot 3\cdot 5\cdot 29}$ |
| 12 | $\frac{691}{2^3\cdot 3^2\cdot 5\cdot 7\cdot 13}$ | | 30 | $\frac{-1721\cdot 1001259881}{2^2\cdot 3^2\cdot 7\cdot 11\cdot 31}$ |
| 14 | $\frac{-1}{2^2\cdot 3}$ | | 32 | $\frac{37\cdot 683\cdot 305065927}{2^6\cdot 3\cdot 5\cdot 17}$ |
| 16 | $\frac{3617}{2^5\cdot 3\cdot 5\cdot 17}$ | | 34 | $\frac{-151628697551}{2^2\cdot 3}$ |
| 18 | $\frac{-43867}{2^2\cdot 3^3\cdot 7\cdot 19}$ | | | |

Field: $L=\mathbb{Q}(\sqrt{2})$.
Ideals: Ramified: 2; Split: $7,17,23,31$; Inert: $3,5,11,13,19,29$.

| $k$ | $\zeta_L(1-k)$ | | $k$ | $\zeta_L(1-k)$ |
|---|---|---|---|---|
| 2 | $\frac{1}{2^2\cdot 3}$ | | 20 | $\frac{283\cdot 617\cdot 211202599\cdot 51060226589}{2^3\cdot 3\cdot 5^2\cdot 11}$ |
| 4 | $\frac{11}{2^3\cdot 3\cdot 5}$ | | 22 | $\frac{131\cdot 593\cdot 169471\cdot 1358111\cdot 31902217001}{2^2\cdot 3\cdot 23}$ |
| 6 | $\frac{19^2}{2^2\cdot 3^2\cdot 7}$ | | 24 | $\frac{11\cdot 19\cdot 103\cdot 977\cdot 3343\cdot 2294797\cdot 678737272814753}{2^4\cdot 3^2\cdot 5\cdot 7}$ |
| 8 | $\frac{24611}{2^4\cdot 3\cdot 5}$ | | 26 | $\frac{657931\cdot 39944352181\cdot 146669017694031181}{2^2\cdot 3}$ |
| 10 | $\frac{2873041}{2^2\cdot 3\cdot 11}$ | | 28 | $\frac{9349\cdot 362903\cdot 474581\cdot 140488497482040347316036311}{2^3\cdot 3\cdot 5\cdot 29}$ |
| 12 | $\frac{13\cdot 691\cdot 3031619}{2^3\cdot 3^2\cdot 5\cdot 7}$ | | 30 | $\frac{79\cdot 1721\cdot 1190311\cdot 1001259881\cdot 301077394625804292874471 9}{2^2\cdot 3^2\cdot 7\cdot 11}$ |
| 14 | $\frac{11\cdot 151\cdot 78007661}{2^2\cdot 3}$ | | 32 | $\frac{37\cdot 89\cdot 683\cdot 39217\cdot 111392753\cdot 305065927\cdot 340337069485949994266 99}{2^6\cdot 3\cdot 5\cdot 17}$ |
| 16 | $\frac{79\cdot 3617\cdot 558366571709}{2^5\cdot 3\cdot 5\cdot 17}$ | | 34 | $\frac{11\cdot 37\cdot 59\cdot 151628697551\cdot 9433401125068736391055674409958357 17}{2^2\cdot 3}$ |
| 18 | $\frac{43867\cdot 194507186357160 01}{2^2\cdot 3^3\cdot 7\cdot 19}$ | | | |

Field: $L = \mathbb{Q}(\sqrt{5})$.
Ideals: Ramified: 5; Split: 11, 19, 29, 31; Inert: 2, 3, 5, 7, 13, 17, 23.

| $k$ | $\zeta_L(1-k)$ | $k$ | $\zeta_L(1-k)$ |
|---|---|---|---|
| 2 | $\frac{1}{2\cdot3\cdot5}$ | 20 | $\frac{283\cdot617\cdot564172514549641}{2^2\cdot3\cdot5^2\cdot11}$ |
| 4 | $\frac{1}{2^2\cdot3\cdot5}$ | 22 | $\frac{107\cdot131\cdot149\cdot593\cdot47058898298437}{2\cdot3\cdot5\cdot23}$ |
| 6 | $\frac{67}{2\cdot3^2\cdot5\cdot7}$ | 24 | $\frac{103\cdot1093\cdot1214221\cdot2294797\cdot36228867817}{2^3\cdot3^2\cdot5\cdot7\cdot13}$ |
| 8 | $\frac{19^2}{2^3\cdot3\cdot5}$ | 26 | $\frac{19\cdot5839\cdot657931\cdot823345533268358047}{2\cdot3\cdot5}$ |
| 10 | $\frac{191\cdot2161}{2\cdot3\cdot5^2\cdot11}$ | 28 | $\frac{2969\cdot9349\cdot362903\cdot2735340507483319678769}{2^2\cdot3\cdot5\cdot29}$ |
| 12 | $\frac{691\cdot1150921}{2^2\cdot3^2\cdot5\cdot7\cdot13}$ | 30 | $\frac{17\cdot1721\cdot13815257\cdot33847091\cdot1001259881\cdot13133142812173}{2\cdot3^2\cdot5^2\cdot7\cdot11\cdot31}$ |
| 14 | $\frac{17\cdot33446579}{2\cdot3\cdot5}$ | 32 | $\frac{37\cdot131\cdot683\cdot305065927\cdot3389247557\cdot5539193421920211463}{2^5\cdot3\cdot5\cdot17}$ |
| 16 | $\frac{457\cdot3617\cdot33092833}{2^4\cdot3\cdot5\cdot17}$ | 34 | $\frac{347\cdot661\cdot3359\cdot271805903\cdot151628697551\cdot39267702302944517}{2\cdot3\cdot5}$ |
| 18 | $\frac{41\cdot43867\cdot317680421579}{2\cdot3^3\cdot5\cdot7\cdot19}$ | | |

Field: $L = \mathbb{Q}(\sqrt{7})$.
Ideals: Ramified: 2, 7; Split: 3, 19, 29, 31; Inert: 5, 11, 13, 17, 23.

| $k$ | $\zeta_L(1-k)$ |
|---|---|
| 2 | $\frac{2}{3}$ |
| 4 | $\frac{113}{3\cdot5}$ |
| 6 | $\frac{2\cdot173\cdot257}{3^2\cdot7}$ |
| 8 | $\frac{37040933}{2\cdot3\cdot5}$ |
| 10 | $\frac{2\cdot13\cdot4073517757}{3\cdot11}$ |
| 12 | $\frac{691\cdot1355989\cdot85309877}{3^2\cdot5\cdot7\cdot13}$ |
| 14 | $\frac{2\cdot23\cdot31126933\cdot500577719}{3}$ |
| 16 | $\frac{3617\cdot1494552660374041255373}{2^2\cdot3\cdot5\cdot17}$ |
| 18 | $\frac{2\cdot29\cdot21529\cdot23801\cdot43867\cdot5432745778374 61}{3^3\cdot7\cdot19}$ |
| 20 | $\frac{283\cdot617\cdot12391\cdot442499227688819077982 4023}{3\cdot5^2\cdot11}$ |
| 22 | $\frac{2\cdot13^2\cdot131\cdot593\cdot773\cdot829\cdot6449\cdot65480409127140961 2853}{3\cdot23}$ |
| 24 | $\frac{73\cdot103\cdot2294797\cdot629518464449268982260011362617 91181}{2\cdot3^2\cdot5\cdot7\cdot13}$ |
| 26 | $\frac{2\cdot2857\cdot657931\cdot1468739855213\cdot1304919704649976009751 0021}{3}$ |
| 28 | $\frac{439\cdot9349\cdot65993\cdot362903\cdot263493529999527230535448728123640 85479}{3\cdot5\cdot29}$ |
| 30 | $\frac{2\cdot1721\cdot1001259881\cdot11476721593\cdot10725729902031170122756826827773671 40637}{3^2\cdot7\cdot11\cdot31}$ |
| 32 | $\frac{37\cdot683\cdot92413\cdot305065927\cdot1566074201\cdot543869790242280163\cdot2888308889549447149987}{2^3\cdot3\cdot5\cdot17}$ |
| 34 | $\frac{2\cdot13\cdot151628697551\cdot2050420934478 97\cdot1788619008252178278652191975327334133823661}{3}$ |

Field: $L = \mathbb{Q}(\zeta_7)^+ = \mathbb{Q}[x]/(x^3 + x^2 - 2x - 1)$
Ideals: Ramified: 7; Split: 13, 29; Inert: 2, 3, 5, 11, 17, 19, 23, 31.

| $k$ | $\zeta_L(1-k)$ |
|---|---|
| 2 | $\frac{-1}{3\cdot7}$ |
| 4 | $\frac{79}{2\cdot3\cdot5\cdot7}$ |
| 6 | $\frac{-7393}{3^2\cdot7}$ |
| 8 | $\frac{142490119}{2^2\cdot3\cdot5\cdot7}$ |
| 10 | $\frac{-1141452324871}{3\cdot7\cdot11}$ |
| 12 | $\frac{691\cdot10903\cdot278995143079}{2\cdot3^2\cdot5\cdot7\cdot13}$ |
| 14 | $\frac{-1033\cdot54105393349620356 89}{3\cdot7^2}$ |
| 16 | $\frac{3617\cdot19387\cdot6997171\cdot399890401961287}{2^3\cdot3\cdot5\cdot7\cdot17}$ |
| 18 | $\frac{-97\cdot43867\cdot9105835027474306843301627809}{3^3\cdot7\cdot19}$ |
| 20 | $\frac{283\cdot617\cdot21766351\cdot5118351012301487009695100128 9}{2\cdot3\cdot5^2\cdot7\cdot11}$ |
| 22 | $\frac{-131\cdot593\cdot751\cdot1657\cdot95131\cdot2557424168676190300514101539043}{3\cdot7\cdot23}$ |
| 24 | $\frac{103\cdot2294797\cdot4000924171430 59\cdot483171322664923 3\cdot8824732711929451}{2^2\cdot3^2\cdot5\cdot7\cdot13}$ |
| 26 | $\frac{-29527\cdot657931\cdot3306506726170474827689 89\cdot6783401807199940111277317}{3\cdot7}$ |
| 28 | $\frac{9349\cdot82471\cdot362903\cdot7430353258315 93\cdot975522475034052090 7\cdot5887531329453854793 73}{2\cdot3\cdot5\cdot7^2\cdot29}$ |
| 30 | $\frac{-1721\cdot3373\cdot1001259881\cdot1189250352889060 9\cdot181878041594305140264558754657075835980477429}{3^2\cdot7\cdot11\cdot31}$ |
| 32 | $\frac{37\cdot683\cdot24847\cdot38575843\cdot125089171\cdot305065927\cdot1270758367\cdot1403876917177358 4013\cdot3181305764066444826301 9}{2^4\cdot3\cdot5\cdot7\cdot17}$ |
| 34 | $\frac{-103\cdot367\cdot151628697551\cdot30924576265171010083636208263230553718869153969291311796245209292857661 91}{3\cdot7}$ |

Field: $L = \mathbb{Q}(\zeta_{11})^+ = \mathbb{Q}[x]/(x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1)$
Ideals: Ramified: 11; Split: 23; Inert: 2, 3, 5, 7, 13, 17, 19, 29, 31.

| $k$ | $\zeta_L(1-k)$ |
|---|---|
| 2 | $\frac{-2^2\cdot 5}{3\cdot 11}$ |
| 4 | $\frac{2\cdot 71\cdot 11941}{3\cdot 5\cdot 11}$ |
| 6 | $\frac{-2^2\cdot 5\cdot 521\cdot 4888380551}{3^2\cdot 7\cdot 11}$ |
| 8 | $\frac{13721\cdot 2520121\cdot 102462575851}{3\cdot 5\cdot 11}$ |
| 10 | $\frac{-2^2\cdot 5\cdot 98178488021\cdot 1560850707193521481}{3\cdot 11}$ |
| 12 | $\frac{2\cdot 691\cdot 1607981\cdot 6134561\cdot 29139491\cdot 37913350779491952174 1}{3^2\cdot 5\cdot 7\cdot 11\cdot 13}$ |
| 14 | $\frac{-2^2\cdot 5\cdot 31\cdot 71\cdot 109841\cdot 4712650115236500312066042412229825266552711}{3\cdot 11}$ |
| 16 | $\frac{31\cdot 3617\cdot 18131\cdot 42641\cdot 2466915721\cdot 165369057873988872947201869480111559682352 31}{2\cdot 3\cdot 5\cdot 11\cdot 17}$ |
| 18 | $\frac{-2^2\cdot 5\cdot 43867\cdot 113011\cdot 8358181640776075276627190359814407768568787649916064923929232288413 81}{3^3\cdot 7\cdot 11\cdot 19}$ |
| 20 | $\frac{2\cdot 131\cdot 283\cdot 617\cdot 821\cdot 48195178319060637293145712194105725623898833632349035199034024825350437319874667 1}{3\cdot 5^2\cdot 11}$ |
| 22 | $\frac{-2^2\cdot 5\cdot 31\cdot 131\cdot 593\cdot 2111\cdot 9811\cdot 4754681\cdot 150743667211\cdot 748530934469196858871937810651748750942524270057139070201532459316162670 1}{3\cdot 11^2\cdot 23}$ |
| 24 | $\frac{2140902620412862787531604307645176100404841968426079695283278792036337157813694947611346966909345714822325560273519654201}{3^2\cdot 5\cdot 7\cdot 11\cdot 13}$ |
| 26 | $\frac{-27262351437064287055837277219600077432474446936390947079576072013633330161797302838862865030118484478282914405623405657211406482602 0}{3\cdot 11}$ |

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