INTRODUCTION

Fix a prime p, and let $\mathbf{C}_p$ denote the completion of an algebraic closure of $\mathbf{Q}_p$. We let ord_p denote the p-adic valuation on $\mathbf{C}_p$, normalized so that $\mathrm{ord}_p(p) = 1$, and let $\mathcal{O}_p$ denote the valuation ring of ord_p.

A p-adic weight is an even $\mathbf{C}_p$-valued character of $\mathbf{Z}_p^{\times}$. The set of p-adic weights is the set of $\mathbf{C}_p$-valued points of a rigid analytic space $\mathcal{W}$ defined over $\mathbf{Q}_p$. In [5] and [6] Coleman and Mazur define, for any p-adic weight $\kappa \in \mathcal{W}(\mathbf{C}_p)$, a $\mathbf{C}_p$-linear subspace of $\mathbf{C}_p[[q]]$ of overconvergent p-adic cusp forms of weight κ and tame level one, which is naturally a Banach space over $\mathbf{C}_p$, and which is equipped with an action of the Hecke operators U_p and T_l (l ranging over primes distinct from p). The operator U_p is completely continuous on this space, and so has a characteristic power series $P(\kappa, T) := \det(1 - TU_p)$ which is an entire power series in the variable T. (In fact, in order to obtain a Banach space, rather than a Frechét space, one must fix a radius of overconvergence. However, the characteristic power series that one obtains is independent of this choice, and so we have suppressed this detail in our discussion.)

In the case $p = 2$, $\mathbf{Z}_2 = \{\pm 1\} \times \Gamma$, where $\Gamma = 1 + 4\mathbf{Z}_2$. We therefore identify 2-adic weights with characters of Γ. Since Γ is topologically generated by the element 5, the function $t(\kappa) = \kappa(5)5^{-14} - 1$ identifies $\mathcal{W}$ with the open unit disc $1 > |t|$ around the origin of $\mathbf{C}_2$.

The object of this thesis is to prove the following Theorem:

Main Theorem. *Let $\mathcal{A}$ denote the annulus $1 > |u| \geq |64|$, which is a rigid analytic space defined over $\mathbf{Q}_2$. We construct a sequence of rigid analytic functions $a_n(u)$ ($n \geq 1$) of norm less than or equal to one defined on $\mathcal{A}$, and a map $\kappa : \mathcal{A} \to \mathcal{W}$, which we write $u \mapsto \kappa(u)$, with the following properties:*

(i) $a_1 = 1$ and $a_2 = u$.

(ii) Let $\mathcal{A}_1$ denote the sub-annulus $1 > |u| > |64|$ of $\mathcal{A}$. The map κ satisfies $|\kappa(u)| = |u|$ for all points $u \in \mathcal{A}_1(\mathbf{C}_2)$, and (consequently) induces an isomorphism between $\mathcal{A}_1$ and the subannulus $1 > |t| > |64|$ of $\mathcal{W}$.

(iii) Let $\mathcal{A}_2$ denote the sub-annulus $|u| = |64|$ of $\mathcal{A}$. When restricted to $\mathcal{A}_2$, the map κ realizes $\mathcal{A}_2$ as a finite degree two cover of the sub-disc $|64| \geq |t| \geq 0$ of $\mathcal{W}$, having exactly two ramification points, which are a conjugate pair of points defined over $\mathbf{Q}_2(\sqrt{3})$.

(iv) If $\kappa \in \mathcal{W}(\mathbf{C}_2)$, then the fibre $\mathcal{A}_\kappa$ is a zero-dimensional rigid analytic space of degree either one or two (as follows from parts (i) and (ii)).

The points $u \in \mathcal{A}_\kappa(\mathbf{C}_2)$ are precisely the reciprocal zeroes of the characteristic power series $P(\kappa, T)$ which have minimal slope, and the multiplicity with which each eigenvalue u occurs as a point of $\mathcal{A}_\kappa$ is equal to the multiplicity with which u occurs as a reciprocal zero of $P(\kappa, T)$.

Furthermore, for each $u \in \mathcal{A}_\kappa$, the q-expansion $f(u) = \sum_{n=1}^{\infty} a_n(u)q^n \in \mathcal{O}_2[[q]]$ is the unique overconvergent 2-adic cusp form of weight $\kappa = \kappa(u)$ which is a normalized eigenform for all the Hecke operators of weight κ and whose U_2-eigenvalue equals u.

(v) If $u, u' \in \mathcal{A}(\mathbf{C}_2)$, then

$$f(u) \equiv f(u') \mod (u - u')\mathcal{O}_2[[q]].$$

(vi) If l is an odd prime, and $u \in \mathcal{A}(\mathbf{C}_2)$, then

$$a_l(u) \equiv 1 + \kappa(u)(l)l^{-1} \begin{cases} \mod (u-2)(u-4)(u-8)(u-32)\mathcal{O}_2, & l \equiv 1 \mod 8 \\ \mod (u-8)(u-32)\mathcal{O}_2, & l \equiv 3 \mod 8 \\ \mod (u-4)(u-8)(u-32)\mathcal{O}_2, & l \equiv 5 \mod 8 \\ \mod (u-2)(u-8)(u-32)\mathcal{O}_2, & l \equiv 7 \mod 8. \end{cases}$$

(vii) If k is an integer and χ is a character of Γ of finite order, let $\kappa_{\chi,k}$ denote the 2-adic weight defined by $\kappa_{\chi,k}(\gamma) = \chi(\gamma)\gamma^k$. Let τ denote the character

$$\tau : \mathbf{Z}_2^\times \to \{\pm 1\} \subset \mathbf{C}_2,$$

and let 2^n be the conductor of the character $\tau^k \kappa_{\chi,k}$ of $\mathbf{Z}_2^\times$. If χ is non-trivial and $k \geq 2$, or if χ is trivial and $k \geq 3$, then for any $u \in \mathcal{A}_{\kappa_{\chi,k}}$ the overconvergent cusp form $f(u)$ of weight $\kappa_{\chi,k}$ is in fact a classical modular form of weight k and nebentypus χ on $\Gamma_1(2^m)$, where $m = n$ if $n > 0$, and $m = 1$ if $n = 0$.

In [6], for any *odd* prime p, Coleman and Mazur construct a rigid analytic curve $\mathcal{C}$, which they call the eigencurve, which, roughly speaking, parameterizes overconvergent Hecke eigenforms of arbitrary p-adic weight.

There is a natural map $\kappa : \mathcal{C} \to \mathcal{W}$, as well as rigid analytic functions a_n for $n \geq 1$ of norm less than or equal to one, such that the points of $\mathcal{C}(\mathbf{C}_p)$ lying over a given weight $\kappa \in \mathcal{W}(\mathbf{C}_p)$ naturally correspond to the overconvergent p-adic Hecke eigenforms of weight κ and finite slope, via the association $c \mapsto \sum_{n=1}^{\infty} a_n(c)q^n \in \mathcal{O}_2[[q]]$.

Once this construction has been extended to the case $p = 2$, it will certainly be true that there is a natural embedding of $\mathcal{A}$ into the eigencurve, whose image is an admissable open, such that the maps κ and a_n on $\mathcal{A}$ are the restrictions to $\mathcal{A}$ of the corresponding maps on $\mathcal{C}$.

Since the construction of Coleman and Mazur gives no hint as to the geometric structure of the eigencurve, or the nature of the universal family of overconvergent forms which it parameterizes, part of the interest of our Main Theorem stems from the fact that it provides an explicit description of a part of the eigencurve for $p = 2$.

The family $f(u)$ parameterized by $\mathcal{A}$ also seems to be quite interesting in its own right. Part *(v)* of the Theorem shows that the family $f(u)$ contains various Kummer families, whose existence is the subject of a conjecture of Gouvea and Mazur [9]. The family $f(u)$ also passes through four "evil twin" Eisenstein series: those of weights four and six on $\Gamma_0(2)$ (for which $u = 8$ and $u = 32$, respectively), that of weight three on $\Gamma_1(4)$ (for which $u = 4$), and that of weight two and non-trivial nebentypus on $\Gamma_1(8)$ (for which $u = 2$). This is of course very much related to the congruence of part *(vi)*.

Part *(vii)* of the Theorem follows from a Theorem of Coleman [3], but we find a simpler proof of this result for the modular forms that we construct.

In [5, Appendix II], Coleman uses an explicit approximation for the matrix of U_2 together with a trace formula to find an estimate for the characteristic power series of U_2 on the space of families of overconvergent 2-adic modular forms which is sufficient to conclude the existence of the analytic family $f(u)$ over the disc $|8| \geq |u| > |16|$. From general principles, he then deduces a weaker version of the congruence of part *(v)* over this disc.

The proof of our Main Theorem relies on a more refined approximation of the matrix of U_2, and does not require a trace formula. The advantage of depending only on an approximation of the matrix of U_2, and not on a trace formula, is that we can then estimate not only the characteristic power series, and thus the minimal slope eigenvalues, of U_2, but also the corresponding eigenvectors. This is the key to proving parts *(v)*, *(vi)* and *(vii)* of the Main Theorem. In particular, let us point out that the congruence of part *(v)* is stronger than the congruence that one would deduce by the arguments of [5, Appendix II] or of [7].

In order to make this approximation, we were forced to redevelop the theory of [5] (in the case when $N = 1$ and $p = 2$) in a completely explicit form, in which all the implied estimates of this theory have become precisely determined. In doing this, it turned out that the theory was largely reduced to an elementary level, depending only on the classical theory of modular forms and modular functions over $\mathbf{C}$, together with some simple 2-adic analysis (the most difficult being that which goes into the construction of the 2-adic L-function, which is required to analyze the Eisenstein series). Having been led onto this path by the necessities of computation, it seemed worthwhile to stick to it. Thus the proof of the Main Theorem that we present is elementary, in above the sense. In particular, it does not depend on the rather subtle theory of the canonical subgroup, which is at the heart of the theory developed in [11], [5], and [6]. One pleasant side-effect of our approach is that the proof of the Main Theorem is largely self-contained.

We give an elementary definition of the space of overconvergent 2-adic modular forms of arbitrary 2-adic weight, and develop the Riesz theory necessary to study the characteristic power series $P(\kappa, T)$ and analytic families of eigenvectors for U. This is done not only in the interest of being self-contained. In order to prove the congruences of the Main Theorem, we must avoid introducing extraneous denominators. Thus the theory of [5] is not immediately applicable, since in this paper Coleman always works over a field, not over a ring of integers.

Let us briefly outline the contents of this thesis. Chapter 1 is of a preliminary nature. Its object is to given an explanation of our Main Theorem in the setting of classical modular forms on $\mathbf{SL}(2, \mathbf{Z})$ and $\Gamma_0(2)$.

Chapter 2 is a series of explicit computations concerning modular functions and modular forms on the groups $\Gamma_0(2)$, $\Gamma_1(4)$ and $\Gamma_1(4) \cap \Gamma_0(8)$.

In Chapter 3 we give an elementary definition of the space of overconvergent 2-adic cusp forms of arbitrary 2-adic weight, and a brief account of the Riesz theory necessary to study the characteristic power series $P(\kappa, T)$ and analytic families of eigenvectors for U_2.

Chapter 4 contains the proof of the Main Theorem. As indicated above, the proof works by making explicit the theory of [5]. Thus we work with a specific operator $\tilde{U}$, and the crux of the proof consists in explicit estimates for the action of this operator. Most of the chapter consists of applying these estimates in various situations, in order to deduce the various parts of the Main Theorem.

Chapter 5 presents some complements to and corollaries of the Main Theorem. We explain how the Main Theorem verifies the Gouvea-Mazur conjecture on the existence of Kummer failies in some instances. We identify the weight zero member of our minimal slope family with a certain 2-adic modular function constructed in [1], and show that the weight two member is the derivative of this modular function. Finally, we make explicit some estimates of the function $\kappa(u)$ and the family $f(u)$ that follow from the computations of chapter 4.

In an appendix we relate the explicit theory of overconvergent 2-adic modular forms that we have developed to the general theory of [5],[6], [8] and [11].