CHAPTER V

COMPLEMENTS AND COROLLARIES

In this chapter we present some corollaries and complements to the Main Theorem. In the first section we explain how our Main Theorem verifies some cases of the Gouvea-Mazur conjecture. In the second section we give explicit formulae for the minimal slope forms $f(u)$ for the values of u lying over κ_0 and κ_2. In the third and final section we state some explicit estimates for the family $f(u)$ and the function $\kappa(u)$.

5.1 Some instances of the Gouvea-Mazur conjecture

As a corollary of our Main Theorem, we can verify some instances of the conjecture of [9] on the existence of Kummer families of constant slope. In this reference, the notion of a Kummer family of modular forms is defined for an odd prime p, but it easily extends to the prime 2: suppose that f_i is a sequence of Hecke eigenforms on some $\Gamma_1(2^n)$ of weights k_i. We say that this sequence forms a Kummer family if $k_i \equiv k_j \mod 2^m$ implies that $f_i \equiv f_j \mod 2^{m+2}$. We say that the Kummer family has slope α if for each f_i, $\mathrm{ord}_2(a_2(f_i)) = \alpha$.

Let us introduce the following notation: if $\kappa \in \mathcal{W}$ such that $1 > |t(\kappa)| > |64|$, part *(ii)* of the Main Theorem shows that there is a unique value $u(\kappa)$ of u lying over κ. Write $f(\kappa) = f(u(\kappa))$.

Proposition 5.1. *(i) The modular forms $f(\kappa_k)$ for $k \geq 3$ and odd form a Kummer family of slope two.*

(ii) The modular forms $f(\kappa_k)$ of weight k on $\Gamma_1(4)$ for $k \geq 4$ and $k \equiv 0 \mod 4$ form a Kummer family of slope three.

(iii) The modular forms $f(\kappa_k)$ of weight k on $\Gamma_0(2)$ for $k \geq 10$ and $k \equiv 2 \mod 8$ form a Kummer family of slope four.

(iv) The modular forms $f(\kappa_k)$ of weight k on $\Gamma_0(2)$ for $k \geq 6$ and $k \equiv 6 \mod 16$ form a Kummer family of slope five.

(v) If χ is a non-trivial even character of conductor 2^n (so that $n \geq 3$), and $k \geq 2$, then the modular forms $f(\kappa_{\chi,k})$ of weight k and nebentypus χ on $\Gamma_1(2^n)$ form a Kummer family of slope 2^{3-n}.

Part *(vii)* of the Main Theorem shows that all these forms are classical. Part *(ii)* shows that $|u(\kappa)| = |t(\kappa)|$, and the latter is computed by Lemma 3.10, proving the assertion about the slopes.

To prove the congruences, note that $|t/u| = 1$ over the annulus $1 > |u| > |64|$, and so t/u is a unit in $\mathbf{Z}_2[[u, 64/u]]$. Write

$$t/u = \sum_{n=0}^{\infty} b_n u^n + \sum_{n=1}^{\infty} c_n (64/u)^n,$$

with each $b_n, c_n \in \mathbf{Z}_2$, and b_0 a unit in $\mathbf{Z}_2$. Suppose that u, u' are two points in this annulus, lying over the points t and t' of $\mathcal{W}$. Then

$$t - t' = (u - u')(\sum_{n=0}^{\infty} b_n \frac{u^{n+1} - u'^{n+1}}{u - u'} - (64/u)(64/u') \sum_{n=1}^{\infty} c_n \frac{(64/u)^{n-1} - (64/u')^{n-1}}{(64/u) - (64/u')}).$$

The inequalities on $|u|$ and $|u'|$ now show that $|t - t'| = |u - u'||b_0| = |u - u'|$. The congruences are proved by this formula together with part *(v)*.

5.2 The weights κ_0 and κ_2

From the Main Theorem it follows that there is a unique $u_0 \in \mathcal{A}(\mathbf{C}_2)$ such that $\kappa(u_0) = \kappa_0$, and a unique $u_2 \in \mathcal{A}(\mathbf{C}_2)$ such that $f(u_2) = \kappa_2$. Then $f(u_0)$ is the unique normalized minimal slope overconvergent cuspidal eigenform of weight κ_0, and $f(u_2)$ is the unique normalized minimal slope overconvergent cuspidal eigenform of weight κ_2. The aim of this Section is to give estimates for $f(u_0)$ and $f(u_2)$ using the modular functions j and $1/j_2$.

Theorem 5.2. *Write*

$$1/j_2 = q \prod_{n=1}^{\infty} (1 + q^n)^{24} = \sum_{n=1}^{\infty} a(n) q^n,$$

and

$$j = 1/q + 744 + \sum_{n=1}^{\infty} c(n) q^n.$$

Also, for any integer $m \geq 1$, write $\sigma_{-1}(m) = \sum_{d|n} d^{-1}$.

(i) For any odd integer $m \geq 1$ and any integer $n \geq 0$,

$$a(m \cdot 2^n) \equiv 3^n \cdot 2^{2n} \sigma_{-1}(m) \mod 2^{3n+5}\mathbf{Z}_2.$$

(ii) For any integer $n \geq 1$,

$$U^n(1/j_2)/a(2^n) \equiv f(u_0) \mod 2^{4n+8}\mathbf{Z}_2[[q]].$$

(Note that it follows from part (i) that the left hand side of this congruence is indeed an element of $\mathbf{Z}_2[[q]]$.)

In particular, since f is an eigenform for all the Hecke operators, we find that for any odd prime l and any integers $m \geq 1$ and $n \geq 0$,

$$a(lm \cdot 2^n)/a(2^n) \equiv (a(l \cdot 2^n)/a(2^n))(a(m \cdot 2^n)/a(2^n)) - l^{-1}(a(m \cdot 2^n/l)) \mod 2^{4n+8}\mathbf{Z}_2.$$

(We define $a(m \cdot 2^n/l) = 0$ when m/l is not an integer.)

(iii) For any odd integer $m \geq 1$ and any integer $n \geq 1$,

$$c(m \cdot 2^n) \equiv -3^{n-1} 2^{3n+8} \sigma_{-1}(m) \mod 2^{3n+13}\mathbf{Z}_2.$$

(iv) For any integer $n \geq 1$,

$$U^n(j - 744)/c(2^n) \equiv f(u_0) \mod 2^{4n+4}\mathbf{Z}_2[[q]].$$

(Note that it follows from part (iii) that the left hand side of this congruence is indeed an element of $\mathbf{Z}_2[[q]]$*.)*

In particular for any odd prime l *and any integers* $m \geq 1$ *and* $n \geq 1$,

$$c(lm \cdot 2^n)/c(2^n) \equiv (c(l \cdot 2^n)/c(2^n))(c(m \cdot 2^n)/c(2^n)) - l^{-1}(c(m \cdot 2^n/l)/c(2^n)) \mod 2^{4n+4}\mathbf{Z}_2.$$

(We define $c(m \cdot 2^n/l) = 0$ *when* m/l *is not an integer.)*

Remarks. *(i)* Part *(iii)* of the Theorem is due to Kolberg ([13], cited in [1]). A weaker version of *(i)*, to the effect that $\mathrm{ord}_2(a(m \cdot 2^n) \geq 3n$, with equality when $m = 1$, was first shown to the author by Frank Calegari (unpublished), and also follows from the results of [1].

(ii) The congruences of part *(iv)* which are introduced by the phrase "In particular" were conjectured by Atkin, and first proved by Akiyama [1], using methods quite similar to ours (but restricted to this particular situation of weight zero).

We now prove the Theorem. Since $|r(\kappa_0)| = |8|^{3/4}$ and $E_{\kappa_0} = 1$, Proposition 3.22 shows that

$$M_{\kappa_0} = (1/j_2)\mathcal{O}_2[[8^{3/4}/j_2]].$$

Define

$$M := 1/j_2\mathbf{Z}_2[[16/j_2]] \subset M_{\kappa_0},$$

and for $n \geq 1$ define

$$M_n = (1/j_2)\mathbf{Z}_2 \oplus \cdots \oplus (16)^n/j_2^{n+1}.$$

Then part *(i)* of Lemma 3.1, together with the formula $U/8 = (1/16)\,\mathrm{trace}\,V$, shows that

$$(U/8)(M) \subset (1/j_2)\mathbf{Z}_2[[256/j_2]] \subset M_n + 16^n M.$$

Thus we may compute the characteristic polynomial $C(T)$ of $U/8$ on M.

The formulae for $U(1/j_2)$ and $U(1/j_2^2)$ show that the matrix of $U/8$ on $M_1/16$ with respect to the basis $1/j_2, 16/j_2^2$ is congruent to

$$\begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix} \mod 16.$$

From this we compute that

$$C(T) \equiv 1 - 3T \mod 16\mathbf{Z}_2\langle T\rangle.$$

Weierstrass preparation shows that we may lift this to a factorization

$$C(T) = (1 - u'T)C'(T)$$

with $u' \equiv 3 \mod 2$ and $C'(T) \in \mathbf{Z}_2\langle T\rangle$ such that $C'(T) \equiv 1 \mod 16\mathbf{Z}_2\langle T\rangle$. Thus Proposition 3.6 applies, and taking $m = 1/j_2$ we compute an element $v \in M$ such that $v \equiv 1/j_2 \mod 16M$ and $U(v) = 8u'v$. Now by part *(iv)* of the Main Theorem, we conclude that $8u' = u_0$ and v is a scalar multiple of $f(u_0)$ (in fact a unit multiple, since both v and f have $1/j_2$ coefficient congruent to one modulo sixteen). Thus $f(u_0) \in M$. For the duration of the proof, let us write $f = f(u_0) \in M$.

Lemma 5.3. *The* $\mathbf{Z}_2$*-module* $M/((U/8)-u')M$ *is torsion-free.*

It suffices to show that $M/((U/8)-u')M$ is 2-torsion free. Suppose that $g \in M$ such that $g \mod ((U/8)-u')M$ is a 2-torsion element of $M/((U/8)-u')M$. Then there exists $h \in M$ such that

$$2g = ((U/8)-u')h.$$

Thus

$$((U/8)-u')h \equiv 0 \mod 2M,$$

and so

$$h \equiv af \mod 2M$$

for some $a \in \mathbf{Z}_2$. Thus $(h-af)/2 \in M$, and we see that

$$((U/8)-u')(h-af)/2 = (2g)/2 = g,$$

and thus

$$g \equiv 0 \mod ((U/8)-u')M.$$

This proves the Lemma.

Lemma 5.4. $(U/8)((U/8)-u')M \subset 16((U/8)-u')M$.

From the above matrix for the action of $U/8$ on $M_1/16$ we conclude that $(U/8)((U/8)-u')$ annhilates $M_1/16$, and so

$$(U/8)((U/8)-u')M \subset 16M.$$

Thus we may define an operator

$$T := (U/8)((U/8)-u')/16 : M \to M.$$

Any element in the image of T reduces modulo $((U/8)-u')M$ to an element of the 16-torsion of $M/((U/8)-u')M$.

Thus the previous Lemma shows that the image of T is contained in $((U/8)-u')M$. The Lemma follows.

Lemma 5.5. $M = f\mathbf{Z}_2 \oplus ((U/8)-u')M$.

First of all consider $U/8$ acting on $M/2M$. We see that

$$(U/8)(1/j_2) \equiv 1/j_2 \mod 2M,$$

while if $n \geq 2$,

$$(U/8)(1/j_2^n) \equiv 0 \mod 2M.$$

Recall also that $u' \equiv 1 \mod 2$ and that $1/j_2 \equiv f \mod 2M$. Thus

$$M/2M = f\mathbf{Z}_2/2 \oplus ((U/8)-u')(M/2M).$$

From this it follows that

$$M = f\mathbf{Z}_2 + ((U/8) - u')M.$$

To show that the sum is direct, we must show that $f\mathbf{Z}_2 \cap ((U/8) - u')M = 0$. Now the previous Lemma shows that f is not in the image of $(U/8) - u'$, since $(U/8)f = u'f$, and u' is a 2-adic unit. Lemma 5.3 shows that $M/((U/8) - u')$ is torsion free, which implies that af is not in the image of $U/8 - u'$ for any non-zero $a \in \mathbf{Z}_2$. This completes the proof of the Lemma.

We now prove parts *(i)* and *(ii)* of the Theorem. Since $1/j_2 \equiv f \mod 16$, we see from the preceeding Lemma that

$$1/j_2 = af + 16g$$

for some $a \equiv 1 \mod 16$ and some $g \in ((U/8) - u')M$. (A slightly more careful calculation shows that in fact $a \equiv 1 \mod 32$, but we won't need this.) Now by Lemma 5.4 $(U/8)^n g = 16^n g_n$ for some $g_n \in M$. Thus

$$(U/8)^n(1/j_2) = au'^n f + 16^{n+1} g_n.$$

Now the coefficient of $1/j_2$ in $(U/8)^n(1/j_2) \in M = 1/j_2\mathbf{Z}_2[[16/j_2]]$ is equal to the coefficient of q when $(U/8)^n(1/j_2)$ is expressed as a power series in q, and thus (from the formula for U acting on q-expansions) is equal to $a(2^n)/8^n$. Since f is normalized, the coefficient of $1/j_2$ in f is equal to one. Let b_n denote the coefficient of $1/j_2$ in g_n. Then we see that

$$a(2^n)/8^n = au'^n + 16^{n+1} b_n.$$

Since $g_n \in M = 1/j_2\mathbf{Z}_2[[16/j_2]]$, we see that

$$g_n \equiv b_n/j_1 \mod 16\mathbf{Z}_2[[q]].$$

Now

$$b_n/j_1 \equiv b_n f \mod 16\mathbf{Z}_2[[q]],$$

since $1/j_1 \equiv f \mod 16M \subset 16\mathbf{Z}_2[[q]]$. Thus

$$(U/8)^n(1/j_2) \equiv au'^n f + 16^{n+1} b_n f \equiv (a(2^n)/8^n) f \mod 16^{n+2}\mathbf{Z}_2[[q]].$$

Since au'^n is a 2-adic unit, we see that $a(2^n)/8^n$ is a 2-adic unit. Thus we may divide by $a(2^n)/8^n$ to conclude that

$$(U/8)^n(1/j_2)/(a(2^n)/8^n) \equiv f \mod 16^{n+2},$$

proving part *(ii)* of the Theorem.

To prove part *(i)* of the Theorem, we note that by part *(vii)* of the Main Theorem, and the fact that $f = f(u_0)$ is an eigenform for all the Hecke operators, for any odd integer $m \geq 1$, $a_m(u_0) \equiv \sigma_{-1}(m) \mod 2^7$. Thus

$$(a(m \cdot 2^n)/8^n)/(a(2^n)/8^n) \equiv \sigma_{-1}(m) \mod 2^7,$$

for any odd integer m and any $n \geq 0$, and so

$$a(m \cdot 2^n) \equiv (a(2^n)/8^n) \cdot 2^{3n}\sigma_{-1}(m) \mod 2^{3n+7}.$$

Now recall that $a(2^n)/8^n = au'^n + 16^{n+1}b_n$, and that $a \equiv 1 \mod 32$. Thus

$$a(2^n)/8^n \equiv u'^n \mod 32.$$

Since $1/j_2 \equiv f \mod 2^8$, we see that $24 \equiv u_0 \mod 2^8$, and so $u' = u_0/8 \equiv 3 \mod 32$. Putting everything together, we find that

$$a(m \cdot 2^n) \equiv 3^n \cdot 2^{3n}\sigma_{-1}(m) \mod 2^{3n+5}.$$

We now turn to proving parts *(iii)* and *(iv)* of the Theorem. We first prove a Lemma:

Lemma 5.6. *(i)* $c(2) = -2^{11} \mod 2^{19}$.
(ii) $U(j-744)/c(2) \in 1/j_2\mathbf{Z}_2[[256/j_2]]$, *and* $U(j-744)/c(2) \equiv f \mod 16M$.

From Lemma 2.2 we see that

$$j_2^2 + (48 + 2^{12}/V(j_2))j_2 - V(j_2) = (j_4^4 + 48j_4(j_4+16) + 2^{12}j_4 - j_4(j_4+16)^3)/(j_4+16)^2 = 0.$$

This is the minimal polynomial of $j_2 \in \mathbf{Z}_2((q))$ over $\mathbf{Z}_2((q^2))$, and so

$$U(j_2) = (1/2)\,\mathrm{trace}\,V\,(1/j_2) = -24 - 2^{11}/j_2.$$

Now Lemma 2.1 shows that

$$j = j_2 + 3 \cdot 256 + 3 \cdot 2^{16}/j_2 + 2^{24}/j_2^2,$$

and so

$$j - 744 = (j_2 + 24) + 3 \cdot 2^{16}/j_2 + 2^{24}/j_2^2.$$

Applying U we find that

$$U(j-744) = -2^{11}/j_2 + 3 \cdot 2^{16}U(1/j_2 + (2^8/3)/j_2^2).$$

Now $(U/8)(1/j_2 + (2^8/3)/j_2^2) \in 1/j_2\mathbf{Z}_2[[256/j_2]]$, and thus

$$U(j-744) \in -2^{11}/j_2 + 2^{19}/j_2\mathbf{Z}_2[[256/j_2]].$$

Comparing the coefficients of $1/j_2$ on both sides, we see that $c(2) \equiv -2^{11} \mod 2^{19}$, proving part *(i)*. Now dividing both sides by $c(2)$ proves part *(ii)*.

From the Lemma it follows in particular that $U(j-744)/c(2) \equiv f \mod 16M$. Thus parts *(iii)* and *(iv)* of the Theorem follow by the same type of argument that proved parts *(i)* and *(ii)*.

Corollary 5.7. *Write* $E/j_2 = \sum_{n=1}^{\infty} b(n)q^n$. *Let* u_2 *be the unique point of* $\mathcal{A}(\mathbf{C}_2)$ *lying over* κ_2. *For any integer* $m \geq 1$ *write* $\sigma_1(m) = \sum_{d|m} d = m\sigma_{-1}(m)$.

(i) For every integer $n \geq 1$, $b(n) = na(n)$. *Thus for any odd integer* $m \geq 1$ *and any integer* $n \geq 0$,

$$b(m \cdot 2^n) \equiv 3^n \cdot 2^{4n} \sigma_{-1}(m) \mod 2^{4n+5}.$$

(ii) For any integer $n \geq 1$,

$$U^n(E/j_2)/b(2^n) \equiv f(u_2) \mod 2^{4n+8}\mathbf{Z}_2[[q]].$$

Thus $f(u_2) = qd/dq(f(u_0))$. *(Note that it follows from part (i) that the left hand side of this congruence is indeed an element of* $\mathbf{Z}_2[[q]]$.*)*

In particular, since $f(u_2)$ *is an eigenform for all the Hecke operators, we see that for any odd prime* l *and any integers* $m \geq 1$, $n \geq 0$,

$$b(lm \cdot 2^n)/b(2^n) \equiv (b(l \cdot 2^n)/b(2^n))(b(m \cdot 2^n)/b(2^n)) - l(b(m \cdot 2^n/l)/b(2^n)) \mod 2^{4n+8}.$$

(We define $b(m \cdot 2^n/l) = 0$ *when* m/l *is not an integer.)*

Recall that $E = j_2(qd/dq(1/j_2))$, so that $E/j_2 = qd/dq(1/j_2)$. Thus part *(i)* follows.

A consideration of q-expansions shows that

$$U^n \circ (qd/dq) = 2^n (qd/dq) \circ U^n.$$

Thus

$$U^n(E/j_2)/b(2^n) = U^n(E/j_2)/(2^n a(2^n)) = U^n(qd/dq(1/j_2))/(2^n a(2^n))$$
$$= 2^n(qd/dq)U^n(1/j_2)/(2^n a(2^n)) = (qd/dq)(U^n(1/j_2)/a(2^n)) \equiv (qd/dq)(f(u_0)) \mod 2^{4n+8}.$$

It remains to show that $qd/dq(f(u_0)) = f(u_2)$. Since $|u_2| = |16|$ (by part *(ii)* of the Main Theorem) and since $|2u_0| = |16|$ also, we see that it is enough to show that $qd/dq(f(u_0)) \subset M_{\kappa_2}$. By Lemma 3.22,

$$M_{\kappa_2} = E^*_{\kappa_2}(1/j_2)\mathcal{O}_2[[8^{3/4}/j_2]] = E(1/j_2)\mathcal{O}_2[[8^{3/4}/j_2]],$$

since $E^*_{\kappa_2} = E$. Now $f(u_0) \subset (1/j_2)\mathcal{O}_2[[16/j_2]]$, and

$$qd/dq(1/j^n) = (E/j_2)(1/j_2^{n-1}) = E/j_2^n.$$

Thus

$$(d/dq)((1/j_2)\mathcal{O}_2[[16/j_2]] \subset (E/j_2)\mathcal{O}_2[[16/j_2]] \subset M_{\kappa_2},$$

and we are done.

5.3 Explicit formulae for $f(u)$ and $\kappa(u)$

In this section we collect explicit estimates for the family $f(u)$ and the function $\kappa(u)$ which ae easily obtained from the work of section 4.

Proposition 5.8. *(i)* $f(u) \equiv q + uq^2 + (-300 + 9t + 24u)q^3 + (1942 + 21t + 32u)q^5 \mod (u^2, q^6)\mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]]$.

(ii) The functions t and u on $\mathcal{A}$ satisfy the relation

$$u^2 + 7tu + (-2^{12} + 26 \cdot 64t + 24t^2) \equiv 0 \mod u^3\mathbf{Z}_2[[u]]\langle 64/u\rangle.$$

(iii) $t/u \equiv 33 + 10(64/u) + 11(64/u)^2 + 14(64/u)^3 + 36(64/u)^4 + 8(64/u)^5 + 48(64/u)^6 + 32(64/u)^7 \mod u\mathbf{Z}_2[[u]]\langle 64/u\rangle$.

By Proposition 4.18,

$$F(u) \equiv 1/j_2 + (u - 64 - 33t)/j_2^2 \mod \mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]].$$

In the course of proving Lemma 4.1, we saw that

$$E^3B^2E^*_{\kappa'} = 1 + (40 + a)q^2 + (-24 + 41a)q^2 + (-9952 + 52a/3 + a^2 + ab)q^3 +$$

$$+(65512 - 29765a/3 + 40a^2 + 40ab)q^4 + (724848 + 277846a/5 - 24a^2 - 24ab + ac)q^5 + \cdots.$$

Now $t \in u\mathbf{Z}_2[[u]]\langle 64/u\rangle$, and $64 = u(64/u) \in u\mathbf{Z}_2[[u]]\langle 64/u\rangle$, and so by Lemma 4.2 we infer that

$$a \equiv 33t \mod u^2\mathbf{Z}_2[[u]]\langle 64/u\rangle,$$

$$b \equiv 0 \mod u^2\mathbf{Z}_2[[u]]\langle 64/u\rangle,$$

and

$$c \equiv 13t \mod u^2\mathbf{Z}_2[[u]]\langle 64/u\rangle.$$

Substituting

$$1/j_2 = q + 24q^2 + 300q^3 + 2624q^4 + 18126q^5 + \cdots$$

into the expression for $F(u)$, multiplying this by the expression for $E^3B^2E^*_{\kappa'}$, and using the preceding congruences, we prove part *(i)*.

In section 4.5 we saw that over the region $|64| \geq |t| \geq 0$, the relation between u and t is described by a quadratic polynomial $Q^*(T)$, and reduces to

$$(u/64)^2 + 7(t/64)(u/64) + (-1 + 26(t/64) + 24(t/64)^2) \equiv 0 \mod 64\mathbf{Z}_2\langle u/64, 64/u\rangle.$$

Multiplying by 2^{12}, we find that

$$u^2 + 7tu + (-2^{12} + 26 \cdot 64t + 24t^2) \equiv 0 \mod 64^3\mathbf{Z}_2\langle u/64, 64/u\rangle.$$

Since

$$64^3\mathbf{Z}_2\langle u/64, 64/u\rangle \cap \mathbf{Z}_2[[u]]\langle 64/u\rangle \subset u^3\mathbf{Z}_2[[u]]\langle 64/u\rangle,$$

part *(ii)* follows.

From part *(ii)* we derive the formula

$$t/u \equiv (-1/7)(1 - (64/u)^2 + 2(13 \cdot (64/u)(t/u) + 12(t/u)^2)) \mod u\mathbf{Z}_2[[u]]\langle 64/u\rangle,$$

which we may use to recursively compute t/u. Iterating this recursive formula (and recalling that $64 = u(64/u) \in u\mathbf{Z}_2[[u]]\langle 64/u\rangle$) yields the formula of part *(iii)*.