CHAPTER IV

THE MAIN THEOREM

In section 1 of this chapter we state our Main Theorem on the existence of the minimal slope family $f(u)$. The rest of the chapter is devoted to its proof. In section 2 we provide some formulae and estimates which provide the crux of the proof. In sections 3, 4 and 5 we make the computations which will prove parts *(i)*, *(ii)*, *(iii)*, *(iv)* and *(v)* of the Theorem. These are drawn together in section 6. In section 7 we compute some explicit members of the family $f(u)$, certain evil Eisenstein series. Sections 8 presents the proof of part (vi), modulo a certain congruence for $1/j_2^3$, which is proved in section 9. Finally, section 10 proves part (vii).

4.1 Statement of the Main Theorem

For any $\kappa \in \mathcal{W}$, we have defined the characteristic power series $P(\kappa, T) \in \mathcal{O}_2\langle T\rangle$ of U acting on the space M_κ, and have seen that it is an entire power series. Thus it factors as a product

$$P(\kappa, T) = \prod_{n=1}^{\infty}(1 - \lambda_n(\kappa)T),$$

with the reciprocal zeroes $\lambda_n(\kappa)$ of $P(\kappa, T)$ being ordered so that

$$|\lambda_1(\kappa)| \geq |\lambda_2(\kappa)| \geq \cdots .$$

We refer to $\mathrm{ord}_2(\lambda_n(\kappa))$ as the slope of $\lambda_n(\kappa)$. The slopes $\mathrm{ord}_2(\lambda_n(\kappa))$ tend to infinity with n — in particular, there are only finitely many $\lambda_n(\kappa)$ for which $\mathrm{ord}_n(\lambda_n(\kappa))$ is minimal among the set of slopes. We refer to these as the reciprocal zeroes of $P(\kappa, T)$ of minimal slope.

Main Theorem. *Let $\mathcal{A}$ denote the annulus $1 > |u| \geq |64|$, which is a rigid analytic space defined over $\mathbf{Q}_2$. We construct an element*

$$f(u) = \sum_{n=1}^{\infty} a_n(u)q^n \in \mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]],$$

and a map $\kappa : \mathcal{A} \to \mathcal{W}$, which we write $u \mapsto \kappa(u)$, with the following properties:

(i) $a_1 = 1$ and $a_2 = u$.

(ii) Let $\mathcal{A}_1$ denote the sub-annulus $1 > |u| > |64|$ of $\mathcal{A}$. The map κ satisfies $|\kappa(u)| = |u|$ for all points $u \in \mathcal{A}_1(\mathbf{C}_2)$, and (consequently) induces an isomorphism between $\mathcal{A}_1$ and the subannulus $1 > |t| > |64|$ of $\mathcal{W}$.

(iii) Let $\mathcal{A}_2$ denote the sub-annulus $|u| = |64|$ of $\mathcal{A}$. When restricted to $\mathcal{A}_2$, the map κ realizes $\mathcal{A}_2$ as a finite degree two cover of the sub-disc $|64| \geq |t| \geq 0$ of $\mathcal{W}$, having exactly two ramification points, which are a conjugate pair of points defined over $\mathbf{Q}_2(\sqrt{3})$.

(iv) If $\kappa \in \mathcal{W}(\mathbf{C}_2)$, then the fiber $\mathcal{A}_\kappa$ is a zero-dimensional rigid analytic space of degree either one or two (as follows from parts (i) and (ii)).

The points $u \in \mathcal{A}_\kappa(\mathbf{C}_2)$ are precisely the reciprocal zeroes of the characteristic power series $P(\kappa, T)$ which are of minimal slope, and the multiplicity with which each eigenvalue u occurs as a point of $\mathcal{A}_\kappa$ is equal to the multiplicity with which u occurs as a reciprocal zero of $P(\kappa, T)$.

Furthermore, for each $u \in \mathcal{A}_\kappa$, the q-expansion $f(u) = \sum_{n=1}^{\infty} a_n(u) q^n \in \mathcal{O}_2[[q]]$ is the unique overconvergent 2-adic cuspform of weight $\kappa = \kappa(u)$ which is an eigenform for all the Hekce operators and whose U-eigenvalue equals u.

(v) If $u, u' \in \mathcal{A}(\mathbf{C}_2)$, then $f(u) \equiv f(u') \mod (u - u')\mathcal{O}_2[[q]]$.

(vi) If l is an odd prime, and $u \in \mathcal{A}(\mathbf{C}_2)$, then

$$a_l(u) \equiv 1 + \kappa(u)(l) l^{-1} \begin{cases} \mod (u-2)(u-4)(u-8)(u-32)\mathcal{O}_2, & l \equiv 1 \mod 8 \\ \mod (u-8)(u-32)\mathcal{O}_2, & l \equiv 3 \mod 8 \\ \mod (u-4)(u-8)(u-32)\mathcal{O}_2, & l \equiv 5 \mod 8 \\ \mod (u-2)(u-8)(u-32)\mathcal{O}_2, & l \equiv 7 \mod 8. \end{cases}$$

(vii) If k is an integer and χ is a character of Γ of finite order, let $\kappa_{\chi,k}$ denote the 2-adic weight defined by $\kappa_{\chi,k}(\gamma) = \chi(\gamma)\gamma^k$. Let τ denote the character

$$\tau : \mathbf{Z}_2^\times \to \{\pm 1\} \subset \mathbf{C}_2,$$

and let 2^n be the conductor of the character $\tau^k \kappa_{\chi,k}$ of $\mathbf{Z}_2^\times$. If χ is non-trivial and $k \geq 2$, or if χ is trivial and $k \geq 3$, then for any $u \in \mathcal{A}_{\kappa_{\chi,k}}$ the overconvergent cusp form $f(u)$ of weight $\kappa_{\chi,k}$ is in fact a classical modular form of weight k and nebentypus χ on $\Gamma_1(2^m)$, where $m = n$ if $n > 0$, and $m = 1$ if $n = 0$.

The proof of this Theorem will occupy the remainder of this chapter.

4.2 Leading term estimates

The following two Lemmas relate to the action of $\tilde{U}$ on $1/j_2^n$ for $n = 1, 2, 3$. Unfortunately, the proofs of both these Lemmas are by direct but rather extensive calculation. These calculations are most easily done using a computer algebra package. We did them using Mathematica.

Since the calculations are straightforward, the details could well have been left to the reader. However, since they are so central to the result that we are trying to prove, we have outlined the steps involved in the calculations, while leaving the actual arithmetic and algebraic manipulations to the reader.

We advise any reader who wishes to check our calculations for themselves to have on hand a computer algebra package to assist them, as well as the backs of many envelopes.

It will be convenient to abbreviate some functions of κ' which appear in these calculations: we will write

$$a = 2/\zeta^*(\kappa'),$$

$$b = (\kappa'(3) - 1)/3 - 2/\zeta^*(\kappa'),$$

and

$$c = (\kappa'(5) - 1)/5.$$

Lemma 4.1. *(i)* $\tilde{U}(1/j_2) = (64+a)/j_2 + (55\cdot 64a/3 + ab)/j_2^2 + (-587\cdot 2^{12}a/15 - 7a^2/3 + 1148ab - a^2b + ac)/j_2^3 + \cdots$.

(ii) $\tilde{U}(1/j_2^2) = 1/j_2 + (47\cdot 64 + 88a)/j_2^2 + (81\cdot 2^{12} + 281\cdot 2^9 a/3 + 88ab)/j_2^3 + \cdots$.

(iii) $\tilde{U}(1/j_2^3) = (112+a)/j_2^2 + (145\cdot 2^{10} + 253\cdot 64a/3 + ab)/j_2^3 + \cdots$.

This Lemma is proved by direct calculation. Recall that

$$\tilde{U}(1/j_2^n) = U((1/j_2^n)E^3B^2E^*_{\kappa'}/V(E^3B^2E^*_{\kappa'})) \in \mathbf{Z}_2[[t, 1/j_2]].$$

A consideration of q-expansions shows that to compute $\tilde{U}(1/j_2^n)$ as a power series in $1/j_2$ modulo $1/j_2^4$, we must compute the product $(1/j_2)^n E^3B^2E^*_{\kappa'}/V(E^3B^2E^*_{\kappa'})$ modulo $1/j_2^7$. Since we are concerned only with $n \geq 1$, it will suffice to compute $E^3B^2E^*_{\kappa'}/V(E^3B^2E^*_{\kappa'})$ modulo $1/j_2^6$.

We begin with the q-expansions

$$E = 1 + 24(q + q^2 + 4q^3 + q^4 + 6q^5 + \cdots),$$

$$B = 1 + 16(-q + 7q^2 - 28q^3 + 71q^4 - 126q^5 + \cdots),$$

and

$$E^*_{\kappa'} = 1 + (2/\zeta^*(\kappa'))(q + q^2 + (1 + 3^{-1}\kappa'(3))q^3 + q^4 + (1 + 5^{-1}\kappa'(5))q^5 + \cdots)$$

$$= 1 + a(q + q^2 + (4/3 + a + b)q^3 + q^4 + (6/5 + c)q^5 + \cdots).$$

From these one obtains the q-expansion

$$E^3B^2E^*_{\kappa'} = 1 + (40+a)q + (-24+41a)q^2 + (-9952 + 52a/3 + a^2 + ab)q^3 +$$

$$+(65512 - 29765a/3 + 40a^2 + 40ab)q^4 + (724848 + 277846a/5 - 24a^2 - 24ab + ac)q^5 + \cdots.$$

Substituting q^2 for q and reducing modulo q^6, we find that

$$V(E^3B^2E^*_{\kappa'}) = 1 + (40+a)q^2 + (-24+41a)q^4 + \cdots.$$

Now performing a division shows that

$$E^3B^2E^*_{\kappa'}/V(E^3B^2E_{\kappa'}) = 1 + (40+a)q + (-64+40a)q^2 + (-11552 - 188a/3 + ab)q^3 +$$

$$+(68096 - 34496a/3 + 40ab)q^4 + (1187888 + 1020178a/15 - 7a^2/3 - 64ab - a^2b + ac)q^5 + \cdots.$$

Recall (from Lemma 2.1) that

$$1/j_2 = q\prod_{n=1}^{\infty}(1+q^n)^{24} = q + 24q^2 + 300q^3 + 2624q^4 + 18126q^5 + \cdots.$$

Reversing this series shows that

$$q = 1/j_2 - 24/j_2^2 + 852/j_2^3 - 35744/j_2^4 + 1645794/j_2^5 \cdots.$$

Substituing this expression into the q-expansion for $E^3B^3E^*_{\kappa'}/V(E^3B^2E^*_{\kappa'})$ yields the formula

$$E^3B^2E^*_{\kappa'}/V(E^3B^2E_{\kappa'}) = 1+(40+a)/j_2+(-2^{10}+16a)/j_2^2+(25\cdot 2^{10}-53\cdot 64a/3+ab)/j_2^3+$$
$$+(-165\cdot 2^{12}+71\cdot 2^{11}a/3-32ab)/j_2^4+(555\cdot 2^{15}-7127\cdot 2^{12}a/15-7a^2/3+380ab-a^2b+ac)/j_2^5+\cdots.$$

Thus we see that

$$\tilde{U}(1/j_2^n) = U(1/j_2^n+(40+a)/j_2^{n+1}+(-2^{10}+16a))/j_2^{n+2}+$$
$$+(25\cdot 2^{10}-53\cdot 64a/3+ab)/j_2^{n+3}+(-165\cdot 2^{12}+71\cdot 2^{11}a/3-32ab)/j_2^{n+4}+$$
$$+(555\cdot 2^{15}-7127\cdot 2^{12}a/15-7a^2/3+380ab-a^2b+ac)/j_2^{n+5}+\cdots).$$

Now recall the formulae for the action of U (modulo $1/j_2^4$) from section 3.1:

$$U(1/j_2) = 24/j_2+2^{11}/j_2^2,$$

$$U(1/j_2^2) = 1/j_2+9\cdot 2^7/j_2^2+3\cdot 2^{16}/j_2^3+\cdots,$$
$$U(1/j_2^3) = 72/j_2^2+15.2^{12}/j_2^3+\cdots,$$
$$U(1/j_2^4) = 1/j_2^2+9\cdot 2^9/j_2^3+\cdots,$$
$$U(1/j_2^5) = 120/j_2^3+\cdots,$$

and

$$U(1/j_2^6) = 1/j_2^3+\cdots.$$

Combining these formulae with the preceeding formula for $\tilde{U}(1/j_2^n)$, in the cases $n=1$, 2 and 3, proves the Lemma (after some further computation!).

Lemma 4.2. *(i)* $a = 2/\zeta^*(\kappa') \equiv 3489t+28t^2 \mod (2^{12}t, 2^6t^2, t^3)\mathbf{Z}_2[[t]]$.
(ii) $b = (\kappa'(3)-1)/3-2/\zeta^*(\kappa') \equiv 61\cdot 64t+21t^2 \mod (2^{12}t, 2^6t^2, t^3)\mathbf{Z}_2[[t]]$.
(iii) $c = (\kappa'(5)-1)/5 \equiv 3277t \mod (2^{12}t, 2^6t^2, t^3)\mathbf{Z}_2[[t]]$.

Let us write

$$2/\zeta^*(\kappa') = \sum_{n=0}^{\infty} a_n t^n \in \mathbf{Z}_2[[t]].$$

In order to prove part *(i)*, we must show that $a_0 = 0$, and approximate a_1 and a_2.

Let us restrict our attention for a moment to $\kappa = \kappa_k$ for integers $k > 14$. Then $\kappa' = \kappa_{k-14}$, and

$$\zeta^*(\kappa') = (1-2^{k-15})\zeta(15-k),$$

where ζ denote the Riemann zeta function. Now as a complex analytic function, $(1-2^{s-1})\zeta(1-s)$ has a simple pole at $s=0$, with residue $-1/2$. The analogous statement is true 2-adic analytically: $\zeta^*(\kappa')$ has a simple pole at $k=14$, with residue $-1/2$, when expanded as a Laurent series in $k-14$. (See [16], Theorem 5.11.)

Now $t = 5^{k-14}-1 = \log 5(k-14)+\cdots \in \mathbf{Z}_2[[k-14]]$, and so when expanded as a power series in t, $\zeta^*(\kappa')$ has a simple pole at $t=0$ with residue $-\log 5/2$.

Thus $2/\zeta^*(\kappa') = (-4/\log 5)t + \cdots \in \mathbf{Z}_2[[t]]$, and we see that indeed $a_0 = 0$, while $a_1 = (-4/\log 5)$.

Now we compute that

$$\log 5 = \log(1+4) = \sum_{n=1}^{\infty} (-1)^{n-1}(4^n)/n \equiv 2463 \cdot 4 \mod 2^{14}.$$

Thus $a_1 = (-4/\log 5) \equiv 3489 \mod 2^{12}$. Now to compute $a_2 \mod 64$, we substitute in a particular value of κ: when $\kappa = \kappa_{30}$, we see that $\kappa' = \kappa_{16}$, and $t = 5^{16} - 1$. Note that $\mathrm{ord}_2(5^{16} - 1) = 6$.

Now one computes (using a table of Bernoulli numbers, for example) that

$$2/\zeta^*(\kappa') = 2/\zeta^*(\kappa_{16}) = 2/\zeta(-15)(1 - 2^{15}) = \frac{1632}{3617(1 - 2^{15})}.$$

Thus

$$16320/3617(1-2^{15}) = 2/\zeta^*(\kappa') = a_1 t + a_2 t^2 + \cdots \equiv 3489 \cdot (5^{16} - 1) + a_2(5^{16} - 1)^2 \mod 2^{18}.$$

Solving for a_2, we find that $a_2 \equiv 28 \mod 64$. This completes the proof of part *(i)* of the Lemma.

To prove part *(ii)*, we must first expand $(\kappa'(3) - 1)/3$ as a power series in t. Let us write $-3 = 5^l$, where $l = \log 5/\log 3 \in \mathbf{Z}_2$. Then

$$(\kappa'(3) - 1)/3 = (\kappa'(-3) - 1)/3 = (\kappa'(5^l) - 1)/3$$

$$= (\kappa'(5)^l - 1)/3 = ((t+1)^l - 1)/3 = (lt + l(l-1)t^2/2)/3 + \cdots .$$

Now one easily computes that $l \equiv 1699 \mod 2^{12}$, and then that

$$(\kappa'(3) - 1)/3 \equiv -799t - 15t^2 \mod (2^{12}t, 2^6t^2, t^3)\mathbf{Z}_2[[t]].$$

Combining this with the result of part *(i)* shows that

$$(\kappa'(3) - 1)/3 - 2/\zeta^*(\kappa') \equiv -799t - 15t^2 - 3489t - 28t^2$$

$$\equiv 61 \cdot 64t + 21t^2 \mod (2^{12}t, 2^6t^2, t^3)\mathbf{Z}_2[[t]],$$

proving part *(ii)* of the Lemma.

By the definition of t, $(\kappa'(5) - 1)/5 = t/5$. Now $1/5 \equiv 3277 \mod 2^{12}$. This proves part *(iii)* of the Lemma.

4.3 The computation for $1 > |u| \geq |16|$

In this section we construct the minimal slope family over the sub-annulus of $\mathcal{A}$ defined by $1 > |u| \geq |16|$. Since the projection to $\mathcal{W}$ is supposed to identify this sub-annulus of $\mathcal{A}$ with the sub-annulus of $\mathcal{W}$ described by $1 > |t| \geq |16|$, we will begin by working over the annulus $\mathcal{W}_2$ described by $1 > |s| \geq |2|$, which maps to $\mathcal{W}$ via $t = s^4$.

Let us abbreviate $R = R(\mathcal{W}_2) = \mathbf{Z}_2[[s]]\langle 2/s\rangle$ for the duration of this section.
Define the R-module
$$M = 1/j_2R \oplus t/j_2^2R[[s^3/j_2]].$$

Note that by Proposition 3.17,

$$E_{\kappa'}^*/V(E_{\kappa'}^*) \in 1 + t/j_2R[[s^3/j_2]],$$

while by part *(i)* of Lemma 3.14,

$$E^3B^2/V(E^3B^2) \in \mathbf{Z}_2[[8/j_2]] \subset R[[s^3/j_2]],$$

since $\mathbf{Z}_2 \subset R$ and $8 = (2/s)^3s^3$. Taking the product of these two equations, we see that

$$e = E_{\kappa'}^*E^3B^2/V(E_{\kappa'}^*E^3B^2) \in E^3B^2/V(E^3B^2) + t/j_2R[[s^3/j_2]], \tag{4.1}$$

and that, less precisely,

$$e \in R[[s^3/j_2]]. \tag{4.2}$$

Equation (4.2) shows that $(t/j_2^2)R[[s^3/j_2]]$ is closed under multiplication by e, and Corollary 3.2 shows that $U((t/j_2^2)R[[s^3/j_2]] \subset (t/j_2)R[[s^6/j_2]]$. Putting these remarks together, we see that $(t/j_2)^2R[[s^3/j_2]]$ is closed under $\tilde{U}/t$, and that in fact

$$(\tilde{U}/t)((t/j_2)^2R[[s^3/j_2]]) \subset (1/j_2)R[[s^6/j_2]]. \tag{4.3}$$

From equation (4.1) we see that

$$e/j_2 \in E^3B^2/V(E^3B^2)j_2 + (t/j_2^2)R[[s^3/j_2]].$$

Now

$$(U/t)(E^3B^2/V(E^3B^2)j_2) = (64/t)/j_2 \in (1/j_2)R,$$

by part *(ii)* of Lemma 3.14 and the fact that $64/t = 16(16/t) \in R$. Combining this result with equation (4.3) we find that

$$(\tilde{U}/t)(1/j_2) \subset (1/j_2)R[[s^6/j_2]]. \tag{4.4}$$

Now for each $n \geq 1$, define

$$M_n = (1/j_2)R \oplus (t/j_2^2)R \oplus (ts^{3(n-1)}/j_2^n)R.$$

Then equations (4.3) and (4.4) show (recalling that $t = s^4$)

$$(\tilde{U}/t)(M) \subset M_n + s^{3n-1}M.$$

Thus Proposition 3.4 applies and allows us to compute the characteristic power series $C(T) \in R\langle T\rangle$ of $\tilde{U}/t$.

In particular, we can also compute the characteristic power series of $\tilde{U} = t(\tilde{U}/t)$ on M, and this is equal to $C(tT)$.

Lemma 4.3. *We have the equality $C(tT) = P(T)$. (Recall that $P(T)$ was defined in the previous chapter to be the characteristic power series of $\tilde{U}$ on $M(\mathcal{X}, s^3)$.)*

We prove this by applying Proposition 3.5. Write $\bar{M} = M(\mathcal{W}_2, s^3)$, and $\bar{M}_n = M(\mathcal{W}_2, s^3)_n$. Now

$$M(\mathcal{W}_2, s^3) = M(\mathcal{W}_2, s^3)_n \oplus (1/j_2^n)R(\mathcal{W}_2)[[8/j_2]],$$

and we let Π_n denote the projection onto the first factor. We take $\varphi = \tilde{U}$. Finally, we note that

$$\bar{M}_{n+1} \subset M_n \subset s\bar{M}_{n+1}.$$

Then all the hypotheses of Proposition 3.5 are satisfied, and the Lemma follows.

Recall that $M_1 = (1/j_2)R$. From Lemma 4.1 we compute that

$$\det(1 - T(\tilde{U}/t)_{M_1/s^2}) = 1 - (64/t + a/t)T.$$

Lemma 4.2 shows that

$$a/t \equiv 1 + 32 \cdot 109 + 28t \mod (2^12, 2^6t, t^2)R$$

$$\equiv 1 \mod (32, 4t, t^2)R.$$

Now $32 = 8(2/s)^2 s^2$, $4t = (4s)s^2$, and $t^2 = (s^6)s^2$, so in particular $a/t \equiv 1 \mod s^2R$. Also $64/t = s^2(2/s)^6 \equiv 0 \mod s^2R$. Thus

$$C(T) \equiv 1 + T \mod s^2R\langle T\rangle.$$

Weierstrass preparation shows that we may factor $C(T) = (1 - u'T)C'(T)$ for some $u' \in R^\times$ such that $u' \equiv 1 \mod s^2R$ and some $C'(T) \in R\langle T\rangle$ such that $C'(T) \equiv 1 \mod s^2$.

We now apply Proposition 3.6 to this situation, taking $m = 1/j_2 \in M_1$. We find an element $v \in M$ such that $(\tilde{U}/t)(v) = u'v$, and such that $v \equiv 1/j_2 \mod s^2$. Let r be the coefficient of $1/j_2$ in $v \in M = (1/j_2)R \oplus (t/j_2^2)R[[s^3/j_2]]$. Then $r \equiv 1 \mod s^2$, and so r is a unit in R. Define $F = r^{-1}v$, and $u = u't$. Then

$$F \in 1/j_2 + (t/j_2^2)R[[s^3/j_2]],$$

and

$$\tilde{U}F = uF.$$

Lemma 4.4. *The element $F \in M$ is in fact an element of*

$$(1/j_2)R(\mathcal{A}_1) \oplus (t/j_2^2)R(\mathcal{A}_1)[[1/j_2]]$$

(i. e. it is actually a function of t, not just of s). In particular $u \in t\mathbf{Z}_2[[t]]\langle 16/t\rangle$. If $\kappa \in \mathcal{W}(\mathbf{C}_2)$ such that $1 > |t(\kappa)| \geq |16|$ then the the corresponding value $u(\kappa)$ of u is the unique recpiprocal zero of $P(\kappa, T)$ having minimal slope, and the element

$$f(\kappa) = F(\kappa)E^3B^2E^*_{\kappa'} \in \mathcal{M}_\kappa$$

spans the $u(\kappa)$-eigenspace of U in $\mathcal{M}_\kappa$.

Since F is constructed from $1/j_2$ by applying $\tilde{U}/t$, and $\tilde{U}/t$ involves only functions of t, not of s, we see that

$$f \in M \cap \mathbf{Z}_2[[t]]\langle 16/t\rangle \subset (1/j_2)\mathbf{Z}_2[[t]]\langle 16/t\rangle \oplus (t/j_2^2)\mathbf{Z}_2[[t]]\langle 16/t\rangle[[1/j_2]].$$

Since $(\tilde{U}/t)(F) = (u/t)F$, we see by comparing coefficients of $1/j_2$ on each side of this equation that $u/t \in \mathbf{Z}_2[[t]]\langle 16/t\rangle$, and so $u \in t\mathbf{Z}_2[[t]]\langle 16/t\rangle$.

Now write

$$f = FE^3B^2E^*_\kappa \in E^3B^2E^*_{\kappa'}M \subset R[[q]].$$

We compute that

$$U(f) = U(FE^3B^2E^*_{\kappa'}) = U(Fe)E^3B^2E^*_{\kappa'} = \tilde{U}(F)E^3B^2E^*_{\kappa'} = uFE^3B^2E^*_{\kappa'} = uf.$$

Thus f is indeed an eigenvector for U with eigenvalue u. Specializing this at the weight κ, we find that indeed $f(\kappa)$ is an eigenvector for U in $\mathcal{M}_\kappa$ with eigenvalue u.

Lemma 4.3 yields the equation

$$P(T) = C(tT) = (1 - u'tT)C'(tT) = (1 - uT)C'(tT),$$

where $C'(T) \equiv 1 \mod s^2$. If we specialize this at the weight κ and recall (by Proposition 3.22) that $P(\kappa, T)$ is the specialization of $P(T)$ at κ, we see that indeed $u(\kappa)$ is the unique reciprocal zero of $P(\kappa, T)$ of minimal slope.

We will now show that $f(\kappa)$ spans the $u(\kappa)$-eigenspace for U in $\mathcal{M}_\kappa$. Recall that

$$\mathcal{M}_\kappa = \mathbf{Q}_2 \otimes_{\mathbf{Z}_2} (E^3B^2E^*_{\kappa'}/j_2)\mathcal{O}_2[[s(\kappa)/j_2]]$$

for some $s(\kappa)$ such that $s(\kappa)^4 = t(\kappa)$. Now

$$\begin{aligned}\mathbf{Q}_2 &\otimes_{\mathbf{Z}_2} (E^3B^2E^*_{\kappa'}/j_2)\mathcal{O}_2[[r(\kappa)^3/j_2]] \\ &= \mathbf{Q}_2 \otimes_{\mathbf{Z}_2} ((E^3B^2E^*_{\kappa'}/j_2)\mathcal{O}_2 \oplus (t(\kappa)E^3B^2E^*_{\kappa'}/j_2^2)\mathcal{O}_2[[s(\kappa)^3/j_2]]).\end{aligned}$$

Thus if $g \in \mathcal{M}_\kappa$, after replacing g by its product with a suffciently large power of 2, we may assume that g lies in

$$(E^3B^2E^*_{\kappa'}/j_2)\mathcal{O}_2 \oplus (t(\kappa)E^3B^2E^*_{\kappa'}/j_2^2)\mathcal{O}_2[[s(\kappa)^3/j_2]].$$

Now suppose that $U(g) = u(\kappa)g$. Let r be the coefficient of q in g. Then

$$g - rf(\kappa) \in (t(\kappa)E^3B^2E^*_{\kappa'}/j_2^2)\mathcal{O}_2[[s(\kappa)^3/j_2]].$$

We wish to show that $g - rf(\kappa) = 0$. Set

$$G := (g - rf)/E^3B^2E^*_{\kappa'}.$$

Then $g - rf(\kappa) = 0$ if and only if $G = 0$. Note that
$$G \in (t(\kappa)/j_2^2)\mathcal{O}_2[[s(\kappa)^3/j_2]].$$
Since $g - rf$ is an eigenvector for U with eigenvalue $u(\kappa)$, G is an eigenvector for $\tilde{U}$ with eigenvalue $u(\kappa)$, or equivalently, an eigenvector for $\tilde{U}/t(\kappa)$ with eigenvalue $u'(\kappa) = u(\kappa)/t(\kappa)$. But the same reasoning that proved equation (4.4) proves that
$$(\tilde{U}/t(\kappa))((t(\kappa)/j_2^2)\mathcal{O}_2[[s(\kappa)^3/j_2]]) \subset (1/j_2)\mathcal{O}_2 \oplus (s(\kappa)^6/j_2^2)\mathcal{O}_2[[s(\kappa)^6/j_2]].$$
In particular
$$(\tilde{U}/t(\kappa))(G) \in (1/j_2)\mathcal{O}_2 \oplus (s(\kappa)^6/j_2^2)\mathcal{O}_2[[s(\kappa)^6/j_2]].$$
But also
$$(\tilde{U}/t(\kappa))(G) = u'G \in (t(\kappa)/j_2^2)\mathcal{O}_2[[s(\kappa)^3/j_2]] = (s(\kappa)^4/j_2)\mathcal{O}_2[[s(\kappa)^3/j_2]],$$
and thus
$$(\tilde{U}/t(\kappa))(G) \in (s(\kappa)^6/j_2^2)\mathcal{O}_2[[s(\kappa)^6/j_2]].$$
Repeating this argument n times, we find that
$$(\tilde{U}/t(\kappa))^n(G) = u'(\kappa)^n G \in (s(\kappa)^{2n+4}/j_2^2)\mathcal{O}_2[[s(\kappa)^6/j_2]].$$
Letting n tend to infinity, and recalling that u' is a unit in R, we see that $G = 0$. Thus $g = rf(\kappa)$ is indeed a scalar multiple of $f(\kappa)$. This completes the proof of the Lemma.

Lemma 4.5. *The function $u \in t\mathbf{Z}_2[[t]]\langle t\rangle$ induces an isomorphism between the annulus $1 > |t| \geq |16|$ and $1 > |u| \geq |16|$.*

This follows from the fact that u' is a unit, so $|u| = |u't| = |t|$.

By virtue of this Lemma we may regard f as a function of u, and we write $f = f(u)$ to emphasize this point. We may also regard κ as a function of u, rather than the other way around, and we write $\kappa = \kappa(u)$.

Summary 4.6. *The function $u \mapsto \kappa(u)$ establishes an isomorphism between the annulus $1 > |u| \geq |16|$ and the sub-annulus $1 > |t| \geq |16|$ of $\mathcal{W}$. For any u such that $1 > |u| \geq |16|$, u is the unique reciprocal zero of $P(\kappa(u),T)$ having minimal slope, and $f(u)$ spans the u-eigenspace for U in $M_{\kappa(u)}$.*

This summarizes the statements of Lemmas 4.4 and 4.5.

4.4 The computation for $|16| \geq |u| \geq |64|$: case 1

In this section we begin the construction of the minimal slope family over the sub-annulus of $\mathcal{A}$ defined by $|16| \geq |u| \geq |64|$. This calculation is a little more involved than that of the previous section, and breaks into two cases, the case when $|16| \geq |t(\kappa)| \geq |64|$, and the case when $|64| \geq |t(\kappa)| \geq 0$.

We deal with the first case in this section. Define $\mathcal{X}$ to be the affinoid cut out by the conditions $|16| \geq |s| \geq |64|$ and $|1 - (64/s)^2| = 1$, which we map to $\mathcal{W}$ via $t = s(1 - (64/s)^2)$. Note that the second condition may be rewritten $|t/s| = 1$. In particular, $|t| \geq |64|$.

Lemma 4.7. *The map $\mathcal{X} \to \mathcal{W}$ given by $s \mapsto s(1-(64/s)^2)$ has for image the sub-annulus $|16| \geq |t| \geq |64|$ of $\mathcal{W}$. It induces an isomorphism of the region $|16| \geq |s| > |64|$ with the region $|16| \geq |t| > |64|$, and induces a degree two unramified covering map from the region $|64| = |s|, |1-(64/s)^2| = 1|$ to the region $|t| = |64|$. It satisfies $|s| = |t|$.*

The points of $\mathcal{X}(\mathbf{C}_2)$ lying over a given value of t are the solutions of the equation $s^2 - ts - 2^{12}$. If $|16| \geq |t| > |64|$, there is a unique solution to this equation lying in $\mathcal{X}(\mathbf{C}_2)$, while if $|t| = |64|$, there are two such points. Since $|t/s| = 1$, we always have that $|s| = |t|$. Since the derivative of this equation is $2s - t$, and since $|t/s| = 1$, we see that the two solutions of this equation never coincide when $|t| = |64|$. The Lemma is proved.

We write $R = R(\mathcal{X}) = \mathbf{Z}_2\langle s/16, 64/s, 1/(1-(64/s)^2)\rangle$. We define

$$M = (1/j_2)R \oplus (s/j_2^2)R[[8/j_2]].$$

Part *(i)* of Proposition 16 shows that

$$E^*_{\kappa'}/V(E^*_{\kappa'}) \in 1 + t/j_2R[[8/j_2]] = 1 + (t/s)s/j_2R[[8/j_2]] \subset 1 + s/j_2R[[8/j_2]] \subset 1 + 8/j_2R[[8/j_2]],$$

since both s/t and $s/8 = 2(s/16)$ are elements of R. Also Lemma 19 shows that

$$E^3B^2/V(E^3B^2) \in 1 + 8/j_2\mathbf{Z}_2[[8/j_2]] \subset 1 + 8/j_2R[[8/j_2]].$$

Thus we obtain two formulae:

$$E^*_{\kappa'}E^3B^2/V(E^*_{\kappa'}E^3B^2) \in 1 + 8/j_2R[[8/j_2]], \tag{4.5}$$

$$E^*_{\kappa'}E^3B^2/V(E^*_{\kappa'}E^3B^2) \in E^3B^2/V(E^3B^2) + s/j_2^2R[[8/j_2]]. \tag{4.6}$$

We first note that equation (4.5) implies that $s/j_2^2R[[8/j_2]]$ is closed under multiplication by $E^3B^2E^*_{\kappa'}/V(E^3B^2E^*_{\kappa'})$, while Corollary 3.2 shows that

$$(U/s)((s/j_2^2)R[[8/j_2]]) = U((1/j_2^2)R[[8/j_2]]) \subset (1/j_2)R[[64/j_2]].$$

Combining these two observations shows that

$$(\tilde{U}/s)(s/j_2^2R[[8/j_2]]) \subset (1/j_2)R[[64/j_2]]. \tag{4.7}$$

Now equation (4.6) shows that

$$E^*_{\kappa'}E^3B^2/V(E^*_{\kappa'}E^3B^2)/j_2 \in E^3B^2/V(E^3B^2)j_2 + s/j_2^2R[[8/j_2]]. \tag{4.8}$$

Part *(ii)* of Lemma 19 shows that

$$(U/s)(E^3B^2/V(E^3B^2)j_2) = (64/s)/j_2 \in 1/j_2R, \tag{4.9}$$

since $64/S \in R$. Combining equations (4.7), (4.8) and (4.9), we find that

$$(\tilde{U}/s)1/j_2 \in (1/j_2)R[[64/j_2]]. \tag{4.10}$$

Now define

$$M_n = (1/j_2)R \oplus (s/j_2^2)R \oplus \cdots \oplus (s8^{n-1}/j_2^{n+1})R \subset M.$$

Then equations (4.7) and (4.10) show that

$$(\tilde{U}/s)(M) \subset M_n + 8^nM.$$

Proposition 3.4 applies, and allows us to compute the characteristic power series of $\tilde{U}/s$ on M, which we denote by $C(T) \in R\langle T\rangle$. We may also compute the characteristic power series for $\tilde{U}$ on M, using the same M_n, and this is of course just $C(sT)$.

Lemma 4.8. *We have the equality $C(sT) = P(T)$.*

The proof of this Lemma is analogous to the proof of Lemma 4.3 in the previous section.

Now $M_1 = (1/j_2)R \oplus (s/j_2^2)R$. By Lemma 4.1 we compute that the matrix of $\tilde{U}$ on $M_1/8$ with respect to the basis $1/j_2$, s/j_2^2 is equal to

$$\begin{pmatrix} (64+a)/s & 1 \\ (55 \cdot 64a + 3ab)/3s^2 & (47 \cdot 64 + 88a)/s \end{pmatrix}.$$

Lemma 4.2 shows that

$$a/s \equiv 1 + 32 \cdot 109(t/s) + 28t(t/s)t \mod (2^{12}(t/s), 2^6 t(t/s), t^2(t/s))R$$

$$\equiv t/s \mod (32(t/s), 4t(t/s), t^2(t/s))R.$$

Now $t = 16(t/s)(s/16) \in 16R$, and so in particular $a/s \equiv t/s \mod 8R$. Thus we see that the preceeding matrix is congruent to

$$\begin{pmatrix} t/s + 64/s & 1 \\ 5 \cdot 64t/s^2 & -64/s \end{pmatrix} \mod 8R.$$

The characteristic polynomial of this matrix is equal to

$$1 - (t/s)T - ((64/s)^2 + 6(64t/s^2))T^2.$$

Thus

$$C(T) \equiv 1 + (t/s)T - ((64/s)^2 + 6(64t/s^2))T^2 \mod 8R.$$

Reducing this congruence modulo two we find in particular that

$$C(T) \equiv 1 - (t/s)T - (64/s)^2T^2 = (1-T)(1 + (64/s)^2 T) \mod 2R.$$

(Recall that $t/s = 1 - (64/s)^2$.) Thus

$$(dC/dT)_{|T=1} \equiv t/s \mod 2R.$$

Since t/s is a unit in R, Newton's method allows us to lift the root $T = 1$ of $C(T) \mod 2R$ to a root of $C(T)$: we may write

$$C(T) = (1 - u'T)C'(T)$$

for some $u' \equiv 1 \mod 2R$ and some $C'(T) \in R\langle T\rangle$ such that $C'(T) \equiv 1 \mod 2R\langle T\rangle$.

We now apply Proposition 3.6 to this situation, taking $m = s/j_2^2 \in M_1$. We conclude that there exists $v \in M$ such that $(\tilde{U}/s)(v) = u'v$ and

$$v \equiv ((64/s)^2 + \tilde{U}/s)(s/j_2^2) \equiv 1/j_2 + (1 + (t/s) + (64/s))(s/j_2^2) \mod 2M.$$

In particular, if r is the coefficient of $1/j_2$ in v, $r \equiv 1 \mod 2R'$, and so we may define $F = r^{-1}v$. We also define $u = u's$. Then

$$F \in 1/j_2 + (s/j_2^2)R[[8/j_2]]$$

and $\tilde{U}(F) = uF$.

Lemma 4.9. *If $\kappa \in \mathcal{W}(\mathbf{C}_2)$ such that $|16| \geq |t(\kappa)| \geq |64|$, then the the values $u(\kappa)$ of the function u corresponding to those $s(\kappa) \in \mathcal{X}(\mathbf{C}_2)$ lying over $t(\kappa)$ are the precisely the reciprocal zeroes of $P(\kappa, T)$ having minimal slope. For any such value of $s(\kappa)$, the element $f(s(\kappa)) = F(s(\kappa))E^3B^2E^*_\kappa \in M_\kappa$ spans the $u(\kappa)$-eigenspace for U in $\mathcal{M}_\kappa$.*

By Lemma 4.8,

$$P(\kappa, T) = C(s(\kappa)T) = (1 - u(\kappa)T)C'(sT).$$

Thus $u(\kappa)$ is indeed a reciprocal zero of $P(\kappa, T)$ having minimal slope. Furthermore, since

$$C(T) \equiv (1-T)(1+(64/s)^2T) \mod 2R\langle T\rangle,$$

we see that when $|t(\kappa)| = |s(\kappa)| > |64|$ there is a unique such reciprocal zero of minimal slope, corresponding to the unique $s(\kappa)$ lying over $t(\kappa)$, while when $|t(\kappa)| = |s(\kappa)| = |64|$, we see that there are two reciprocal zeroes, corresponding to the two values $s(\kappa)$ lying over $t(\kappa)$. This proves the first statement of the Lemma.

Since

$$F(s(\kappa)) \in (1/j_2) + (s(\kappa)/j_2^2)\mathcal{O}_2[[8/j_2]] \subset (1/j_2)\mathcal{O}_2[[8/j_2]],$$

certainly

$$f(s(\kappa)) = F(s(\kappa))E^3B^2E^*_{\kappa'} \in M_\kappa \subset \mathcal{M}_\kappa.$$

Also, since $\tilde{U}(F(s(\kappa))) = u(\kappa)F(s(\kappa))$, we see that

$$U(f(s(\kappa)) = U(F(s(\kappa))e)E^3B^2E^*_{\kappa'} = uf(s(\kappa)).$$

To see that $f(s(\kappa))$ spans the $u(\kappa)$-eigenspace of U in $\mathcal{M}_\kappa$, we argue as in the proof of Lemma 4.4.

As in the proof of that Lemma, it suffices to show that if

$$g \in E^3B^2E^*_{\kappa'}((1/j_2)\mathcal{O}_2 \oplus (s(\kappa)/j_2^2)\mathcal{O}_2[[8/j_2^2]],$$

and $U(g) = ug$, then $g = rf(s(\kappa))$, where r is the coefficient of q in g. We define $G = (g - rf(s(\kappa)))/E^3B^2E^*_{\kappa'}$. Then $(\tilde{U}/s(\kappa))(G) = u'G$. As in the proof of Lemma 4.4,

$$G \in (s(\kappa)/j_2^2)\mathcal{O}_2[[8/j_2]].$$

Unfortunately, this is not quite enough to conclude that $G = 0$. However, we can strengthen the argument of Lemma 4.4 slightly, to conclude that in fact

$$G \in (8s(\kappa)/j_2^3)\mathcal{O}_2[[8/j_2]].$$

Once we show this, by iterating $(\tilde{U}/s(\kappa))$ on G and using the fact that u is a unit together with the fact (which is proved in exactly the same way as equation (4.7)) that

$$(\tilde{U}/s(\kappa)))((8s(\kappa)/j_2^3)\mathcal{O}_2[[8/j_2]]) \subset (1/j_2)\mathcal{O}_2 \oplus (s(\kappa)/j_2^2)\mathcal{O}_2 \oplus 8(8s(\kappa)/j_2^3)\mathcal{O}_2[[8/j_2]],$$

we conclude that $G = 0$.

We are left to show that $G \in (8s(\kappa)/j_2^3)\mathcal{O}_2[[8/j_2]]$. This is equivalent to showing that both the coefficient of q and of q^2 in $g - rf(s(\kappa))$ vanish. But by the choice of r the coefficient of q in $g - rf(\kappa)$ vanishes, and since $g - rf(s(\kappa))$ is an eigenvector for U with non-zero eigenvalue, we see that its coefficient of q^2 must also vanish. This completes the proof that $f(s(\kappa))$ spans the $u(\kappa)$-eigenspace of U in $\mathcal{M}_\kappa$, and thus completes the proof of the Lemma.

Lemma 4.10. *The function $u \in R$ induces an isomorphism between the affinoid $\mathcal{X}$ and the affinoid $|16| \geq |u| \geq |64|, |(1-(64/u)^2)| = 1$, such that $|s| = |u|$.*

This follows from the fact that u' is a unit, so that $|u| = |u's| = |s|$.

By virtue of this Lemma, we may regard f as a function of u, and we write $f = f(u)$ to emphasize this point. We may also regard κ as a function of u, and write $\kappa = \kappa(u)$.

Summary 4.11. *The function $u \mapsto \kappa(u)$ from the region $|16| \geq |u| \geq |64|, |1-(64/u)^2| = 1$ induces an isomorphism from the region $|16| \geq |u| > |64|$ to the region $|16| \geq |t| > |64|$ of $\mathcal{W}$, and induces an unramified degree two covering map from the region $|u| = |64|, |1-(64/u)^2| = 1$ to the region $|t| = |64$ of $\mathcal{W}$. It satisfies $|u| = |t(\kappa(u))|$. For any κ such that $|16| \geq |t(\kappa)| \geq |64|$, the points u lying over κ are precisely the reciprocal zeroes of $P(\kappa, T)$ having minimal slope, and $f(u)$ spans the u-eigenspace of U in $\mathcal{M}_{\kappa(u)}$.*

This summarizes the statements of Lemmas 4.9 and 4.10.

4.5 The computation for $|16| \geq |u| \geq |64|$: case 2

We now consider the second case of the calculation over $|16| \geq |u| \geq |64|$. Let $\mathcal{X}'$ denote the sub-disc $|64| \geq |t| \geq 0$ of $\mathcal{W}$. Let

$$R' = R(\mathcal{X}') = \mathbf{Z}_2\langle t/64\rangle.$$

Define

$$M' = (1/j_2)R' \oplus (64/j_2^2)R'[[8/j_2]],$$

and for each $n \geq 1$ define

$$M'_n = (1/j_2)R' \oplus (64/j_2^2)R' \oplus \cdots \oplus (8^{n+1}/j_2^{n+1})R'.$$

Then the arguments made in the previous section for R, M, M_n and $\tilde{U}/s$ work equally well, if we replace R, M, M_n and $\tilde{U}/s$ respectively by R', M', M'_n and $\tilde{U}/64$. In particular, we see that

$$(\tilde{U}/64)(M') \subset M'_n + 8^n M'.$$

Thus we may compute the characteristic power series of $\tilde{U}/64$ on M, which we denote by $D(T)$. We may also compute the characteristic power series of $\tilde{U}$ on M', and this will simply be $D(64T)$.

Lemma 4.12. *We have the equality $D(64T) = P(T)$ as elements of $R'\langle T\rangle$.*

This is proved by the same argument as Lemma 4.3.

Lemma 4.1 shows that the matrix of $\tilde{U}/64$ on $M'_2/64M'_2$ with respect to the basis $1/j_2$, $64/j_2^2$, $512/j_2^3$ is equal to

$$\begin{pmatrix} 1 + a/64 & 1 & 0 \\ 55a/3 \cdot 64 + ab/2^{12} & 47 + 88a/64 & 14 + 8a/64 \\ \begin{matrix}(-587 \cdot 2^{12}a/15 - 7a^2/3+ \\ +1148ab - a^2b + ac)/2^{15}\end{matrix} & 81 \cdot 8 + 281a/3 + 88ab/2^9 & 145 \cdot 16 + 253a/3 + ab/64 \end{pmatrix}.$$

If we recall that $t = 64(t/64) \in 64R'$, Lemma 4.2 shows that

$$a \equiv 3489 \cdot 64(t/64) + 28 \cdot 2^{12}(t/64)^2 \mod 2^{18}R',$$

that

$$b \equiv 61 \cdot 2^{12}(t/64) \mod 2^{18}R',$$

and that

$$c \equiv 3277 \cdot 64(t/64) \mod 2^{18}R'.$$

Thus

$$a^2 \equiv 3905 \cdot 2^{12}(t/64)^2 + 7 \cdot 2^{21}(t/64)^3 \mod 2^{24}R',$$

$$ab \equiv 29 \cdot 2^{18}(t/64)^2 \mod 2^{24}R',$$

$$a^2 b \equiv 61 \cdot 2^{24}(t/64)^3 \mod 2^{30}R',$$

and

$$ac \equiv 1517 \cdot 2^{12}(t/64)^2 + 11 \cdot 2^{20}(t/64)^3 \mod 2^{24}R'.$$

Substituting these congruences into the above matrix and reducing modulo $64R'$, one finds that this matrix is congruent to

$$\begin{pmatrix} 1+33t/64 & 1 & 0 \\ 29(t/64) & 47+24(t/64) & 14+8(t/64) \\ 24(t/64) & 8 & 16 \end{pmatrix} \mod 64R'.$$

Computing the characteristic polynomial of this matrix shows that

$$D(T) \equiv 1 + 7(t/64)T + (-1 + 26(t/64) + 24(t/64)^2)T^2 \mod 64R'[T].$$

Now $-1 + 26(t/64) + 24(t/64)^2$ is a unit in R', and so by Weierstrass preparation we may factor

$$D(T) = Q(T)D'(T)$$

with $Q(T) \in R'[[T]]$ a quadratic polynomial such that

$$Q(T) \equiv 1 + 7(t/64)T + (-1 + 26(t/64) + 24(t/64)^2)T^2 \mod 64R'[T],$$

and $D'(T) \in R'\langle T\rangle$ such that

$$D'(T) \equiv 1 \mod 64R'\langle T\rangle.$$

Let us write $Q^*(T) = T^2 Q(1/T)$.

Let $\mathcal{A}_2$ denote the sub-annulus of $\mathcal{A}$ defined by $|u| = |64|$.

Lemma 4.13. *The equation $Q^*(u/64) = 0$ makes the annulus $\mathcal{A}_2$ a finite cover of degree two of $\mathcal{X}'$, with two ramification points, which are a conjugate pair of points defined over $\mathbf{Q}_2(\sqrt{3})$.*

Let us write

$$0 = Q^*(u/64) = (u/64)^2+7((t/64)+64a(t/64))(u/64)+(-1+26(t/64)+24(t/64)^2+64b(t/64)))$$

for some $a(t/64), b(t/64) \in R'$. Then we may write

$$t/64 = (1/7)(64/u) - (1/7)(u/64) - (2/7)(32a(t/64)+$$

$$+13(t/64)(64/u) + 12(t/64)^2(64/u) + 32b(t/64)(64/u)).$$

This yields a recursive formula which allows us to compute $t/64 \in \mathbf{Z}_2\langle 64/u, u/64\rangle$. Thus the equation $Q(u/64) = 0$ does indeed define a map from $\mathcal{A}_2$ to $\mathcal{X}'$, which is evidently finite of degree two.

The ramification points are determined by setting the discriminant of $Q^*(T)$ equal to zero. This discriminant is equal to

$$d(t/64) := (7(t/64) + 64a(t/64))^2 - 4(-1 + 26(t/64) + 24(t/64)^2 + 64b(t/64))$$

$$= (t/64)^2 - 8(t/64) + 4 + 32(-3(t/64) + 3(t/64)^2 + 4(t/64)a(t/64) + 128a(t/64)^2 + 8b(t/64))$$

$$\equiv (t/64)^2 - 8(t/64) + 4 \mod 32R'.$$

Weierstrass preparation shows that $d(t/64) = d_1(t/64)d_2(t/64)$ with d_1 a quadratic polynomial in $\mathbf{Z}_2[t/64]$ such that

$$d_1 \equiv (t/64)^2 - 8(t/64) + 4 \mod 32\mathbf{Z}_2[t/64],$$

and $d_2(t/64) \in 1+32R'$. Thus the zeroes of $d(t/64)$ coincide with the zeroes of $d_1(t/64)$. The congruence for d_1 shows that the discriminant of $d_1(t/64)$ is congruent to $8^2-4\cdot 4 = 48 = 3\cdot 16$ mod 128. Thus the zeroes of $d_1(t/64)$ are defined over $\mathbf{Q}_2(\sqrt{3})$. This completes the proof of the Lemma.

Now let

$$R'' := R(\mathcal{A}_2) = \mathbf{Z}_2\langle u/64, 64/u\rangle,$$

and define

$$M'' = (1/j_2)R'' \oplus (64/j_2)^2 R''[[8/j_2]],$$

and for each $n \geq 1$,

$$M''_n = (1/j_2)R'' \oplus (64/j_2)^2 R'' \oplus \cdots \oplus (8^{n+1}/j^{n+1})R'' = R'' \otimes_{R'} M'_n.$$

Then using the same arguments as used in the previous section for R, M and M_n, we see that

$$(\tilde{U}/64)(M'') \subset M''_n \oplus 8^n M'',$$

and so we may compute the characteristic power series of $\tilde{U}/64$ on M''. Furthermore, the argument of Lemma 3.8 shows that this is simply $D(T) \in R'\langle T\rangle$ regarded as as an element of $R''\langle T\rangle$.

Now recall $D(T) = Q(T)D'(T)$. In $R''[T]$, $Q(T)$ factors as

$$Q(T) = (1 - (u/64)T)(1 + (7(t/64) + (u/64))T),$$

since $Q^*(u/64) = 0$. Thus

$$D(T) = (1 - (u/64)T)(1 + (7(t/64) + (u/64))T)D'(T),$$

and we may apply Proposition 3.6. We take $m = 64/j_2^2 \in M_1''$, and remark that the matrix of $\tilde{U}/64$ on M_1'' is congruent to

$$\begin{pmatrix} 1 + (t/64) & 1 \\ (t/64) & 1 \end{pmatrix} \quad \mathrm{mod}\ 2R''.$$

Thus by Proposition 3.6 we conclude that there is an element $v \in M''$ such that $(\tilde{U}/64)(v) = (u/64)v$ and

$$v \equiv (7(t/64) + (u/64) + \tilde{U}/64))(64/j_2^2) \equiv 1/j_2 + (1 + (t/64) + (u/64))(64/j_2^2) \mod 2M''.$$

Let r be the coefficient of $1/j_2$ in v. We see that $r \equiv 1 \mod 2R''$, and so may define $F = r^{-1}v$. Then

$$F \in 1/j_2 + (64/j_2^2)R''[[8/j_2]],$$

and

$$(\tilde{U}/64)(F) = (u/64)F.$$

Lemma 4.14. *If $\kappa \in \mathcal{W}(\mathbf{C}_2)$ such that $|64| \geq |t(\kappa)| \geq 0$, then the values of $u \in \mathcal{A}_2(\mathbf{C}_2)$ lying over κ are precisely the reciprocal zeroes of $P(\kappa, T)$ having minimal slope. Furthermore, the mutliplicity with which u occurs in the fiber of $\mathcal{A}_2$ over κ is equal to the multiplicity with which u occurs as a reciprocal zero of $P(\kappa, T)$. For any u lying over κ the element $f(u) = F(u)E^3B^2E_\kappa^* \in \mathcal{M}_\kappa$ spans the u-eigenspace of U in $\mathcal{M}_\kappa$.*

The equations $P(\kappa, T) = D(64T) = Q(64/T)D'(T)$ with $D'(T) \equiv 1 \mod 64R'\langle T\rangle$, together with the fact that $Q^*(T) = 0$ is the equation of $\mathcal{A}_2$ over $\mathcal{X}'$, prove the first two sentences.

The fact that $f(u)$ spans the u-eigenspace for U in $\mathcal{M}_\kappa$ is proved in an identical fashion to the analogous statement of Lemma 4.9.

Summary 4.15. *The map $u \mapsto \kappa(u)$ from $|u| = |64|$ to the sub-disc $|64| \geq |t| \geq 0$ of $\mathcal{W}$ is finite of degree two, with two ramification points which are conjugate elements of $\mathbf{Q}_2(\sqrt{3})$. For any κ such that $|64| \geq |t(\kappa| \geq 0$, the points of u in the fiber of this map over κ coincide with the reciprocal zeroes of $P(\kappa, T)$ having minimal slope. Furthermore, the multiplicity with which a point appears in the fiber is equal to the multiplicity with which it appears as a reciprocal zero of $P(\kappa, T)$. For any such u, $f(u)$ spans the u-eigenspace of $\tilde{U}$ in M_κ.*

This summarizes the results of Lemmas 4.13 and 4.14.

4.6 Proof of parts *(i), (ii), (iii), (iv)* and *(v)* of the Main Theorem

If we consider Summaries 4.6, 4.11 and 4.15, we see that we have computed maps

$$\kappa(u) \in \mathbf{Z}_2[[u]]\langle 16/u\rangle,$$

$$\kappa(u) \in \mathbf{Z}_2\langle u/16, 64/u, 1/(1-(64/u)^2)\rangle[[q]],$$

and

$$\kappa(u) \in \mathbf{Z}_2\langle u/64, 64/u\rangle,$$

and elements

$$f(u) \in \mathbf{Z}_2[[u]]\langle 16/u\rangle[[q]],$$

$$f(u) \in \mathbf{Z}_2\langle u/16, 64/u, 1/(1-(64/u)^2)\rangle[[q]],$$

and

$$f(u) \in \mathbf{Z}_2\langle u/64, 64/u\rangle[[q]].$$

To see that these notations are not at odds with each other, we first observe that for any u in one of the regions where these maps κ are defined, the value $\kappa(u)$ is uniquely determined by u as being that weight for which u is a reciprocal zero of $P(\kappa, T)$ having minimal slope. Thus the different maps κ glue to yield a single map

$$\kappa(u) \in \mathbf{Z}_2[[u]]\langle 64/u\rangle.$$

Now for any u in the domain for one of the functions $f(u)$, $f(u)$ is uniquely determined as the basis vector for the u-eigenspace in $\mathcal{M}_{\kappa(u)}$ whose coefficient of q equals one. Thus the elements $f(u)$ glue to yield a single element

$$f(u) \in \mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]].$$

Write $f(u) = \sum_{n=1}^{\infty} a_n(u)q^n$.

Part *(i)* follows, since we have constructed $f(u)$ so that $a_1(u) = 1$, which implies, since $U(f(u)) = uf(u)$, that $a_2(u) = u$. Part *(ii)* follows from Summaries 4.6 and 4.11, while part *(iii)* follows from Summary 4.15. Taken together, all three Summaries prove every claim of part *(iv)*, with the exception of the claim that $f(u)$ is an eigenform for all the Hecke operators. This follows from Proposition 3.25 together with the fact that the Hecke operators on $\mathcal{M}_\kappa$ commute among themselves, since $f(u)$ spans the u-eigenspace of $\tilde{U}$ on $\mathcal{M}_\kappa$. We will give a proof of this claim which is independent of Proposition 3.25 in section 4.10.

In order to prove part *(v)*, we prove the following Lemma:

Lemma 4.16. *The functions $a_n(u)$, for $n \geq 1$, lie in the subring $\mathbf{Z}_2 + u\mathbf{Z}_2[[u]]\langle 64/u\rangle$ of $\mathbf{Z}_2[[u]]\langle 64/u\rangle$.*

To prove this, write

$$F(u) = f(u)/E^3B^2E^*_{\kappa(u)'}.$$

Then from the explicit construction of $F(u)$ given in the previous three sections we see that

$$F(u) \in 1/j_2\mathbf{Z}_2 \oplus (u/j_2^2)\mathbf{Z}_2[[u]]\langle 64/u\rangle[[1/j_2]].$$

(This follows from the definitions of the spaces M of section 4.3 and 4.4, and the space M'' of 4.5, together with the fact that F lies in these three spaces.) Now

$$1/j_2 \oplus (u/j_2^2)\mathbf{Z}_2[[u]]\langle 64/u\rangle[[1/j_2]] \subset \mathbf{Z}_2[[q]] + u\mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]] = (\mathbf{Z}_2 + u\mathbf{Z}_2[[u]]\langle 64/u\rangle)[[q]].$$

Thus

$$F(u) \in (\mathbf{Z}_2 + u\mathbf{Z}_2[[u]]\langle 64/u\rangle)[[q]].$$

Also

$$E^3B^2 \in \mathbf{Z}_2[[q]] \subset (\mathbf{Z}_2 + u\mathbf{Z}_2[[u]]\langle 64/u\rangle)[[q]],$$

and (using part *(i)* of Lemma 4.2)

$$E^*_{\kappa(u)'} \in 1 + (2/\zeta^*)\mathbf{Z}_2[[t]][[q]] \subset 1 + t\mathbf{Z}_2[[t]][[q]]$$

$$\subset 1 + u\mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]] \subset (\mathbf{Z}_2 + u\mathbf{Z}_2[[u]]\langle 64/u\rangle)[[q]].$$

Hence

$$f(u) = F(u)E^3B^2E^*_{\kappa(u)'} \in (\mathbf{Z}_2 + u\mathbf{Z}_2[[u]]\langle 64/u\rangle)[[q]].$$

This proves the Lemma.

Now for some fixed n, let us write

$$a_n(u) = a + u(\sum_{m=0}^{\infty} b_m u^m + \sum_{m=1}^{\infty} c_m (64/u)^m)$$

$$= a + \sum_{m=0}^{\infty} b_m u^{m+1} + 64\sum_{m=1}^{\infty} c_m (64/u)^{m-1} \in \mathbf{Z}_2 + u\mathbf{Z}_2[[u]]\langle 64/u\rangle.$$

Then for any pair $u, u' \in \mathcal{A}(\mathbf{C}_2)$,

$$a_n(u) - a_n(u')$$
$$= (u-u')(\sum_{m=0}^{\infty} b_m \frac{u^{m+1} - u'^{m+1}}{(u-u')} - \frac{64\cdot 64}{uu'}\sum_{m=1}^{\infty} c_m \frac{(64/u)^{m-1} - (64/u')^{m-1}}{(64/u) - 64/u'}) \in (u-u')\mathcal{O}_2.$$

This proves part *(v)*.

4.7 Evil Eisenstein Series

Suppose that χ is an even character of conductor 2^n. By the evil Eisenstein series of weight k and nebentypus χ, we mean the modular form

$$E^{\mathrm{evil}}_{\chi,k} = \sum_{n=1}^{\infty}(\sum_{\substack{d|n \\ n/d \text{ odd}}} \chi(n/d)\tau(d)^k d^{k-1})\, q^n.$$

(This terminology was coined in [9].) When χ is trivial, we denote this form simply by E_k^{evil}. If χ is trivial and k is even, then E_k^{evil} is a weight k modular form on $\Gamma_0(2)$ (indeed, $E_k^{\text{evil}} = (\zeta(1-k)/2)(E_k - V(E_k))$, where E_k is the weight k Eisenstein series of level one). If χ is trivial and k is odd, then E_k^{evil} is a weight k modular form on $\Gamma_1(4)$. Finally, if χ is non-trivial, then $E_{\chi,k}^{\text{evil}}$ is a weight k modular form of nebentypus χ on $\Gamma_1(2^n)$. Note that for any k and χ, $E_{\chi,k}^{\text{evil}}$ is a normalized eigenform for all the Hecke operators. In particular, the U-eigenvalue of $E_{k,\chi}^{\text{evil}}$ is equal to 2^{k-1}.

Proposition 4.17. *(i) Let χ be the unique non-trivial even character of conductor eight. Then $\kappa(2) = \kappa_{\chi,2}$, and $f_2 = E_{\chi,2}^{\text{evil}}$.*

(ii) We have $\kappa(4) = \kappa_3$, and $f_4 = E_3^{\text{evil}}$.

(iii) We have $\kappa(8) = \kappa_4$, and $f_8 = E_4^{\text{evil}}$.

(iv) We have $\kappa(32) = \kappa_6$, and $f_{32} = E_6^{\text{evil}}$.

Let us presume for a moment that for those k and χ considered in the Proposition, $E_{\chi,k}^{\text{evil}} \in \mathcal{M}_{\kappa_{\chi,k}}$. For each such value of k and χ, $|t(\kappa_{\chi,k})| = |2^{k-1}| > |64|$, and so by part *(ii)* there is a unique value of u lying over $\kappa_{\chi,k}$, and $|u| = |t(\kappa_{\chi,k})| = |2^{k-1}|$. By part *(iv)*, u is the unique reciprocal zero of $P(\kappa_{\chi,k}, T)$ of minimal slope.

Assuming that $E_{\chi,k}^{\text{evil}} \in \mathcal{M}_{\kappa_{\chi,k}}$, we see that since since $U(E_{\chi,k}^{\text{evil}}) = 2^{k-1}E_{\chi,k}^{\text{evil}}$, Lemma 3.23 shows that 2^{k-1} is reciprocal zero of $P(\kappa_{\chi,k}, T)$. But u and 2^{k-1} have the same slope, and thus uniqueness of u shows that $u = 2^{k-1}$. Then the characterization of $f(u)$ as the basis of the u-eigenspace for U, together with the fact that $f(u)$ and $E_{\chi,k}^{\text{evil}}$ both have their coefficient of q equal to one, shows that $f(u) = E_{\chi,k}^{\text{evil}}$. This proves the Proposition, once we have justified our assumption.

We have to show that $E_{\chi,k}^{\text{evil}} \in \mathcal{M}_{\kappa_{\chi,k}}$ for those χ and k considered in the statement of the Proposition. In the three cases when χ is trivial, this follows from Proposition 3.24. Thus it remains to prove this for the evil Eisenstein series $E_{\chi,2}^{\text{evil}}$, where χ is the unique non-trivial even character of conductor eight.

We must show that

$$E_{\chi,2}^{\text{evil}}/E_{\kappa_{\chi,2}}^{*} \in \mathbf{C}_2 \otimes_{\mathcal{O}_2} (1/j_2)\mathcal{O}_2[[2^{3/4}/j_2]].$$

Now

$$E_{\chi,2}^{\text{evil}}/E_{\kappa_{\chi,2}}^{*} = (E_{\chi,2}^{\text{evil}})^2/(E_{\chi,2}^{\text{evil}}E_{\kappa_{\chi,2}}^{*}),$$

and the numerator and denominator of this ratio are both weight four forms on $\Gamma_0(8)$. Thus, using the discussion of section 2.3, we see that each can be written as a degree four homogeneous polynomial in θ and $V(\theta)$. It is a relatively simple computation to determine these polynomials – it requires a knowledge of the first five terms of the q-expansion of each of these modular forms.

We compute

$$(E_{\chi,2}^{\text{evil}})^2 = \theta V(\theta)(\theta - V(\theta))^2/16,$$

and

$$E_{\chi,2}^{\text{evil}}E_{\kappa_{\chi,2}}^{*} = \theta V(\theta)(\theta - V(\theta))(2V(\theta) - \theta)/4.$$

Thus

$$E^{\text{evil}}_{\chi,2}/E^*_{\kappa_{\chi,2}} = (\theta - V(\theta))/4(2V(\theta) - \theta) = (1/4)((\theta/V(\theta) - 1)/(2 - \theta/V(\theta)).$$

Now since part *(iii)* of Lemma 2.7 shows that $\theta/V(\theta) \in 1 + 4/j_2\mathbf{Z}[[4/j_2]]$, we conclude that

$$E^{\text{evil}}_{\chi,2}/E^*_{\kappa_{\chi,2}} \in j_2\mathbf{Z}[[4/j_2]] \subset (1/j_2)\mathcal{O}_2[[2^{3/4}/j_2]].$$

Thus the Proposition is proved.

4.8 Proof of part *(vi)* of the Main Theorem

In this section we prove part *(vi)* of the Main Theorem. The argument is an extension of that used to prove part *(v)*. We begin by proving the following approximation for the q-expansion of $F(u) = f(u)/E^3B^2E^*_{\kappa'} \in \mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]]$:

Proposition 4.18. $F(u) \equiv 1/j_2 + (u - 64 - 33t)/j_2^2 \mod u^2\mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]]$.

We have seen, in the previous section, that

$$F(u) \in \mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]] = \mathbf{Z}_2[[u]]\langle 64/u\rangle[[1/j_2]].$$

Let us write

$$F(u) = \sum_{n=1}^{\infty} \alpha_n/j_2^n,$$

for some $\alpha_n \in \mathbf{Z}_2[[u]]\langle 64/u\rangle$. By construction of $F(u)$, we know that $\alpha_1 = 1$. Thus to prove the Proposition, we must show that

$$\alpha_2 \equiv (u - 64 - 33t) \mod u^2\mathbf{Z}_2[[u]]\langle 64/u\rangle,$$

and that

$$\alpha_n \in u^2\mathbf{Z}_2[[u]]\langle 64/u\rangle$$

for $n > 2$.

Now $\mathbf{Z}_2\langle u/64, 64/u\rangle \subset \mathbf{Z}_2[[u]]\langle 64/u\rangle$, and

$$2^{12}\mathbf{Z}_2\langle u/64, 64/u\rangle \cap \mathbf{Z}_2[[u]]\langle 64/u\rangle \subset u^2\mathbf{Z}_2[[u]]\langle 64/u\rangle.$$

Thus it suffices to show that

$$\alpha_2 \equiv u - 64 - 33t \mod 2^{12}\mathbf{Z}_2\langle u/64, 64/u\rangle,$$

and that

$$\alpha_n \equiv 0 \mod 2^{12}\mathbf{Z}_2\langle u/64, 64/u\rangle,$$

for $n > 2$.

To prove this, we can work in the space

$$M'' = 1/j_2\mathbf{Z}_2\langle u/64, 64/u\rangle \oplus 64/j_2^2\mathbf{Z}_2\langle u/64, 64/u\rangle[[8/j_2]]$$

which was considered in section 4.5. In that section we showed that $F(u) \in M''$, and that M'' is closed under $\tilde{U}/64$. In fact, we saw that

$$\tilde{U}/64(M'') \subset 1/j_2\mathbf{Z}_2[[u]]\langle 64/u\rangle[[64/j_2]].$$

From this it follows immediately that

$$a_n \equiv 0 \mod 2^{12}\mathbf{Z}_2\langle u/64, 64/u\rangle$$

when $n > 3$.

Proving the necessary congruence for $n = 2$ is a matter of calculation:

$$f(u) = q + uq^2 + \cdots ,$$

and

$$E^3B^2E^*_{\kappa'} = (1 + 40q + \cdots)(1 + (2/\zeta^*(\kappa'))q + \cdots) = 1 + (40 + 2/\zeta^*(\kappa'))q + \cdots .$$

Thus

$$F(u) = f(u)/E^3B^2E^*_{\kappa} = q + (u - 40 - 2/\zeta^*(\kappa'))q^2 + \cdots = 1/j_2 + (u - 64 - 2/\zeta^*(\kappa'))/j_2^2 + \cdots .$$

Part *(i)* of Lemma 4.2 shows that

$$2/\zeta(\kappa) \equiv 33t \mod (64t, 64t^2, t^3)\mathbf{Z}_2[[t]].$$

Now

$$t = 64(t/64) \in 64\mathbf{Z}_2\langle u/64, 64/u\rangle,$$

and so

$$2/\zeta(\kappa) \equiv 33t \mod 2^{12}\mathbf{Z}_2\langle u/64, 64/u\rangle.$$

Hence

$$\alpha_2 = (u - 64 - 2/\zeta^*(\kappa')) \equiv (u - 64 - 33t) \mod 2^{12}\mathbf{Z}_2\langle u/64, 64/u\rangle,$$

proving the required congruence.

Corollary 4.19. $(f(u) - uV(f(u)))/E^3B^2E^*_{\kappa'} \equiv 1/j_2 - (64 + 33t)/j_2^2 + 24u/j_2^3$ mod $u^2\mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]]$.

The preceeding Proposition computes $F(u) = f(u)/E^3B^2E^*_{\kappa'}$ modulo $u^2\mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]]$. Thus to prove the Corollary, we have to compute $uV(f(u))/E^3B^2E^*_{\kappa'}$ modulo $u^2\mathbf{Z}_2[[u]]\langle 64/u\rangle$, or equivalently, compute $V(f(u))/E^3B^2E^*_{\kappa'}$ modulo $u\mathbf{Z}_2[[u]]\langle 64/u\rangle$. Now

$$V(f(u))/E^3B^2E^*_{\kappa} = V(f(u)/E^3B^2E^*_{\kappa'})V(E^3B^2E_{\kappa'})/E^3B^2E^*_{\kappa'}$$

$$= V(F(u))V(E^3B^2E^*_{\kappa'})/E^3B^2E^*_{\kappa'}.$$

From the Proposition we see that

$$F(u) \equiv 1/j_2 + u(1+64/u-33t/u)/j_2^2 \equiv 1/j_2 \mod u\mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]].$$

Part *(i)* of Lemma 3.1 shows that $V(1/j_2) \in 1/j_2^2 + (16/j_2^3)\mathbf{Z}[[16/j_2]]$. A consideration of q-expansions allows us to compute that

$$V(1/j_2) = 1/j_2^2 - 48/j_2^3 + \cdots,$$

and thus

$$V(1/j_2) \equiv 1/j_2^2 - 48/j_2^3 \mod 64\mathbf{Z}_2[[q]] \subset u\mathbf{Z}_2[[u]]\langle 64/u\rangle.$$

Thus

$$V(F(u)) \equiv V(1/j_2) \equiv 1/j_2^2 - 48/j_2^3 \mod u\mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]].$$

Now

$$V(E^*_{\kappa'})/E^*_{\kappa'} \equiv 1 \mod t\mathbf{Z}_2[[t]][[q]] \subset u\mathbf{Z}_2[[u]]\langle 64/u\rangle,$$

while

$$V(E^3B^2)/E^3B^2 \equiv 1 - 40/j_2 \mod 64\mathbf{Z}_2[[q]] \subset u\mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]].$$

Thus

$$\begin{aligned} V(f(u))/E^3B^2E^*_\kappa &\equiv (1/j_2^2 - 48/j_2^3)(1-40/j_2) \equiv 1/j_2^2 - 24/j_2^3 + 64(-1/j_2^3 + 30/j_2^4) \\ &\equiv 1/j_2^2 - 24/j_2^3 \mod u\mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]]. \end{aligned}$$

(For the final congruence, observe that $64 = u(64/u) \in u\mathbf{Z}_2[[u]]\langle 64/u\rangle$.) Now we compute that

$$\begin{aligned} (f(u) - uV(f(u)))/E^3B^2E^*_{\kappa'} &\equiv 1/j_2 + (u - 64 - 33t)/j_2^2 - u(1/j_2^2 - 24/j_2^3) \\ &= 1/j_2 - (64+33t)/j_2^2 + 24u/j_2^3 \mod \mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]]. \end{aligned}$$

This proves the Corollary.

Lemma 4.20. $2/\zeta^*(\kappa) \equiv -24/(1-2^{13}) \mod t\mathbf{Z}_2[[t]]$.

Since $2/\zeta^*(\kappa) \in \mathbf{Z}_2[[t]]$, and $t(\kappa_{14}) = 0$, we see that

$$2/\zeta^*(\kappa) - 2/\zeta^*(\kappa_{14}) \in t\mathbf{Z}_2[[t]].$$

Now a computation gives

$$2/\zeta^*(\kappa_{14}) = -24/(1-2^{13}),$$

and the Lemma is proved.

We now make some definitions:

$$g(u) := (\zeta^*(\kappa)/2)(E^*_{\kappa(u)} - V(E^*_{\kappa(u)})) = \sum_{n \text{ odd}} \Big(\sum_{d|n} \kappa(u)(d)d^{-1}\Big)\, q^n,$$

and

$$G(u) = g_\kappa/E^3B^2E^*_{\kappa'}.$$

Proposition 4.21. $G(u) \equiv 1/j_2 - (64+33t)/j_2^2 + (256\cdot 5+16t)/j_2^3 \mod u^2\mathbf{Z}_2[[u]]\langle 64/u\rangle[[q]]$.

By the same reasoning as in the proof of Proposition 4.18, it suffices to show that

$$G(u) \equiv 1/j_2 - (64+33t)/j_2^2 + (256\cdot 5+16t)/j_2^3 \mod 2^{12}\mathbf{Z}_2\langle u/64, 64/u\rangle[[q]].$$

Now Lemma 3.20 shows that

$$E^*_\kappa/E^3B^2E^*_{\kappa'} \in 1 + (64/j_2)\mathbf{Z}_2\langle u/64, 64/u\rangle[[64/j_2]],$$

and so by part *(i)* of Lemma 3.1

$$V(E^*_\kappa/E^3B^2E^*_{\kappa'}) \in 1 + (8/j_2)\mathbf{Z}_2\langle u/64, 64/u\rangle[[8/j_2]].$$

By part *(i)* of Lemma 3.14 and Proposition 3.15 we see that

$$E^3B^2E^*_{\kappa'}/V(E^3B^2E^*_{\kappa'}) \in 1 + 8/j_2\mathbf{Z}_2\langle u/64, 64/u\rangle[[8/j_2]],$$

and so

$$V(E^*_\kappa)/E^3B^2E^*_{\kappa'} = V(E^*_\kappa/E^3B^2E^*_{\kappa'})V(E^3B^2E^*_{\kappa'})/E^3B^2E^*_{\kappa'} \in 1+8/j_2\mathbf{Z}_2\langle u/64, 64/u\rangle[[8/j_2]].$$

Thus

$$(2/\zeta^*(\kappa))G(u) = (E^*_\kappa - V(E^*_\kappa))/E^3B^2E^*_{\kappa'} \in 8/j_2\mathbf{Z}_2\langle u/64, 64/u\rangle[[8/j_2]].$$

Dividing through by $2/\zeta^*(\kappa)$ we find (using Lemma 4.20) that

$$G(u) \in 1/j_2\mathbf{Z}_2\langle u/64, 64/u\rangle[[8/j_2]].$$

Thus to prove the Proposition, it suffices to explicitly compute the first four terms of $G(u)$ as a power series in $1/j_2$.

For the duration of the proof we will adopt the notation of section 4.2, so that $a = 2/\zeta^*(\kappa)$ and $b = (\kappa'(3)-1)/3 - 2/\zeta^*(\kappa')$. Note that

$$1+\kappa(3)3^{-1} = 1 + 3^{13} + 3^{14}((\kappa'(3)-1)/3) = 1 + 3^{13} + 3^{14}(a+b).$$

We begin by computing the first four terms of $G(u)$ as a power series in q:

$$\begin{aligned} G(u) &= g(u)/E^3B^2E^*_{\kappa'} \\ &= \frac{q + (1+3^{13}+a+b)q^3 + \cdots}{1 + (40+a)q + (-24+41a)q^2 + (-9952+52a/3+a^2+ab)q^3 + \cdots} \\ &= q - (40+a)q^2 + (1595948 + 40a + a^2 + b)q^3 - \\ &\quad -(63828928 + 4787848a/3 + 40a^2 + a^3 + 40b + 2ab)q^4 + \cdots. \end{aligned}$$

We now substitute

$$q = 1/j_2 - 24/j_2^2 + 852/j_2^3 - 35744/j_2^4 + \cdots,$$

to find that

$$\begin{aligned} G(u) &= 1/j_2 - (64+a)/j_2^2 + (6245\cdot 256 + 88a + a^2 + b)/j_2^3 - \\ &\quad -(10917\cdot 2^{14} + 18763\cdot 256a/3 + 112a^2 + a^3 + 112b + 2ab)/j_2^4 + \cdots. \end{aligned}$$

Thus Lemma 4.2, combined with the observation that $t = 64(t/64) \in 64\mathbf{Z}_2\langle u/64, 64/u\rangle$, shows that

$$G(u) \equiv 1/j_2 - (64+33t)/j_2^2 + (5\cdot 256 + 24t)/j_2^3 + \cdots \mod 2^{12}\mathbf{Z}_2\langle u/64, 64/u\rangle[[q]],$$

proving the Proposition.

Corollary 4.22. $f(u) - uV(f(u)) - g(u) \equiv 4u(6 - 6t - 5 \cdot 64/u)/j_2^3$ mod $(32u, u^2)\mathbf{Z}_2[[u]]\langle 64/u \rangle$.

From Corollary 4.19 and Proposition 4.21 we see that

$$(f(u) - uV(f(u)) - g(u))/E^3B^2E^*_{\kappa'} = (24u - 24t - 256 \cdot 5)/j_2^3$$

$$= 4u(6 - 6t/u - 5 \cdot 64/u)/j_2^3 \mod u^2\mathbf{Z}_2[[u]]\langle 64/u \rangle[[q]].$$

We observed in the proof of part *(v)* in the previous section that

$$E^*_\kappa \equiv 1 \mod u\mathbf{Z}_2[[u]]\langle 64/u \rangle[[q]],$$

while

$$E^3B^2 \in 1 + 8\mathbf{Z}_2[[q]].$$

Thus

$$E^3B^2E^*_{\kappa'} \equiv 1 \mod (u, 8)\mathbf{Z}_2[[u]]\langle 64/u \rangle[[q]],$$

and so

$$f(u) - uV(f(u)) - g(u) \equiv 4u(6 - 4t/u - 5 \cdot 64/u)/j_2^3 \mod (32u, u^2)\mathbf{Z}_2[[u]]\langle 64/u \rangle[[q]],$$

proving the Corollary.

The following Proposition will be proved in the next section:

Proposition 4.23. *Write* $1/j_2^3 = \sum_{n=3}^\infty b_n q^n$. *Then the coefficients* b_n *satisfy the following congruences:*

$$b_n \equiv 0 \begin{cases} \mod 8 & n \equiv 0, 1, 2, 4, 6 \mod 8 \\ \mod 4 & n \equiv 5 \mod 8 \\ \mod 2 & n \equiv 7 \mod 8 \end{cases}.$$

Corollary 4.24. *If l is an odd prime, then*

$$a_l(u) - (1 + \kappa(u)(l)l^{-1}) \in \mathbf{Z}_2[[u]] + 2^{8+\nu}\mathbf{Z}_2\langle 64/u \rangle,$$

where

$$\nu = \begin{cases} 3, & l \equiv 1 \mod 8 \\ 0, & l \equiv 3 \mod 8 \\ 2, & l \equiv 5 \mod 8 \\ 1, & l \equiv 7 \mod 8. \end{cases}$$

Note that

$$f(u) - uV(f(u)) - g(u) = \sum_{n \text{ odd}} (a_n(u) - \sum_{d|n} \kappa(u)(d)d^{-1})\, q^n.$$

Thus Corollary 4.22 and Proposition 4.23 taken together show that for odd primes l,

$$a_l(u) - (1 + \kappa(u)(l)l^{-1}) \equiv 0 \mod (2^{2+\nu}u, u^2)\mathbf{Z}_2[[u]]\langle 64/u \rangle,$$

where

$$\nu = \begin{cases} 3, & l \equiv 1 \mod 8 \\ 0, & l \equiv 3 \mod 8 \\ 2, & l \equiv 5 \mod 8 \\ 1, & l \equiv 7 \mod 8. \end{cases}$$

The Corollary follows by noting that

$$(2^{2+\nu}u, u^2)\mathbf{Z}_2[[u]]\langle 64/u\rangle \subset \mathbf{Z}_2[[u]] + 2^{8+\nu}\mathbf{Z}_2\langle 64/u\rangle, \tag{4.11}$$

Lemma 4.25. *(i)* $a_l(2) - (1+\kappa(2)(l)l^{-1}) = 0$, *for odd primes* l *such that* $l \equiv 1,7 \mod 8$.
(ii) $a_l(4) - (1+\kappa(4)(l)l^{-1}) = 0$, *for odd primes* l *such that* $l \equiv 1,5 \mod 8$.
(iii) $a_l(8) - (1+\kappa(8)(l)l^{-1}) = 0$ *for all odd primes* l.
(iv) $a_l(32) - (1+\kappa(32)(l)l^{-1}) = 0$ *for all odd primes* l.

This follows directly from Proposition 4.17.

Part *(vii)* of the Main Theorem follows from Corollary 4.24, Lemma 4.25, and the following result:

Proposition 4.26. *Suppose that* $a_1, \ldots, a_m$ *are integers such that* $0 < a_i \leq 6$ *for each* i*, and that*

$$p(u) \in \mathbf{Z}_2[[u]] + 2^b\mathbf{Z}_2\langle 64/u\rangle$$

vanishes at 2^{a_i}*, for* $i = 1, \ldots, m$*. Then*

$$p(u)/(u-2^{a_1})\cdots(u-2^{a_m}) \in \mathbf{Z}_2[[u]] + 2^{b-a_1-\cdots-a_m}\mathbf{Z}_2\langle 64/u\rangle.$$

We may assume that $m = 1$, because the general case will then follow by induction. Write $a = a_1$. Write

$$p(u) = p_1(u) + 2^b p_2(64/u),$$

with $p_1(u) \in \mathbf{Z}_2[[u]]$ and $p_2(u) \in \mathbf{Z}_2\langle 64/u\rangle$. Then we may write

$$p_1(u) = (u-2^a)q_1(u) + p_1(2^a),$$

with $q_1(u) \in \mathbf{Z}_2[[u]]$, and

$$p_2(u) = ((64/u) - 2^{6-a})q_2(64/u) + p_2(64/2^a),$$

with $q_2(64/u) \in \mathbf{Z}_2\langle 64/u\rangle$. Now

$$p_1(2^a) + 2^b p_2(64/2^a) = p(2^a) = 0$$

by assumption. Thus

$$p(u) = (u-2^a)(q_1(u) - 2^{b-a}(64/u)q_2(64/u)) \in (u-2^a)(\mathbf{Z}_2[[u]] + 2^{b-a}\mathbf{Z}_2\langle 64/u\rangle).$$

This proves the Proposition.

4.9 A congruence for $1/j_2^3$

In this section we prove Proposition 4.23:

Proposition 4.23. *Write $1/j_2^3 = \sum_{n=3}^{\infty} b_n q^n$. Then the coefficients b_n satisfy the following congruences:*

$$b_n \equiv 0 \begin{cases} \bmod 8 & n \equiv 0,1,2,4,6 \bmod 8 \\ \bmod 4 & n \equiv 5 \bmod 8 \\ \bmod 2 & n \equiv 7 \bmod 8 \end{cases}.$$

Recall that

$$1/j_2 = q \prod_{n=1}^{\infty} (1+q^n)^{24},$$

and so

$$1/j_2^3 = q^3 \prod_{n=1}^{\infty} (1+q^n)^{72} = q^3 \prod_{n=1}^{\infty} (1+q^n)^{8} \prod_{n=1}^{\infty} (1+q^n)^{64}.$$

Now

$$(1+q^n)^{16} \equiv (1+q^{16n}) \mod 2,$$

and so

$$(1+q^n)^{64} = (1+q^n)^{16\cdot 4} \equiv (1+q^{16n})^4 \mod 8.$$

Thus the product $\prod_{n=1}^{\infty}(1+q^n)^{64}$ is congruent to a power series in the variable q^{16} modulo eight, and so it suffices to prove the claim of the proposition for the coefficients of the product $q^3 \prod_{n=1}^{\infty}(1+q^n)^8$.

Since

$$(1+q^n)^2 \equiv 1+q^{2n} \mod 2,$$

we see that

$$(1+q^n)^8 = (1+q^n)^{2\cdot 4} \equiv (1+q^{2n})^4 = 1 + 4q^{2n} + 6q^{4n} + 4q^{6n} + q^{8n} \mod 8.$$

Thus

$$q^3 \prod_{n=1}^{\infty} (1+q^n)^8 \equiv q^3 \prod_{n=1}^{\infty} (1 + 4q^{2n} + 6q^{4n} + 4q^{6n} + q^{8n}) \mod 8.$$

From this congruence we immediately infer the claim of the Proposition for all n except those such that $n \equiv 1 \mod 8$.

This remaining case is dealt with by the following Lemma:

Lemma 4.27. *If $n \equiv 3 \mod 4$, then the coefficient of q^n in the infinite product $\prod_{n=1}^{\infty}(1+q^n)^4$ is divisible by eight.*

Note that this Lemma does indeed complete the proof of the Proposition: upon substituting q^2 for q and then multiplying by q^3, the Lemma shows that for $n \equiv 1 \mod 8$, the coefficient of q^n in the product $q^3 \prod_{n=1}^{\infty}(1+q^{2n})^4$ is divisible by eight, and this is precisely the result we required to complete the proof of the Proposition.

To prove Lemma 4.27, first note that $(1+q^n)^4 \equiv (1-q^n)^4 \mod 8$. Using this congruence, we write

$$\prod_{n=1}^{\infty}(1+q^n)^4 \equiv \prod_{n \text{ odd}} (1+q^n)^4 \prod_{n \text{ even}} (1-q^n)^4 = (\prod_{n \text{ odd}} (1+q^n)^2 \prod_{n \text{ even}} (1-q^n))^2 \prod_{n=1}^{\infty} (1-q^{2n})^2.$$

Now let $\Theta(q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$. Then a well-known result of Jacobi shows that

$$\Theta(q) = \prod_{n \text{ odd}} (1+q^n)^2 \prod_{n \text{ even}} (1-q^n).$$

Thus we see that

$$\prod_{n=1}^{\infty}(1+q^n)^4 \equiv \Theta(q)^2 \prod_{n=1}^{\infty}(1-q^{2n})^2 \mod 8.$$

Now $(1-q^{2n})^2 \equiv (1+q^{2n})^2 \mod 4$, and so

$$\prod_{n=1}^{\infty}(1-q^{2n})^2 \equiv \prod_{n \text{ odd}}(1-q^{2n})^2 \prod_{n \text{ even}}(1+q^{2n}) \prod_{n=1}^{\infty}(1-q^{4n}) = \Theta(q^2)\prod_{n=1}^{\infty}(1+q^{4n}) \mod 4.$$

Thus

$$\prod_{n=1}^{\infty}(1-q^{2n})^2 = \Theta(q^2)\prod_{n=1}^{\infty}(1+q^{4n}) + 4f(q^2)$$

for some $f(q^2) \in \mathbf{Z}_2[[q^2]]$. Substituting this into the congruence of the preceding paragraph, we find that

$$\prod_{n=1}^{\infty}(1+q^n)^4 \equiv \Theta(q)^2\Theta(q^2)\prod_{n=1}^{\infty}(1+q^{4n}) + 4\Theta(q)^2 f(q^2)$$

$$\equiv \Theta(q)^2\Theta(q^2)\prod_{n=1}^{\infty}(1+q^{4n}) + 4f(q^2) \mod 8$$

(because $\Theta(q) \equiv 1 \mod 2$).

Thus Lemma 4.27 follows from the following result:

Lemma 4.28. *If $n \equiv 3 \mod 4$, then the coefficient of q^n in the power series $\Theta(q)^2\Theta(q^2)$ is divisible by eight.*

For any integer n, let r_n denote the cardinality of the set

$$S(n) = \{(a,b,c) \in \mathbf{Z}^3 | a^2 + b^2 + 2c^2 = n\}.$$

Then

$$\Theta(q)^2\Theta(q^2) = \sum_{n=0}^{\infty} r_n q^n.$$

We must show that if $n \equiv 3 \mod 4$ then $r_n \equiv 0 \mod 8$.

If $n \equiv 3 \mod 8$, and if $a^2 + b^2 + 2c^2 = n$, then certainly c is non-zero, and also at least one of a or b is non-zero. Thus we may decompose $S(n)$ as a disjoint union of sets

$$S(n)_1 = \{(a,b,c) \in S(n) | a = 0,\, b, c \neq 0\},$$

$$S(n)_2 = \{(a,b,c) \in S(n) | b = 0,\, a, c \neq 0\},$$

and

$$S(n)_3 = \{(a,b,c) \in S(n) | a, b, c \neq 0\}.$$

Then clearly $S(n)_1$ has cardinality divisible by four (consider the transformations $(0,b,c) \mapsto (0,\pm b,\pm c)$), $S(n)_2$ has the same cardinality as $S(n)_1$ (consider the transformation $(a,0,c) \mapsto (0,a,c)$), and $S(n)_3$ has cardinality divisible by eight (consider the transformations $(a,b,c) \mapsto (\pm a,\pm b,\pm c)$). Thus

$$r_n = \#S(n) = \#S(n)_1 + \#S(n)_2 + \#S_3(n) = 2\#S(n)_1 + \#S(n)_3$$

has cardinality divisible by eight, and the Lemma is proved.

4.10 Proof of part *(vii)* of the Main Theorem

Let us note first that part *(vii)* of the Theorem follows from the main result of [3] and [4], with the exception of the cases $\chi = 1$ and $k = 3$, 4, or 5, and $k = 2$ and χ of conductor eight, for which the corresponding value of u has slope $k-1$. However, these are precisely the cases which are covered by Proposition 4.17.

However, as we remarked in our introduction, we can find a different proof of this result in our situation. To illustrate, consider the case of weight κ_k, with $k \geq 14$. Then we see that $\kappa' = \kappa_{k-14}$, and so $E^3B^2E^*_{\kappa'}/j_2 = ABE^3E_{\kappa_{k-14}}$ and $E^3B^2E^*_{\kappa'}/j_2^2 = A^2E^3E^*_{\kappa_{k-14}}$ are classical modular forms on either $\Gamma_0(2)$ or $\Gamma_1(4)$, depending on the parity of k. We have seen in sections 4.3, 4.4 and 4.5 that the minimal slope eigenform of weight k with eigenvalue u is obtained (up to a scalar multiple) by acting polynomial combinations of the operator U/u on either $E^3B^2E^*_{\kappa'}/j_2$ or $E^3B^2E^*_{\kappa'}/j_2^2$, and then taking a limit. Now the theory of [2] shows that applying U/u to classical modular forms yields classical modular forms. Thus the eigenform $f(u)$, being a limit of classical modular forms of fixed weight, must be classical.

To treat those κ_k for $3 \leq k \leq 13$, which are excluded in the above discussion, is a finite check.

First note that if k is in this range, then $|t(\kappa_k)| = 4|k-14| > |64|$, so by part *(i)* there is a unique point $u \in \mathcal{A}(\mathbf{C}_2)$ lying over κ_k.

In the examples of chapter 1, for each even weight k in this range we present a normalized classical eigenform of weight k, for k even and $4 \leq k \leq 12$, whose U-eigenvalue has the same slope as u. It follows from Proposition 3.24 that this classical form is in $\mathcal{M}_\kappa$, and thus equals $f(u)$, by virtue of the characterization of u and $f(u)$ provided by part *(iv)*.

The case of odd weights k with $3 \leq k \leq 13$ can be checked similarly. We won't do this here.

The same argument as made above treats the weights $\kappa_{k,\chi}$ for non-trivial χ, provided that $k \geq 15$ (since in this case, $E^*_{\kappa_{\chi,k-14}}$ is a classical modular form). To treat the cases $14 \geq k \geq 1$ and χ non-trivial, one must modify the construction of the minimal slope family, by making a different choice of e: if κ is a 2-adic weight, (re)define κ' as $\kappa'(\gamma) = \gamma^{-1}\kappa(\gamma)$. Now define $e = \theta E^*_{\kappa'}$, so that

$$\tilde{U}(f) = U(f\theta E^*_{\kappa'}/V(\theta E^{\kappa'})).$$

Let t now denote the coordinate $t(\kappa) = 5^{-1}\kappa(5) - 1 = \kappa'(5) - 1$ on $\mathcal{W}$. Let $\mathcal{X}$ denote the annulus $1 > |s| > |\sqrt{2}|$, defined over $\mathbf{Q}_2(\sqrt{2})$, and let

$$R = R(\mathcal{X}) = \mathbf{Z}_2[\sqrt{2}][[s, \sqrt{2}/s]].$$

We map $\mathcal{X}$ to $\mathcal{W}$ via $t = s^4$. The image of $\mathcal{X}$ is the sub-annulus $1 > |t| > |4|$ of $\mathcal{W}$. Recall that j_8 is the uniformizer of $X_0(8)$ constructed in section 2, and define

$$x = 1/(j_8 + 8).$$

Then $x \in 1/j_8 + (8/j_8^2)\mathbf{Z}_2[[8/j_8]] \subset 1/j_2 + (4/j_2^2)\mathbf{Z}_2[[4/j_2]]$. (The last equality follows by part *(iii)* of Lemma 2.4.)

Define

$$M = (1/j_2)R \oplus (t/j_2^2)R[[s^3/j_2]],$$

and for $n \geq 1$, define

$$M_n = xR \oplus (t/j_2^2)R \oplus (s^{3n-2}/j_2^n)R \subset M.$$

An argument similar to that which proves Proposition 3.19 shows that

$$\theta E^*_{\kappa'} M = E^*_\kappa M \subset M_\kappa.$$

Now part *(iii)* of Lemma 2.7 together with an analogue of Proposition 3.17 shows that

$$\theta E^*_{\kappa'}/V(\theta E^*_{\kappa'}) \in 1 + (t/j_2)R(\mathcal{X})[[s^3/j_2]].$$

Combined with Corollary 3.2, this shows that M is closed under the action of $\tilde{U}/t$, and that in fact

$$(\tilde{U}/t)(M) \subset M_n + s^{3n-1}M.$$

Thus we may apply Proposition 3.4 and compute the characteristic power series $C(T)$ of $\tilde{U}/t$ on M.

A (much simpler!) computation analogous those of Lemmas 4.1 and 4.2 shows that $\tilde{U}/t$ acts on M_1/s^2 via multiplication by $(1 + 4/t)$, and so

$$C(T) \equiv 1 - (1 + 4/t)T \mod s^2 R\langle T\rangle.$$

The element $1 + 4/t$ is a unit in R, and so by Weierstrass preparation we may lift this factorization $\mod s^2$ to a factorization

$$C(T) = (1 - u'T)C'(T),$$

with $u' \equiv 1 + 4/t \mod s^2 R$ and $C'(T) \in R\langle T\rangle$ satisfying $C'(T) \equiv 1 \mod s^2 R\langle T\rangle$. We now apply Proposition 3.6, taking m to be the basis element $x \in M_1$, to obtain an element $F' \in M$, whose coefficient of $1/j_2$ is equal to 1, and such that $\tilde{U}(F') = tu'F'$. Then

$$f'(\kappa) = \theta E^*_{\kappa'} F'$$

is a family of U-eigenvectors of minimal slope in M_κ, and so must be equal to the family $f(u)$ by part *(iv)* of the Main Theorem.

Thus we have a new construction of the minimal slope family over the annulus $1 > |t| > |4|$, which shows that for u lying over this annulus, $f(u)$ is obtained by acting U/u on the modular form $\theta E^*_{\kappa_{k-1,\chi}} x$. But this is a classical modular form provided $k \geq 1$: x has a pole at $\underline{1/2}$

(since $j_8(\underline{1/2}) = -8$ by Lemma 2.3), but this is cancelled by the zero of θ at $\underline{1/2}$! The same argument as above now shows that $f_{\kappa_{k,\chi}}$ is classical. This completes the proof of part *(vii)*.

As a final remark, let us observe that from part *(vii)* we may conclude that $f(u)$ is an eigenform for all the Hecke operators.

For *(vii)* shows that for infinitely many values of u, $f(u)$ is a classical modular form of weight $\kappa(u)$ on $\Gamma_1(4)$. Such a classical modular form on $\Gamma_1(4)$ is contained in $\mathcal{M}_{\kappa(u)}$, by Proposition 3.24. Since $f(u)$ spans the u-eigenspace for U in $\mathcal{M}_{\kappa(u)}$, in particular $f(u)$ must span the u-eigenspace for U in the space of classical modular forms of weight $\kappa(u)$. Since the Hecke operators commute, this eigenspace is preserved by the Hecke operators, and thus for these values of u, $f(u)$ is an eigenform for all the Hecke operators.

So for any prime $l \neq 2$, $T_l(f(u))$ and $a_l(u)f(u)$ are two rigid analytic q-expansions which agree for infinitely many values of u, and thus they agree for all values of u. This gives a proof that indeed $f(u)$ is an eigenform for the full Hecke algebra, and completes the proof of the Main Theorem.