CHAPTER III

OVERCONVERGENT MODULAR FORMS

In this chapter we set up the foundations on which our proof of the Main Theorem will rely. In section 1 we study the operators U and V acting on various subspaces of $R[[q]]$ (R a commutative ring). In section 2 we develop a theory of characteristic power series for endomorphisms of a module over a complete ring, and in section 3 we apply this theory to certain subspaces of $R[[q]]$ for rings R of rigid analytic functions. In section 4 we define the space of 2-adic weights and the Eisenstein family, which is paramaterized by this space. In section 5 we define the operator $\tilde{U}$ on $\mathbf{Z}_2[[t]][[q]]$ and obtain some key estimates for this operator. The detailed study of this operator is the basis of the proof of the Main Theorem. In the final section, we define the spaces of overconvergent 2-adic modular forms of arbitrary weight, and define the characteristic power series $P(\kappa, T)$ of the operator U on these spaces.

3.1 The action of U and V on q-expansions

If f is a classical modular form or modular function whose q-expansion coefficients lie in R for some subring $R \subset \mathbf{C}$, then we identify f with its q-expansion in $R[[q]]$. In particular, we regard $1/j$, $1/j_2$, $1/j_4$ and $1/j_8$ as elements of $\mathbf{Z}[[q]]$, and so as elements of $R[[q]]$ for any ring R, via the natural morphism $\mathbf{Z}[[q]] \to R[[q]]$. If we let f denote one of these modular functions, then since $f \in q + q^2\mathbf{Z}[[q]]$, we see that $R[[q]] = R[[f]]$. More generally, for any $r \in R$, we see that $R[[rf]] \subset R[[q]]$. Throughout this chapter we will be working with such rings, for various choices of R, f and r.

For any ring R, we let $V : R[[q]] \to R[[q]]$ denote the endomorphism induced by the substitution $q \mapsto q^2$. Then V induces an isomorphism of $R[[q]]$ with the subring $R[[q^2]]$ of $R[[q]]$. Since $R[[q]]$ is free of rank 2 over $R[[q^2]]$, we may define a map $\operatorname{trace} V : R[[q]] \to R[[q]]$ as the composition of the inverse of V with the trace from $R[[q]]$ to $R[[q^2]]$. Explicitly,

$$\operatorname{trace} V(\sum_{n=0}^{\infty} r_n q^n) = 2\sum_{n=0}^{\infty} r_{2n} q^n.$$

We define $U : R[[q]] \to R[[q]]$ via the formula

$$U(\sum_{n=0}^{\infty} r_n q^n) = \sum_{n=0}^{\infty} r_{2n} q^n.$$

Thus $2U = \operatorname{trace} V$. If 2 is invertible in R, then we may also express this as $U = (1/2)\operatorname{trace} V$. For any $f, g \in R[[q]]$, we have the basic formula

$$U(fV(g)) = U(f)g.$$

Lemma 3.1. *(i) If r divides 16 in R then $V(R[[r^2/j_2]]) \subset R[[r/j_2]]$, and the right side is free of rank two over the left side. Thus*

$$\operatorname{trace} V(R[[r/j_2]]) \subset R[[r^2/j_2]].$$

(ii) If r divides 8 in R then

$$U(\mathcal{O}_2[[r/j_2]]) \subset \mathcal{O}_2[[r^2/j_2]]$$

Write $16 = rr'$. Then part *(ii)* of Lemma 2.5 shows that

$$V(r^2/j_2) = (r/j_4)^2(1 + \sum_{n=2} r_n(rr'/j_4)^{n-1}) \in R[[r/j_4]].$$

Part *(ii)* of Lemma 2.4 shows that

$$r/j_4 \in r/j_2 + r^2r'/j_2^2\mathbf{Z}_2[[rr'/j_2]] \subset r/j_2 + (r/j_2^2)R[[r/j_2]].$$

Thus

$$V(r^2/j_2) \in (r/j_2)^2 + (r/j_2)^3R[[r/j_2]].$$

This shows that $V(R[[r^2/j_2]]) \subset R[[r/j_2]]$, and that in fact

$$R[[r/j_2]] = R[[V(r^2/j_2)]] \oplus (r/j_2)R[[V(r^2/j_2]],$$

proving part *(i)*.

Part *(ii)* of this Lemma is in the spirit of [11, Lemma 3.11.7]. To prove it, first observe that since $R[[r/j_2]] = R[[V(r^2/j_2)]] \oplus (r/j_2)R[[V(r^2/j_2]]$, and since

$$U(fV(g)) = U(f)g$$

for any $f, g \in R[[q]]$, it is enough to show that $U(r/j_2) \in R[[r^2/j_2]]$.

If we replace r by $2r$ in the statement of part *(i)*, we see that

$$(\operatorname{trace} V)(2r/j_2) \in R[[4r^2/j_2]].$$

Thus

$$(\operatorname{trace} V)(2r/j_2) = \sum r_n(4r^2/j_2)^n$$

for some $r_n \in R$. Since $2r/j_2$ has no constant term (as a power series in q), the same is true of $(\operatorname{trace} V)(2r/j_2)$, and so $r_0 = 0$. Thus

$$U(r/j_2) = (1/4)\operatorname{trace} V(2r/j_2) = (r^2/j_2)(\sum_{n=1}^{\infty} r_n(4r^2/j_2)^{n-1}) \in R[[r^2/j_2]].$$

Part *(ii)* follows. (Note that in this argument we assumed that we could divide by 2, but this is fine, because it is clearly enough to work in the universal case, when $R = \mathbf{Z}[r, r']/(rr' - 8)$ is an integral domain.)

Corollary 3.2. *Let n be a positive integer and define m so that $n = 2m - p$ with $0 \le p \le 1$. If $r \in R$ divides 8, then $U((1/j_2)^n R[[r/j_2]]) \subset r^p (1/j_2)^m R[[r^2/j_2]]$.*

A consideration of q-expansions shows that

$$U((1/j_2)^n R[[1/j_2]]) = U(q^n R[[q]]) \subset q^m R[[q]] = (1/j_2)^m R[[1/j_2]].$$

Comparing this with part *(ii)* of the Lemma shows that

$$U((r/j_2)^n R[[r/j_2]]) \subset (r^2/j_2)^m R[[r^2/j_2]].$$

Dividing by r (see the remark in the proof of the Lemma about working in the universal case), the Corollary is proved.

This Lemma and its Corollary will be extremely useful in the argument to come, but we will also need explicit formulae for the action of U on $R[[1/j_2]]$. We have chosen to work with $1/j_2$ rather than $1/j$ because Atkin-Lehner theory [2] shows that U preserves the field of modular functions on $X_0(2)$, so that $U(1/j_2^n)$ will be some rational function of $1/j_2$. In fact, since $1/j_2^n$ has no poles except at the cusp $\underline{0}$, the same is true of $U(1/j_2^n)$, and so $U(1/j_2^n)$ will be a polynomial in $1/j_2$.

The proof of Lemma 3.1 shows that

$$\mathbf{Z}[[1/j_2]] = \mathbf{Z}[[1/V(j_2)]] \oplus 1/j_2\mathbf{Z}[[1/V(j_2)]].$$

In order to compute $U(1/j_2^n)$ for all n, we must find the minimal polynomial of $1/j_2$ over $\mathbf{Z}[[1/V(j_2)]]$. Concretely, we have to write $1/j_2^2$ as a $\mathbf{Z}[[1/V(j_2)]]$-linear combination of 1 and $1/j_2$.

To do this, it is convenient to work in terms of the uniformizer j_4 of $X_0(4)$, since both $1/j_2$ and $V(1/j_2)$ are functions on $X_0(4)$.

Now recall the equations that we have already derived: from Lemma 2.2,

$$1/j_2 = (j_4 + 16)/j_4^2 = 1/j_4 + 16/j_4^2,$$

and thus

$$1/j_2^2 = (j_4 + 16)^2/j_4^4 = 1/j_4^2 + 32/j_4^3 + 2^8/j_4^4.$$

Also from part *(ii)* of Lemma 2.5,

$$V(1/j_2) = (1/j_4^2)(1/(1 + 16/j_4)) = 1/j_4^2 - 16/j_4^3 + 2^8/j_4^4 - 2^{12}/j_4^5 + \cdots .$$

We have to write $1/j_2^2$ as a $\mathbf{Z}[[1/V(j_2)]]$-linear combination of 1 and $1/j_2$. We do this by systematically subtracting $\mathbf{Z}$-multiples of $1/V(j_2^n)$ and $1/V(j_2^n)j_2$, for $n = 1, 2, 3, \ldots$. Now

$$\begin{aligned} 1/j_2^2 - 1/V(j_2) &= ((j_4 + 16)^3 - j_4^3)/j_4^4(j_4 + 16) \\ &= (48j_4^2 + 3 \cdot 2^8 j_4 + 2^{12})/j_4^4(j_4 + 16) = 48/j_4^3 + 2^{12} j_4^5 + \cdots . \end{aligned}$$

Also, by Lemma 2.2,

$$1/V(j_2)j_2 = (1/j_4 + 16j_4^2)/(j_4^2/(j_4 + 16)) = 1/j_4^3,$$

and therefore

$$\begin{aligned} 1/j_2^2 - 1/V(j_2) - 48/V(j_2)j_2 &= (48j_4^2 + 3 \cdot 2^8 j_4 + 2^{12} - 48j_4(j_4 + 16))/j_4^4(j_4 + 16) \\ &= 2^{12}/j_4^4(j_4 + 16) = 1/V(j_2^2)j_2. \end{aligned}$$

Thus we have proved the following Lemma:

Lemma 3.3. *(i)* $1/j_2^2 - (48/V(j_2) + 2^{12}/V(j_2^2))1/j_2 - V(1/j_2) = 0$ *is the minimal polynomial of* $1/j_2$ *over* $\mathbf{Z}[[1/V(j_2)]]$.

(ii) $U(1/j_2) = (1/2\,\mathrm{trace}\,V)(1/j_2) = 24/j_2 + 2^{11}/j_2^2$.

(iii) The polynomials $U(1/j_2^n) \in \mathbf{Z}[[1/j_2]]$ *satisfy the recursion*

$$U(1/j_2^{n+2}) - (48/j_2 + 2^{12}/j_2^2)U(1/j_2^{n+1}) - (1/j_2)U(1/j_2^n) = 0.$$

We have just shown *(i)*, and *(ii)* follows from the relation $U = (1/2)\,\mathrm{trace}\,V$. Part *(iii)* follows in the usual way: let us formally factor the polynomial

$$X^2 - (48/j_2 + 2^{12}/j_2^2)X - 1/j_2 = (X - \alpha_1)(X - \alpha_2).$$

Then

$$U(1/j_2^n) = (\alpha_1^n + \alpha_2^n)/2,$$

and so the $U(1/j_2^n)$ satisfy the recursion

$$U(1/j_2^{n+2}) - (48/j_2 + 2^{12}/j_2^2)U(1/j_2^{n+1}) - (1/j_2)U(1/j_2^n) = 0$$

as claimed.

Beginning with the values $U(1) = 1$ and $U(1/j_2) = 24/j_2 + 2^{11}/j_2^2$, we compute for example that

$$U(1/j_2^2) = 1/j_2 + 9 \cdot 2^7/j_2^2 + 3 \cdot 2^{16}/j_2^3 + 2^{23}/j_2^4,$$

$$U(1/j_2^3) = 72/j_2^2 + 15 \cdot 2^{12}/j_2^3 + 27 \cdot 2^{19}/j_2^4 + 9 \cdot 2^{27}/j_2^5 + 2^{35}/j_2^6,$$

$$U(1/j_2^4) = 1/j_2^2 + 9 \cdot 2^9/j_2^3 + 105 \cdot 2^{15}/j_2^4 + 7 \cdot 2^{27}/j_2^5 + 27 \cdot 2^{32}/j_2^6 + 3 \cdot 2^{41}/j_2^7 + 2^{47}/j_2^8,$$

$$U(1/j_2^5) = 15 \cdot /j_2^3 + 35 \cdot 2^{13}/j_2^4 + 189 \cdot 2^{20}/j_2^5 + 225 \cdot 2^{28}/j_2^6 + 275 \cdot 2^{35}/j_2^7 +$$

$$+ 45 \cdot 2^{44}/j_2^8 + 15 \cdot 2^{51}/j_2^9 + 2^{59}/j_2^{10},$$

$$U(1/j_2^6) = 1/j_2 + 81 \cdot 2^7/j_2^4 + 135 \cdot 2^{17}/j_2^5 + 693 \cdot 2^{24}/j_2^6 + 891 \cdot 2^{32}/j_2^7 +$$

$$+ 1287 \cdot 2^{39}/j_2^8 + 273 \cdot 2^{48}/j_2^9 + 135 \cdot 2^{55}/j_2^{10} + 9 \cdot 2^{64}/j_2^{11} + 2^{71}/j_2^{12}.$$

Apparently this Lemma was first proved by Kolberg, who also found closed form solutions to the recurrence. (See [13], cited in [1].)

3.2. Characteristic power series

In this section we work out a simple Riesz theory in a rather general setting. We will relate this to the Riesz theory of [5] in the situations that will interest us.

Proposition 3.4. *Let R be a ring complete with respect to an ideal I, and let M be an R-module. Let φ be an endomorphism of M over R, and suppose that M has an increasing sequence of R-submodules*

$$M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$$

such that

(i) each M_n is a free R-module of finite rank;

(ii) each M_n is a direct summand of M_{n+1};

(iii) for every n, $M_n \cap I^n M = I^n M_n$;

(iv) $\varphi(M) \subset M_n + I^n M$ for all n.

It follows from these assumptions that for each n, M_n/I^n is a free R/I^n-module of finite rank, and that φ induces an endomorphism of M_n/I^n, which we denote by $\varphi_{|M_n/I^n}$. Define

$$P_n(T) = \det(1 - T\varphi_{|M_n/I^n}).$$

Then

$$P(T) := \lim_{n\to\infty} P_n(T) \in R\langle T\rangle$$

exists, and is defined to be the characteristic power series of φ. (Here $R\langle T\rangle$ denotes that subring of $R[[T]]$ consisting of power series whose coefficients tend to zero I-adically.)

First of all, let us note that hypothesis *(iii)* shows that the natural map $M_n/I^n \to M/I^n$ is an injection, which we use to identify M_n/I^n with the image of M_n in M/I^n. The map φ induces an endomorphism of M/I^n, which by hypothesis *(iv)* has image in M_n/I^n. By restricting to M_n/I^n, we obtain the map $\varphi_{|M_n/I^n}$.

Now to prove the Proposition, it suffices to show that for all n,

$$P_{n+1}(T) \equiv P_n(T) \mod I^n R[T].$$

Now $P_{n+1}(T) \in R/I^{n+1}[T]$ is defined to be $\det(1 - T\varphi_{|M_{n+1}/I^{n+1}})$. Thus $P_{n+1}(T) \mod I^n$ is equal to $\det(1 - T\varphi_{|M_{n+1}/I^n})$. But M_n/I^n is a direct summand of M_{n+1}/I^n (by hypothesis *(ii)*) and $\varphi(M_{n+1}/I^n) \subset M_n/I^n$ (by hypothesis *(iii)*). Basic properties of determinants now show that

$$\det(1 - T\varphi_{|M_{n+1}/I^n}) = \det(1 - T\varphi_{|M_n/I^n}) = P_n(T).$$

Thus indeed

$$P_{n+1}(T) \equiv P_n(T) \mod I^n R[T],$$

and the Proposition is proved.

Although we have defined $P(T)$ to be the characteristic power series of φ, we have not shown that it is independent of the choice of M_n that we have made. (And the generality of the situation considered in the Lemma gives no reason to think it should be independent of this choice.) Thus, whenever we apply the Proposition, we will always be clear as to the choice of the system M_n that is being made.

This being said, there are some circumstances when we will want to compare the characteristic power series computed with respect to different choices of M_n, or even of M. The following result will suffice for all the situations that we consider:

Proposition 3.5. *Assume that R is an integral domain complete with respect to an ideal I, and let K be the field of fractions of R. Let $M \subset \bar{M}$ be R-modules, and let φ be an endomorphism of $\bar{M}$ over R such that $\varphi(M) \subset M$. Suppose given an increasing sequence of sub-modules of $\bar{M}$,*

$$\bar{M}_1 \subset \bar{M}_2 \subset \bar{M}_n \subset \cdots ,$$

as well as an increasing sequence of sub-modules of M,

$$M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots ,$$

such that

(i) each $\bar{M}_n$ (respecitvely each M_n) is a free R-module of finite rank;

(ii) each $\bar{M}_n$ (respecitvely M_n) is a direct summand of $\bar{M}_{n+1}$ (respectively M_{n+1});

(iii) there is an integer k such that $M_n \subset \bar{M}_{n+k}$ for all n, and the induced maps $K \otimes_R M_n \to K \otimes_R \bar{M}_{n+k}$ are all isomorphisms;

(iv) $\varphi(\bar{M}) \subset \bar{M}_n + I^n\bar{M}$ for all n (respectively $\varphi(M) \subset M_n + I^n M$ for all n).

(v) there is a projection (i. e. idempotent endomorphism) Π_n on $\bar{M}$ with image $\bar{M}_n$, which thus realizes $\bar{M}_n$ as a direct summand of $\bar{M}$, with the additional property that $\Pi_{n+k}(M) = M_n$, so that Π_{n+k} also realizes M_n as a direct summand of M.

These hypotheses allow us apply Proposition 3.4 to compute the characteristic power series $\bar{P}(T)$ of φ on $\bar{M}$, using the sub-modules $\bar{M}_n$, as well as the characteristic power series $P(T)$ of φ on M, using the sub-modules M_n.

We have the equality $P(T) = \bar{P}(T)$.

To see that Proposition 3.4 applies to both the cases of $\bar{M}$ and M, note that hypotheses *(i)*, *(ii)* and *(iv)* simply state that the corresponding hypothesis of Proposition 3.4 is satisfied for both $\bar{M}$ and M, while hypothesis *(v)* implies that each $\bar{M}_n$ is a direct summand of $\bar{M}$, and that each M_n is a direct summand of M, and thus hypothesis *(iii)* of Proposition 3.4 also holds for both $\bar{M}$ and M. Thus we can indeed compute both $P(T)$ and $\bar{P}(T)$.

By definition

$$\bar{P}(T) = \lim_{n\to\infty} \det(1 - T\varphi_{|\bar{M}_n/I^n}),$$

while

$$P(T) = \lim_{n\to\infty} \det(1 - T\varphi_{|M_n/I^n}).$$

But hypotheses *(v)* shows that

$$\det(1 - T\varphi_{|\bar{M}_n/I^n}) \equiv \det(1 - T(\Pi_n \circ \varphi)_{|\bar{M}_n}) \mod I^n,$$

and that

$$\det(1 - T\varphi_{|M/I^n}) \equiv \det(1 - T(\Pi_{n+k} \circ \varphi)_{|M_n}) \mod I^n.$$

By hypothesis *(iv)*,

$$\det(1 - T(\Pi_{n+k} \circ \varphi)_{|M_n}) = \det(1 - T(\Pi_{n+k} \circ \varphi)_{|K\otimes_R M_n}) = \det(1 - T(\Pi_{n+k} \circ \varphi)_{|K\otimes_R \bar{M}_{n+k}}).$$

Thus

$$P(T) = \lim_{n\to\infty} \det(1 - T\varphi_{|M_n/I^n}) = \lim_{n\to\infty} \det(1 - T(\Pi_{n+k} \circ \varphi)_{|M_n})$$

$$= \lim_{n\to\infty} \det(1 - T(\Pi_{n+k}\circ\varphi)_{|K\otimes_R \bar{M}_{n+k}}) = \lim_{n\to\infty} \det(1 - T\varphi_{|\bar{M}_{n+k}/I^{n+k}}) = \bar{P}(T).$$

This proves the Proposition.

The following Proposition allows us to construct (under appropriate hypotheses) an eigenvector of an endomorphism using a factorization of its characteristic power series:

Proposition 3.6. *Assume that R, I, M, φ and $M_1 \subset M_2 \subset \cdots M_n \subset \cdots$ satisfy the hypotheses of Proposition 3.4, and assume in addition that M is I-adically complete.*

Let $P_n(T) \in R/I^n[T]$ and $P(T) \in R\langle T\rangle$ be as defined in the previous Proposition. Also, define d_n to be the rank of the free R-module M_n, and define

$$Q_n(T) = \det(T - \varphi_{|M_n/I^n}) \in R/I^n[T],$$

so that $Q_n(T) = T^{d_n}P_n(1/T)$.

Suppose that $P(T)$ factors as

$$P(T) = (1 - uT)P'(T),$$

with $u \in R^\times$ and $P'(T) \in R\langle T\rangle$. Write

$$P'_n(T) := P'(T) \mod I^n \in R/I^n[T],$$

so that $P_n(T) = (1-uT)P'_n(T)$. Then $Q_n(T) = (T-u)Q'_n(T)$, where $Q'_n(T) = T^{d_n-1}P'_n(1/T)$.

Finally, choose $m \in M_1$ and define $v_1 = Q'_1(\varphi)(m \mod I) \in M_1/I$. Note that $(\varphi - u)(v_1) = Q_1(\varphi)(m \mod I) \equiv 0 \mod I$.

Then there exists $v \in M$ such that $v \equiv v_1 \mod I$, and $Tv = uv$.

Define $v_n = u^{-d_n}Q'_n(\varphi)(m) \in M/I^n$. We will show that $\varphi(v_n) \equiv uv_n \mod I^n$, and that $v_{n+1} \equiv v_n \mod I^n$. Since M is I-adically complete, we may then define $v = \lim_{n\to\infty} v_n \in M$, and the Proposition will be proved.

By definition,

$$(\varphi - u)v_n = u^{-d_n}(\varphi - u)Q'_n(\varphi)(m) = u^{-d_n}Q_n(\varphi)(m) \equiv 0 \mod I^n,$$

since $m \in M_1 \subset M_n$ and $Q_n(T) = \det(T - \varphi_{|M_n/I^n})$. Thus $\varphi(v_n) \equiv uv_n \mod I^n$.

Now $Q'_{n+1}(T) \equiv T^{(d_{n+1}-d_n)}Q'_n(T) \mod I^n$, and so

$$v_{n+1} = u^{-d_{n+1}}Q'_{n+1}(\varphi)(m)$$

$$\equiv u^{-d_{n+1}}\varphi^{(d_{n+1}-d_n)}Q_n(\varphi)(m) \equiv u^{(d_n-d_{n+1})}\varphi^{(d_{n+1}-d_n)}v_n \mod I^n.$$

But we have just computed that $\varphi(v_n) \equiv uv_n \mod I^n$. Thus we see that indeed $v_{n+1} \equiv v_n \mod I^n$, and the Proposition is proved.

The final Proposition of this section is a kind of converse to the previous Proposition; it shows that (under appropriate hypotheses) the existence of an eigenvector for an endomorphism implies a factorization of its characteristic power series.

Proposition 3.7. *Suppose that R, I, M, φ and $M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$ satisfy the hypotheses of Proposition 3.4, so that we may compute the characteristic power series $P(T)$ of φ on M.*

Suppose in addition that there exists $m \in M$ and $r \in R$ such that $\varphi(m) = rm$ for $r \in R$, and such that for any sequence r_n of elements of R, if the sequence $r_n rm$ converges to zero I-adically in M, then the sequence r_n converges to zero I-adically in R. Then there exists $P'(T) \in R\langle T\rangle$ such that $P(T) = (1 - rT)P'(T)$.

Since $1 - rT$ is a unit in $R[[T]]$, we may certainly factor $P(T) = (1 - rT)P'(T)$ in the ring $R[[T]]$. We must show that $P'(T) \in R\langle T\rangle$.

As in the the statement and proof of Proposition 3.6, let us write $Q_n(T) = \det(T - \varphi_{|M_n/I^n}) \in R/I^n[T]$. We may use the division algorithm in the ring $R/I^n[T]$ to write $Q_n(T) = (T - r)Q_n'(T) + Q_n(r)$. Let d_n be the rank of M_n/I^n, and thus the degree of $Q_n(T)$. Now

$$P_n(T) = \det(1 - T\varphi_{|M_n/I^n}) = T^{d_n} Q_n(1/T) = (1 - rT)T^{d_n - 1}Q_n'(1/T) + Q_n(r)T^{d_n}.$$

From this equation we see that if we could show that $\lim_{n\to\infty} Q_n(r) = 0$, we would deduce that $P'(T) = \lim_{n\to\infty} T^{d_n-1}Q_n'(1/T)$ is the I-adic limit of polynomials, and so is an element of $R\langle T\rangle$. Thus we are reduced to showing that the first limit vanishes.

By assumption,

$$rm = \varphi(m) \in M_n + I^n M,$$

and the theory of characteristic polynomials shows that $Q_n(\varphi)$ annihilates M_n/I^n. Thus in particular

$$Q_n(\varphi)(rm) = Q_n(r)rm \equiv 0 \mod I^n.$$

Thus the sequence $Q_n(r)rm$ converges to zero I-adically in M, and so by assumption, the sequence $Q_n(r)$ conveges to zero I-adically in R, and we are done.

3.3. Applications to rings of modular functions

We will now describe the situation to which we will apply the above results. Let $\mathcal{X}$ be a reduced rigid analytic space over $\mathbf{Q}_2$, and let $\mathcal{R}(\mathcal{X})$ be the ring of bounded rigid analytic functions on $\mathcal{X}$. Let $R(\mathcal{X})$ denote the subring of $\mathcal{R}(\mathcal{X})$ consisting of rigid analytic functions on $\mathcal{X}$ of norm less than or equal to one. (Then $\mathcal{R}(\mathcal{X}) = \mathbf{Q}_2 \otimes_{\mathbf{Z}_2} R(\mathcal{X})$.) Suppose that $r \in R(\mathcal{X})$ is a rigid analytic function on $\mathcal{X}$ such that $1 > |r(x)| \geq |8|$ for $x \in \mathcal{X}(\mathbf{C}_2)$. This hypothesis on r shows that $R(\mathcal{X})$ is r-adically complete. Also, suppose given $e \in R(\mathcal{X})[[r/j_2]]$.

Define $M(\mathcal{X}, r) = (1/j_2)R(\mathcal{X})[[r/j_2]]$. Then $M(\mathcal{X}, r)$ is closed under U (by Corollary 3.2), and also under multiplication by e. Let $\tilde{U}$ be the endomorphism of $M(\mathcal{X}, r)$ defined by $\tilde{U}(f) = U(ef)$. Let $M(\mathcal{X}, r)_1 = 0$, and

$$M(\mathcal{X}, r)_n = 1/j_2 R(\mathcal{X}) \oplus \cdots \oplus r^{n-2}/j_2^{n-1} R(\mathcal{X}),$$

for $n \geq 2$. Then Corollary 3.2 shows that

$$\tilde{U}(M(\mathcal{X}, r)) \subset M(\mathcal{X}, r)_n + r^n M(\mathcal{X}, r),$$

and so Proposition 3.4 applies, yielding the characteristic power series of $\tilde{U}$ on $M(\mathcal{X}, r)$, which we denote by $P_{M(\mathcal{X},r)}(T) \in R(\mathcal{X})\langle T\rangle$.

The following Lemma shows that the characteristic power series of $\tilde{U}$ is functorial with respect to $\mathcal{X}$. Suppose that $\psi : \mathcal{Y} \to \mathcal{X}$ is a rigid analytic map. Then ψ induces a homomorphism $\psi' : R(\mathcal{X}) \to R(\mathcal{Y})$, which in turn induces a map $\psi'' : R(\mathcal{X})\langle T\rangle \to R(\mathcal{Y})\langle T\rangle$. (Here $R(\mathcal{X})$ is being considered with its r-adic topology, and $R(\mathcal{Y})$ with its $\psi'(r)$-adic topology.)

Lemma 3.8. $P_{M(\mathcal{Y},\psi'(r))}(T) = \psi''(P_{M(\mathcal{X},r)}(T))$.

This is just a matter of examining the definitions:

$$P_{M(\mathcal{Y},\psi'(r))}(T) = \lim_{n\to\infty} \det(1 - T\tilde{U}_{|M(\mathcal{Y},\psi'(r))_n/\psi'(r)^n}).$$

However,

$$M(\mathcal{Y},\psi'(r))_n/\psi'(r)^n = R(\mathcal{Y}) \otimes_{R(\mathcal{X})} M(\mathcal{X},r)_n/r^n,$$

and so

$$\det(1 - T\tilde{U}_{|M(\mathcal{Y},\psi'(r))_n/\psi'(r)^n}) = \psi''(\det(1 - T\tilde{U}_{|M(\mathcal{X},r)_n/r^n}).$$

Thus

$$P_{M(\mathcal{Y},\psi'(r))}(T) = \lim_{n\to\infty} \psi''(\det(1 - T\tilde{U}_{|M(\mathcal{X},r)_n/r^n}))$$

$$= \psi''(\lim_{n\to\infty} \det(1 - T\tilde{U}_{|M(\mathcal{X},r)_n/r^n})) = \psi''(P_{M(\mathcal{X},r)}(T)),$$

proving the Lemma.

Finally, we will want to compare the characteristic power series of $\tilde{U}$ that we have just defined with that defined in [5].

Lemma 3.9. *Suppose that $\mathcal{X}$ is affinoid. Then $\tilde{U}$ is a completely continuous operator on $(1/j_2)(\mathcal{R}(\mathcal{X})\langle r/j_2\rangle$, and the characteristic power series $P_{M(\mathcal{X},r)}(T)$ of $\tilde{U}$ on $M(\mathcal{X},r)$, as defined above in the sense of Proposition 3.4, is equal to the characteristic power series of $\tilde{U}$ on $(1/j_2)\mathcal{R}(\mathcal{X}_{/\mathbf{C}_2})\langle r/j_2\rangle$, as defined in the sense of* [5].

First note that by Corollary 3.2,

$$\tilde{U}((1/j_2)\mathcal{R}(\mathcal{X})\langle r/j_2\rangle) \subset (1/j_2)\mathcal{R}(\mathcal{X})[[r^2/j_2]] \subset (1/j_2)\mathcal{R}(\mathcal{X})\langle r/j_2\rangle.$$

Thus $(1/j_2)\mathcal{R}(\mathcal{X})\langle r/j_2\rangle$ is closed under $\tilde{U}$. Write

$$(1/j_2)\mathcal{R}(\mathcal{X})\langle r/j_2\rangle = \mathbf{Q}_2 \otimes M(\mathcal{X},r)_n \oplus (1/j_2^n)\mathcal{R}(\mathcal{X})\langle r/j_2\rangle,$$

and let Π_n denote the projection onto the first direct summand. Then we see that in fact

$$\tilde{U} = \lim_{n\to\infty} \Pi_n \circ \tilde{U}.$$

Thus $\tilde{U}$ is the limit of operators with finite dimensional range, and so is a completely continuous operator on $(1/j_2)\mathcal{R}(\mathcal{X})\langle r/j_2\rangle$. The characteristic power series of $\tilde{U}$, in the sense of [5], is defined to be the limit of the determinants

$$\det(1 - T(\Pi_n \circ \tilde{U})_{|\mathcal{R}(\mathcal{X}_{/\mathbf{C}_2})\otimes_{R(\mathcal{X})} M(\mathcal{X},r)_n}).$$

But we observe that

$$\det(1 - T(P_N \circ \tilde{U})_{|\mathcal{R}(\mathcal{X}_{/\mathbf{C}_2}) \otimes_{R(\mathcal{X})} M(\mathcal{X},r)_n}) \equiv \det(1 - T\tilde{U}_{|M(\mathcal{X},r)_n/r^n}) \mod r^n R(\mathcal{X})[T],$$

and thus see that this definition of the characteristic polynomial does indeed agree with that made in Proposition 3.4. Thus the Lemma is proved.

Finally, suppose that $\mathcal{X}$ is not necessarily affinoid. Consider the system of admissable affinoids $\mathcal{X}'$ of $\mathcal{X}$, directed by inclusion. Then $R(\mathcal{X})$ is the inverse limit of the $R(\mathcal{X}')$, and Lemma 3.8 shows that $P_{M(\mathcal{X},r)}(T)$ is the inverse limit of the $P_{M(\mathcal{X}',r)}(T)$. Thus Lemma 3.9 shows that $P_{M(\mathcal{X},r)}(T)$ is the inverse limit of characteristic power series computed in the sense of [5]. In particular, if $x \in \mathcal{X}(\mathbf{C}_2)$, then $x \in \mathcal{X}'(\mathbf{C}_2)$ for some $\mathcal{X}'$, and so the power series in $\mathcal{O}_2\langle T\rangle$ obtained by specializing $P_{M(\mathcal{X},r)}(T)$ at the point x is an *entire* function of T.

3.4 The space $\mathcal{W}$ of 2-adic weights and Eisenstein series

By a 2-adic weight we mean a continuous even $\mathbf{C}_2$-valued character κ of $\mathbf{Z}_2^\times$. Let $\Gamma = 1 + 4\mathbf{Z}_2$. Then

$$\mathbf{Z}_2^\times = \{\pm 1\} \times \Gamma,$$

and so giving an even character of $\mathbf{Z}_2$ is equivalent to giving a character of Γ. If κ is a 2-adic weight, we therefore regard κ either as an even character of $\mathbf{Z}_2^\times$ or as a character of Γ, interchangeably. We write τ for the projection $\mathbf{Z}_2 \to \{\pm 1\} \subset \mathbf{C}_2$.

Since Γ is a procyclic group, topologically generated by the element 5, any continuous character of Γ is determined by its value on 5, and in fact $\kappa \mapsto \kappa(5)5^{-14} - 1$ establishes a bijection between the set of 2-adic weights and the open unit disc around the origin in $\mathbf{C}_2$. We use the map t to identify $\mathcal{W}$ with the disc $1 > |t| \geq 0$, and we denote points of $\mathcal{W}(\mathbf{C}_2)$ interchangeably by κ, $t(\kappa)$ or simply t, depending on which is convenient. We also use t to identify $R(\mathcal{W})$ with $\mathbf{Z}_2[[t]]$.

Now suppose that χ is some character of Γ of finite order, and $k \in \mathbf{C}_2$ with $|1/2| > |k|$ (in particular, k could be an integer). We define the 2-adic weight $\kappa_{k,\chi}$ to be the character

$$\kappa_{k,\chi} : \gamma \mapsto \chi(\gamma)\gamma^{k-1}$$

of Γ (the right side is evaluated using the Binomial Theorem). (In [6], this is referred to as an accessible weight with coordinates (χ, k).) If χ is the trivial character of Γ, we write simply κ_k for $\kappa_{k,\chi}$.

Lemma 3.10. *(i) $k \mapsto t(\kappa_k) = 5^{k-14} - 1$ induces an isomorphism of the disc $|1/2| > |k|$ with the disc $|2| > |t|$ such that $|t| = 4|k - 14|$;*

(ii) if χ is a non-trivial finite order character of conductor 2^r (i. e. having kernel $1 + 2^r\mathbf{Z}_2$, which implies that necessarily $r \geq 3$) such that $\chi(5) = \zeta$ for some 2^{r-2} th primitive root of unity ζ, then $k \mapsto t(\sigma_{k,\chi}) = \chi(5)5^{k-14} - 1$ induces an isomorphism between the disc $|1/2| > |k|$ with the disc $|2| > |t - (1 - \zeta)|$, such that $|t - (1 - \zeta)| = 4|k - 14|$.

This Lemma follows immediately from the definitions.

The Eisenstein family is defined to be the following element of $\mathbf{Z}_2[[t]][[q]]$:

$$E_\kappa^* = 1 + \frac{2}{\zeta^*(\kappa)} \sum_{n=1}^{\infty} (\sum_{\substack{d \mid n \\ d \text{ odd}}} \kappa(d) d^{-1}) q^n.$$

In this formula $\zeta^*(\kappa)$ is an appropriately normalized version of the Kubota-Leopoldt 2-adic L-function: if $\kappa = \kappa_{k,\chi}$, with $k \geq 1$ an integer and χ a finite order character of Γ, then

$$\zeta^*(\kappa_{k,\chi}) = L(\tau^k \chi, 1-k),$$

where L denotes the Kubota-Leopoldt 2-adic L-function.

Proposition 3.11. *(i)* $2/\zeta^*(\kappa) \in \mathbf{Z}_2[[t]]$. *Thus*

$$E_\kappa^* \in \mathbf{Z}_2[[t]][[q]].$$

(ii) For any integer $k \geq 0$ and finite order character χ of Γ, the Eisenstein series

$$E_{\kappa_{k,\chi}}^* = 1 + \frac{2}{\zeta^*(\kappa_{k,\chi})} \sum_{n=1}^{\infty} (\sum_{\substack{d \mid n \\ d \text{ odd}}} \chi(d) d^{k-1}) q^n$$

is a classical modular form on $\Gamma_1(2^n)$ of weight k and nebentypus $\tau^k\chi$, for an appropriate value of n. (If $\tau^k\chi$ is a non-trivial Dirichlet character, one can choose n so that 2^n is the conductor of $\tau^k\chi$; otherwise one can choose $n = 1$.)

For a proof that $2/\zeta^*(\kappa) \in \mathbf{Z}_2[[t]]$, see [12] or [16]. When d is odd, the function $\kappa \mapsto \kappa(d)d^{-1}$ is a function of norm 1 on $\mathcal{W}$, and so certainly belongs to $\mathbf{Z}_2[[t]]$. So the fact that $2/\zeta_\kappa^* \in \mathbf{Z}_2[[t]]$ certainly implies that $E_\kappa^* \in \mathbf{Z}_2[[t]][[q]]$. This proves part (i). Part (ii) is well known.

Let us remark that, in the notation of chapter 2, $E_{\kappa_1}^* = \theta$ and $E_{\kappa_2}^* = E$.

Proposition 3.12. *(i)* $E_{\kappa_{4l}}^*/B^l \in 1 + (128l/j_2)\mathbf{Z}_2\langle l\rangle[[64/j_2]]$.
(ii) $E_{\kappa_{2l}}^*/E^l \in 1 + (32l/j_2)\mathbf{Z}_2\langle l\rangle[[16/j_2]]$.
(iii) $E_{\kappa_{2l+1}}^*/\theta E^l \in 1 + (16/j_2)\mathbf{Z}_2\langle l\rangle[[16/j_2]]$.

We give the details of the proof of part *(i)*. The proofs of parts *(ii)* and *(iii)* are similar, and so we simply sketch those modifications to the proof of part *(i)* that must be made in order to prove these latter two statements.

Observe that if we apply the Binomial Theorem to part *(ii)* of Lemma 2.7 we see that

$$(B/V(B))^l \in 1 + (16l/j_2)\mathbf{Z}_2\langle l\rangle[[8/j_2]] = 1 + (16l/j_2)\mathbf{Z}_2\langle l\rangle[[8/j_2]]. \tag{3.1}$$

Let N_+ denote the $\mathbf{Z}_2\langle l\rangle$ module

$$N_+ = (16l/j_2)\mathbf{Z}_2\langle l\rangle[[8/j_2]],$$

and N denote the direct sum

$$N = \mathbf{Z}_2\langle l\rangle \oplus N_+ = \mathbf{Z}_2\langle l\rangle \oplus (16l/j_2)\mathbf{Z}_2\langle l\rangle[[8/j_2]] \subset \mathbf{Z}_2\langle l\rangle[[q]].$$

Define the endomorphism $\bar{U}$ of $\mathbf{Z}_2\langle l\rangle[[q]]$ via the formula

$$\bar{U}(f) = U(f(B/V(B))^l).$$

Corollary 3.2 shows that

$$\bar{U}(N_+) = \bar{U}((16l/j_2)\mathbf{Z}_2\langle l\rangle[[8/j_2]]) \subset (128l/j_2)\mathbf{Z}_2\langle l\rangle[[64/j_2]], \tag{3.2}$$

while

$$\begin{aligned}\bar{U}(1) &= U((B/V(B))^l) \in U(1 + (16l/j_2)\mathbf{Z}_2\langle l\rangle[[8/j_2]]) \\ &= 1 + U((16l/j_2)\langle l\rangle[[8/j_2]]) \subset 1 + (128l/j_2)\mathbf{Z}_2\langle l\rangle[[64/j_2]]. \end{aligned} \tag{3.3}$$

Thus we see in particular that $\bar{U}(N_+) \subset 8N_+$, and $\bar{U}(N) \subset N$.

Lemma 3.13. *There is a unique $F(l) \in 1+8N_+$ such that $\bar{U}(F(l)) = F(l)$, and furthermore, if $f \in N$ such that $\bar{U}(f) = f$, then $f = rF$ where $r \in \mathbf{Z}_2\langle l\rangle$ is the constant term of f.*

We see from equation (3.3) that

$$\bar{U}(1) \equiv 1 \mod 8N_+.$$

Let us write $F_1 = 1$. Suppose that $F_n \in 1 + 8N_+$ such that $\bar{U}(F_n) \equiv 1 \mod 8^n N_+$. We will construct $\bar{U}(F_{n+1}) \in 1 + 8N_+$ such that $\bar{U}(F_{n+1}) \equiv F_{n+1} \mod 8^{n+1}N_+$, and such that $F_{n+1} \equiv F_n \mod 8^n N_+$.

To construct F_{n+1}, write $\bar{U}(F_n) - F_n = 8^n f$ for some $f \in N_+$, and define $F_{n+1} = F_n + 8^n f$. Then

$$\bar{U}(F_{n+1}) = F_n + 8^n f + 8^n \bar{U}(f) \equiv F_n + 8^n f = F_{n+1} \mod 8^{n+1}N_+,$$

since $f \in N_+$, and so $\bar{U}(f) \in 8N_+$.

Now we define

$$F(l) = \lim_{n\to\infty} F_n \in 1 + 8N.$$

By construction, $\bar{U}(F(l)) = F(l)$.

Suppose that $f \in N$ such that $U(f) = f$. Let r be the constant term of f. Then $f - rF \in N_+$, and so $\bar{U}^n(f - rF) \in 8^n N_+$. But both F and f are fixed by $\bar{U}$, and so

$$f - rF = \lim_{n\to\infty} \bar{U}^n(f - rF) = 0,$$

showing that $f = rF$. This proves the second claim of the Lemma, as well as the uniqueness of $F(l)$, completing the proof of the Lemma.

Now suppose that l_0 is a positive integer. Then we may repeat all the above arguments after specializing l to l_0, and we find that $F(l_0) \in 1 + (16l_0/j_2)\mathbf{Z}_2[[8/j_2]]$ is, up to scaling, the unique power series $f \in \mathbf{Z}_2 \oplus (16l_0/j_2)\mathbf{Z}_2[[8/j_2]]$ such that $U(f(B/V(B))^{l_0}) = f$.

Now Lemma 2.6 shows that $E^*_{\kappa 4l_0}$ is a homogeneous polynomial of degree l_0 in A and B with coefficients in $\mathbf{Q}$, and so

$$rE^*_{\kappa 4l_0}/B^{l_0} \in \mathbf{Z}_2 \oplus (16l_0/j_2)\mathbf{Z}_2[[8/j_2]]$$

for some $r \in \mathbf{Z}_2$. (Recall that $1/j_2 = A/B$.) We compute that

$$U((rE^*_{\kappa_{4l_0}}/B^{l_0})(B/V(B))^{l_0}) = U(rE_*)\kappa_{4l_0}/V(B)^{l_0}) = U(rE^*_{\kappa_{4l_0}})/B^{l_0} = rE^*_{\kappa_{4l_0}}/B^{l_0}.$$

Thus by the preceeding Lemma $rE^*_{\kappa_{4l_0}}/B^{l_0}$ is a mutliple of F_{l_0}, and a consideration of constant terms shows that in fact $rE^*_{\kappa_{4l_0}}/B^{l_0} = rF(l_0)$, and thus $E^*_{\kappa_{4l_0}}/B^{l_0} = F(l_0)$.

Now $E^*_{\kappa_{4l}}/B^l$ and $F(l)$ are both elements of $\mathbf{Z}_2\langle l\rangle[[q]]$, which yield the same q-expansion when evaluated at any non-negative integer l_0. Since the positive integers are infinite in number, we see that this equation holds as an identity in rigid analytic functions of l.

Since by construction $F(l) \in 1 + 16l/j\mathbf{Z}_2\langle l\rangle[[8/j]]$, equation (3.3) shows that

$$E^*_{\kappa_{4l}}/B^l = F(l) = \bar{U}(F(l)) \in 1 + 128l/j\mathbf{Z}_2\langle l\rangle[[64/j]].$$

This proves part *(i)* of the Proposition.

Part *(ii)* is proved in a similar fashion. We notice that applying the Binomial Theorem to part *(i)* of Lemma 2.7 yields the formula

$$(E/V(E))^l \in 1 + (8l/j_2)\mathbf{Z}_2\langle l\rangle[[4/j_2]]. \tag{3.4}$$

We then take $N_+ = (8l/j_2)\mathbf{Z}_2\langle l\rangle[[4/j_2]]$ and $N = \mathbf{Z}_2\langle l\rangle \oplus N_+$, and arguing as above, we prove part *(ii)*.

To prove part *(iii)*, we multiply equation (3.4) by part *(iii)* of Lemma 2.7 to obtain

$$(\theta/V(\theta))(E/V(E))^l \in 1 + (4/j_2)\mathbf{Z}_2\langle l\rangle[[4/j_2]].$$

Taking $N_+ = (4/j_2)\mathbf{Z}_2\langle l\rangle[[4/j_2]]$ and $N = \mathbf{Z}_2\langle l\rangle \oplus N_+$, a similar argument to that of part *(i)* completes the proof.

3.5 Defining $\tilde{U}$

In this section we prepare to apply the theory developed in sections 3.2 and 3.3. We begin by defining e, which in turn is used to define $\tilde{U}$.

All the spaces $\mathcal{X}$ that we consider will lie over weight space $\mathcal{W}$. Recall that t denotes the function on $\mathcal{W}$ defined by $t(\kappa) = \kappa(5)5^{-14} - 1$, and that $R(\mathcal{W}) = \mathbf{Z}_2[[t]]$.

If $\kappa \in \mathcal{W}(\mathbf{C}_2)$, define $\kappa' \in \mathcal{W}(\mathbf{C}_2)$ via the formula

$$\kappa'(\gamma) = \kappa(\gamma)\gamma^{-14}.$$

The map $\kappa \mapsto \kappa'$ is a rigid analytic isomorphism of $\mathcal{W}$ with itself.

Now define $e \in \mathbf{Z}_2[[t,q]]$ via the formula

$$e = E^3B^2E^*_{\kappa'}/V(E^3B^2E^*_{\kappa'}).$$

We then define the $\mathbf{Z}_2[[t]]$-linear endomorphism $\tilde{U}$ of $\mathbf{Z}_2[[t,q]] = \mathbf{Z}_2[[t,1/j_2]]$ via the formula

$$\tilde{U}(f) = U(fe) = U(fE^3B^2E^*_{\kappa'}/V(E^3B^2E^*_{\kappa'})).$$

In order to study $\tilde{U}$ we will need to establish some properties of e.

We begin by studying the modular function $E^3B^2/V(E^3B^2)$.

Lemma 3.14. *(i)* $E^3B^2/V(E^3B^2) \in 1 + 8/j_2\mathbf{Z}_2[[8/j_2]]$.
(ii) $U(E^3B^2/V(E^3B^2)j_2) = 64/j_2$.

We see that part *(i)* follows from parts *(i)* and *(ii)* of Lemma 2.7. Part *(ii)* is equivalent to the result

$$U(E^3B^2/j_2) = 64E^3B^2/j_2.$$

To prove this, we begin by observing that the space of weight 14 cusp forms on $\Gamma_0(2)$ is two dimensional, and thus spanned by $\Delta E = q + q^3 + \cdots$ and $V(\Delta)E = q^2 + 24q^3 + \cdots$, since these are obviously linearly independent. Atkin-Lehner theory [2] shows that this space has a basis of newforms, since there are no weight 14 cuspforms of level one. If $\sum_{n=1}^{\infty} a_n q^n$ is a weight 14 newform on $\Gamma_0(2)$, normalized so that $a_1 = 1$, a consideration of q-expansions shows that necessarily

$$\sum_{n=1}^{\infty} a_n q^n = \Delta E + a_2 V(\Delta)E.$$

Thus a normalized newform of weight 14 is uniquely determined by its U eigenvalue a_2. Since there are two distinct newforms of weight 14, there must correspondingly be two distinct values of a_2. But a_2 equals either 64 or -64 by Atkin-Lehner theory, so we see that both values occur. In particular, $\Delta E + 64V(\Delta)E$ is a normalized newform on $\Gamma_0(2)$ with U eigenvalue 64. Now Lemma 2.6 shows that both Δ and $V(\Delta)$ may be written as degree three homogeneous polynomials in A and B. Since they are both cuspforms, they must both be divisible by the product AB. It is now a simple calculation of the first two terms of their q-expansions to see that $\Delta = AB^2$ and $V(\Delta) = A^2B$.Thus

$$\Delta E + 64V(\Delta)E = AB^2E + 64A^2BE = E(1 + 64/j_2)B^3/j_2 = E^3B^2/j_2.$$

Part *(ii)* of the Lemma follows.

We now study the element $E^*_{\kappa'}/V(E^*_{\kappa'})$ of $\mathbf{Z}_2[[t,q]]$. We will study it first when restricted to the subdisc $|16| \geq |t| \geq 0$ of $\mathcal{W}$, which we denote by $\mathcal{W}_1$.

By Lemma 3.10, $\mathcal{W}_1(\mathbf{C}_2)$ consists of characters $\kappa = \kappa_k$ with $k - 14 = 4l \in 4\mathcal{O}_2$. The functions t and l are related by the equation $l = (\log(1+t))/4\log 5$. We have

$$\kappa'_k = \kappa_{k-14} = \kappa_{4l}.$$

The ring $R(\mathcal{W}_1)$ is equal to $\mathbf{Z}_2\langle t/16\rangle = \mathbf{Z}_2\langle l\rangle$.

Proposition 3.15. $E^*_{\kappa'}/V(E^*_{\kappa'}) \in 1 + (t/j_2)R(\mathcal{W}_1)[[8/j_2]]$.

In the course of the proof it will be more convenient to use the variable l then t; in terms of l, the assertion to be proved is that

$$E^*_{\kappa_{4l}}/V(E^*_{\kappa_{4l}}) \in 1 + (16l/j_2)\mathbf{Z}_2\langle l\rangle[[8/j_2]].$$

Write $F(l) = E^*_{\kappa_{4l}}/B^l$. Then part *(i)* of Lemma 3.12 shows that

$$F(l) \in 1 + 128l/j_2\mathbf{Z}_2\langle l\rangle[[64/j_2]].$$

Now part *(i)* of Lemma 3.1 shows that

$$V(F(l)) \in 1 + 16l/j_2\mathbf{Z}_2\langle l\rangle[[8/j_2]].$$

Also recall that equation (3.1) shows that

$$(B/V(B))^l \in 1 + (16l/j_2)\mathbf{Z}_2\langle l\rangle[[8/j_2]].$$

Multiplying the previous two formulae, we find that

$$E^*_{\kappa_{4l}}/V(E^*_{\kappa_{4l}}) = (F(l)/V(F(l)))(B/V(B))^l \in 1 + 16l/j_2\mathbf{Z}_2\langle l\rangle[[8/j_2]].$$

This completes the proof of the Proposition.

Corollary 3.16. *If $\kappa \in \mathcal{W}(\mathbf{C}_2)$ and $|16| \geq |t| \geq 0$, then*

$$E^*_{\kappa'}/V(E^*_{\kappa'}) \in 1 + (t(\kappa)/j_2)\mathcal{O}_2[[8/j_2]].$$

This follows from the Proposition, by evaluating the power series

$$E^*_{\kappa'}/V(E^*_{\kappa'}) \in 1 + (t/j_2)\mathbf{Z}_2\langle t/16\rangle[[8/j_2]]$$

at the point $t = t(\kappa) \in 16\mathcal{O}_2$.

Having studied $E^*_{\kappa'}/V(E^*_{\kappa'})$ over the sub-disc $|16 \geq |t| \geq 0$ of $\mathcal{W}$, we now study it over the sub-annulus $1 > |t| \geq |16|$ of $\mathcal{W}$. The ring of rigid analytic functions on this annulus of norm less than or equal to one is equal to $\mathbf{Z}_2[[t]]\langle 16/t\rangle$.

Actually, we will work over the annulus $1 > |s| \geq |2|$, which we map to $\mathcal{W}$ via the map $s \mapsto t = s^4$. Let us denote this annulus by $\mathcal{W}_2$. Then $R(\mathcal{W}_2) = \mathbf{Z}_2[[s]]\langle 2/s\rangle$, which is a $\mathbf{Z}_2[[t]]$ algebra via $t \mapsto s^4$.

Proposition 3.17. $E^*_{\kappa'}/V(E^*_{\kappa'}) \in 1 + t/j_2R(\mathcal{W}_2)[[s^3/j_2]]$.

This is actually a formal consequence of Proposition 3.15. The quotient $E^*_{\kappa'}/V(E^*_{\kappa'})$ is an element of $\mathbf{Z}_2[[t]][[q]] = \mathbf{Z}_2[[t]][[1/j_2]]$. Thus we can write

$$E^*_{\kappa'}/V(E^*_{\kappa'}) = \sum_{n=0}^{\infty}(\sum_{i=0}^{\infty} a_{i,n}t^i)(1/j_2)^n,$$

for some $a_{i,n} \in \mathbf{Z}_2$. Part *(i)* shows that $a_{i,0} = 0$ if $i > 0$, and that $a_{0,n} = 0$ if $n > 0$. Thus we rewrite this as

$$E^*_{\kappa'}/V(E^*_{\kappa'}) = 1 + (t/j_2)\sum_{n=1}^{\infty}(\sum_{i=1}^{\infty} 2^{4i-1-3n}a_{i,n}(t/16)^{i-1})(8/j_2)^{n-1},$$

and note that part *(i)* also shows that $2^{4i-1-3n}a_{i,n} \in \mathbf{Z}_2$ for all $i > 0$ and $n > 0$.

Let us rewrite this again:

$$E^*_{\kappa'}/V(E^*_{\kappa'}) = 1 + t/j_2 \sum_{n=1}^{\infty} (\sum_{i=1}^{\infty} s^{-3(n-1)} a_{i,n} t^{i-1})(s^3/j_2)^{n-1}.$$

Now if $1 \leq i \leq (3n+1)/4$, then (recalling that $s^4 = t$)

$$s^{-3(n-1)} a_{i,n} t^{i-1} = (2^{4i-3n-1} a_{n,i})(2/s)^{(3n+1-4i)} \in \mathbf{Z}_2[[s]]\langle 2/s \rangle,$$

while if $(3n+1)/4 < i$, then

$$s^{-3(n-1)} a_{i,n} t^{i-1} = a_{n,i} s^{(4i-1-3n)} \in \mathbf{Z}_2[[s]]\langle 2/s \rangle.$$

Thus indeed

$$E^*_{\kappa'}/V(E^*_{\kappa'}) \in 1 + \mathbf{Z}_2[[s]]\langle 2/s \rangle,$$

and the Proposition is proved.

Corollary 3.18. *If $\kappa \in \mathcal{W}$ and $1 > |t(\kappa)| \geq |16|$, then*

$$E^*_{\kappa}/V(E^*_{\kappa}) \in 1 + (t(\kappa)/j_2)\mathcal{O}_2[[t(\kappa)^{3/4}/j_2]].$$

This follows from the Proposition by evaluating at any point $s = t(\kappa)^{1/4}$ (of course it doesn't matter which fourth root we choose).

3.6 Overconvergent modular forms and the characteristic power series of $\tilde{U}$

We can now define the spaces of overconvergent 2-adic cusp forms for any weight $\kappa \in \mathcal{W}(\mathbf{C}_2)$. The ideas of this section all come from [5].

If $\kappa \in \mathcal{W}(\mathbf{C}_2)$ such that $1 > t(\kappa) \geq |16|$, choose some $r(\kappa) \in \mathbf{C}_2$ such that $|r(\kappa)| = |t(\kappa)|^{3/4}$. If $\kappa \in \mathcal{W}(\mathbf{C}_2)$ such that $|16| \geq |t| \geq 0$, define $r(\kappa) = 8$. (Since $|8| = |16|^{3/4}$, the two definitions are compatible when $|t| = |16|$.)

Now for any $\kappa \in \mathcal{W}(\mathbf{C}_2)$, define

$$M_\kappa = (E^*_\kappa/j_2)\mathcal{O}_2[[r(\kappa)/j_2]] \subset \mathcal{O}_2[[q]],$$

and define

$$\mathcal{M}_\kappa = \mathbf{Q}_2 \otimes_{\mathbf{Z}_2} M_\kappa \subset \mathbf{C}_2[[q]].$$

We call $\mathcal{M}_\kappa$ the space of overconvergent 2-adic cusp forms of weight κ. (See the appendix for a discussion of the general context in which this definition should be viewed.)

The following alternative description of M_κ will be the one which we use more frequently below:

Lemma 3.19. $M_\kappa = (E^3 B^2 E^*_{\kappa'}/j_2)\mathcal{O}_2[[r(\kappa)/j_2]]$.

This is clearly equivalent to the claim that $E^*_\kappa/E^3B^2E^*_{\kappa'} \in 1 + (r(\kappa)/j_2)\mathcal{O}_2[[r(\kappa)/j_2]]$, which follows from the following Lemma.

Lemma 3.20. *(i)* $E^*_\kappa/E^3B^2E^*_{\kappa'} \in 1 + 64/j_2R(\mathcal{W}_1)[[64/j_2]]$.
(ii) $E^*_\kappa/E^3B^2E^*_{\kappa'} \in 1 + s^{3/2}/j_2R(\mathcal{W}_2)[[s^{3/2}/j_2^2]]$.

We will prove part *(i)*. Part *(ii)* follows by a similar argument. First we note that by Lemma 3.1, part *(ii)*, together with Proposition 3.15 ,

$$\tilde{U}(R(\mathcal{W}_1)[[8/j_2]]) \subset R(\mathcal{W}_1)[[64/j_2]].$$

Thus an argument analogous to that of the proof of part *(i)* of Proposition 3.12 shows that there is a unique fixed element of $\tilde{U}$ in $1+(8/j)R(\mathcal{W}_1)[[8/j_2]]$, which must equal $E_\kappa/E^3B^2E^*_{\kappa'}$. Thus

$$E_\kappa/E^3B^2E^*_{\kappa'} = \tilde{U}(E_\kappa/E^3B^2E^*_{\kappa'}) \in \tilde{U}(R(\mathcal{W}_1)[[8/j_2]]) \subset R(\mathcal{W}_1)[[64/j_2]],$$

proving part *(i)*. As noted, part *(ii)* is proved similarly.

Now we define $M_{1,\kappa} = 0$ and

$$M_{n,\kappa} = (E^3B^2E^*_{\kappa'}/j_2)((r(\kappa)/j_2)R \oplus \cdots \oplus (r(\kappa)/j_2)^{n-2}R)$$

for $n \geq 2$.

Proposition 3.21. *For any $\kappa \in \mathcal{W}(\mathbf{C}_2)$, M_κ is closed under U. Furthermore, for each $n \geq 1$,*

$$U(M_\kappa) \subset M_{n,\kappa} + r(\kappa)^n M_\kappa.$$

Let us write $\tilde{U}_\kappa$ for the endomorphism of $\mathcal{O}_2[[q]]$ obtained by specializing $\tilde{U}$ to weight κ. If $f \in \mathcal{O}_2[[q]]$, then

$$U((E^3B^2E^*_{\kappa'})f) = U((E^3B^2E^*_\kappa/V(E^3B^2E^*_{\kappa'}))f)E^3B^2E^*_{\kappa'} = \tilde{U}_\kappa(f).$$

Define

$$M'_\kappa = (E^3B^3E^*_{\kappa'})^{-1}M_\kappa = (1/j_2)\mathcal{O}_2[[r(\kappa)/j_2]],$$

and for each $n \geq 1$, define

$$M'_{n,\kappa} = (E^3B^3E^*_{\kappa'})^{-1}M_{n,\kappa} \subset M'_\kappa.$$

We see that it suffices to show that M'_κ is closed under $\tilde{U}_\kappa$, and that $\tilde{U}(M'_\kappa) \subset M'_{n,\kappa} + r(\kappa)^n M'_\kappa$. Now Corollaries 3.16 and 3.18 show that M'_κ is closed under multiplication by $E^3B^2E^*_{\kappa'}/V(E^3B^2E^*_{\kappa'})$, while corollary 3.2 shows that

$$U(M'_\kappa) \subset M'_{n,\kappa} + r(\kappa)^n M'_\kappa.$$

Thus

$$\tilde{U}(M'_\kappa) = U((E^3B^2E^*_{\kappa'}/V(E^3B^2E^*_{\kappa'}))M'_\kappa) \subset U(M'_\kappa) \subset M'_{n,\kappa} + r(\kappa)^n M'_\kappa,$$

proving the Proposition.

We can now apply Proposition 3.4 to this situation. We write $P(\kappa, T)$ for the characteristic power series of U on M_κ.

We wish to show that the coefficients of $P(\kappa, T)$ vary rigid analytically as κ varies.

Proposition 3.22. *Recall that $\mathcal{W}_1$ denotes the sub-disc $|16| \geq |t|$ of $\mathcal{W}$, while $\mathcal{W}_2$ denote the annulus $1 > |s| \geq |2|$, which maps to $\mathcal{W}$ via $t = s^4$. Then $P_{M(\mathcal{W}_1,8)}(T)$ (a priori an element of $\mathbf{Z}_2\langle t\rangle[[T]] \supset \mathbf{Z}_2[[t]][[T]]$) and $P_{M(\mathcal{W}_2,s^3)}(T)$ (a priori an element of $\mathbf{Z}_2[[s]]\langle 2/s\rangle[[T]] \supset \mathbf{Z}_2[[t]][[T]]$) are both elements of $\mathbf{Z}_2[[t]][[T]]$, and are equal to the same element, which we denote simply by $P(T) \in \mathbf{Z}_2[[t]][[T]]$.*

The power series $P(\kappa, T) \in \mathcal{O}_2[[T]]$ is obtained by specializing $P(T)$ at the point $t = t(\kappa)$. It is an entire power series in T.

Let $\mathcal{X}$ denote the region $|s| = |2|$. Then $\mathcal{X}$ is naturally embedded in $\mathcal{W}_2$, while the map from $\mathcal{W}_2$ to $\mathcal{W}$, when restricted to $\mathcal{X}$, has image lying in $\mathcal{W}_2$. We will apply Lemma 3.8 to both the map $\mathcal{X} \to \mathcal{W}_1$ and the map $\mathcal{X} \to \mathcal{W}_2$. Via these maps we regard both $R(\mathcal{W}_1) = \mathbf{Z}_2\langle t/16\rangle$ and $R(\mathcal{W}_2) = \mathbf{Z}_2[[s]]\langle 2/s\rangle$ as subrings of $\mathbf{Z}_2\langle s/2, 2/s\rangle = R(\mathcal{X})$.

From Lemma 3.8 we conclude that $P_{M(\mathcal{W}_1,8)}(T) = P_{M(\mathcal{X},8)}(T)$ as elements of the ring $\mathbf{Z}_2\langle s/2, 2/s\rangle[[T]]$, while $P_{M(\mathcal{W}_2,s^3)}(T) = P_{M(\mathcal{X},s^3)}(T)$ as elements of the ring $\mathbf{Z}_2\langle s/2, 2/s\rangle[[T]]$. But $s^3/8$ is a unit in $\mathbf{Z}_2\langle s/2, 2/s\rangle$, and so $P_{M(\mathcal{X},s^3)}(T) = P_{M(\mathcal{X},8)}(T)$. Thus we conclude that $P_{M(\mathcal{W}_1,8)}(T) = P_{M(\mathcal{W}_2,s^3)}(T)$. But the first has coefficients in $\mathbf{Z}_2\langle t/16\rangle$, while the second has coefficients in $\mathbf{Z}_2[[s]]\langle 2/s\rangle$. Since the intersection of these two rings is equal to $\mathbf{Z}_2[[t]]$, we see that the first part of the Proposition is proved.

To prove that $P(\kappa, T)$ is obtained by specialzing $P(T) = P_{M(\mathcal{W}_1,8)}(T) = P_{(M(W_2,s^3)}(T)$ at the point $t = t(\kappa)$, note that the proof of Proposition 3.21 shows that for any particular value of κ, $P(\kappa, T)$ is equal to the characteristic power series of the endomorphism $\tilde{U}_\kappa$ of M'_κ. But if $1 > |t(\kappa)| \geq |16|$ (respectively $|16| \geq |t(\kappa)| \geq |0|$), then this is precisely the specialization of the endomorphism $\tilde{U}$ acting on $M(\mathcal{W}_2, s^3(\kappa))$ (respectively $M(\mathcal{W}_1, 8)$) at a point $s(\kappa) \in \mathcal{W}_2(\mathbf{C}_2)$ lying over $t(\kappa)$ (respectively at the point $t(\kappa) \in \mathcal{W}_1(\mathbf{C}_2)$). The result now follows by specializing the computation of $P(T)$ to the particular value of κ under consideration.

Finally, the fact that $P(\kappa, T)$ is entire follows from the remark made following the proof of Lemma 3.8. This completes the proof of the Proposition.

This Proposition is a special case ($p = 2$ and $N = 1$) of the theory developed in [5]. The idea of using $\tilde{U}$ to show that $P(\kappa, T)$ depends analytically on κ is due entirely to Coleman.

Lemma 3.23. *If $f \in \mathcal{M}_\kappa$ is non-zero such that $U(f) = \lambda f$, then there exists $P'(T) \in \mathcal{O}_2\langle T\rangle$ such that*

$$P(\kappa, T) = (1 - \lambda T)P'(T).$$

If $\lambda = 0$, there is nothing to prove, so we may as well assume that $\lambda \neq 0$. By multiplying f by a suitable power of 2, we may assume that f is non-zero and lies in M_κ. Now it is certainly true that $m = f$ and $r = \lambda$ satisfy the hypothesis of Proposition 3.7 (where of course I is taken to be the principal ideal generated by $r(\kappa)$), and so the result follows.

We now prove that classical modular forms on $\Gamma_1(4)$ are also overconvergent modular forms.

Proposition 3.24. *Suppose that f is a modular form on $\Gamma_1(4)$ of weight k, whose q-expansion coefficients lie in $\overline{\mathbf{Q}}$, and which has a vanishing constant term in its q-expansion. Then $f \in \mathcal{M}_{\kappa_k}$.*

We must show that

$$f/E^3B^2E^*_{\kappa_{k-14}} \in \mathbf{C}_2 \otimes_{\mathcal{O}_2} (1/j_2)\mathcal{O}_2[[r(\kappa)/j_2]].$$

We consider two cases.

Suppose first that $k-14=2l$. Specializing part *(ii)* of Proposition 3.12 to our particular value of l shows that

$$E^*_{\kappa_{2l}}/E^l \in 1+(32/j_2)\mathbf{Z}_2[[16/j_2]].$$

Now $f/E^3B^2E^*_{\kappa_{2l}} = (f/E^{7+l})(E^4/B^2)(E^l/E^*_{\kappa_{2l}})$. From the discussion of modular forms on $\Gamma_1(4)$ in section 2.3, we conclude that f is a degree l+7 homogeneous polynomial in E and $V(E)$. Part *(ii)* of Lemma 2.7 shows that $E/V(E) \in 1+(8/j_2)\mathbf{Z}[[8/j_2]]$. Thus $f/E^{l+7} \in \overline{\mathbf{Q}} \otimes_{\mathbf{Z}} Z[[8/j_2]]$. In fact, since we assumed that f has vanishing constant term,

$$f/E^{l+7} \in (1/j_2)\overline{\mathbf{Q}} \otimes_{\mathbf{Z}} \mathbf{Z}[[8/j_2]].$$

By part *(iii)* of Lemma 2.6, $E^2 = 64A+B = B(1+64/j_2)$, and so

$$E^4/B^2 = (1+64/j_2)^2.$$

Thus indeed

$$f/E^3B^2E^*_{\kappa'} = (f/E^{7+l})(E^4/B^2)(E^l/E^*_{\kappa_{2l}}) \in (1/j_2)\mathbf{C}_2 \otimes_{\mathcal{O}_2} \mathcal{O}_2[[8/j_2]].$$

Since $|r(\kappa)| \geq |8|$, we are done with this case.

Suppose now that $k-14=1+2l$ is odd. Specializing part *(iii)* of Proposition 3.12 to our particular value of l shows that

$$E^*_{\kappa_{2l+1}}/\theta E^l \in 1+(16/j_2)\mathcal{O}_2[[16/j_2]].$$

The disucssion of modular forms on $\Gamma_1(4)$ in section 2.3 shows that f/θ is a modular form of weight $2l+14$ on $\Gamma_1(4)$. It now follows by the previous case that

$$f/E^3B^2E^*_{\kappa_{2l+1}} = (f/\theta E^3B^2E^*_{\kappa_{2l}})(\theta E^l/E^*_{\kappa_{2l+1}})(E^*_{\kappa_{2l}}/E^l) \in \mathbf{C}_2 \otimes_{O_2} (1/j_2)\mathcal{O}_2[[8/j_2]].$$

Since $|r(\kappa)| = |4|^{3/4}$, this completes the proof of the Proposition.

Suppose that l is an odd prime, and $\kappa \in \mathcal{W}$. We define the weight κ action of the Hecke operator T_l on $\mathbf{C}_2[[q]]$ via the formula

$$T_l : \sum_{n=0}^{\infty} a_n q^n \mapsto \sum_{n=0}^{\infty} a_{ln} q^n + \kappa(l) l^{-1} \sum_{n=1}^{\infty} a_n q^{ln}.$$

We say that $f \in \mathcal{M}_\kappa$ is an "eigenform for all Hecke operators" if f is an eigenvector for U and for the weight κ action of T_l for every odd prime l. We say that f is normalized if in addition the coefficient of q in f equals 1.

We mention the following result simply for the sake of completeness.

Propostion 3.25. *For any $\kappa \in \mathcal{W}(\mathbf{C}_2)$ and for any odd prime l, the space $\mathcal{M}_\kappa$ is closed under the weight κ action of the operator T_l.*

This result is proved in [5] and [6]. It requires more sophisticated methods then those that we are using. The relation between these references and our presentation is discussed in the appendix.