CHAPTER II

MODULAR FUNCTIONS AND MODULAR FORMS

In this chapter we perform explicit computations on $X_0(2^n)$ for $n = 1, 2$ and 3. In section 1 we compute uniformizers for each of these modular curves (these three curves are of genus zero). In section 2 we derive some relations among the q-expansions of these uniformizers. In section 3 we calculate modular forms on $\Gamma_0(2)$, $\Gamma_1(4)$ and $\Gamma_1(4) \cap \Gamma_0(8)$. We make no claim to originality with regard to any of the material of this chapter – it all certainly falls under the heading "classical". However, it is convenient to collect the results we need in one place, and to fix our notation, in preparation for the computations of the subsequent chapters.

2.1 Uniformizers

We adopt the following notation: j_{2^n}, for $n = 1, 2$ or 3, will denote that uniformizer of $X_0(2^n)$ which has a pole at the infinite cusp, a zero at the zero cusp, and whose q-expansion begins $j_{2^n} = 1/q + \cdots$. As usual, j with no subscript will denote the uniformizer E_4^3/Δ of $X_0(1)$, $j = 1/q + 744 + \cdots$.

We regard $X_0(2^n)$ simply as the quotient of the extended upper half-plane by the action of $\Gamma_0(2^n)$. We let

$$B_1 : X_0(2^n) \to X_0(2^{n-1})$$

denote the map

$$z \mod \Gamma_0(2^n) \mapsto z \mod \Gamma_0(2^{n-1}),$$

while $B_2 : X_0(2^n) \to X_0(2^{n-1})$ denotes the map

$$z \mod \Gamma_0(2^n) \mapsto 2z \mod \Gamma_0(2^{n-1}).$$

If $r \in \mathbf{P}^1(\mathbf{Q})$, we denote the corresponding cusp on any modular curve by $\underline{r}$. As always, q denotes the function on the upper half-plane $z \mapsto e^{2\pi i z}$. Any function f on the upper half-plane which is invariant under $z \mapsto z + 1$ may be expanded as a series in integral powers of q, which we will refer to as the q-expansion of f.

Lemma 2.1. *The function*

$$j_2(z) = \Delta(z)/\Delta(2z) = 1/(q \prod_{n=1}^{\infty} (1 + q^n)^{24})$$

is a uniformizer for $X_0(2)$. It satisfies the equation

$$j_2 \circ w_2 = 2^{12}/j_2.$$

The map $B_1 : X_0(2) \to X_0(1)$ is described by the equation

$$j \circ B_1 = (j_2 + 256)^3 / j_2^2.$$

It is ramified at the point $512 = j_2(i)$ *over the point* $1728 = j(i)$ *with degree two, at the point* $-256 = j_2((-1+\sqrt{-3})/2)$ *over the point* $0 = j((-1+\sqrt{-3})/2)$ *with degree three, and at the point* $0 = j_2(\underline{0})$ *over the point* $\infty = j(\underline{\infty})$ *with degree two.*

The map $B_2 : X_0(2) \to X_0(1)$ *is described by the equation*

$$j \circ B_2 = (j_2 + 16)^3 / j_2.$$

It is ramified at the point $8 = j_2(i/2)$ *over the point* $1728 = j(i)$ *with degree two, at the point* $-16 = j_2((-1+\sqrt{-3})/4)$ *over* $0 = j((-1+\sqrt{-3})/2)$ *with degree three, and at the point* $\infty = j_2(\underline{\infty})$ *over the point* $\infty = j(\underline{\infty})$ *with degree two.*

The map from the upper half-plane to the affine j-line is ramified over two points: the point $j = 1728$, corresponding to the $\mathbf{SL}(2,\mathbf{Z})$ orbit of $z = i$, with ramification degree two, and the point $j = 0$, corresponding to the $\mathbf{SL}(2,\mathbf{Z})$ orbit of $z = (-1+\sqrt{-3})/2$, with ramification degree three.

The map from the upper half-plane to $Y_0(2)$ is ramified over one point, corresonding to the $\Gamma_0(2)$ orbit of $z = (1+i)/2$, with ramification degree two.

Thus the finite degree three map $B_{1|Y_0(2)}$ from $Y_0(2)$ to the affine j-line is ramified at the $\Gamma_0(2)$ orbit of $z = i$, with ramificaton degree two, and is totally ramified over the point $j = 0$.

There are two cusps on $X_0(2)$, the cusp $\underline{\infty}$ and the cusp $\underline{0}$. The cusp $\underline{\infty}$ is unramified over the point $j = \infty$ with respect to B_1, while B_1 makes the cusp $\underline{0}$ ramified of degree two over $j = \infty$.

As in the statement of the Lemma, we define

$$j_2(z) = \Delta(z)/\Delta(2z) = 1/(q \prod_{n=1}^{\infty} (1+q^n)^{24}).$$

Then j_2 a modular function on $\Gamma_0(2)$. Since Δ has no zeroes on the upper half-plane, it has no zeroes or poles on $Y_0(2)$. A consideration of q-expansions shows that j_2 has a simple pole at $\underline{\infty}$. Thus j_2 is a uniformizer on $X_0(2)$. Since Δ is a modular form of weight twelve and level one, we compute that

$$j_2 \circ w_2(z) = j_2(-1/2z) = \Delta(-1/2z)/\Delta(-1/z) = 2^{12}\Delta(2z)/\Delta(z) = 2^{12}/j(z).$$

In particular $j_2(\underline{0}) = 2^{12}/j_2(\underline{\infty}) = 2^{12}/\infty = 0$.

If we pull back j via B_1 to a function on $X_0(2)$, it has a pole of order one at $\underline{\infty}$, a pole of order two at $\underline{0}$, a zero of order three at $j_2((-1+\sqrt{-3})/2)$, and no other zeroes or poles. Thus $j \circ B_1$ is a scalar multiple of $(j_2 - a)^3/j_2^2$, where $a = j_2((-1+\sqrt{-3})/2)$. Now

$$(j_2 - a)^3/j_2^2 = 1/q - (24 + 3a) + \cdots,$$

while

$$j = 1/q + 744 + \cdots.$$

Since B_1 preserves q-expansions, we see that $a = -256$ and that

$$j \circ B_1 = (j_2 + 256)^3/j_2^2.$$

From this equation we see that $-256 = j_2((-1+\sqrt{-3})$ is the point lying over $0 = j((-1+\sqrt{-3})/2)$. We also see that

$$j{\circ}B_1 - 1728 = (j_2+64)(j_2-512)^2/j_2^2,$$

and thus, from the above description of the ramification of B_1, that $j_2(i) = 512$ and $j_2((1+i)/2) = -64$. We have now verified the asserted formula for the map B_1 and also the claims about its ramification.

The maps B_1 and B_2 are related by the equation $B_2 = B_1{\circ}w_2$. This allows us to compute the ramification points of B_2 from those of B_1. We also compute

$$j{\circ}B_2 = ((j_2+256)^3/j_2^2){\circ}w_2 = (2^{12}/j_2+256)^3/(2^{12}/j_2)^2 = (j_2+16)^3/j_2.$$

This completes the proof of the Lemma.

Lemma 2.2. *The function*

$$j_4 = 1/(q\prod_{n=1}^{\infty}(1+q^n)^8(1+q^{2n})^8)$$

is a uniformizer on $X_0(4)$, satisfying

$$j_4{\circ}w_4 = 2^8/j_4$$

as well as $j_4(\underline{1/2}) = -16$. The map $B_1 : X_0(4) \to X_0(2)$ is described by the equation

$$j_2{\circ}B_1 = j_4^2/(j_4+16).$$

It is ramified at $-32 = j_4((1+i)/2)$ over $-64 = j_2((1+i)/2)$, and at $0 = j_4(\underline{0})$ over $0 = j_2(\underline{0})$. The map $B_2 : X_0(4) \to X_0(2)$ is described by the equation

$$j_2{\circ}B_2 = 16j_4 + j_4^2.$$

It is ramified at $-16 = j_4((1+i)/4)$ over $-64 = j_2((1+i)/2)$, and at $\infty = j_4(\underline{\infty})$ over $\infty = j_2(\underline{\infty})$.

The map B_1 makes the curve $X_0(4)$ a degree two cover of $X_0(2)$. There are three cusps on $X_0(4)$: $\underline{\infty}$, $\underline{1/2}$ and $\underline{0}$. The cusps $\underline{\infty}$ and $\underline{1/2}$ are both unramified for B_1, which maps them to the cusp $\underline{\infty}$ of $X_0(2)$, while $\underline{0}$ is ramified of degree two over the cusp $\underline{0}$ of $X_0(2)$ with respect to B_1. The map from the upper half-plane to $Y_0(4)$ is unramified, and so the map $B_{1|Y_0(4)}$ from $Y_0(4)$ to $Y_0(2)$ is ramified with degree two at the $\Gamma_0(4)$ orbit of $(1+i)/2$ (since $(1+i)/2$ is ramified under the map from the upper half-plane to the j-line, while $j_2((1+i)/2) = -64$ is not a ramification point of the map $B_{1|Y_0(2)}$ from $Y_0(2)$ to the j-line).

From this ramification data we see using Riemann-Hurwitz that $X_0(4)$ has genus zero. Let j_4 be a uniformizer of $X_0(4)$ which has a pole at $\underline{\infty}$, a zero at $\underline{0}$, and is normalized so that

$j_4 = 1/q + \cdots$. When we pull back j_2 to $X_0(4)$ via B_1, we obtain a function with simple poles at $\underline{\infty}$ and $\underline{1/2}$ and a double zero at $\underline{0}$, with no other zeroes or poles. Thus

$$j_2 \circ B_1 = j_4^2/(j_4 - a),$$

where $a = j_4(\underline{1/2})$. Also, $j_2 \circ B_1 + 64$ has a double zero at $j_4((1+i)/2)$, because we have seen that B_1 is ramified at this point. Now

$$j_2 \circ B_1 + 64 = (j_4^2 + 64 j_4 - 64a)/(j_4 - a),$$

and so $j_4^2 + 64j_4 - 64a$ must be a perfect square. Thus $j_4(\underline{1/2}) = a = -16$, and we find that $j_2 \circ B_1 = j_4^2/(j_4+16)$, and also that $j_2 \circ B_1 = (j_2+32)^2/(j_4+16)$, so that $j_4((1+i)/2) = -32$. This completes our computation of the formulae for the map B_1 and its ramification points.

To obtain the product formula for j_4, note that $\Delta(z)/\Delta(4z)$ has a pole of order three at $\underline{\infty}$ and a zero of order three at $\underline{0}$. A comparison of q-expansions shows that

$$j_4^3 = \Delta(z)/\Delta(4z) = 1/(q^3 \prod_{n=1}^{\infty} (1+q^n)^{24}(1+q^{2n})^{24}),$$

and so

$$j_4 = (\Delta(z)/\Delta(4z))^{1/3} = 1/(q \prod_{n=1}^{\infty} (1+q^n)^{8}(1+q^{2n})^{8}).$$

To compute $j_4 \circ w_4$, note that $j_4 \circ w_4$ has a simple zero at $\underline{\infty}$ and a simple pole at $\underline{0}$, since $\underline{\infty}$ and $\underline{0}$ are interchanged by w_4. Thus $j_4 | w_4 = b/j_4$ for some scalar b. Since $w_4(\underline{1/2}) = \underline{1/2}$, we find that

$$-16 = j_4(\underline{1/2}) = j_4 \circ w_4(\underline{1/2}) = b/j_4(\underline{1/2}) = -b/16,$$

and hence $b = 2^8$. Thus $j_4 \circ w_4 = 2^8/j_4$.

Finally, combining this formula with the formula $B_2 = w_2 \circ B_1 \circ w_4$, we may compute the ramification points of B_2, and also see that

$$j_2 \circ B_2 = (2^{12}(j_4+16)/j_4^2) \circ w_4 = j_4^2 + 16 j_4.$$

This completes the proof of the Lemma.

Lemma 2.3. *The function*

$$j_8 = 1/(q \prod_{n=1}^{\infty} (1+q^n)^4(1+q^{2n})^2(1+q^{4n})^4)$$

is a uniformizer of $X_0(8)$, *satisfying*

$$j_8 \circ w_8 = 32/j_8,$$

as well as $j_8(\underline{1/4}) = -4$ *and* $j_8(\underline{1/2}) = -8$. *The map* $B_1 : X_0(8) \to X_0(4)$ *is described by*

$$j_4 \circ B_1 = j_8^2/(j_8+4).$$

It is ramified at $-8 = j_8(\underline{1/2})$ *over* $-16 = j_4(\underline{1/2})$*, and at* $0 = j_8(\underline{0})$ *over* $0 = j_4(\underline{0})$*. The map* $B_2 : X_0(8) \to X_0(4)$ *is described by*

$$j_4 \circ B_2 = j_8^2 + 8j_8.$$

It is ramified over $-4 = j_8(\underline{1/4})$ *over* $-16 = j_4(\underline{1/2})$*, and at* $\infty = j_8(\underline{\infty})$ *over* $\infty = j_4(\underline{\infty})$*.*

The map B_1 makes the curve $X_0(8)$ a degree two cover of $X_0(4)$. It has four cusps: $\underline{\infty}$, $\underline{1/4}$, $\underline{1/2}$, and $\underline{0}$. The cusps $\underline{\infty}$, $\underline{1/4}$ are both mapped to the cusp $\underline{\infty}$ of $X_0(4)$ by B_1, the cusp $\underline{1/2}$ is ramified of degree two over the cusp $\underline{1/2}$ of $X_0(4)$ with respect to B_1, and the cusp $\underline{0}$ is ramified of degree two over the cusp $\underline{0}$ of $X_0(4)$ with respect to B_1. Since the map from the upper half-plane to $Y_0(4)$ is unramified, we see that $B_{1|Y_0(8)}$ is unramified.

Riemann-Hurwitz now shows that $X_0(8)$ has genus zero. Let j_8 be a uniformizer which has a pole at $\underline{\infty}$ and a zero at $\underline{0}$, normalized so that $j_8 = 1/q + \cdots$.

When we pull back j_4 to $X_0(8)$ via B_1, we obtain a function with simple poles at $\underline{\infty}$ and $\underline{1/4}$, and a zero of order two at $\underline{0}$. Thus $j_4 \circ B_1 = j_8^2/(j_8 - a)$, where $a = j_8(\underline{1/4})$. Also $(j_4 + 16) \circ B_1$ has a double zero at $\underline{1/2}$, because B_1 is ramified at this point. But

$$(j_4 + 16) \circ B_1 = (j_8^2 + 16j_8 - 16a)/(j_8 - a),$$

and so $j_8^2 + 16j_8 - 16a$ must be a perfect square. Thus $j_8(\underline{1/4}) = a = -4$, and so

$$j_4 \circ B_1 = j_8^2/(j_8 + 4).$$

We also find that $j_4 \circ B_1 + 16 = (j_8 + 8)^2/(j_8 + 4)$, so that $j_8(\underline{1/2}) = -8$. This completes our computation of the formulae for B_1 and its ramification points.

A routine calculation of zeroes and poles shows that

$$j_8 = (\Delta(z)^2 \Delta(4z)/\Delta(2z)\Delta(8z)^2)^{1/12} = 1/(q \prod_{n=1}^{\infty} (1 + q^n)^4 (1 + q^{2n})^2 (1 + q^{4n})^4).$$

To compute $j_8 \circ w_8$, note that a consideration of zeroes and poles shows that $j_8 \circ w_8 = a/j_8$ for some scalar b. Now $w_8(\underline{1/2}) = \underline{1/4}$, and so

$$-4 = j_8(\underline{1/4}) = j_8 \circ w_8(\underline{1/2}) = b/j_8(\underline{1/2}) = -b/8.$$

Thus $b = 2^5$, and $j_8 \circ w_8 = 2^5/j_8$.

Finally, using this last formula together with the formula $B_2 = w_4 \circ B_1 \circ w_8$, we may compute the ramification points of B_2, and we find that

$$j_4 \circ B_2 = (2^8(j_8 + 4)/j_8^2) \circ w_2 = j_8^2 + 8j_8.$$

This completes the proof of the Lemma.

2.2. Formal computations in $\mathbf{Z}[[q]]$

In this brief section we will work in the ring $\mathbf{Z}[[q]]$. The q-expansions of each of the modular functions $1/j$, $1/j_2$, $1/j_4$ and $1/j_8$ lie in this ring, and we will identify each of these functions with the corresponding elements of $\mathbf{Z}[[q]]$. The following Lemma states some relations holding between these elements of $\mathbf{Z}[[q]]$.

Lemma 2.4. *We have the following formulae:*
(i) $1/j \in 1/j_2 + (256/j_2^2)\mathbf{Z}[[256/j_2]] \subset \mathbf{Z}[[q]]$.
(ii) $1/j_4 \in 1/j_2 + (16/j_2^2)\mathbf{Z}[[16/j_2]] \subset \mathbf{Z}[[q]]$.
(iii) $1/j_8 \in 1/j_2 + (4/j_2^2)\mathbf{Z}[[8/j_2]] \subset \mathbf{Z}[[q]]$.

From Lemma 2.1, using the fact that B_1 preserves q-expansions, we see that as elements of $\mathbf{Z}[[q]]$,

$$1/j = (1/j_2)/(1+256/j_2)^3 = (1/j_2)(1+\sum_{n=1}^{\infty} a_n(256/j_2)^n) \in 1/j_2 + (256/j_2^2)\mathbf{Z}[[256/j_2]],$$

proving part *(i)* of the Lemma. (There is an elementary formula for the a_n, but we will not need this; all we need is that they are integers.)

From Lemma 2.2 we obtain the equation

$$1/j_2 = 1/j_4 + 16/j_4^2$$

(as elements of $\mathbf{Z}[[q]]$). We may invert this equation to write

$$1/j_4 = 1/j_2(1+\sum_{n=1}^{\infty} a''_n(16/j_2)^n) \in 1/j_2 + (16/j_2^2)\mathbf{Z}[[16/j_2]].$$

This proves part *(ii)*.

Finally, Lemma 2.3 yields the equation

$$1/j_4 = 1/j_8 + 4/j_8^2.$$

Combining this with Lemma 2.2, we see that

$$1/j_2 = 1/j_4 + 16/j_4^2 = 1/j_8 + 20/j_8^2 + 128/j_8^3 + 256/j_8^4.$$

Inverting this equation proves part *(iii)* of the Lemma.

We denote by V the endomorphism of $\mathbf{Z}[[q]]$ induced by the substitution $q \mapsto q^2$. Note that B_2 induces the endomorphism V on q-expansions.

Lemma 2.5. *The following equations hold in* $\mathbf{Z}[[q]]$*:*
(i) $V(1/j) = (1/j_2)^2/(1+16/j_2)^3 \in (1/j_2^2) + (16/j_2^3)\mathbf{Z}_2[[16/j_2]] \subset \mathbf{Z}[[q]]$.
(ii) $V(1/j_2) = (1/j_4)^2/(1+16/j_4) \in (1/j_4^2) + (16/j_4^3)\mathbf{Z}_2[[16/j_4]] \subset \mathbf{Z}[[q]]$.
(iii) $V(1/j_4) = (1/j_8)^2/(1+8/j_8) \in (1/j_8)^2 + (8/j_8^3)\mathbf{Z}_2[[8/j_8]] \subset \mathbf{Z}[[q]]$.

These equations for the action of V follow immediately from the formulae for the action of B_2 given in Lemmas 2.1, 2.2 and 2.3.

2.3. Formulae for modular forms

We begin by studying $\Gamma_0(2)$. Suppose that $k = 2l$ is an even non-negative integer; we may regard a weight k modular form as an l-fold differential on $X_0(2)$ (via the Kodaira-Spencer mapping of [11, A1.3.17]). Concretely, if f is a weight k modular form on $\Gamma_0(2)$, then $f(2\pi i dz)^{\otimes l}$ descends to a rational section of $\Omega^{\otimes l}_{X_0(2)}$. It may have a pole at the image of $z = (1+i)/2$, because the map from the upper half-plane to $Y_0(2)$ is ramified at this point. It may also have poles at the cusps. We can be precise: $f(2\pi i dz)^{\otimes l}$ has poles of order at most l at the cusps, since $(2\pi i dz) = dq/q$, and also has a pole of order at most $l/2$ at $j_2((1+i)/2) = -64$, since the map from the upper half-plane to $X_0(2)$ is ramified of degree two over this point.

Let us define $\omega^{\otimes 4}$ by

$$\Omega^{\otimes 2}_{X_0(2)}(2(\underline{\infty}) + 2(\underline{0}) + (-64)).$$

Note that this line bundle on $X_0(2)$ has degree one, and so there is no line bundle ω of which it is the fourth tensor power; the point is that if we were to pull back $\omega^{\otimes 4}$ to a higher level modular curve which was a fine moduli space, then $\omega^{\otimes 4}$ would pull back to the fourth power of the line bundle ω canonically defined on this higher level modular curve (as is discussed in the appendix). For this reason we will write the l^{th} tensor power of $\omega^{\otimes 4}$ as $\omega^{\otimes 4l}$.

Now if $l = 2m$ is even, we see that modular forms of weight $k = 2l = 4m$ on $\Gamma_0(2)$ correspond to global sections of the line bundle ω^{4m} over $X_0(2)$, while if $l = 2m+1$ is odd, we see that modular forms of weight $k = 2l = 4m+2$ on $\Gamma_0(2)$ correspond to sections of the line bundle $\omega^{4m} \otimes \Omega_{X_0(2)}(\underline{\infty} + \underline{0})$ on $X_0(2)$.

Now E_4 and $V(E_4)$ are linearly independent weight four modular forms on $\Gamma_0(2)$, and so must form a basis for the global sections of the degree one line bundle $\omega^{\otimes 4}$. Thus any modular form of weight $4m$ on $\Gamma_0(2)$ is a degree m homogeneous polynomial in E_4 and $V(E_4)$.

The line bundle $\Omega_{X_0(2)}((\underline{\infty}) + (\underline{0}))$ has degree zero, and so its global sections are spanned by a single element. We write E for the differential

$$E = -dj_2/j_2 = -d\log j_2 = (1 + 24\sum_{n=1}^{\infty}(\sum_{d|n,(d,2)=1} d\,)q^n)\,dq/q$$

which is a section, and which we see is in fact the weight two Eisenstein series on $\Gamma_0(2)$. Thus any modular form of weight $4m+2$ on $\Gamma_0(2)$ is the product of E with a degree m homogeneous polynomial in E_4 and $V(E_4)$.

Since $\omega^{\otimes 4}$ has degree one, it has a one-dimensional subspace of global sections which vanish at $\underline{\infty}$. Define

$$A = (E_4 - V(E_4))/240 = \sum_{n=1}^{\infty}(\sum_{d|n,(n/d,2)=1} d^3)q^n.$$

Then A is a basis for this subspace. Since A vanishes at $\underline{\infty}$, it has no other zeroes, and thus divides any modular form on $\Gamma_0(2)$ which vanishes at $\underline{\infty}$.

There is also a one-dimensional subspace of global sections of $\omega^{\otimes 4}$ which vanish at $\underline{0}$. Define $B = j_2 A$. Then B is a basis of this subspace. From its q-expansion we see that

$$B = (-E_4 + 16V(E_4))/15 = 1 + 16(-q + 7q^2 - 28q^3 + 71q^4 - 126q^5 + \cdots).$$

Since B vanishes at $\underline{0}$, it has no other zeroes, and so divides any modular form which vanishes at $\underline{0}$. In particular, the product AB is the unique cuspform of weight 8 on $\Gamma_0(2)$, and it divides any other cuspform on $\Gamma_0(2)$.

Now A and B are a basis for the global sections of $\omega^{\otimes 4}$, and so any weight $4m$ modular form on $\Gamma_0(2)$ is a degree m homogeneous polynomial in A and B. The pair A, B has an advantage over the pair $E_4, V(E_4)$ which is expressed in the following Lemma.

Lemma 2.6. *(i) Let A be the unique weight four modular form on $\Gamma_0(2)$ whose q-expansion begins $A = q + \cdots$, and let $B = j_2 A$. Then any modular form f of weight $4m$ on $\Gamma_0(2)$ may be written uniquely in the form $f = F(A, B)$, with F a homogeneous polynomial of degree m. The modular form f has its q-expansion coefficients in a ring $R \subset \mathbf{C}$ if and only if the polynomial F has coefficients in R.*

(ii) Let E be the weight two Eisenstein series on $\Gamma_0(2)$, $E = 1 + 24q + \cdots$. A modular form f of weight $4m+2$ on $\Gamma_0(2)$ may be written uniquely in the form $f = F(A, B)E$, with F a homogeneous polynomial of degree m. The modular form f has its q-expansion coefficients in a subring $R \subset \mathbf{C}$ if and only if when we write $g = F(A, B)E$, the degree m homogeneous polynomial F has coefficients in R.

(iii) $E^2 = 64A + B$.

The first claim of part it (i) simply restates what we have already shown. The second claim is immediate, since the polynomial F is computed recursively as follows: if

$$f = \sum_{n=1}^{\infty} a_n q^n,$$

then one writes

$$f = a_0 B^m + ((f - a_0 B^m)/A)A.$$

Now $(f - a_0 B^m)/A$ has weight $4(m-1)$, so the result follows by induction, since A and B have integral q-expansion coefficients, and A is normalized to have its first q-expansion coefficient equal to one.

The first claim of part *(ii)* again restates what we have already shown, while the second claim follows from part *(i)*, since E has integral q-expansion coefficients, and constant term 1, so that f has q-expansion coefficients in R if and only if the same is true of f/E.

To prove part *(iii)*, note that E^2 is a section of $\Omega_{X_0(2)}(\underline{\infty} + \underline{0})^{\otimes 2}$, and so as a section of $\omega^{\otimes 4}$, E^2 has a zero at $j_2 = -64$. This observation shows that E^2 is a scalar multiple of $A(j_2 + 64) = 64A + B$. A comparison of constant terms shows that in fact $E^2 = 64A + B$. Thus part *(iii)* is proved, and with it, the Lemma.

We now turn to studying modular forms on $\Gamma_1(4)$. The modular curve $X_1(4)$ maps isomorphically to the modular curve $X_0(4)$. The $\Gamma_1(4)$ moduli problem has the advantage over the $\Gamma_0(4)$ moduli problem that on elliptic curves, it is representable, which is to say that $Y_1(4)$ is a fine moduli space over $\mathbf{C}$. However, on generalized elliptic curves it is not representable, and $X_1(4)$ is not a fine moduli space. Thus the sheaf ω on $Y_1(4)$ does not extend to $X_1(4)$, although the sheaf $\omega^{\otimes 2}$ does. The problem cusp is the cusp $\underline{1/2}$: the matrix equation

$$\begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

shows that any modular form of odd weight on $\Gamma_1(4)$ necessarily vanishes at the cusp $\underline{1/2}$. For if f has weight k on $\Gamma_1(4)$, then

$$\lim_{z\to i\infty} f|_k \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \lim_{z\to i\infty} f|_k \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

(since f has weight k on $\Gamma_1(4)$)

$$= \lim_{z\to i\infty} f|_k \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (-1)^k \lim_{z\to i\infty} f|_k \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Thus if k is odd,

$$\lim_{z\to i\infty} f|_k \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = 0,$$

i. e. f vanishes at the cusp $\underline{1/2}$.

The situation is better for even weight modular forms on $\Gamma_1(4)$. Note that

$$B_1^*\omega^{\otimes 4} = B_1^*\Omega^{\otimes 2}_{X_0(2)}(2(\underline{\infty}) + 2(\underline{0}) + (-64)) = \Omega^{\otimes 2}_{X_1(4)}(2(\underline{\infty}) + 2(\underline{1/2}) + 2(\underline{0})),$$

where $B_1 : X_1(4) = X_0(4) \to X_0(2)$ has the same meaning as in section 2.1. This follows from the fact that B_1 is ramified over $\underline{0}$ and over $j_2 = -64$.

Thus it is consistent with our earlier notation to write

$$\omega^{\otimes 2} = \Omega_{X_1(4)}(\underline{\infty} + \underline{1/2} + \underline{0}).$$

This line bundle has degree one. If f is a modular form of even weight $2l$, then $f(z)(2\pi i z)^l$ yields a section of $\omega^{\otimes 2}$, and this creates a bijection between such modular forms and such sections. Since $\omega^{\otimes 2}$ has degree one, it has a two dimensional space of global sections. Since a consideration of q-expansions shows that E and $V(E)$ are linearly independent, they must span the space of of global sections of $\omega^{\otimes 2}$, and thus a weight $2l$ modular form is a degree l homogeneous polynomial in E and $V(E)$.

The theta function of the quadratic form $x_1^2 + x_2^2$

$$\theta = 1 + 4q + 4q^2 + 8q^5 + \cdots$$

is a well-known modular form of weight one on $\Gamma_1(4)$. It has nebentypus τ, where τ is the unique non-trivial character of conductor four. This function is also the weight one Eisenstein series with character τ:

$$\theta = 1 + (2/L(\tau,0)) \sum_{n=1}^{\infty} (\sum_{d|n} \tau(d)) q^n.$$

As we have observed, any modular form of odd weight must vanish at the cusp $\underline{1/2}$. In particular θ vanishes at $\underline{1/2}$, so the same is true of θ^2. However, a weight two modular form has exactly one zero ($\omega^{\otimes 2}$ has degree one) and so θ has no other zeroes. Thus if g is a modular form of weight $2l+1$, g/θ is a modular form of weight $2l$. Thus any modular form of weight $2l+1$ is a product of θ and a modular form of weight $2l$. In particular, any weight one modular form is a scalar multiple of θ. This shows that θ equals the weight one Eisenstein series, if one didn't already know this fact.

To complete our discussion, we should express θ^2 in terms of E and $V(E)$. A consideration of q-expansions yields the following formula, due to Jacobi: $\theta^2 = (E + 2V(E))/3$.

The quotient of the extended upper half-plane by $\Gamma_1(4) \cap \Gamma_0(8)$ maps isomorphically onto $X_0(8)$. However, the $\Gamma_1(4) \cap \Gamma_0(8)$ moduli problem is representable, and when thought of as representing this moduli problem, $X_0(8)$ is a fine moduli space. In particular, it has a line bundle ω, the line bundle of relative differentials of the the universal (generalized) elliptic curve over $X_0(8)$. The Kodaira-Spencer isomorphism

$$\omega^{\otimes 2} = \Omega_{X_0(8)}(\underline{\infty} + \underline{1/4} + \underline{1/2} + \underline{0})$$

shows that ω has degree one. In particular its space of global sections is two dimensional. Since θ and $V(\theta)$ are linearly independent global sections, they must form a basis for this space. Thus a modular form of weight k on $\Gamma_1(4) \cap \Gamma_0(8)$ is a degree k homogeneous polynomial in θ and $V(\theta)$.

We finish this section by proving the following useful Lemma:

Lemma 2.7. *(i)* $E/V(E) \in 1 + (8/j_2)\mathbf{Z}[[8/j_2]]$.
(ii) $B/V(B) \in 1 + (16/j_2)\mathbf{Z}[[16/j_2]]$.
(iii) $\theta/V(\theta) \in 1 + (4/j_8)\mathbf{Z}[[4/j_8]]$.

To prove part *(i)*, we regard E and $V(E)$ as modular forms on $\Gamma_0(4)$. The proof of Lemma 2.2 shows that $j_2 + 64 = (j_4 + 32)^2/(j_4 + 16)$, and thus as a section of the degree one line bundle $\omega^{\otimes 2}$ on $X_0(4)$, E vanishes at $j_4 = -32$. Thus the rational function $(E - V(E))/E$ on $X_0(4)$ has a zero at $\underline{\infty}$ and a pole at $j_4 = -32$, and thus is a scalar multiple of $1/(j_4 + 32)$. A consideration of q-expansions shows that $(E - V(E))/E = 24/(j_4 + 32)$. Rearranging yields

$$E/V(E) = (j_4{+}32)/(j_4{+}8) = (1{+}32/j_4)/(1{+}8/j_4) \in 1{+}(8/j_4)\mathbf{Z}[[8/j_4]] = 1{+}(8/j_2)\mathbf{Z}[[8/j_2]].$$

(The last equation follows from Lemma 2.4, part *(ii)*.)

To prove part *(ii)*, use Lemma 2.6, which shows that $B = E^2/(1{+}64A/B) = E^2/(1{+}64/j_2)$. Thus

$$\begin{aligned} B/V(B) &= E^2(1 + 64/V(j_2))/V(E^2)(1 + 64/j_2) \\ &= (1 + 32/j_4)^2(1 + 64/V(j_2))/(1 + 8/j_4)^2(1 + 64/j_2). \end{aligned}$$

Now

$$1 + 64/V(j_2) = 1 + 64/j_4(j_4 + 16) = (j_4 + 8)^2/j_4(j_4 + 16) = (1 + 8/j_4)^2/(1 + 16/j_4).$$

Thus

$$B/V(B) = (1{+}32/j_4)^2/(1{+}16/j_4)(1{+}64/j_2) \in 1{+}(16/j_4)\mathbf{Z}[[16/j_4]] = 1{+}(16/j_2)\mathbf{Z}[[16/j_2]].$$

(Again, the last equation follows from part *(ii)* of Lemma 2.4.)

To prove part *(iii)*, regard θ and $V(\theta)$ as modular forms on $\Gamma_1(4) \cap \Gamma_0(8)$. Lemma 2.3 shows that $j_8(\underline{1/2}) = -8$, and so as a section of the degree one line bundle ω on $X_0(8)$, we see that θ vanishes at $j_8 = -8$. Thus the rational function $(\theta - V(\theta))/\theta$ has a zero at $\underline{\infty}$ and a pole at $j_8 = -8$, and so is a scalar multiple of $1/(j_8 + 8)$. A consideration of q-expansions shows that $(\theta - V(\theta))/\theta = 4/(j_8 + 8)$. Rearranging, we get

$$\theta/V(\theta) = (j_8 + 8)/(j_8 + 4) = (1 + 8/j_8)/(1 + 4/j_8) \in 1 + 4/j_8\mathbf{Z}[[4/j_8]].$$

This completes the proof of the Lemma.