CHAPTER I

EXPLANATION AND EXAMPLES

In this first chapter, we present some consequences of our Main Theorem in the language of classical modular forms on $\mathbf{SL}(2, \mathbf{Z})$ and $\Gamma_0(2)$, in order to provide motivation for what follows. The Theorems which we state below do not incorporate all the implications that our Main Theorem has for classical modular forms, because they make no reference to forms on $\Gamma_1(2^n)$ for $n > 1$ with non-trivial nebentypus. However, they should provide a helpful context within which to regard the work of the subsequent chapters.

Before stating the Theorems, it will be convenient to introduce some terminology.

Suppose that V is a finite dimensional vector space of dimension n over $\overline{\mathbf{Q}}$, and that T is a linear operator on V. Let $\lambda_1, \ldots, \lambda_n$ be the list of eigenvalues of T, where each eigenvalue is listed with its correct multiplicity. We write $s_i = \mathrm{ord}_2(\lambda_i)$, and refer to s_i as the slope of the eigenvalue λ_i, and more generally as a slope of T. (We should really modify the term "slope" in our definition by the adjective "2-adic", since we could equally well consider the slopes for other primes p. However, since we work only with the prime 2, we don't do this.) If $v \in V$ is an eigenvector for T with eigenvalue λ_i, we also refer to s_i as the slope of v. If s is a slope of T, the multiplicity of s is the number of indices i such that $s_i = s$. We refer to the least of the s_i $(i = 1, \ldots, n)$ as the minimal slope of T. If v is an eigenvector of T whose slope is minimal, we say that v is a minimal slope eigenvector of T.

For any positive even integer k, the space of cusp forms of level one and weight k is finite dimensional over $\mathbf{C}$. The cusp forms whose q-expansion coefficients lie in $\overline{\mathbf{Q}}$ form a finite dimensional vector space over $\overline{\mathbf{Q}}$ underlying this $\mathbf{C}$-vector space, which we will denote by V_k. For any prime l, the Hecke operator T_l restricts to an operator on V_k. Its action is most easily described via q-expansions: if $f = \sum_{n=1}^{\infty} a_n q^n \in V_k$, then

$$T_l(f) = \sum_{n=1}^{\infty} a_{nl} q^n + l^{k-1} \sum_{n=1}^{\infty} a_n q^{nl}.$$

One of the primary objects of this thesis is to determine the value and multiplicity of the minimal slope of the Hecke operator T_2 acting on the space V_k, for all weights k.

Here is a table of the slopes of T_2 on V_k, for even integers k such that $12 \leq k \leq 50$. The minimal slopes are printed in boldface:

k	Slopes of T_2 on V_k
12	**3**
14	no cusp forms
16	**3**
18	**4**
20	**3**
22	**5**
24	**3**, 7
26	**4**
28	**3**, 8
30	**6, 6**
32	**3**, 7
34	**4**, 8
36	**3**, 9, 9
38	**5**, 8
40	**3**, 7, 11
42	**4**, 9, 12
44	**3**, 8, 11
46	**6, 6**, 13
48	**3**, 7, 12, 15
50	**4**, 8, 13

It is a Theorem of Hatada [10], which the reader will observe is verified by the above table for $k \leq 50$, that the minimal slope of T_2 on V_k is at least three, and is greater than three unless $k \equiv 0 \mod 4$.

A close examination of this table suggests a more precise result than that of Hatada, and in fact we have the following consequence of our Main Theorem:

Theorem 1.1. *If k is an even integer such that $k \geq 12$ and $k \neq 14$, then*

(i) if $k \equiv 0, 4, 8$ or $12 \mod 16$, then the minimal slope of T_2 on V_k is three, and it occurs with multiplicity one;

(ii) if $k \equiv 2$ or $10 \mod 8$, then the minimal slope of T_2 on V_k is four, and it occurs with multiplicity one;

(iii) if $k \equiv 6 \mod 16$, then the minimal slope of T_2 on V_k is five, and it occurs with multiplicity one;

(iv) if $k \equiv 14 \mod 16$, then the minimal slope of T_2 on V_k is six, and it occurs with multiplicity two.

Furthermore, in case (iv), the two eigenvalues of minimal slope are distinct.

Note that any integer k falls into exactly one of these four categories (the restriction $k \geq 12$ and $k \neq 14$ being made simply to ensure that V_k is non-trivial), and so Theorem 1.1 describes the value and multiplicity of the minimal slope T_2 on V_k for every weight k.

Theorem 1.1 shows that for any fixed k such that $V_k \neq 0$, either there is a unique minimal slope eigenvalue for T_2 on V_k, or there are two distinct minimal slope eigenvalues for T_2 on V_k. Thus we see that if λ is a minimal slope eigenvalue of T_2 on V_k then there is a unique (up to scaling) T_2-eigenvector with eigenvalue λ. Furthermore, since the Hecke operators on

V_k commute among themselves, we see that this eigenvector is in fact an eigenform for all the Hecke operators. We can pin down this eigenvector uniquely by insisting that it be normalized (which is to say, have its first q-expansion coefficient equal to one).

Our Main Theorem does more than just compute the minimal slopes for each weight. It shows that the Hecke eigenvalues of these minimal slope T_2-eigenvectors, for the Hecke operators T_l with l an *odd* prime, depend rigid-analytically on k. This statement is rather imprecise as it stands, because for those values of k such that $k \equiv 14 \mod 16$, there are two minimal slope eigenforms, and so these Hecke eigenvalues will be multi-valued functions of k. In order to make a precise statement, as well as to incorporate the behaviour of the T_2-eigenvalue, it is necessary to transfer our attention from modular forms on $\mathbf{SL}(2, \mathbf{Z})$ to modular forms on $\Gamma_0(2)$.

If k is any even integer, the space of modular forms of weight k on $\Gamma_0(2)$ which vanish at the cusp infinity is a finite dimensional vector space over $\mathbf{C}$, and the space of such modular forms whose q-expansion coefficients lie in $\overline{\mathbf{Q}}$ is a finite dimensional $\overline{\mathbf{Q}}$-vector space underlying this $\mathbf{C}$-vector space, which we denote by V_k' . Of course V_k' contains V_k as a subspace. If l is an odd prime, the Hecke operator T_l on V_k extends to an operator on V_k', given by the same formula on q-expansions. However, the Hecke operator T_2 does not so extend. Rather, one finds that the Atkin operator U_2 acts on V_k' (see [2]). This operator is defined as follows: if $f = \sum_{n=1}^{\infty} a_n q^n$, then $U_2(f) = \sum_{n=1}^{\infty} a_{2n} q^n$.

If $k < 4$, then V_k' is trivial. Otherwise, V_k' splits into a direct sum

$$V_k' = V_{k,Eis}' \oplus V_{k,cusp}',$$

where $V_{k,Eis}$ is one dimensional and is spanned by the evil Eisenstein series

$$\sum_{n=1}^{\infty} \Big(\sum_{\substack{d|n \\ n/d \text{ odd}}} d^k \Big) q^n$$

(which is a Hecke eigenform, and in particular is a U_2-eigenvector of slope $k-1$), while $V_{k,cusp}'$ is the subspace of V_k' consisting of the cusp forms, i. e. those modular forms which vanish at the cusp zero as well as the cusp infinity.

The theory of new forms developed in [2] shows that $V_{k,cusp}'$ splits as a direct sum $V_{k,old}' \oplus V_{k,new}'$, and that each summand is closed under the action of U_2.

Let us recall the definition of $V_{k,old}'$. Given any cusp form $f(\tau)$ of level one and weight k, lying in V_k, we can form a pair of forms $f(\tau)$ and $f(2\tau)$ on $\Gamma_0(2)$, which both lie in V_k', and span a two dimensional subspace of V_k'. Now suppose that $f(\tau)$ is a T_2-eigenform, with eigenvalue equal to λ.

The equations

$$U_2(f(\tau)) = T_2(f(\tau)) - 2^{k-1} f(2\tau) = \lambda f(\tau) - 2^{k-1} f(2\tau)$$

and

$$U_2(f(2\tau)) = f(\tau)$$

show that the two dimensional subspace spanned by $f(\tau)$ and $f(2\tau)$ is closed under the action of U_2, and that the characteristic polynomial of U_2 on this two dimensional space is equal to

$x^2 - \lambda x + 2^{k-1}$. In particular, if $\mathrm{ord}_2(\lambda) < (k-1)/2$, we see that in this two dimensional space there is a unique (up to scaling) U_2-eigenform whose eigenvalue α satisfies $\mathrm{ord}_2(\alpha) = \mathrm{ord}_2(\lambda)$.

The space $V'_{k,old}$ is by definition the sum of these two dimensional subspaces of V'_k spanned by $f(\tau)$ and $f(2\tau)$, as $f(\tau)$ ranges over all elements of V_k. Since V_k has a basis of T_2-eigenforms, we see that it is enough to consider those f which are T_2-eigenforms, and thus we see that $V'_{k,old}$ is indeed closed under U_2.

The space $V'_{k,new}$ is defined to be the orthogonal complement in V'_k to $V'_{k,old}$ under the Petersson inner product. It is shown in [2] that $V'_{k,new}$ has a basis of U_2-eigenvectors, on each of which the eigenvalue of U_2 is equal to $\pm 2^{k/2-1}$.

The following Lemma is easily derived from the preceding description of the action of U_2 on V'_k:

Lemma 1.2. *Let k be an integer such that $k \geq 12$ and $k \neq 14$. Let λ be a minimal slope eigenvalue for T_2 on V_k. Let f be a normalized minimal slope eigenvector for T_2 in V_k whose eigenvalue is equal to λ (recall that f is uniquely determined). Let α be the unique root of $x^2 - \lambda x + 2^{k-1}$ such that $\mathrm{ord}_2(\alpha) = \lambda$. (Theorem 1.1, together with our assumptions on k, shows that $\mathrm{ord}_2(\lambda) < k/2 - 1 < (k-1)/2$, and so there is indeed a unique such α.) Then $f(\tau) - (2^{k-1}/\alpha) f(2\tau)$ is a normalized minimal slope eigenvector for U_2 in V'_k, with eigenvalue α, which is also an eigenform for the Hecke operators T_l for each odd prime l, with the same eigenvalues as f.*

This construction establishes a bijection between the normalized minimal slope eigenvectors for T_2 in V_k and the normalized minimal slope eigenvectors for U_2 in V'_k.

We will refer to the normalized minimal slope eigenvectors for U_2 on V'_k as the minimal slope eigenforms of weight k. This is reasonable terminology, since we have seen (at least when $k \geq 12$ and $k \neq 14$) that they are indeed eigenforms for all the Hecke operators.

We now have the following Theorem on the rigid analytic variation of the minimal slope eigenforms:

Theorem 1.3. *There is a formal q-expansion $f(u) = \sum_{n=1}^{\infty} a_n(u) q^n$, whose coefficients $a_n(u)$ are rigid analytic functions of u converging on the annulus $|8| \geq |u| \geq |64|$ in $\mathbf{C}_2$, defined over $\mathbf{Q}_2$ and of norm less than or equal to one, and a rigid analytic function $k(u)$ mapping this annulus to the disc $|2| \geq |k| \geq 0$ in $\mathbf{C}_2$, with the following properties:*

(i) $a_1 = 1$ and $a_2 = u$.

(ii) The function $k(u)$ induces an isomorphism between the annulus $|8| \geq |u| > |64|$ and the annulus $|k - 14| > |16|$, and on this annulus, $|u| = |4(k-14)|$.

(iii) The function $k(u)$ induces a finite map of degree two between the annulus $|u| = |64|$ and the disc $|16| \geq |k| \geq 0$, with exactly two ramification points, which are a conjugate pair of elements in $\mathbf{Q}_2(\sqrt{3})$.

(iv) If k is an even integer such that $k \geq 4$ then the q-expansions $f(u)$ for the one or two values of u lying over k are the q-expansions of modular forms on $\Gamma_0(2)$, which are precisely the minimal slope eigenforms of weight k.

(v) If $u, u' \in \mathbf{C}_2$ such that $|8| \geq |u|, |u'| \geq |64|$, then

$$f(u) \equiv f(u') \mod (u - u')\mathcal{O}_2[[q]].$$

(vi) If $u \in \mathbf{C}_2$ such that $|8| \geq |u| \geq |64|$ and l is an odd prime then

$$a_l(u) \equiv 1 + l^{k(u)-1} \begin{cases} \bmod 8(u-8)(u-32) & l \equiv 1 \mod 8 \\ \bmod (u-8)(u-32) & l \equiv 3 \mod 8 \\ \bmod 4(u-8)(u-32) & l \equiv 5 \mod 8 \\ \bmod 2(u-8)(u-32) & l \equiv 7 \mod 8. \end{cases}$$

In the remainder of this chapter, we will elaborate on the statement of this Theorem.

First let us point out that parts *(ii)*, *(iii)* and *(iv)* of Theorem 1.3, combined with the reasoning of the preceding Lemma, have Theorem 1.1 as a corollary.

Example. Let us give some examples of minimal slope eigenforms. Via part *(iv)* of Theorem 1.3, this will allow us to compute $f(u)$ and $k(u)$ for some particular values of u.

Note first that the restriction $k \geq 4$ in part *(iv)* is certainly necessary, for otherwise V'_k is empty. Furthermore, for $k = 4$ and $k = 6$, $V'_k = V'_{k,Eis}$, while if $k \geq 8$, then $k-1$ (which is the slope of U_2 on the one dimensional space $V'_{k,Eis}$) is greater than the minimal slope for U_2 as given by Theorem 1.3, and so the minimal slope eigenforms are necessarily cusp forms.

We now list the minimal slope eigenforms of weights k between four and fourteen.

$k = 4$: In this case V'_k is spanned by the Eisenstein series

$$\sum_{n=1}^{\infty} \Big(\sum_{\substack{d \mid n \\ n/d \text{ odd}}} d^3 \Big) q^n = q + 8q^2 + 28q^3 + 64q^4 + 126q^5 + \cdots,$$

which in particular must be the minimal slope eigenform of weight 4. Its U_2-eigenvalue is $u = 8$, and thus $k(8) = 4$ and

$$f(8) = 8 + 8q^2 + 28q^3 + 64q^4 + 126q^5 + \cdots.$$

Note that this formula for $f(8)$ also follows from part *(vi)* of Theorem 1.3.

$k = 6$: In this case V'_k is spanned by the Eisenstein series

$$\sum_{n=1}^{\infty} \Big(\sum_{\substack{d \mid n \\ n/d \text{ odd}}} d^5 \Big) q^n = q + 32q^2 + 244q^3 + 1024q^4 + 3126q^5 + \cdots,$$

which in particular is the minimal slope eigenvector for U_2 on V'_6. It has U_2-eigenvalue $u = 32$, and thus $k(32) = 6$ and

$$f(32) = q + 32q^2 + 244q^3 + 1024q^4 + 3126q^5 + \cdots.$$

This formula for $f(32)$ also follows from part *(vi)* of Theorem 1.3.

$k = 8$: There is a unique normalized cusp form of weight 8 on $\Gamma_0(2)$, whose q-expansion is equal to

$$q \prod_{n=1}^{\infty} (1-q^n)^8 (1-q^{2n})^8 = q - 8q^2 + 12q^3 + 64q^4 - 210q^5 + \cdots.$$

In particular, this is the minimal slope eigenform of weight 8, and it has U_2-eigenvalue $u = -8$. Thus in the notation of Theorem 1.3, we see that $k(-8) = 8$, and that

$$f(-8) = q - 8q^2 + 12q^3 + 64q^4 - 210q^5 + \cdots.$$

$k = 10$: There is a unique normalized cusp form of weight 10 on $\Gamma_0(2)$, whose q-expansion begins $q+16q^2-156q^3+256q^4+870q^5+\cdots$, which in particular is the minimal slope eigenform of weight 10, and it has U_2-eigenvalue $u = 16$. Thus $k(16) = 10$, and

$$f(16) = q + 16q^2 - 156q^3 + 256q^4 + 870q^5 + \cdots .$$

$k = 12$: The is a unique normalized cusp form of weight 12 on $\mathbf{SL}(2, \mathbf{Z})$, usually denoted Δ, whose q-expansion is

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \cdots .$$

Let $x^2 + 24x + 2^{11} = (x - \alpha)(x - \beta)$, where $\mathrm{ord}_2(\alpha) = 3$, $\mathrm{ord}_2(\beta) = 8$. (Note that $\alpha \in \mathbf{Z}_2$. The first few terms in the 2-adic expansion of α are $\alpha = 8+32+64+128+512+\cdots$.) Then Lemma 1.2 implies that

$$\Delta(\tau) - \beta\Delta(2\tau) = q + \alpha q^2 + 252q^3 + \alpha^2 q^4 + 4830q^5 \cdots$$

is the minimal slope eigenform of weight 12, and it has U_2-eigenvalue $u = \alpha$. So we see that $k(\alpha) = 12$, and that

$$f(\alpha) = q + \alpha q^2 + 252q^3 + \alpha^2 q^4 + 4830q^5 \cdots .$$

$k = 14$: The space of modular forms on $\Gamma_0(2)$ of weight 14 is spanned by two newforms, whose q-expansions begin

$$q - 64q^2 - 1836q^3 + 4096q^4 + 3990q^5 + \cdots$$

and

$$q + 64q^2 + 1236q^3 + 4096q^4 - 57450q^5 + \cdots .$$

These are the two minimal slope eigenforms of weight 14, whose U_2 eigenvalues are respectively $u = -64$ and $u = 64$. Thus we see that $k(-64) = k(64) = 14$, and that

$$f(-64) = q - 64q^2 - 1836q^3 + 4096q^4 + 3990q^5 + \cdots$$

and

$$f(64) = q + 64q^2 + 1236q^3 + 4096q^4 - 57450q^5 + \cdots .$$

Example. The following examples should serve to illustrate the congruences stated in parts *(v)* and *(vi)* of Theorem 1.3:

(a) Let $u = 16$ and $u' = -8$, so that $u - u' = 24$. Then

$$f(16) - f(-8) = 24(q^2 - 7q^3 + 8q^4 + 45q^5 + \cdots).$$

(b) Let $u = 64$ and $u' = -64$, so that $u - u' = 128$. Then

$$f(64) - f(-64) = 128(q^2 + 24q^3 - 480q^5 + \cdots).$$

Remark. Since the functions $a_n(u)$ are rigid analytic on the annulus $|8| \geq |u| \geq |64|$, with norm less than or equal to one, they are elements of the ring $\mathbf{Z}_2\langle u/8, 64/u\rangle$. From this information alone, we could deduce that

$$a_n(u) - a_n(u') \equiv 0 \mod 64(u-u')/uu'.$$

Part *(v)* provides a strengthening of this congruence. It is equivalent to the statement that each function $a_n(u)$ lies in the subring $\mathbf{Z}_2 + u\mathbf{Z}_2\langle u/8, 64/u\rangle$ of the ring $\mathbf{Z}_2\langle u/8, 64/u\rangle$.

Remark. From part *(ii)* we deduce that $k(u) - 14 \in (u/4)\mathbf{Z}_2\langle u/8, 64/u\rangle$. In particular, if u and u' are two points of $\mathbf{C}_2$ such that $|8| \geq |u|, |u'| \geq |64|$, then

$$k(u) \equiv k(u') \mod (u-u')/4.$$

Remark. If $u \in \mathbf{C}_2$ such that $|8| \geq |u| \geq |64|$, and if we apply part *(v)* with $u' = 8$, and use the formula for $f(8)$ computed above, we find that

$$a_n(u) \equiv 1 + l^3 \mod (u-8).$$

On the other hand, it follows from the preceding Remark that

$$4(k(u)-4) = 4(k(u)-k(8)) \equiv 0 \mod (u-8),$$

which shows that

$$l^{k(u)-1} - l^3 = l^3(l^{k(u)-4} - 1) \equiv 0 \mod (u-8).$$

Thus from the parts of Theorem 1.1 other than part *(vi)* we can deduce that

$$a_n(u) \equiv 1 + l^{k(u)-1} \mod (u-8).$$

Similarly, using the formula for $f(32)$ computed above, we can deduce from the parts of Theorem 1.1 other than part *(vi)* that

$$a_n(u) \equiv 1 + l^{k(u)-1} \mod (u-32).$$

The congruence of part *(vi)* provides a strengthening of these two congruences. (Note that since $\mathbf{Z}_2\langle u/8, 64/u\rangle$ is not a UFD, even the case of part *(vi)* when $l \equiv 3 \mod 8$ is not an automatic consequence of the preceding parts.)

In the case when $k(u) = 12$, so that $f(u)$ is an old form attached to Δ, the congruences of part *(vi)* follow from the congruences stated on page 4 of [15]. Thus part (*vi*) of Theorem 1.3 provides a generalization of (some of) these congruences for Δ to the whole minimal slope family $f(u)$.