APPENDIX

The object of this appendix to briefly review some of the foundational results of [5], [6], [8] and [11] in the context of the prime 2, and to compare them with the theory developed in the body of the thesis.

In sections 1 and 2, we review the definition of overconvergent 2-adic modular forms of integral weight. This is the theory developed in [8] and [11]. In section 3 we review the definition of overconvergent modular forms of 2-adic weight. This is the theory developed in [5] and [6].

A.1 The curves $X_0(2^n)$, canonical subgroups, and the Deligne-Tate map

For any non-negative integer n, the modular curve $X_0(2^n)$ is defined over $\mathbf{Q}$, and is a coarse moduli space for the problem of classifying cyclic 2^n-isogenies between (generalized) elliptic curves over $\mathbf{Q}$-algebras. To any cyclic 2^n isogeny we may associate the elliptic curve which is the domain of the isogeny, yielding a map $\pi_{2^n} : X_0(2^n) \to X_0(1)$.

We base change these curves and maps to $\mathbf{Q}_2$, where we study them using rigid analysis.

The curve $X_0(1)$ is of genus zero, and is uniformized by the modular invariant j. The point $j = \infty$ corresponds to the degenerate elliptic curve over $\mathbf{C}_2$. The points $\infty > |j| > 1$ correspond to elliptic curves over $\mathbf{C}_2$ which are Tate curves, i. e. elliptic curves having multiplicative reduction. The points satisfying $|j| = 1$ correspond to elliptic curves with good ordinary reduction. The points with $1 > |j|$ correspond to elliptic curves with good supersingular reduction. (It is well known that $j = 0$ is the unique supersingular modulus modulo two.)

If E is an elliptic curve over $\mathbf{C}_2$ with $|j(E)| \geq 1$, then the reduction of E modulo two has a unique connected subgroup scheme of order 2^n, for any $n \geq 0$, which is the kernel of the n^{th} power of Frobenius. These subgroup schemes lift uniquely to subgroup schemes of E, and yield a compatible family of sections of π_{2^n}, for each $n \geq 0$, over the disc $|j| \geq 1$. (We call this a disc, because we include the point $j = \infty$.)

The theory of the canonical subgroup shows that each of these sections extends some way into the disc $1 > |j|$. To be correct about this, one should first of all pass from $X_0(1)$ to $X(3)$, say, which is a fine moduli scheme. There is a weight one form on $\Gamma(3)$, call it A, which is a lifting of the Hasse invariant (which itself is a modular form of level one, but only defined modulo two), by [11, p. 31].

Suppose that E is an elliptic curve over $\mathbf{C}_2$ with some level three structure and that ω is a differential on E with non-zero reduction modulo two. Then if $|A(E,\omega)| > |2|^{1/(2^{n-2}\cdot 3)}$, E has a canonical subgroup which is cyclic of order 2^n [8], [11]. (Note that since any two choices of A are congruent modulo two, when $n \geq 1$ the region cut out by this inequality is independent of the choice of A.) This yields a map from the region $|A(E,\omega)| > |2|^{1/(2^{n-2}\cdot 3)}$ of $X(3)$ to $X_0(2^n)$. Since the canonical subgroup is independent of the level three structure on E, this map factors through the image of this region in the j-line. We will determine this region explicitly.

Let E_4 denote the weight four Eisenstein series of level 1,

$$E_4 = 1 + 240 \sum_{n=1}^{\infty} (\sum_{d|n} d^3)\, q^n.$$

Then $E_4 \equiv 1 \mod 8$ (in fact modulo 16, but we don't need that here), and so E_4 is a lifting over $\mathbf{Z}_2$ of the fourth power of the Hasse invariant (which is well-defined modulo eight). Thus $|A(E,\omega)| > |2|^{1/(2^{n-2}\cdot 3)}$ if and only if $|E_4(E,\omega)| > |2|^{1/(2^{n-4}\cdot 3)}$. Now recall the formula $j = E_4^3/\Delta$, where

$$\Delta = q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

is the unique normalized weight twelve cusp form of level one. Since $\Delta(E,\omega)$ is a 2-adic unit if E has good reduction and ω has non-zero reduction modulo two, we see that the image of the region $|A(E,\omega)| > |2|^{1/(2^{n-2}\cdot 3)}$ in the j-line is the region $|j| > |2|^{1/2^{n-4}}$. Thus over this region, the canonical subgroup gives rise to a section of π_{2^n}, which we will call the canonical section and denote σ_{2^n}.

If r is an element of $\mathbf{C}_2$ such that $1 \geq |r| \geq |256|$, we write D_r for the open disc $|j| > |r|$. If $|r| \geq 2^{1/2^{n-4}}$ for some n, then D_r lies in the domain of the canonical section σ_{2^n}. Our Lemma 2.4 may be regarded as yielding explicit formulae for these canonical sections for $n = 1, 2, 3$. One minor difference is that we work with j_2 rather than with j. This is simply because j_2 is easier to compute with. Since the canonical section σ_2 identifies the disc $|j| > |256|$ in $X_0(1)$ with the disc $|j_2| > |256|$ in $X_0(2)$, as long as we are restricting our attention to the disc D_r, for $|r| > |256|$, we are free to work with either choice of uniformizer.

The Deligne-Tate map is the map φ from D_{16} to D_{256} defined in modular terms by taking an elliptic curve E to its quotient by its canonical subgroup. In symbols,

$$\varphi = \pi_2 \circ w_2 \circ \sigma_2,$$

where w_2 denotes the Atkin-Lehner involution of $X_0(2)$: w_2 sends any 2-isogeny $E \to E'$ to the dual isogeny $E' \to E$. Katz shows that the map φ is finite and flat of degree two, as a map between rigid analytic spaces [11, Theorem 3.10.1,III]. More generally, he shows that if $1 \geq |r| \geq |16|$, φ induces a finite flat map of degree two between D_r and D_r^2.

When $r = 1$, D_1 is the disc $|j| > 1$ parameterizing the Tate curves $\mathbf{G}_m/q^{\mathbf{Z}}$. Since $j = 1/q + 744 + 196884q + \cdots$, we see that $q = 1/j + 744/j^2 + 750420/j^3 + \cdots$, and so q is a uniformizer of this disc. Now the canonical subgroup of the Tate curve $\mathbf{G}_m/q^{\mathbf{Z}}$ is μ_2, and so φ, when restricted to D_1, is simply the map $q \mapsto q^2$, which we have called V. (In [8] this map is called Frob, and in [11] it is called φ.)

The ring of bounded rigid analytic functions on D_r is $\mathbf{C}_2 \otimes \mathcal{O}_2[[r/j]]$. If $1 \geq |r| \geq |16|$, then the result of Katz shows that $V(\mathbf{C}_2 \otimes \mathcal{O}_2[[r^2/j]]) \subset \mathbf{C}_2 \otimes \mathcal{O}_2[[r/j]]$, and that in fact $\mathbf{C}_2 \otimes \mathcal{O}_2[[r/j]]$ is locally free of rank two over $V(\mathbf{C}_2 \otimes \mathcal{O}_2[[r^2/j]])$.

These statements shed light on our Lemma 3.1, but Lemma 3.1 is a stronger result, because it gives an analogous result to this but "without denominators". This should be contrasted with the situation for general p, where according to Katz the map φ is in general finite but not flat when one works without denominators; see [11, statement of Theorem 3.10.1,I]).

A.2 The curves $X_1(n)$ and modular forms

For any non-negative integer n, the modular curve $X_1(2^n)$ is defined over $\mathbf{Q}$, and is a coarse moduli space for the problem of classifying 2^n-torsion points on (generalized) elliptic curves over $\mathbf{Q}$-algebras. The curve $X_1(2^n)$ has a natural action of the group $(\mathbf{Z}/2^n)^\times$ (usually referred to as the group of diamond operators): if $a \in (\mathbf{Z}/2^n)^\times$, then the action of a takes a pair (E, P) (P a point of exact order 2^n on E) to the pair (E, aP). The quotient of $X_1(2^n)$ by the action of $(\mathbf{Z}/2^n)^\times$ is $X_0(2^n)$.

If $n \geq 3$ then $X_1(2^n)$ is actually a fine moduli space, and so has a line bundle ω which is the pushforward to $X_1(2^n)$ of the sheaf of relative differentials (with log poles at the singular points) of the universal (generalized) elliptic curve over $X_1(2^n)$. This line bundle is equivariant for the action of the group $(\mathbf{Z}/2^n)^\times$. For any integer k, a classical modular form on $\Gamma_1(2^n)$ of weight k is a global section of $\omega^{\otimes k}$ over $X_1(2^n)$.

Now suppose that $1 > |r| \geq 2^{1/2^{n-4}}$. Then the canonical section identifies D_r with an open disc in $X_0(2^n)$. Let $\tilde{D}_r$ denote the preimage of this open disc in $X_1(2^n)$. Let k be an integer. By an overconvergent modular of weight k on $\Gamma_1(2^n)$ we mean a rigid analytic section of $\omega^{\otimes k}$ over $\tilde{D}_r$ for some r in the above range. Note that any classical modular forms is overconvergent. Although an overconvergent modular form is a section and not a function, we may talk about its absolute value at a point: if f is an overconvergent modular form, and E is an elliptic curve with appropriate level structure, let ω be a holomorphic differential on E which has non-zero reduction modulo p. Then define $|f(E)| := |f(E, \omega)|$.

Let us fix some r and k as in the preceeding paragraph. Since the disc $\tilde{D}_r$ is (by construction) closed under the action of $(\mathbf{Z}/2^n)^\times$ (with quotient equal to D_r), and since $\omega^{\otimes k}$ is $(\mathbf{Z}/2^n)^\times$-equivariant, we see that the diamond operators act on the space of sections of $\omega^{\otimes k}$ on $\tilde{D}_r$. Thus we can decompose this space into a direct sum of subspaces on which $(\mathbf{Z}/2^n)^\times$ acts by a character. If f is a section on which $(\mathbf{Z}/2^n)^\times$ acts by the character χ, we say that f has nebentypus χ.

If $n = 0, 1$ or 2, then $X_1(2^n)$ is not a fine moduli space and the line bundle ω does not descend to $X_1(2^n)$. However, we can still define classical and overconvergent modular forms of weight k on $\Gamma_1(2^n)$, for example by introducing an auxilary $\Gamma(3)$ level structure, and defining them as modular forms on $\Gamma_1(2^n) \cap \Gamma(3)$ which are independent of the $\Gamma(3)$ level structure. The curves $X_0(2^n)$ are not fine moduli spaces either, for any value of n, and so we define classical and overconvergent modular forms on $\Gamma_0(2^n)$ as modular forms on $\Gamma_1(2^n)$ with trivial nebentypus, or again by introducing auxilary $\Gamma(3)$ level structure.

Over the disc $\tilde{D}_1$, the universal elliptic curve is equal to the Tate curve $\mathbf{G}_m/q^{\mathbf{Z}}$. As we recalled above, the variable q uniformizes the disc D_1. The map from $\tilde{D}_1$ to D_1 is unramified, and so we may regard q as a uniformizing parameter in a neighbourhood of the cusp infinity in $\tilde{D}_1$. If t denotes the variable on $\mathbf{G}_m$, then dt/t provides a trivialization of ω over $\tilde{D}_1$. If f is any overconvergent modular form, we define the q-expansion of f to be the power series in $\mathbf{C}_2[[q]]$ obtained by expanding the rigid analytic function $f/(dt/t)$ on $\tilde{D}_1$ in powers of the uniformizer q. Since $\tilde{D}_r$ is connected if $1 > |r|$, the principle of rigid analytic continuation shows that an overconvergent modular form is uniquely determined by its q-expansion.

Now suppose that we have chosen some n, and some $r \in \mathbf{C}_p$ such that $1 \geq |r| \geq |2|^{1/2^{n-3}}$. Let (E,P) be a point of $\tilde{D}_r$, so that P is a point of E generating the canonical subgroup of E of order 2^n. Because of our choice of r, E in fact has a canonical subgroup of order 2^{n+1}; let Q be a generator of this subgroup such that $2Q = P$. The point Q is well-determined up to addition of $2^{n-1}P$ (the unique point of order two in the cyclic group generated by P). Thus the pair $(E/2^{n-1}P, \overline{Q})$ (where $\overline{Q}$ is the image of Q in the quotient elliptic curve $E/2^{n-1}P$) is well-defined independent of the choice of Q. This defines a lift of the Deligne-Tate map to a map $\tilde{D}_r \mapsto \tilde{D}_{r^2}$.

If f is an overconvergent modular form of weight k on $\tilde{D}_r^2$ we can use this lifted Deligne-Tate map to define an overconvergent modular form $V(f)$ on $\tilde{D}_r$: let (E,P) be a point in $\tilde{D}_{r^2}$, choose Q as above, and consider the isogeny $\pi : E \to E/2^{n-1}P$. Then define

$$V(f)(E,P) = 2^{-k}\pi^* f(E/2^{n-1}P, \overline{Q}).$$

(Recall that $f(E/2^{n-1}P, Q)$ is a k-fold differential on $E/2^{n-1}P$, which we are pulling back via π to obtain a k-fold differential on E). On q-expansions this is again the map $q \mapsto q^2$. (To see this, consider Tate curve $\mathbf{G}_m/q^{\mathbf{Z}}$. The isogeny π has kernel μ_2, and so π is the map $\mathbf{G}_m/q^{\mathbf{Z}} \mapsto \mathbf{G}_m/q^{2\mathbf{Z}}$ given by $t \mapsto t^2$. Then $2^{-k}\pi^*(f(q)(dt/t)^{\otimes k}) = f(q^2)(dt/t)^{\otimes k}$, proving our assertion.)

If $|j(E)| \geq 1$, so that E has ordinary reduction (either good or multiplicative) then $|V(f)(E,P)| = |f(E/2^{n-1}P, \overline{Q})|$. However, if $1 > |j(E)|$, then

$$|V(f)(E,P)| = |f(E/2^{n-1}P, \overline{Q}|/|j(E)|^{k/12}.$$

(Recall that $|j(E))|^{1/12} = |A(E)|$, if E has good reduction and A is a lifting of the Hasse invariant, and see [11, Theorem 3.3 and its proof].) Thus when $k \neq 0$, V does not generallly preserve the norms of overconvergent modular forms on $\tilde{D}_r$ if $1 > |r|$. However, we do have the following Lemma:

Lemma 3. *Suppose that n is a non-negative integer, that $1 > |r| \geq |2|^{1/2^{n-3}}$, and that f is an overconvergent modular form of some weight k and nebentypus χ defined on the region $\tilde{D}_{r^2}$ of $X_1(2^n)$ which has no zeroes on this region. Then $f/V(f) \in 1 + r/j\mathcal{O}_2[[r/j]]$, which is to say that $f/V(f)$ is a bounded rigid analytic function on D_r which is of norm one and which assumes the value one at $j = \infty$.*

Since f has no zeroes on $\tilde{D}_{r^2}$, $V(f)$ has no zeroes on $\tilde{D}_r$. Thus $f/V(f)$ is a well-defined, zero-free rigid analytic function on $\tilde{D}_r$. Since f has nebentypus χ, the same is true of $V(f)$. Thus $f/V(f)$ has trivial nebentypus, and so descends to a function on D_r (which is the quotient of $\tilde{D}_r$ by the action of the group $(\mathbf{Z}/2^n)^\times$). Now write $f/V(f) = \sum_{n=0}^\infty a_n(r/j)^n$. Since f does not vanish at $j = \infty$, f and $V(f)$ have the same non-zero constant term in their q-expansion, and so $a_0 = 1$. Now choose s such that $1 \geq |s| > |r|$, and restrict $f/V(f)$ to the *closed* disc $|j| \geq |s|$. Then $f/V(f) = 1 + \sum_{n=1}^\infty a_n(r/s)^n(s/j)^n$. Since $f/V(f)$ has no zeroes on this closed disc, an argument using Weierstrass preparation shows that $a_n(r/s)^n \in \mathcal{O}_2$ for all n. Now letting $|s|$ tend to $|r|$, we see that $a_n \in \mathcal{O}_2$. This proves the Lemma.

Note that this has our Lemma 2.7 as a particular consequence.

One defines the operator U as $U = (1/2)\,\mathrm{trace}\,V$, which takes overconvergent modular forms on $\Gamma_1(2^n)$ of weight k on $\tilde{D}_r$ to modular forms on $\Gamma_1(2^n)$ of weight k on $\tilde{D}_{r^2}$, if $|r| \leq |16|$.

One can also define the Hecke operator T_l for each odd prime l in the usual way, by considering the curves $X_1(2^n;l)$ corresponding to the $\Gamma_1(2^n) \cap \Gamma_0(l)$ moduli problem. Since l-isogenies don't affect the valuation of the Hasse invariant, we find that for any r (such that $|r| \geq |256|$) and k, T_l preserves the space of overconvergent modular forms of weight k defined on $\tilde{D}_r$. The operators U_2 and T_l are given by the usual formulas on q-expansions: if $f = \sum_{n=0}^{\infty}$ is a section of ω^k on $\tilde{D}_r$ which has nebentypus χ, then

$$U_2(f) = \sum_{n=0}^{\infty} a_{2n} q^n$$

and for each odd prime l

$$T_l(f) = \sum_{n=0}^{\infty} a_{ln} q^n + \chi(k) l^{k-1} \sum_{n=0}^{\infty} a_n q^{ln}.$$

Because the operator $\tilde{U}$ increases the radius of overconvergence, it is a completely continuous operator, and so has a characteristic power series on the space of overconvergent modular forms of weight k, which we might denote $P_k(T)$. (Technically, we should consider overconvergent forms over some *closed* disc of fixed radius inside $\tilde{D}_{256}$, in order to obtain a Banach space to which we can apply the theory of [14]. However, the power series obtained is independent of the choice of such a disc.) Since U commutes with the diamond operators, this power series will factor into a product of the characteristic power series obtained by restricting U to each eigenspace for the diamond operators. If χ is a character of $(\mathbf{Z}/2^n)^{\times}$, we will denote this power series by $P_{\chi,k}(T)$. There is no need to specify the level under consideration in this power series, because if χ has conductor 2^m for $m < n$ then any overconvergent form on $\Gamma_1(2^n)$ with nebentypus χ may be descended to an overconvergent form on $\Gamma_1(2^m)$, via the canonical section.

A.3 Overconvergent forms of 2-adic weight

The idea of defining overconvergent modular forms of non-integral p-adic weight is due to Coleman. If $\kappa \in \mathcal{W}(\mathbf{C}_2)$ and $f(q) \in \mathbf{C}_2[[q]]$, we say that f is an overconvergent 2-adic modular form of weight κ if f/E_{κ}^{*} is the q-expansion of an overconvergent modular form of weight zero on $\Gamma_0(1)$. In other words, we are requiring that $f \in E_{\kappa}^{*}\mathbf{C}_2 \otimes \mathcal{O}_2[[r/j_2]]$ for some value of r.

Suppose that $\kappa = \kappa_{k,\chi}$ for some integer k and some finite order character χ of Γ. One shows that $E_{\kappa_{k,\chi}}^{*}$ is an overconvergent form of weight k and nebentypus χ in the sense of the previous section. Since $E_{\kappa_{k,\chi}}^{*} \equiv 1$ in q-expansion, one can show that $E_{\kappa_{k,\chi}}$ is zero-free on $\tilde{D}_r$

for some r with $1 > |r|$. (In fact, one can take $r = r(\kappa)^2$, in the notation of section 3.6). Thus we see that a q-expansion f is the q-expansion of an overconvergent modular form of weight $\kappa_{\chi,k}$ in the sense just defined, if and only if it is an overconvergent modular form of weight k and nebentypus χ in the sense of the preceding section. In particular, any classical form of weight k and nebentypus χ is overconvergent of weight $\kappa_{\chi,k}$.

The key point in the theory of [5] is to show that for any κ, $E^*_\kappa/V(E^*_\kappa)$ is an overconvergent modular form of weight zero.

Knowing this, one then sees that if f is overconvergent of weight κ, then

$$U(f) = U(fE^*_\kappa/V(E^*_\kappa))E^*_\kappa$$

is again overconvergent, and that U is acting completely continuously. One computes a characteristic power series $P(\kappa, T)$. If $\kappa = \kappa_{\chi,k}$ then $P(\kappa, T) = P_{\chi,k}(T)$. Since the E^*_κ vary rigid analytically, we find that $P(\kappa, T)$ rigid analytically interpolates the characteristic powers series $P_{\chi,k}(T)$.

The method of proving that $E^*_\kappa/V(E^*_\kappa)$ is an overconvergent modular function is identical to the proofs of our Propostions 3.13 and 3.16, except that we have taken care to obtain precise estimates.

In Chapter 3, we give a stricter definition of overconvergent modular form of weight κ than that given above: we insisted that $f/E^*_\kappa \in \mathbf{C}_2 \otimes \mathcal{O}_2[[r(\kappa)/j_2]]$ for a particular value of $r(\kappa)$. With this definition, it is not necessarily true that a classical modular form is overconvergent, because for large n, the canonical section σ_{2^n} is not defined on the disc $D_{r(\kappa)}$.This is why our Proposition 3.24 is restricted to modular forms of small level. However, working with this restricted definition of overconvergent modular form is harmless as far as studying the spectral theory of U is concerned, since U increases the radius of overconvergence.

In [5] and [6], it is also proved (by studying the curves $X_1(2^n; l)$) that the overconvergent modular forms of any weight κ are preserved by the weight κ action of the Hecke operators T_l (l and odd prime). We have quoted this result as Proposition 3.22 in section 3.6. The key point of the proof is the observation that an acting by an l-isogeny on an elliptic curve in characteristic two does not affect the valuation of the Hasse invariant.

In [3] and [4], it is proved, using a subtle analysis of the De Rham cohomology on $X_1(2^n)$, that an overconvergent modular form of (integral) weight κ and nebentypus χ is classical, if it is a U-eigenvector with eigenvalue of slope less then $k-1$. This immediately implies part (vii) of our Main Theorem, with the exception of the four points $u = 2, 4, 8, 32$ corresponding to the evil Eisenstein series.