OPTIMAL QUOTIENTS OF MODULAR JACOBIANS

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Fix a field K of characteristic zero, and an algebraic closure $\bar{K}$ of K. Let G be an algebraic group over K equipped with a commutative, finite rank $\mathbb{Z}$-algebra of endomorphisms $\mathbb{T}$. If I is an ideal of $\mathbb{T}$, then we let $G[I]$ denote the algebraic subgroup of G consisting of elements annihilated by I, and $G[I]^0$ denote the connected component of the identity of $G[I]$. The $\bar{K}$-valued points of the quotient $G[I]/G[I]^0$ form a finite $\mathbb{T}$-module, and the object of this paper is to introduce a simple technique that gives information about the support of this $\mathbb{T}$-module, and to apply this technique to obtain some theorems concerning optimal quotients of Jacobians of modular curves.

The technique itself is explained in the completely elementary section 1, and encapsulated in corollary 1.3. We devote the rest of this introduction to explaining its applications to optimal quotients, after first recalling the necessary notation and terminology.

Modular forms, modular Jacobians, and Hecke operators. If N is a positive integer, we let $J_0(N)$ denote the Jacobian of the modular curve $X_0(N)$. We let $S(N)$ denote the space of weight two cusp forms on $\Gamma_0(N)$, and let $\mathbb{T}$ denote the finite rank, commutative $\mathbb{Z}$-algebra of endomorphisms of $S(N)$ generated by the Hecke operators. The algebra $\mathbb{T}$ acts faithfully as endomorphisms of $J_0(N)$, and there is a $\mathbb{T}$-equivariant isomorphism of $S(N)$ with the cotangent space at the origin of $J_0(N)$ [5, p. 444].

Newforms and optimal quotients. If f is a normalized newform in $S(N)$, we let S_f denote the subspace of $S(N)$ spanned by f and its algebraic conjugates, and let $S_f^{\perp}$ denote the orthogonal complement of S_f in $S(N)$ with respect to the Petersson inner product. Each of S_f and $S_f^{\perp}$ are closed under the action of $\mathbb{T}$; let I_f (respectively $I_f^{\perp}$) denote the annihilator in $\mathbb{T}$ of S_f (respectively $S_f^{\perp}$). Note that I_f is also the kernel of the homomorphism $\mathbb{T} \to \mathbb{C}$ induced by the action of the Hecke operators on the eigenform f. Since S is the direct sum of S_f and $S_f^{\perp}$, we find that $I_f \cap I_f^{\perp} = 0$, while $I_f + I_f^{\perp}$ has finite index in $\mathbb{T}$. Let us denote the sum $I_f + I_f^{\perp}$ by C_f; it is the *ideal of fusion* of f in $\mathbb{T}$, which measures the congruences between f and Hecke eigenforms lying in the space $I_f^{\perp}$.

Write $A_f = J_0(N)/I_f$ and $B_f = I_f J_0(N)$; there is a short exact sequence of abelian varieties

$$0 \to B_f \to J_0(N) \to A_f \to 0.$$

One refers to A_f as the *optimal quotient* of $J_0(N)$ corresponding to f. Dualizing this exact sequence (and taking into account the autoduality of the Jacobian $J_0(N)$)

we obtain the short exact sequence

$$0 \to \hat{A}_f \to J_0(N) \to \hat{B}_f \to 0.$$

Composing the injection $\hat{A}_f \to J_0(N)$ with the surjection $J_0(N) \to A_f$ yields an isogeny $\theta_f : \hat{A}_f \to A_f$ (this is the polarization of $\hat{A}_f$ induced by the canonical polarization of $J_0(N)$).

Maximal ideal in $\mathbb{T}$ and Galois representations. If $\mathfrak{m}$ is a maximal ideal of $\mathbb{T}$, then attached canonically to $\mathfrak{m}$ is a semi-simple representation

$$\rho_{\mathfrak{m}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T}/\mathfrak{m}).$$

We say that $\mathfrak{m}$ is *Eisenstein* if $\rho_{\mathfrak{m}}$ is reducible. We say that $\mathfrak{m}$ satisfies *multiplicity one* if $J_0(N)[\mathfrak{m}]$ is two dimensional over the residue field $\mathbb{T}/\mathfrak{m}$. (The motivation for this definition is that if $\mathfrak{m}$ is non-Eisenstein, so that $\rho_{\mathfrak{m}}$ is irreducible, then $J_0(N)[\mathfrak{m}]$ is isomorphic to a direct sum of copies of $\rho_{\mathfrak{m}}$ [1], and so $\mathfrak{m}$ satisfies multiplicity one if and only if $J_0(N)[\mathfrak{m}]$ is isomorphic to a single copy of $\rho_{\mathfrak{m}}$.) Note that the $\mathbb{T}_{\mathfrak{m}} \otimes \mathbb{Q}$-module $T_{\mathfrak{m}} J_0(N) \otimes \mathbb{Q}$ is free of rank two, and so $\mathfrak{m}$ satisfies multiplicity one if and only if $T_{\mathfrak{m}} J_0(N)$ is free of rank two over $\mathbb{T}_{\mathfrak{m}}$. In particular, such $\mathfrak{m}$ are *good* for $J_0(N)$, in the sense of section 1.

For any prime p, we say that a maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ is *p-finite* if the representation $\rho_{\mathfrak{m}}$ is *finite at p* [7, p. 189]. Now suppose that p exactly divides N. We say that $\rho_{\mathfrak{m}}$ is p-old if there is a maximal ideal $\mathfrak{n}$ in the Hecke ring of level N/p for which $\rho_{\mathfrak{n}}$ is equivalent to $\rho_{\mathfrak{m}}$. (If $\mathfrak{m}$ is p-old then it is necessarily p-finite.) Continue to assume that p exactly divides N; then $J_0(N)$ has semi-abelian reduction over $\mathbb{Z}_p$. Let $G_0(N)_{/\mathbb{Q}_p}$ denote the semi-abelian variety over $\mathbb{Q}_p$ which rigid-analytically uniformizes $J_0(N)$ over $\mathbb{Q}_p$, and let $T_0(N)_{/\mathbb{Q}_p}$ denote the maximal torus in $G_0(N)_{/\mathbb{Q}_p}$. By functoriality of rigid-analytic uniformization, there is an action of $\mathbb{T}$ on $G_0(N)_{/\mathbb{Q}_p}$, and so on $T_0(N)_{/\mathbb{Q}_p}$. We say that $\mathfrak{m}$ satisfies *toric multiplicity one at p* if $T_{\mathfrak{m}} T_0(N)_{/\mathbb{Q}_p}$ is free of dimension one over $\mathbb{T}_{\mathfrak{m}}$. Such $\mathfrak{m}$ are *good* for $T_0(N)_{/\mathbb{Q}_p}$, in the sense of section 1.

Having filled in the necessary background, we can now discuss the applications of corollary 1.3 to optimal quotients. There are two problems that we will analyze, and we describe each problem, and the corresponding result that we obtain, in turn.

Problem I: Fix a normalized newform f in $S(N)$. We wish to understand the relation between the kernel of the polarization $\theta : \hat{A}_f \to A$ and the ideal of fusion C_f of f. More precisely, a consideration of cotangent spaces shows that B_f is isomorphic to the connected component of $J_0(N)[I_f^{\perp}]$, and that $\hat{A}_f$ is isomorphic to the connected component of $J_0(N)[I_f]$. Thus

$$\ker(\theta) = \hat{A}_f \cap B_f \subset J_0(N)[I_f] \cap J_0(N)[I_f^{\perp}] = J_0(N)[C_f],$$

and we wish to analyze the cokernel of this inclusion. (This problem is related to, but not the same as, that of understanding the relation between the modular degree of a strong Weil curve and the modulus of congruence of the corresponding eigenform, as discussed for example in [11].) This problem is addressed by the following result, which follows immediately from corollary 1.3, upon taking the algebraic group G to be $J_0(N)$, and the ideal of endomorphisms I to be alternately I_f and $I_f^{\perp}$.

Theorem A. *Let f be a normalized new form in $S(N)$, and let $\theta : \hat{A}_f \to A_f$ be the polarization induced by the canonical polarization of $J_0(N)$. Then the cokernel of the inclusion $\ker(\theta) \subset J_0(N)[C_f]$ is supported at maximal ideals of $\mathbb{T}$ which do not satisfy multiplicity one.*

We should point out that the cokernel of the inclusion $\ker(\theta) \subset J_0(N)[C_f]$ need not vanish. We illustrate this by an example. Calculations of Kilford and Stein show that when $N = 431$ or 503 there is a maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ having residue characteristic two which does not satisfy multiplicity one. Furthermore, there is a strong Weil curve E arising from a newform f of level N for which the $\mathfrak{m}$-adic component of C_f is equal to $\mathfrak{m}$. Thus $J[\mathfrak{m}] \subset J[C_f]$. On the other hand, $J[\mathfrak{m}]$ cannot be contained in $\ker\theta$, since $J[\mathfrak{m}] \cap E = E[2]$, and the fact that $\mathfrak{m}$ does not satisfy multiplicity one shows that the inclusion $E[2] \subset J[\mathfrak{m}]$ cannot be an equality.

Problem II: Suppose that p is a prime which exactly divides N. Then $J_0(N)$ and A_f both have semi-abelian reduction at p. Let $\Phi_{J_0(N)}$ (respectively Φ_{A_f}) denote the connected component group of the special fibre at p of the Néron model of $J_0(N)$ (respectively of A_f). The surjection $J_0(N) \to A_f$ induces a morphism $\Phi_{J_0(N)} \to \Phi_{A_f}$. The second problem we wish to address is that of understanding the cokernel of this map. (This problem has been raised by William Stein in the particular case that $N = p$ [8].) It is addressed by the following result.

Theorem B. *Assume as in the preceding discussion that the prime p exactly divides N. Let I be an ideal of $\mathbb{T}$ such that $\mathbb{T}/I$ is torsion free, and let A denote the quotient $J_0(N)/I$. (For example I could be the ideal I_f corresponding to a normalized new form f in $S(N)$, in which case A is equal to A_f.) Then any maximal ideal of $\mathbb{T}$ in the support of the cokernel of the map of connected component groups $\Phi_{J_0(N)} \to \Phi_A$ is either p-old, or else does not satisfy toric multiplicity one.*

Theorem B is proved by first applying corollary 1.3 with the algebraic group G taken to be $G_0(N)_{/\mathbb{Q}_p}$ and the ideal I taken to be I_f, to obtain information about the support of the $\mathbb{T}$-module $G_0(N)_{/\mathbb{Q}_p}[I_f]/(G_0(N)_{/\mathbb{Q}_p}[I])^0$. An analysis of the relation between rigid analytic uniformizations of abelian varieties with semi-abelian reduction and the component groups of their special fibres, carried out in section 2, then relates $G_0(N)_{/\mathbb{Q}_p}[I_f]/(G_0(N)_{/\mathbb{Q}_p}[I])^0$ to the cokernel of the map $\Phi_{J_0(N)} \to \Phi_{A_f}$.

In the situation of problem II, there is an approach to analyzing the support of the $\mathbb{T}$-module Φ_A using Ribet's lowering-the-level theorem, which we learned from [8]. The following argument is a variation of the observations of [8, §4].

Theorem C. *Let A be an abelian variety subquotient of $J_0(N)$ on which $\mathbb{T}$ acts, and let Φ_A denote the connected component group of the special fibre of the Néron model of A. Then Φ_A is a finite $\mathbb{T}$-module which is supported on the union of the Eisenstein and the p-finite locus of $\mathbb{T}$.*

Proof. Let us fix a maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ of residue characteristic ℓ. As explained in [8, p. 4], we have the diagram of group schemes over $\mathbb{Q}_p$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\Gamma_{\hat{A}}/\ell, \mu_\ell) \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & \Phi_A[\ell] \arrow[r] \arrow[d] & 0, \\
0 \arrow[r] & \mathrm{Hom}(\Gamma_{\hat{A}}/\ell, \mu_\ell) \arrow[r] & A[l] \arrow[r] & \Gamma_A/\ell \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact, and in which E is the maximal finite subgroup scheme of $A[l]$. This diagram may be localized at $\mathfrak{m}$ to yield the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}((\Gamma_{\hat{A}})_{\mathfrak{m}}/\ell, \mu_\ell) \arrow[r] \arrow[d, equal] & E_{\mathfrak{m}} \arrow[r] \arrow[d] & \Phi_A[\ell]_{\mathfrak{m}} \arrow[r] \arrow[d] & 0 \, . \\ 0 \arrow[r] & \mathrm{Hom}((\Gamma_{\hat{A}})_{\mathfrak{m}}/\ell, \mu_\ell) \arrow[r] & A[l]_{\mathfrak{m}} \arrow[r] & (\Gamma_A)_{\mathfrak{m}}/\ell \arrow[r] & 0 \end{tikzcd}$$

Now suppose that $\mathfrak{m}$ is not Eisenstein. Then $A[\ell]_{\mathfrak{m}}$ has a filtration $F^\bullet$ as a $\mathbb{T}_{\mathfrak{m}}/\ell\,[\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})]$-module whose successive quotients F^i/F^{i+1} are isomorphic to $\rho_{\mathfrak{m}}$.

Suppose in addition that $\mathfrak{m}$ is not p-finite. Since $\mathrm{Hom}((\Gamma_{\hat{A}})_{\mathfrak{m}}/\ell, \mu_\ell)$ and $(\Gamma_A)_{\mathfrak{m}}/\ell$ are both finite at p, the short exact sequence

$$0 \to (F^i \cap \mathrm{Hom}((\Gamma_{\hat{A}})_{\mathfrak{m}}/\ell, \mu_\ell))/(F^{i+1} \cap \mathrm{Hom}((\Gamma_{\hat{A}})_{\mathfrak{m}}/\ell, \mu_\ell)) \\ \to F^i/F^{i+1} \to F^i/(F^{i+1} + F^i \cap \mathrm{Hom}((\Gamma_{\hat{A}})_{\mathfrak{m}}/\ell, \mu_\ell)) \to 0$$

which exists for each i shows that the filtration induced by $F^\bullet$ on each of these $\mathbb{T}_{\mathfrak{m}}/\ell$-modules has its successive quotients being one-dimensional $\mathbb{T}/\mathfrak{m}$-vector spaces. (Note that these successive quotients are precisely the kernel and cokernel in the preceding short exact sequence.) Thus each of these $\mathbb{T}_{\mathfrak{m}}/\ell$-modules is of the same length; that is, they are both of length equal to the length of the filtration $F^\bullet$. Now E is also finite at p, and so the submodule $(F^i \cap E)/(F^{i+1} \cap E)$ of $F^i/F^{i+1} = \rho_{\mathfrak{m}}$ must also be one dimensional. Thus the inclusion

$$(F^i \cap \mathrm{Hom}((\Gamma_{\hat{A}})_{\mathfrak{m}}/\ell, \mu_\ell))/(F^{i+1} \cap \mathrm{Hom}((\Gamma_{\hat{A}})_{\mathfrak{m}}/\ell, \mu_\ell)) \to (F^i \cap E)/(F^{i+1} \cap E)$$

must be an equality, and so E is also of length equal to the length of the filtration $F^\bullet$. Hence the inclusion $\mathrm{Hom}((\Gamma_{\hat{A}})_{\mathfrak{m}}/\ell, \mu_\ell) \to E_{\mathfrak{m}}$ is an isomorphism, proving that $\Phi_A[\ell]_{\mathfrak{m}} = 0$. This establishes the proposition. □

The lowering-the-level results of [5] (as strengthened by the main result of [4]), shows that if $\mathfrak{m}$ is a non-Eisenstein, p-finite maximal ideal in $\mathbb{T}$ of residue characteristic ℓ, then $\mathfrak{m}$ is necessarily p-old, provided that ℓ is odd and either ℓ^2 does not divide N or $\ell \not\equiv 1 \pmod{p}$. Combining this together with theorems B and C we see that the only possible non-p-old maximal ideals that can appear in support of the cokernel of the map $\Phi_{J_0(N)} \to \Phi_A$ are either Eisenstein maximal ideals not satisfying toric multiplicity one, or p-finite maximal ideals not satisfying toric multiplicity one, having residue characteristic ℓ which does not satisfy the preceding conditions.

If $N = p$, then certainly ℓ^2 cannot divide N, and (as Ribet observes in [6, prop. 2.2]) the main result of [10] establishes lowering-the-level for non-Eisenstein maximal ideals also in the case that ℓ equals two. The results of [3] show that all the Eisenstein maximal ideals satisfy toric multiplicity one (as observed in the remarks following the statement of [2, thm. 0.5]). Finally, there are no p-old maximal ideals in this situation. Thus we obtain the following corollary, which confirms a conjecture of William Stein [8, conj. 4.1].

Corollary. *Let $N = p$ be prime, let I be an ideal of $\mathbb{T}$ such that $\mathbb{T}/I$ is torsion free, and let A denote the quotient $J_0(N)/I$. Then the map of connected component groups $\Phi_{J_0(N)} \to \Phi_A$ is surjective.*

1. CONNECTED COMPONENT GROUPS OF ALGEBRAIC GROUPS AND THE PROOF OF THEOREM A

If G is a smooth commutative algebraic group over a field K of characteristic zero, then for any integer n we let $G[n]$ denote the the algebraic subgroup of G consisting of the n-torsion elements of G. Since we are in characteristic zero, multiplication by n is a smooth morphism from G to itself, and consequently $G[n]$ is a finite algebraic group (since a smooth morphism from a variety to itself must have finite fibres).

If n divides n', then $G[n] \subset G[n']$. We will be especially interested in the case when we fix a prime ℓ, and consider the ℓ-power torsion $G[\ell^n]$ of G. We write $G[\ell^\infty]$ for the ind-finite algebraic subgroup of G formed by the $G[\ell^n]$ as n ranges over all positive integers.

The formation of $G[\ell^\infty]$ is evidently functorial in G. If G is connected, then multiplication by ℓ is surjective from G to itself, and so $G[\ell^\infty]$ is an ℓ-divisible group.

For any G of the type we are considering, we will let G^0 denote the geometrically connected component of the identity of G. The quotient G/G^0 is a finite algebraic group, *the group of connected components of* G.

Let us remark that if we choose an algebraic closure $\bar{K}$ of K, then the $\bar{K}$-valued points of any finite algebraic group over K form a finite group, and we will not distinguish between a finite algebraic group over K and its group of $\bar{K}$-valued points. (This is permissible, since the finite algebraic group can be recovered up to isomorphism from its group of $\bar{K}$-valued points, equipped with its natural $\mathrm{Gal}(\bar{K}/K)$-action.)

We will require the following easily proved fact:

Fact 1.1. *For any G as above and any prime ℓ, the inclusion $G^0[\ell^\infty] \to G[\ell^\infty]$ identifies $G^0[\ell^\infty]$ with the maximal ℓ-divisible subgroup of $G[\ell^\infty]$. The cokernel of this inclusion is canonically isomorphic to the ℓ-Sylow subgroup of G/G^0.*

For the remainder of this section, let us fix a group G as above. Furthermore, let $\mathbb{T}$ denote a finite commutative $\mathbb{Z}$-algebra of endomorphisms of G. (As indicated in the introduction, in the applications $\mathbb{T}$ will be a Hecke algebra, as the notation suggests.) If I is any ideal in $\mathbb{T}$, we let $G[I]$ denote the algebraic subgroup of G consisting of elements annihilated by I.

If ℓ is any prime number then the ℓ-adic completion of $\mathbb{T}$ factors as the direct product of completions $\mathbb{T}_\mathfrak{m}$, where $\mathfrak{m}$ ranges over the maximal ideals of $\mathbb{T}$ of residue characteristic ℓ. Thus any object on which this ℓ-adic completion acts similarly factors. For example, $G[\ell^\infty]$ factors into such a product of ℓ-divisible subgroups. We let $G[\mathfrak{m}^\infty]$ denote the factor corresponding to the maximal ideal $\mathfrak{m}$. Similarly the ℓ-adic Tate module $T_\ell G(N)_{/\mathbb{Q}_p}$ (by which we refer to the Pontrjagin dual $\mathrm{Hom}(G[\ell^\infty], \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ of $G[\ell^\infty]$; this is not quite the usual ℓ-adic Tate module if G/G^0 has a non-trivial ℓ-Sylow subgroup) factors into such a product, and we let $T_\mathfrak{m} J_0(N)$ denote the factor corresponding to the maximal ideal $\mathfrak{m}$. We say that a maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ is *good* if $T_\mathfrak{m} G$ is a free $\mathbb{T}_\mathfrak{m}$-module.

Lemma 1.2. *Suppose that $\mathfrak{m}$ is good, and that I is an ideal in $\mathbb{T}$ such that the quotient $\mathbb{T}/I$ is torsion free (as a $\mathbb{Z}$-module). Then the ind-finite group $G[\mathfrak{m}^\infty, I]$ (which denotes of course the I-torsion subgroup of the ind-finite group $G[\mathfrak{m}^\infty]$) is ℓ-divisible.*

Proof. This follow by Pontrjagin duality. More precisely,

$$G[\mathfrak{m}^\infty, I] = \mathrm{Hom}_{\mathbb{T}}(\mathbb{T}/I, \mathrm{Hom}(T_{\mathfrak{m}}G, \mathbb{Q}_\ell/\mathbb{Z}_\ell)) = \mathrm{Hom}_{\mathbb{T}}(T_{\mathfrak{m}}G/I, \mathbb{Q}_\ell/\mathbb{Z}_\ell),$$

and so $G[\mathfrak{m}^\infty, I]$ is ℓ-divisible if and only if its Pontrjagin dual $T_{\mathfrak{m}}G/I$ is torsion free. But since $\mathfrak{m}$ is good, this latter group is free over $\mathbb{T}_{\mathfrak{m}}/I$, which is torsion free by virtue of our assumption on I. □

Corollary 1.3. *Let I be an ideal in $\mathbb{T}$ such that $\mathbb{T}/I$ is torsion free, and write $H = G[I]$. Then the connected component group H/H^0 is supported at maximal ideals of $\mathbb{T}$ which are not good.*

Proof. This follows from fact 1.1 and lemma 1.2. □

We can now prove theorem A.

Proof of theorem A. We use the notation introduced in the discussion of problem I. As observed in that discussion, we have $\hat{A}_f = J_0(N)[I_f]^0$ and $B_f = J_0(N)[I_f^\perp]^0$. Thus

$$\ker(\theta) = J_0(N)[I_f]^0 \cap J_0(N)[I_f^\perp].$$

Theorem A is follows immediately from this observation, corollary 1.3 (taking G over K to be $J_0(N)$ over $\mathbb{Q}$ and I to be alternately I_f and $I_f^\perp$), and the fact that multiplicity one maximal ideals are good for $J_0(N)$. □

2. Exact sequences of abelian varieties with semi-abelian reduction and the proof of theorem B

In this section we assume that K is complete with respect to a discrete valuation, and has perfect residue field. If A is an abelian variety over K having semi-abelian reduction, in the sense that the connected component of the special fibre of its Néron model over the ring of integers of K is an extension of an abelian variety by a torus, then A has a uniformization in the category of rigid analytic spaces over K in the form of a short exact sequence

$$0 \to \Gamma_A \to G_A \to A \to 0,$$

in which Γ_A is a finite rank $\mathbb{Z}$-module equipped with a continuous $\mathrm{Gal}(\bar{K}/K)$-action, G_A is a semi-abelian variety, and the maximal abelian variety quotient of G_A is isomorphic to the maximal quotient of A having good reduction. The formation of Γ_A and G_A is functorial in the category of abelian varieties A having semi-abelian reduction.

Let $0 \to C \to B \to A \to 0$ be an exact sequence of abelian varieties over K, and suppose that B (and hence also C and A) have semi-abelian reduction.

Functoriality of uniformization yields a diagram with exact rows and columns

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
\Gamma_B & \longrightarrow & \Gamma_A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
G_B & \longrightarrow & G_A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
B & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

(the exactness of the top row follows because the kernel of the map $B \to A$ is connected; see [9, prop. 4.12]). We can fill this diagram in to obtain the following diagram in which all rows and columns are short exact:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma'_C & \longrightarrow & \Gamma_B & \longrightarrow & \Gamma_A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G'_C & \longrightarrow & G_B & \longrightarrow & G_A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

The algebraic group G'_C need not be connected; functoriality of uniformization yields a map $G_C \to G'_C$, which induces an isomorphism $G_C \xrightarrow{\sim} (G'_C)^0$, as well as a map $\Gamma_C \to \Gamma'_C$, whose cokernel is finite and is canonically isomorphic to the connected component group $G'_C/(G'_C)^0$.

We may dualize our exact sequence of abelian varieties to obtain the exact sequence of dual abelian varieties $0 \to \hat{A} \to \hat{B} \to \hat{C} \to 0$, and a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma'_{\hat{A}} \arrow[r] \arrow[d] & \Gamma_{\hat{B}} \arrow[r] \arrow[d] & \Gamma_{\hat{C}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G'_{\hat{A}} \arrow[r] \arrow[d] & G_{\hat{B}} \arrow[r] \arrow[d] & G_{\hat{C}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hat{A} \arrow[r] \arrow[d] & \hat{B} \arrow[r] \arrow[d] & \hat{C} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & .
\end{tikzcd}$$

Again we obtain an isomorphism $G_{\hat{A}} \xrightarrow{\sim} (G'_{\hat{A}})^0$, and a short exact sequence

$$0 \to \Gamma_{\hat{A}} \to \Gamma'_{\hat{A}} \to G_{\hat{A}}/(G'_{\hat{A}}) \to 0.$$

The monodromy pairing yields a short exact sequence of complexes (the complexes are in the horizontal direction; the exact sequence is in the vertical direction)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma_C \arrow[r] \arrow[d] & \Gamma_B \arrow[r] \arrow[d] & \Gamma_A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\Gamma_{\hat{C}}, \mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{Hom}(\Gamma_{\hat{B}}, \mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{Hom}(\Gamma_{\hat{A}}, \mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Phi_C \arrow[r] \arrow[d] & \Phi_B \arrow[r] \arrow[d] & \Phi_A \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & .
\end{tikzcd}$$

Taking the associated long exact sequence of cohomology, and using the preceding diagrams, we obtain the exact sequence

$$\begin{aligned} 0 \to \ker(\Phi_C \to \Phi_B) \to \Gamma'_C/\Gamma_C \to 0 \to \ker(\Phi_B \to \Phi_A)/im(\Phi_C \to \Phi_B) \\ \to 0 \to \mathrm{Ext}^1(\Gamma'_{\hat{A}}/\Gamma_{\hat{A}}, \mathbb{Z}) \to \mathrm{coker}(\Phi_B \to \Phi_A) \to 0. \end{aligned}$$

Recall the isomorphisms $\Gamma'_C/\Gamma_C \xrightarrow{\sim} G'_C/G_C$ and $\Gamma'_{\hat{A}}/\Gamma_{\hat{A}} \xrightarrow{\sim} G'_{\hat{A}}/G_{\hat{A}}$. Also recall that if M is any finite group, there is an isomorphism

$$\mathrm{Ext}^1(M, Z) \xrightarrow{\sim} \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}) =: M^*.$$

Using these isomorphisms, we may reorganize the information contained in the preceding exact sequence to obtain the exact sequence

$$0 \to G'_C/G_C \to \Phi_C \to \Phi_B \to \Phi_A \to (G'_{\hat{A}}/G_{\hat{A}})^* \to 0. \tag{2.1}$$

There is a particular deduction that we wish to make from the exact sequence (2.1). We will state it as a lemma, after introducing the relevant assumptions and notation.

Suppose that $\mathbb{T}$ is an algebra of endomorphisms of B, that I is an ideal in $\mathbb{T}$, and that $C = IB$ and $A = B/I$. Note that by functoriality, $\mathbb{T}$ acts on $G_{\hat{B}}$.

Lemma 2.2. *In the preceding situation, the cokernel of the map $\Phi_B \to \Phi_A$ is a quotient of the group $(G_{\hat{B}}[I]/(G_{\hat{B}}[I])^0)^*$.*

Proof. Since I annihilates A, we see that $IG'_{\hat{A}} \subset \Gamma'_{\hat{A}}$. Now $IG'_{\hat{A}}$ is an algebraic group, while $\Gamma'_{\hat{A}}$ is a lattice. Thus $IG'_{\hat{A}} = 0$, and we deduce that $G'_{\hat{A}} \subset G_{\hat{B}}[I]$. A consideration of cotangent spaces shows that

$$(G_{\hat{B}}[I])^0 = (G'_{\hat{A}})^0 = G_{\hat{A}}.$$

Thus we have an injection

$$G'_{\hat{A}}/G_{\hat{A}} \to G_{\hat{B}}[I]/(G_{\hat{B}}[I])^0,$$

and hence a surjection

$$(G_{\hat{B}}[I]/(G_{\hat{B}}[I])^0)^* \to (G'_{\hat{A}}/G_{\hat{A}})^*.$$

The lemma now follows, since the exact sequence (2.1) identifies $(G'_{\hat{A}}/G_{\hat{A}})^*$ with the cokernel of the map $\Phi_B \to \Phi_A$. $\square$

We can now prove theorem B.

Proof of theorem B. As in the statement of the theorem, we let I be an ideal in the Hecke algebra $\mathbb{T}$ such that $\mathbb{T}/I$ is torsion-free, and let $A = J_0(N)/I$. Lemma 2.2 (and the self-duality of the Jacobian $J_0(N)$) implies that any maximal ideal in the support of the cokernel of the map

$$\Phi_{J_0(N)} \to \Phi_A$$

is in the support of $Ext^1(G_0(N)_{/\mathbb{Q}_p}[I]/(G_0(N)_{/\mathbb{Q}_p}[I])^0, \mathbb{Z})$, and so in the support of $G_0(N)_{/\mathbb{Q}_p}[I]/(G_0(N)_{/\mathbb{Q}_p}[I])^0$. Now corollary 1.3 (taking $G_{/K}$ to be $G_0(N)_{/\mathbb{Q}_p}$) implies that any such $\mathfrak{m}$ is not good for $G_0(N)_{/\mathbb{Q}_p}$. If $\mathfrak{m}$ is not p-old, then the map $T_{\mathfrak{m}}T_0(N)_{/\mathbb{Q}_p} \to T_{\mathfrak{m}}G_0(M)_{\mathbb{Q}_p}$ is an isomorphism, and so such an $\mathfrak{m}$ must not be good for $T_0(N)_{/\mathbb{Q}_p}$ either. In particular, it must not satisfy toric multiplicity one. $\square$

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