# RATIONALITY OF $\mathcal{R}_3$

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In this note we shall prove that the moduli space $\mathcal{R}_3$ of curves of genus 3 with a non-trivial divisor class of order 2 is rational. Our contribution is just a final touch to the previous work on this problem ([**1**],[**4**]). Our proof is geometric and thus is very different from a recent proof of rationality of $\mathcal{R}_3$ given by P. Katsylo [**8**]. It is known that this space is birationally isomorphic to the moduli space of $\mathcal{R}_4^{be}$ of bi-elliptic curves of genus 4 with one of the three non-trivial 2-torsion divisor classes coming from the elliptic quotient. The construction of this isomorphism goes back to Arthur Coble ([**5**], §14). Its modern version can be found in [**9**]. To be a little more self-contained, we shall recall the original Coble's construction. We show that $\mathcal{R}_4^{be}$ is canonically birationally isomorphic to the moduli space $\mathcal{M}_4^{be}$ of genus 4 bi-elliptic curves. The latter variety is known to be rational [**1**].

**(1.1)** Let $W \subset \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$ be a smooth 3-fold given by a general equation of degree $(2,1,1)$:

$$\sum_{i,j=0}^{1} \sum_{k,r=0}^{2} a_{ij;kr} t_i \tau_j x_k x_r = 0.$$

Its projection $p_i : W \to \mathbf{P}^1 \times \mathbf{P}^1$ is a conic bundle with the discriminant curve $X$ of bi-degree $(3,3)$. The curve $X$ is a canonical curve of genus 4 which also comes with a non-trivial 2-torsion divisor class, the corresponding unramified 2-cover parametrizes the irreducible components of reducible conics $p_1^{-1}(x), x \in X$. The projection $p_2 : W \to \mathbf{P}^2$ is also a conic bundle; its fibres are divisors on $\mathbf{P}^1 \times \mathbf{P}^1$ of bi-degree $(1,1)$. Its discriminant curve is a non-singular plane quartic $C$. The unramified 2-cover of $C$ splits since the irreducible components of the degenerate conic $p_1^{-1}(c), c \in C$, are ordered. The restriction of the projectin $p_2 : W \to \mathbf{P}^1 \times \mathbf{P}^1$ over a fibre $F$ of one of the projections $\pi_i : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1, i = 1,2$, is a pencil of conics in $\mathbf{P}^2$. Its four base points are on $C$. In fact, for every base point $x \in \mathbf{P}^2$ of this pencil, $p_1(p_2^{-1}(x))$ is a singular conic on $\mathbf{P}^1 \times \mathbf{P}^1$ that contains $F$ as one of its reducible components.

**(1.2)** Conversely, given a nonsingular curve $X$ of bi-degree $(3,3)$ and a non-trivial 2-torsion divisor class $\varepsilon \in \mathrm{Pic}(X)$, one constructs the variety $W$ as above with $(X, \varepsilon)$ equal to the discriminant invariant of the conic bundle $W \to \mathbf{P}^1 \times \mathbf{P}^1$. This is done similarly to the standard construction of a representation of a plane curve as a discriminant curve of a net of quadrics (see [**2,10**]). For any point $a = (\alpha_0, \alpha_1 : \beta_0, \beta_1) \in \mathbf{P}^1 \times \mathbf{P}^1$ one defines the polar $P_a(X)$ of $X$ with respect to $a$ by the formula:

$$P_a(X) = \sum_{i,j=0}^{1} \alpha_i \beta_j \frac{\partial^2 F}{\partial t_i \partial \tau_j} = 0,$$

where $F = F(t_0, t_1; \tau_0, \tau_1)$ is a bi-homogeneous equation of $F$. The set of polars of $X$ generates a 3-dimensional linear system in $|\mathcal{O}_X(2)|$ where $X$ is considered to be embedded in $\mathbf{P}^3$. Now we can view any divisor $D \in |\mathcal{O}_X(2)|$ as a conic in the space $\mathbf{P}^2 = |\mathcal{O}_X(K_X + \varepsilon)|^* = |\mathcal{O}_X(1) \otimes \mathcal{O}_X(\varepsilon)|^*$. This defines a map:

$$P : \mathbf{P}^1 \times \mathbf{P}^1 \to |\mathcal{O}_{\mathbf{P}^2}(2)|, \quad a \to P_a(F).$$

It is given by a divisor $W$ of type $(1,1,2)$ on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$.

Consider the Prym-Wirtinger map

$$\phi_\varepsilon : X \to |\mathcal{O}_X(K_X + \varepsilon)|^*$$

given by the complete linear system $|K + X + \varepsilon|$. The image of $X$ is a plane sextic if $X$ is not bi-elliptic and a plane cubic otherwise (see [**3**]). The conic $P(a)$ cuts out on $\phi_\varepsilon(X)$ the polar divisor $P_a(X) \cap X$. We have a natural map

$$S^2(H^0(X, \mathcal{O}_X(K_X + \varepsilon)) \to H^0(X, \mathcal{O}_X(2K_X)) \cong H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(2,2)).$$

It defines a divisor $R$ of tri-degree $(2,2,2)$ on $\mathbf{P}^1 \times \mathbf{P}^1 \times \check{\mathbf{P}}^2$. For each point $x \in X \subset \mathbf{P}^1 \times \mathbf{P}^1$ the conic $R_x = (\{x\} \times \check{\mathbf{P}}^2) \cap R \subset \check{\mathbf{P}}^2$ is equal to the line $\phi_\varepsilon(x)$ taken with multiplicity 2 (cf.[**2**], 6.23.2). One proves that for any $a \in \mathbf{P}^1 \times \mathbf{P}^1$ the conic $P(a)$ is dual to the conic $R_a$. In paricular, $X$ is equal to the set of points $a$ for which $P(a)$ is a singular conic. This shows that the discriminant curve of the conic bundle $W$ is equal to $X$.

**(1.3)** The correspondence $(X, \varepsilon) \to W$ defines a birational isomorphism

$$\mathcal{R}_4 \cong |\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2}(1,1,2)|/\mathrm{Aut}(\mathbf{P}^1 \times \mathbf{P}^1) \times \mathrm{Aut}(\mathbf{P}^2).$$

Now let $\mathcal{K}_3$ denote the coarse moduli space of isomorphism classes of pairs $(C, g^1_4)$, where $C$ is a curve of genus 3 and $g^4_1$ is a pencil of divisors of degree 4 on $C$. It is clear that any $g^1_4$ on $C$ is of the form $|K_C + \alpha|$, where $\alpha$ is a non-trivial divisor class of degree 0. This shows that $\mathcal{K}_3$ is birationally isomorphic to the universal family $\mathcal{J}_3$ of Jacobian varieties of curves of genus 3 fibered over $\mathcal{M}_3$. Let $\mathcal{K}\mathrm{um}_3$ be the universal family of the corresponding Kummer varieties, the quotients of the Jacobian varieties by the inversion involution $a \to -a$.

**(1.4)** Let us now show that there is a birational isomorphism

$$\mathcal{K}\mathrm{um}_3 \cong \mathcal{R}_3.$$

It suffices to establish a birational isomorphism

$$\mathcal{K}\mathrm{um}_3 \cong |\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2}(1,1,2)|/\mathrm{Aut}(\mathbf{P}^1 \times \mathbf{P}^1) \times \mathrm{Aut}(\mathbf{P}^2).$$

Given a point in $\mathcal{K}\mathrm{um}_3$ it defines a genus 3 curve $C$ and a non-trivial divisor class $\alpha$ of degree 0 on $C$. The class $\alpha$ defines two $g^1_4$'s $|K_C + \alpha|$ and $|K_C - \alpha|$. We have a natural map:

$$|K_C + \alpha| \times |K_C - \alpha| \cong \mathbf{P}^1 \times \mathbf{P}^1 \to |2K_C| \cong |\mathcal{O}_{\mathbf{P}^2}(2)|, \quad (D_1, D_2) \to D_1 + D_2,$$

where $C \subset \mathbf{P}^2$ is its canonical embedding. This defines a conic bundle $W$ which is a divisor of degree $(1,1,2)$ on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$. Conversely, given $W$ as before, the discriminant curve of the projection $W \to \mathbf{P}^2$ is a plane quartic. The restriction of $W \to \mathbf{P}^1 \times \mathbf{P}^1$ over a fibre $\pi_i^{-1}(x)$ of $p_i : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ defines a pencil $L_i$ of conics on $\mathbf{P}^2$ with four base points on $C$ (see (1.1). The residual 4 points form a $g_4^1$ on $C$. This gives us two $g_4^1$'s complementary to each other in $2K_C$. This gives the inverse map

$$|\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2}(1,1,2)|/\mathrm{Aut}(\mathbf{P}^1 \times \mathbf{P}^1) \times \mathrm{Aut}(\mathbf{P}^2) \to \mathcal{K}\mathrm{um}_3.$$

**(1.5)** One can show, by using the trigonal conctruction of Recillas [**9**], that the map

$$\mathrm{Pr} : \mathcal{R}_3 \to \mathcal{M}_3$$

obtained by composing the birational isomorphism

$$\phi : \mathcal{R}_3 \to \mathcal{K}\mathrm{um}_3$$

with the natural projection $\mathcal{K}\mathrm{um}_3 \to \mathcal{M}_3$ is equal to the Prym map $(X,\eta) \to \mathrm{Prym}(X,\eta)$.

**2.** Let $\mathcal{R}_3$ be identified with the subvariety of $\mathcal{K}\mathrm{um}_3$ parametrizing the pairs $(C, g_4^1)$, where $g_4^1$ is equal to $|K_C + \varepsilon|, 2\varepsilon = 0$. Its pre-image under the map $\phi$ is isomorphic to $\mathcal{R}_3$. We shall see that it is equal to the variety of bi-elliptic pairs $(X,\eta)$.

**(2.1)** Let $X$ be a canonical curve of genus 4 on $\mathbf{P}^1 \times \mathbf{P}^1$. Suppose $X$ is bi-elliptic. Recall that this means that there is an involution $\tau : X \to X$ such that $E = X/(\tau)$ is an elliptic curve. The involution $\tau$ is induced by an automorphism $\tilde{\tau}$ of $\mathbf{P}^1 \times \mathbf{P}^1$ (since $X$ is canonically embedded in $\mathbf{P}^3$ and all non-singular quadrics in $\mathbf{P}^3$ are projectively isomorphic). Since the two $g_3^1$'s of $X$ induced by the rulings of $\mathbf{P}^1 \times \mathbf{P}^1$ are switched under $\tau$, we obtain that $\tilde{\tau}$ switches the two families of rulings of the quadric. This easily implies that $\tilde{\tau}$ is conjugate to the switch involution of $\mathbf{P}^1 \times \mathbf{P}^1$. Thus we may assume that $\tilde{\tau}$ is this involution. Then the factor $X/(\tau)$ can be identified with a cubic curve $E$ in $\mathbf{P}^1 \times \mathbf{P}^1/(\tilde{\tau}) \cong \mathbf{P}^2$. Suppose $\eta$ is a non-trivial 2-torsion divisor class on X that comes from the elliptic curve $E$. Then the pair $(X,\eta)$ is invariant with respect to the involution $\tilde{\tau}$, and the associated conic bundle $W \subset \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$ is invariant with respect to the involution $\tilde{\tau} \times 1$. It follows from the construction that the two $g_4^1$'s on the quartic discriminant curve $C$ coincide. Since they complement each other in the bi-canonical linear system of $C$, each of them is equal to $|K_C + \varepsilon|$, where $2\varepsilon = 0$. This shows that $\mathcal{R}_3^{be}$ is contained in $\phi^{-1}(\mathcal{R}_3)$. Conversely, if we construct $W$ from a curve $C$ with two equal $g_4^1$'s, then $W$ must be invariant with respect to the involution $\tilde{\tau} \times 1$. Hence its discriminant invariant $(X,\eta)$ must be $\tilde{\tau}$-invariant, and therefore comes from a pair $(E,\varepsilon)$, where $E$ is an elliptic curve and $\varepsilon$ is a non-trivial 2-torsion point on $E$. This proves the existence of a birational isomorphism

$$\mathcal{R}_3 \cong \mathcal{R}_3^{be}.$$

**3.** Finally we prove the rationality of $\mathcal{R}_3^{be}$, hence of $\mathcal{R}_3$. The variety $\mathcal{R}_3^{be}$ is a triple cover of the moduli space $\mathcal{M}_4^{be}$ of bi-elliptic curves of genus 4. The latter has a simple description.

Each generic $X \in \mathcal{M}_4^{be}$ is uniquely determined by the isomorphism class of the following data: $(E, \mathcal{L}, s)$, where $E$ is an elliptic curve, $\mathcal{L}$ is a degree 3 invertible sheaf on $E$, and $s \in H^0(E, \mathcal{L}^{\otimes 2})$. The isomorphism between triples $(E, \mathcal{L}, s)$ and $(E', \mathcal{L}', s')$ is induced by the isomorphisms between $E$ and $E'$. If we use $\mathcal{L}$ to embed $E$ in $\mathbf{P}^2$, this data is equivalent to the data $(E, Q)$, where $E$ is a cubic and $Q$ is a conic on $\mathbf{P}^2$ that cuts out in $E$ the divisor of zeroes of $s$. Here the isomorphism is induced by a projective transformation of $\mathbf{P}^2$. In this way we obtain a birational isomorphism

$$\mathcal{M}_4^{be} \cong V = |\mathcal{O}_{\mathbf{P}^2}(3)| \times |\mathcal{O}_{\mathbf{P}^2}(2)|/\mathrm{PGL}(3),$$

where the group acts diagonally. Similarly, we have a birational isomorphism

$$\mathcal{R}_4^{be} \cong \overline{|\mathcal{O}_{\mathbf{P}^2}(3)|} \times |\mathcal{O}_{\mathbf{P}^2}(2)|/\mathrm{PGL}(3),$$

where $\overline{|\mathcal{O}_{\mathbf{P}^2}(3)|}$ is the variety of pairs $(E, \eta), E \in |\mathcal{O}_{\mathbf{P}^2}(3)|, \eta \in {}_2Pic(E) \setminus \{0\}$. There is a well-known birational PGL(3)-equivariant isomorphism

$$\overline{|\mathcal{O}_{\mathbf{P}^2}(3)|} \cong |\mathcal{O}_{\mathbf{P}^2}(3)|.$$

It is defined by assigning to a plane cubic the Hessian invariant of the net of polar cubics (see **[7]** for details). This shows that

$$\mathcal{R}_3^{be} \cong \mathcal{M}_4.$$

It remains to use that the right-hand space is rational **[1]**.

## References


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