RATIONAL SURFACES WITH MANY NODES

I. DOLGACHEV, M. MENDES LOPES, R. PARDINI

1. Introduction

Let X be a smooth rational projective algebraic surface over an algebraically closed field k of characteristic $\neq 2$. It is known that for any nodal curve C on X there exists a birational morphism $f : X \to X'$ such that the image of C is an ordinary double point (a *node*). Let $n(X)$ be the maximal number of disjoint nodal curves on X. After blowing down all of them we obtain a rational normal surface X' with $n(X)$ nodes. The Picard number $\rho(X')$ of X' is equal to the Picard number $\rho(X)$ of X minus $n(X)$. Since X' is projective, $\rho(X') = \rho(X) - n(X) \geq 1$. In this paper we study the limit cases, namely $\rho(X') = 1$ or 2. More precisely, we prove that $\rho(X') = 1$ is possible only if X' is isomorphic to a quadric cone and we describe all the X''s such that $\rho(X') = 2$.

The question of the number of nodes on an algebraic surface is a very old one and has a long history, but, to our knowledge, this particular problem has not been considered. Our interest in this question arose in the course of investigating complex surfaces of general type with $p_g = 0$ admitting a double plane construction, and in the last section of this paper, working over $\mathbb{C}$, we give an application to such surfaces with $K^2 = 8, 9$. More precisely, we extend some of the results of the previous sections for surfaces with $p_g = q = 0$ and non-negative Kodaira dimension and then we consider surfaces S of general type with $p_g = 0$ with an involution σ. We show that if $K^2 = 9$, then S does not admit an involution σ and we list all the possibilities for the quotient surface $S/ < \sigma >$ if $K^2 = 8$.

One of our main tools is the code associated to a set of nodal curves (see Section 2), which has already been considered by A. Beauville in [B].

Research of the first author is partially supported by a NSF grant, the second author is a member of CMAF and of the Departamento de Matemática da Faculdade de Ciências da Universidade de Lisboa and the third author is a member of GNSAGA of CNR.

Notations and conventions. As already explained, we work over any algebraically closed field k of characteristic $\neq 2$ in sections 2 and 3, whilst in section 4 we work over $\mathbb{C}$.

The multiplicative group of k is denoted by $\mathbb{G}_m$. For any abelian group A we denote by ${}_2A$ the kernel of the homomorphism $[2]\colon A \to A$, $a \mapsto 2a$.

All varieties are projective algebraic. We do not distinguish between line bundles and divisors on a smooth variety, and use additive and multiplicative notation interchangeably. Linear equivalence is denoted by $\equiv$ and numerical equivalence by $\sim$. The intersection product of divisors (line bundles) A and B is denoted by $A \cdot B$. We denote by $\kappa(X)$ the Kodaira dimension of a variety X and by $\rho(X)$ the Picard number of X, that is the rank of the Néron–Severi group of X. A *nodal* curve on a surface is a smooth rational curve C such that $C^2 = -2$. The remaining notation is standard in algebraic geometry.

2. Nodal curves, binary codes and covers

In this section all varieties are defined over an algebraically closed field of characteristic $\neq 2$.

Recall that a binary code is a subspace V of a $k-$dimensional vector space W over $\mathbb{F}_2$ equipped with a basis $(e_1, \dots, e_k)$. The dimension of W (identified in the sequel with $\mathbb{F}_2^k$) is called the *length* of V. For each $v \in V$ the number of nonzero coordinates of v with respect to the basis is called the *weight* of v.

Two codes $V_1, V_2 \subset \mathbb{F}_2^k$ are *isomorphic* if there exists a permutation of the coordinates of $\mathbb{F}_2^k$ mapping V_1 onto V_2.

We say that a code V is *reduced* if there is no $1 \le i \le k$ such that $V \subset \{x_i = 0\}$. To every code V one can associate a reduced code V', simply by deleting the "useless" coordinates. The dimension and the weights of V and V' are the same, while V' has (possibly) smaller length. We say that two codes $V_1 \subset \mathbb{F}_2^{k_1}$, $V_2 \subset \mathbb{F}_2^{k_2}$ are *essentially isomorphic* if the corresponding reduced codes are isomorphic.

We mention here a code that plays an important part in what follows. Given an integer n, consider the code of even vectors $V = \{\sum x_i = 0\} \subset \mathbb{F}_2^n$. V has dimension $n-1$ and its weights are all even. We define the *code of doubly even vectors* $DE(n)$ to be the image of V via the injection $\mathbb{F}_2^n \to \mathbb{F}_2^{2n}$ defined by $(x_1 \dots x_n) \mapsto (x_1 x_1 \dots x_n x_n)$. So $DE(n)$ has length $2n$, dimension $n-1$ and all its weights are divisible by 4.

It is possible to associate to a linear code $V \subset \mathbb{F}_2^n$ a lattice Γ_V in the Euclidean space $\mathbb{R}^n$ (see, for example, [Eb]). One considers the canonical homomorphism $p : \mathbb{Z}^n \to \mathbb{F}_2^n$ and takes Γ_V to be $\frac{1}{\sqrt{2}} p^{-1}(V)$.

For example, the code V of even vectors in $\mathbb{F}_2^n$ defines the root lattice of type D_n. The code of doubly even vectors $DE(n)$ defines the root lattice D_{2n} (loc. cit., p.25).

Binary codes arise naturally in the theory of algebraic surfaces, as follows. Consider a smooth projective surface Y and k disjoint nodal curves $C_1, \dots C_k$ of Y. Let $\mathcal{C}$ be the subgroup of $\mathrm{Pic}(Y)$ generated by the curves C_i, which is a free abelian group of rank k. Let $\varphi : \mathcal{C}/2\mathcal{C} \to \mathrm{Pic}(Y)/2\,\mathrm{Pic}(Y)$ be the natural homomorphism of $2-$elementary abelian groups. We call the kernel V of φ the *(binary) code associated to the* C_i and denote its dimension by r. Here we take for a basis of $W := \mathcal{C}/2\mathcal{C}$ the classes of the curves C_i modulo $2\mathcal{C}$. The lattice Γ_V is isomorphic to the smallest primitive sublattice containing the C_i of the lattice $\mathrm{Num}(X)$ of divisors of X modulo numerical equivalence.

We say that a curve C_i *appears in* V if V is not contained in $\{x_i = 0\}$ and we denote by m the number of C_i that appear in V (so m is the length of the reduced code associated to V). The vector $v = (x^1 \dots x^k) \in \mathbb{F}_2^k$ is in V if and only if there exists $L_v \in \mathrm{Pic}(Y)$ such that $2L_v \equiv \sum x^i C_i$ (when it is convenient, we identify $0, 1 \in \mathbb{F}_2$ with the integers $0, 1$). Notice that $K_Y \cdot L_v = 0$ and thus L_v^2 is even by the adjunction formula. Then the weight $w(v)$ of v is equal to $-2L_v^2$ and so it is divisible by 4. Notice that L_v is uniquely determined by v if and only if ${}_2\,\mathrm{Pic}(Y) = 0$.

The following result is analogous to the construction of the Galois cover of a surface Y associated to a torsion subgroup of $\mathrm{Pic}(Y)$.

Proposition 2.1. *Let Y be a smooth projective surface with ${}_2\,\mathrm{Pic}(Y) = 0$, let $C_1 \dots C_k$ be disjoint nodal curves of Y and let V, L_v be defined as above.*

Then there exists a unique smooth connected Galois cover $\pi\colon Z \to Y$ such that:

i) the Galois group of π is $G := Hom(V, \mathbb{G}_m)$;

ii) the branch locus of π is the union of the C_i that appear in V;

iii) $\pi_* \mathcal{O}_Z = \oplus_{v \in V} L_v^{-1}$, and G acts on L_v^{-1} via the character $v \in V \cong Hom(G, \mathbb{G}_m)$.

Proof. For $v \in V$ and $g \in G$, we define $\epsilon_v(g) \in \{0, 1\}$ by $(-1)^{\epsilon_v(g)} = g(v)$. We fix a basis $v_1 \dots v_r$ of V and we write ϵ_j for ϵ_{v_j}, $j = 1 \dots r$.

By Proposition 2.1 of [Pa], in order to determine $\pi\colon Z \to Y$ we have to assign the (reduced) *building data*, namely:

1) for every nonzero $g \in G = \mathrm{Hom}(V, \mathbb{G}_m)$ an effective divisor D_g;

2) for every $j = 1 \dots r$ a line bundle M_j

in such a way that the following relations are satisfied:

$$2M_j \equiv \sum_{g\in G} \epsilon_j(g) D_g, \quad j = 1 \dots r. \tag{2.1}$$

For $i = 1 \dots k$ we denote by $\psi_i \colon W = \mathbb{F}_2^k \to \mathbb{G}_m$ the homomorphism defined by $(x^1 \dots x^k) \to (-1)^{x_i}$. We define D_g to be the sum of the C_s such that $\psi_s|_V = g$. Notice that the D_g are disjoint and that $D := \sum_g D_g$ is the union of the C_i that appear in V. If we write $v_j = (x_j^1 \dots x_j^k)$, and we identify again $0, 1 \in \mathbb{F}_2$ with the integers $0, 1$, then it is not difficult to check that relations (2.1) can be rewritten as:

$$2M_j \equiv \sum_i x_j^i C_i \ , \quad j = 1 \dots r. \tag{2.2}$$

So equations 2.1 can be solved uniquely by setting $M_j = L_{v_j}$, $j = 1 \dots r$. The corresponding cover $\pi \colon Z \to Y$ satisfies conditions i) and ii) of the statement. In addition, Z is smooth by Proposition 3.1 of [Pa], since D is smooth, and it is connected since the set of $g \in G$ that $D_g \neq 0$ generates G. In order to complete the proof we have to check that for every $v = (x_v^1 \dots x_v^k) \in V$ the eigensheaf M_v^{-1} of $\pi_* \mathcal{O}_Z$ on which G acts via the character v is L_v^{-1}. By Theorem 2.1 of [Pa], we have $2M_v \equiv \sum_g \epsilon_v(g) D_g$. This equation can be rewritten as $2M_v \equiv \sum_i x_v^i C_i$, and thus $2L_v = 2M_v$ in $\mathrm{Pic}(Y)$. The equality $L_v = M_v$ follows since ${}_2\mathrm{Pic}(Y) = 0$. $\square$

Remark 1. Write $U := Y \setminus \cup_i C_i$. Then there is an isomorphism $\psi \colon V \to_2 \mathrm{Pic}(U)$ and the restriction to U of the cover $\pi \colon Z \to Y$ is the $G-$torsor corresponding to ψ under the natural map $H^1(U, G) \to \mathrm{Hom}(V, \mathrm{Pic}(U))$.

Remark 2. The proof of Proposition 2.1 shows that if one removes the assumption ${}_2\mathrm{Pic}(Y) = 0$ then the cover $\pi \colon Z \to Y$ exists but it is not determined uniquely. Also, if one assumes $char(k) = 2$, then the proof shows the existence of a purely inseparable cover with a $G-$action.

Let $\eta \colon Y \to \Sigma$ be the map that contracts the curves C_i that appear in V to singular points of type A_1. The inverse image in Z of a curve C_i that appears in V is a disjoint union of 2^{r-1} $(-1)-$curves. Blowing down all these $(-1)-$curves, we obtain a smooth surface $\bar{Z}$ and a $G-$cover $\bar{\pi} \colon \bar{Z} \to \Sigma$ branched precisely over the singularities of Σ.

Then we have the following commutative diagram:

$$\begin{array}{ccc} Z & \longrightarrow & \bar{Z} \\ {\scriptstyle \pi}\big\downarrow & & {\scriptstyle \bar{\pi}}\big\downarrow \\ Y & \xrightarrow{\eta} & \Sigma, \end{array}$$

We close this section by computing the invariants of Z and $\bar{Z}$.

Lemma 2.2. *With the same assumptions and notations as in Proposition 2.1 (in particular, r is the dimension of V and m is the number of the C_i's that appear in V) one has:*

$$c_2(Z) = 2^r c_2(Y) - m2^r.$$

Proof. If the base field is $\mathbb{C}$, then the formula follows easily by topological considerations. We give an algebraic proof, valid for fields of characteristic $\neq 2$.

Denote by D the branch divisor of π (which is the union of m disjoint nodal curves), and by $R = \pi^{-1}D$ the ramification divisor. Consider the following exact sequence of sheaves on Z:

$$(2.3) \qquad 0 \to \pi^*\Omega^1_Y \xrightarrow{j} \Omega^1_Z \to \mathcal{K} \to 0$$

where the cokernel $\mathcal{K}$ is a torsion sheaf supported on R. Consider a ramification point $P \in Z$ and let R' be the irreducible component of R containing P. The subgroup $H \subset G$ consisting of the elements that induce the identity on R' is isomorphic to $\mathbb{Z}_2$ (cf. [Pa], Lemma 1.1). The surface $W := Z/H$ is smooth, since the fixed locus of H is purely $1-$dimensional, and π factorizes as $Z \xrightarrow{\alpha} W \xrightarrow{\beta} Y$. Let $Q = \alpha(P)$ and $D' = \alpha(R')$. The map β is étale in a neighbourhood of Q, and thus $\beta^*\Omega^1_Y \hookrightarrow \Omega^1_W$ is an isomorphism locally near Q. It follows that the inclusion $\pi^*\Omega^1_Y \hookrightarrow \alpha^*\Omega^1_W$ is an isomorphism locally around P. There exists an open neighbourhood U of Q in W such that $Z|_U$ is defined in $U \times \mathbb{A}^1$ by the equation $z^2 = b$, where b is a local equation for D' and z is the affine coordinate in $\mathbb{A}^1$. Notice that z is a local equation for R' around P. Let x be a function on W such that x, b are local parameters on W around Q. Then the map j of sequence (2.3) can be written locally as $(dx, db) \mapsto (dx, 2zdz)$. It follows that the cokernel $\mathcal{K}$ is naturally isomorphic to the conormal sheaf of R, $\mathcal{O}_R(-R)$. A standard computation with Chern classes gives $c_2(Z) = 2^r c_2(Y) + 2R^2 + \pi^*K_Y \cdot R = 2^r c_2(Y) + 2^{r-1}D^2 + 2^{r-1}K_Y \cdot D = 2^r c_2(Y) - m2^r$.

□

Proposition 2.3. *Under the same assumptions and notation as above the following holds:*

$$\kappa(Z) = \kappa(\bar{Z}) = \kappa(Y);$$
$$K_Z^2 = 2^r K_Y^2 - m2^{r-1} \quad K_{\bar{Z}}^2 = 2^r K_Y^2;$$
$$\chi(Z, \mathcal{O}_Z) = \chi(\bar{Z}, \mathcal{O}_{\bar{Z}}) = 2^r \chi(\mathcal{O}_Y) - m2^{r-3}.$$

Proof. We have $K_{\bar{Z}} = \bar{\pi}^* K_\Sigma$, since $\bar{\pi}$ is unramified in codimension 1 and Σ is normal, and therefore $K_{\bar{Z}}^2 = 2^r K_\Sigma^2 = 2^r K_Y^2$. The formula for K_Z^2 follows immediately. Since χ is a birational invariant, it is enough to compute it for Z. Then the formula for $\chi(Z, \mathcal{O}_Z)$ follows from Lemma 2.2 and Noether's formula.

If $\kappa(Z) = -\infty$, then we have $\kappa(Y) = -\infty$ (π is separable since the characteristic of k is $\neq 2$). So assume that $\kappa(Z) \geq 0$ and denote by $\tilde{Z}$ the minimal model of Z and $\bar{Z}$. Then G acts biregularly on $\tilde{Z}$. We denote by $\tilde{\pi}\colon \tilde{Z} \to \tilde{\Sigma} := \tilde{Z}/G$ the quotient map. The surface $\tilde{\Sigma}$ has canonical singularities and it is birational to Y and Σ. Denote by $\epsilon\colon \tilde{Y} \to \tilde{\Sigma}$ the minimal resolution. We have $K_{\tilde{Z}} = \tilde{\pi}^* K_{\tilde{\Sigma}}$ and thus $K_{\tilde{\Sigma}}$ and $K_{\tilde{Y}} = \epsilon^* K_{\tilde{\Sigma}}$ are nef. So $\tilde{Y}$ is minimal and, in addition, $K_{\tilde{Y}} \sim 0$ iff $K_{\tilde{Z}} \sim 0$ and $K_{\tilde{Y}}^2 = 0$ iff $K_{\tilde{Z}}^2 = 0$. This remark shows that $\kappa(\tilde{Y}) = \kappa(\tilde{Z})$. □

3. Rational surfaces with many nodes.

Throughout this section we assume that Y is a smooth rational surface and $C_1, ..., C_k$ are disjoint nodal curves of Y. As before, we let V be the code associated to the C_i, r its dimension and m the number of the C_i that appear in V . The group $\mathrm{Pic}(Y)$ is free abelian of rank $\rho(Y) = 10 - K_Y^2$ and the intersection form on $\mathrm{Pic}(Y)$ induces a non degenerate $\mathbb{F}_2-$valued bilinear form on $\mathrm{Pic}(Y)/2\,\mathrm{Pic}(Y)$. Since $C_i^2 = -2$ and the C_i are disjoint, the image of $\mathcal{C}/2\mathcal{C}$ is a totally isotropic subspace of $\mathrm{Pic}(Y)$. Thus the dimension r of V satisfies $r \geq k - [\frac{\rho(Y)}{2}]$. As a corollary of the results in the previous section we have the following

Lemma 3.1. *If $r \geq 4$, then $m \geq 8$.*

Proof. Consider the cover $\pi\colon Z \to Y$ of Proposition 2.1 associated to V and the corresponding cover of Σ, $\bar{\pi}\colon \bar{Z} \to \Sigma$. By Proposition 2.3, $\bar{Z}$ is ruled and thus $\chi(\bar{Z}) \leq 1$. The result follows by using the formula for $\chi(Z, \mathcal{O}_Z)$ of Proposition 2.3. □

Theorem 3.2. *Let $C_1 \dots C_k$ be disjoint nodal curves on a rational surface Y, let V be the code associated to $C_1 \dots C_k$ and assume that the length m of the reduced code V' of V is $m \geq 8$. Denote by $\eta\colon Y \to \Sigma$*

the map that contracts to nodes the C_i that appear in V. Then there exists a fibration $\beta\colon \Sigma \to \mathbb{P}^1$ such that:

(i) *the general fibre of β is $\mathbb{P}^1$;*
(ii) *$m = 2n$ is even and β has n double fibres, each containing two nodes of Σ;*
(iii) *the code V is essentially isomorphic to $DE(n)$.*

Proof. Let $\pi\colon Z \to Y$ the cover of Proposition 2.1 and let $\bar{\pi}\colon \bar{Z} \to \Sigma$ the corresponding cover of Σ. By Proposition 2.3, the surface $\bar{Z}$ is ruled and has irregularity $q(\bar{Z}) = 1 + m2^{r-3} - 2^r > 0$. Denote by $\alpha\colon \bar{Z} \to C$ the Albanese pencil. By the canonicity of the Albanese map, the group G preserves the divisor class of a fibre. Consider the canonical homomorphism $G \to \mathrm{Aut}(C)$: if it is not injective, then there exists $g \in G$ that maps each fibre of α to itself. Hence a general fibre, being isomorphic to $\mathbb{P}^1$, has 2 fixed points of g and the ramification locus for the action of G has components of dimension 1, a contradiction since the $G-$cover is branched precisely over the singularities of Σ. Thus we have a commutative diagram:

$$\begin{array}{ccc} \bar{Z} & \xrightarrow{\bar{\pi}} & \Sigma \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ C & \xrightarrow{p} & \mathbb{P}^1, \end{array}$$

where $p\colon C \to \mathbb{P}^1$ is a $G-$cover. The general fibre of β is $\mathbb{P}^1$, since it is isomorphic to the general fibre of α. Since the genus of C is equal to $q(\bar{Z})$, by the Hurwitz formula the branch locus of p consists of $n = m/2$ points (the inverse image of a branch point consists of 2^{r-1} simple ramification points). The cover $\bar{\pi}\colon \bar{Z} \to \Sigma$ is obtained from p by base change and normalization, thus the fibres of β over the branch points $y_1 \dots y_n$ of p are of the form $f_i = 2\delta_i$, $i = 1 \dots n$, and $\cup_i \delta_i$ contains all the nodes of Σ. We claim that each double fibre contains at least one node. Indeed, otherwise δ_i would be contained in the smooth part of Σ and so it would be a Cartier divisor with $\delta_i^2 = 0$, $K_\Sigma \cdot \delta_i = -1$, a contradiction to the adjunction formula.

Set $\beta' = \beta \circ \eta$. Then for every i, one can write $\beta'^* y_i = 2A_i + \sum_s C_{i,s}$ and it follows that for every choice of $h \neq j$ the divisor $\sum_s C_{h,s} + \sum_t C_{j,t}$ is divisible by 2 in $\mathrm{Pic}(Y)$, namely it corresponds to a vector of V. Since the weights of V are all divisible by 4, it follows easily that each δ_i contains precisely 2 nodes of Σ. So it is possible to relabel the C_i in such a way that $\beta'(C_{2j-1}) = \beta'(C_{2j}) = y_j$ for $j = 1 \dots n$, and that $C_{2j} + C_{2j-1} + C_{2h} + C_{2h-1}$ is divisible by 2 in $\mathrm{Pic}(Y)$ for every choice of j, h. This shows that V is essentially isomorphic to the code $DE(n)$. □

Next we apply the above results to describe rational surfaces with "many" disjoint nodal curves. We start by describing an example.

Example 1. Consider a minimal ruled rational surface $\mathbf{F}_e$ and a point $y \in \mathbf{F}_e$. If one blows up y, then the total transform of the ruling of $\mathbf{F}_e$ containing y is the union of two $(-1)-$curves E and E' that intersect transversely in a point y_1. If one blows up also y_1, then the strict transforms of E and E' are disjoint nodal curves. By repeating this procedure n times at points lying on different rulings of $\mathbf{F}_e$, one obtains a rational surface Y containing $2n$ disjoint nodal curves. One has $\rho(Y) = 2n+2$ and it is easy to check that the code V associated to this collection of curves is $DE(n)$. We will call Y the *standard example* of a rational surface with $\rho(Y) - 2$ disjoint nodal curves.

Theorem 3.3. *Let Y be a smooth rational surface and let $C_1 \dots C_k$ be disjoint nodal curves of Y. Then:*

(i) *$k \leq \rho(Y) - 1$, and equality holds if and only if $Y = \mathbf{F}_2$;*
(ii) *if $k = \rho(Y) - 2$ and $\rho(Y) \geq 5$, then Y is the standard example. In particular $k = 2n$ is even and the code V is $DE(n)$.*

Proof. The group $\mathrm{Pic}(Y)$ is free abelian of rank $\rho := \rho(Y)$. The intersection form on $\mathrm{Pic}(Y)$ extends to a nondegenerate bilinear form of signature $(1, \rho(Y) - 1)$ on $N^1(Y) := \mathrm{Pic}(Y) \otimes \mathbb{R}$. The subspace of $N^1(Y)$ spanned by the classes of the C_i has dimension k and the intersection form is negative definite there, thus we get $k < \rho$.

We start by proving (ii). As before, we let $m \leq k$ be the number of nodal curves that appear in the code V. Recall that the dimension r of V is $\geq \rho - 2 - [\frac{\rho}{2}] = [\frac{\rho+1}{2}] - 2$. So, for $\rho \geq 11$, we have $r \geq 4$ and thus $m \geq 8$ by Lemma 3.1.

Assuming then that $\rho \geq 11$, we can apply Theorem 3.2. Thus m is even, say $m = 2n$, and there exists a morphism $\beta\colon \Sigma \to \mathbb{P}^1$ such that the general fibre of β is $\mathbb{P}^1$ and β has n double fibres, occurring at points $y_1 \dots y_n$ of $\mathbb{P}^1$. Each double fibre contains precisely 2 nodes of Σ, and the code V is $DE(n)$. So we have $n - 1 = \dim V \geq [\frac{\rho+1}{2}] - 2$, namely $\rho - 2 \geq m = 2n \geq 2[\frac{\rho+1}{2}] - 2$. It follows that ρ is even and $\rho - 2 = m = 2n$. In particular, $m = k$, i.e. all the C_i appear in V.

Set $\beta' = \beta \circ \eta$, denote by F the cohomology class on Y of a fibre of β' and let

$$T = \{L \in N^1(Y) : L \cdot F = 0\}.$$

A basis of T is given by F and the classes of $C_1 \dots C_{2n}$, since these are independent classes and $\dim T = \rho - 1 = 2n+1$. On the other hand, it is well known that, if one removes a component from each reducible fibre of β', then F and the classes of the remaining components of the reducible fibres are independent. It follows that the $F_i := \beta'^* y_i$, $i = 1, \dots n$ are the only reducible fibres of Y. As in the proof of Theorem 3.2, it is possible to relabel the C_i in such a way that for each i one has $F_i = \lambda_i C_{2i-1} + \mu_i C_{2i} + 2\nu_i D_i$, with D_i irreducible and such that $D_i^2 < 0$. From $K_Y \cdot F = -2$, we get $\nu_i = 1$, $K_Y \cdot D_i = -1$, and thus $D_i^2 = -1$, namely D_i is a (-1)-curve. The curve D_i has nonempty intersection with both C_{2i-1} and C_{2i}, since F_i is connected. So the equality

$$0 = D_i \cdot F = D_i \cdot (2D_i + \lambda_i C_{2i-1} + \mu_i C_{2i}) = -2 + \lambda_i D_i \cdot C_{2i-1} + \mu_i D_i \cdot C_{2i}$$

gives:

$$\lambda_i = \mu_i = D_i \cdot C_{2i-1} = D_i \cdot C_{2i} = 1.$$

Blowing down $D_1 \dots D_n$ one obtains a smooth surface ruled over $\mathbb{P}^1$ with precisely n reducible rulings, each consisting of two (-1)-curves intersecting transversely. Blowing down a $(-1)-$curve of each ruling, we obtain a ruled surface $\mathbf{F_e}$. So Y is the standard example.

In order to complete the proof of (ii), we have to describe the cases $5 \leq \rho \leq 10$. In addition we may assume $m < 8$, since for $m = 8$ (and $\rho = 10$) one can apply the argument above to show that Y is the standard example. Since $m < 8$ all the elements of V have weight 4 and it is easy to check that the only (numerical) possibilities for the pair (k, r) are: $(4,1)$, $(6,2)$, $(7,3)$ and $(8,3)$. One has $m = k$ in all cases but the last one, where $m = 7$.

Consider the first three cases. Let $Z \to Y$ be the Galois cover considered in Proposition 2.1 and $\bar{Z} \to \Sigma$ the corresponding cover of Σ. By Proposition 2.3, $\bar{Z}$ is a surface satisfying $\kappa(\bar{Z}) = \kappa(Y)$, $K^2_{\bar{Z}} = 8$, $\chi(\bar{Z}) = 1$. So $\bar{Z}$ is rational and $K^2_{\bar{Z}} = 8$ implies that $\bar{Z} = \mathbf{F}_e$. Denote by t the trace of $g \in G \setminus \{1\}$ on the l-adic cohomology $H^2(\bar{Z}, \mathbb{Q}_l) \cong \mathbb{Q}_l^2$. It is equal to 0 or 2. Applying the (l-adic) Lefschetz fixed point formula (see [SGA5], (4.11.3), cf. the next section for the analogous statement for the complex cohomology) we see that $t = 0$ is impossible and hence g acts identically on $H^2(\bar{Z}, \mathbb{Q}_l)$. In particular, given the ruling (or a ruling if $e = 0$) $f \colon \bar{Z} \to \mathbb{P}^1$ the action of the Galois group G of $\bar{Z} \to \Sigma$ descends to an action on $\mathbb{P}^1$ and there is an induced fibration $h \colon \Sigma \to \mathbb{P}^1/G = \mathbb{P}^1$. The same argument as in the proof of Theorem 3.2 shows that the action of G on $\mathbb{P}^1$ is faithful. Thus each element g of G fixes precisely two fibres of h, each containing two fixed points of g. Since $\mathrm{Aut}(\mathbb{P}^1)$ does not contain a subgroup isomorphic to $\mathbb{Z}_2^3$ we can rule out immediately the case $(n, r) = (7, 3)$. In the remaining two cases, the cover $\mathbb{P}^1 \to \mathbb{P}^1/G$ is branched over $n = m/2$ points, and over each of these points h has an irreducible double fibre containing 2 nodes. It follows easily that Y is the standard example.

Finally consider the case $(8, 3)$, $m = 7$ (the code is essentially isomorphic to the Hamming code defined by the root lattice of type E_7). By Proposition 2.3 the G-cover $\bar{Z} \to \Sigma$ is a smooth ruled surface with invariants $K^2_{\bar{Z}} = 0$, $\chi(\bar{Z}) = 1$. Thus $\bar{Z}$ is rational. The preimage of the nodal curve of Y not appearing in V is a set $D_1 \dots D_8$ of disjoint nodal curves on which $G = \mathbb{Z}_2^3$ acts transitively. The code $\tilde{V} \subset \mathbb{F}_2^8$ associated to $D_1 \dots D_8$ is acted on by G, and therefore all the nodes appear in $\tilde{V}$, namely $\tilde{V}$ has $m = 8$. Thus $\bar{Z}$ is a standard example with $\rho = 10$. If there is only one pencil with rational fibres $f \colon \bar{Z} \to \mathbb{P}^1$ such that the D_i are contracted by f, then one argues as in case $(7, 3)$ and obtains a contradiction by showing the existence of a $\mathbb{Z}_2^3-$cover $\mathbb{P}^1 \to \mathbb{P}^1$. So assume that there are two pencils with rational fibres $f_j \colon \bar{Z} \to \mathbb{P}^1$, $j = 1, 2$ such that the D_i are contracted both by f_1 and f_2. Denote by F_j, $j = 1, 2$, the class in $N^1(\bar{Z}) := \mathrm{Pic}(\bar{Z}) \otimes \mathbb{R}$ of a smooth fibre of f_j. Considering the intersection form, one sees immediately that the classes of F_1, F_2, $D_1 \dots D_8$ are a basis of $N^1(\bar{Z})$. Consider a nonzero $g \in G$. The surface $Z' := \bar{Z}/ < g >$ is a rational surface with s singular points of type A_1, that are the images of the fixed points of g on $\bar{Z}$. By the standard double cover formulas:

$$1 = \chi(\bar{Z}) = 2\chi(Z') - s/4 = 2 - s/4$$

and so $s = 4$. Denote by t the trace of g on the l-adic cohomology $H^2(\bar{Z}, \mathbb{Q}_l)$. Applying again the Lefschetz fixed point formula we get $t = 2$. The action of g on $H^2(\bar{Z}, \mathbb{Q}_l)$ preserves the subspace $< D_1 \dots D_8 >$ generated by the fundamental classes of the divisors $D_1, \dots, D_8$, and thus it preserves also its orthogonal subspace spanned by the classes of F_1, F_2. The trace of g on $< D_1 \dots D_8 >$ is zero. It follows that g is the identity on $< F_1, F_2 >$, namely every $g \in G$ preserves both pencils. Thus we can apply again the argument above to one of the pencils and the proof of (ii) is complete.

Finally we prove (i). Assume that $k = \rho(Y) - 1$. The code V has length $\rho - 1$, dimension $r \geq [\frac{\rho+1}{2}] - 1$ and all the weights divisible by 4. Thus if $\rho \geq 9$, then $m \geq 8$ by Lemma 3.1 and one can argue as in case (ii) and show that Y is the surface constructed in the standard example and V is essentially isomorphic to $DE(n)$, with $n = \rho/2 - 1$. In particular, $r = n - 1 = \rho/2 - 2$, contradicting $r \geq [\frac{\rho+1}{2}] - 1$. So assume $\rho \leq 8$. If $\rho = 2$, then $K^2_Y = 8$ and Y is a minimal ruled surface $\mathbf{F}_2$. If $\rho > 2$, the only numerical possibility is $\rho = 8$, $r = 3$. Let Y

be a surface corresponding to this possibility. We have $K_Y^2 = 2$. Up to a permutation, we may assume that $C_1 \dots C_4$ is an even set. The corresponding double cover $Y' \to Y$ is a smooth rational surface (same proof as Proposition 2.3), with $K_{Y'}^2 = 0$. The inverse images of $C_1 \dots C_4$ are $(-1)-$ curves, while the inverse images of C_5, C_6, C_7 are three pairs of disjoint nodal curves. Blowing down the $(-1)-$curves, one obtains a rational surface Y'' with $\rho(Y'') = 6$ and containing 6 disjoint nodal curves. This is impossible, and the proof is complete. □

Remark 3. If $\rho(Y) \leq 4$ then it is easy to see that the maximal number k of disjoint nodal curves is equal as follows:

$$(k, \rho) = (0,1), \quad (1,2), \quad (1,3), \quad (2,4).$$

The last case is the standard example.

4. An application

Throughout this section we assume that the ground field is $\mathbb{C}$. We apply the previous results to study involutions (i.e. automorphisms of order 2) on minimal surfaces of general type with $p_g = 0$ and $K_S^2 = 8$ or 9.

We start by extending the results of section 3 to complex surfaces with $p_g = q = 0$ and nonnegative Kodaira dimension. The use of Miyaoka's formula is a key ingredient for the proof below and explains the assumption $k = \mathbb{C}$ in this section.

Proposition 4.1. *Let Y be a surface with $p_g(Y) = q(Y) = 0$ and $\kappa(Y) \geq 0$, and let $C_1 \dots C_k \subset Y$ be disjoint nodal curves. Then:*
i) $k \leq \rho(Y) - 2$;
ii) if $k = \rho(Y) - 2$, then Y is minimal.

Proof. Assume first that Y is minimal. In this case we can apply Miyaoka's formula ([M], section 2): $3c_2(Y) - K_Y^2 \geq \frac{9}{2}k$, and i) follows immediately using $0 \leq K_Y^2 \leq 9$ and Noether's formula.

Now assume that Y is not minimal and let $\bar{Y}$ be the minimal model of Y. We use induction on $\nu := \rho(Y) - \rho(\bar{Y})$. Let $E \subset Y$ be an irreducible $(-1)-$curve and let Y' be the surface obtained by blowing down E. If E does not intersect any of the C_i, then Y' contains k disjoint nodal curves and induction gives: $k \leq \rho(Y') - 2 = \rho(Y) - 3$. So assume, say, $C_1 \cdot E = \alpha > 0$. Then the image C_1' of C_1 in Y' is an irreducible curve such that $(C_1')^2 = -2 + \alpha^2$, $C_1' \cdot K_{Y'} = -\alpha$. Now necessarily $\alpha = 1$. In fact suppose that $\alpha \geq 2$. Then $C_1'^2 > 0$ and therefore the image of C_1' in the minimal model $\bar{Y}$ of Y is a curve C_1''. Since $C_1'' \cdot K_{\bar{Y}} \leq C_1' \cdot K_{Y'} < 0$ and $K_{\bar{Y}}$ is nef because

$\kappa(Y') \geq 0$, we have a contradiction. Therefore C_1' is a $(-1)-$curve. In addition, $E \cdot C_i = 0$ for $i > 1$, since otherwise Y' would contain a pair of irreducible $(-1)-$curves with nonempty intersection, which is impossible again because $\kappa(Y') \geq 0$. Now blowing down C_1' we obtain a surface Y'' containing a set of $k-1$ disjoint irreducible nodal curves. Using induction again, we have $k-1 \leq \rho(Y'') - 2 = \rho(Y) - 4$ and the proof is complete. □

Let S be a surface admitting an involution σ. Let k be the number of isolated fixed points of σ and let D be the 1-dimensional part of the fixed-point locus. The divisor D is smooth (possibly empty). If we consider the blow-up X of the set of isolated fixed points, then the involution σ lifts to an involution on X (which we still denote by σ) and the quotient $Y := X/ < \sigma >$ has k disjoint nodal curves C_i.

We recall the following two well-known formulas:

(Holomorphic Fixed Point Formula) (see [AS], pg.566):

$$\sum_{i=0}^{2}(-1)^i \mathrm{Trace}(\sigma | H^i(S, \mathcal{O}_S)) = \frac{k - D \cdot K_S}{4}$$

(Topological Fixed Point Formula) (see [Gr], (30.9)):

$$\sum_{i=0}^{4}(-1)^i \mathrm{Trace}(\sigma | H^i(S, \mathbb{C})) = k + e(D),$$

where $e(D) = -D^2 - D \cdot K_S$ is the topological Euler characteristic of D.

Lemma 4.2. *Let S be a surface with $p_g(S) = q(S) = 0$ and let σ be an automorphism of S of order 2. Let D be the divisorial part of the fixed locus of σ, let k be the number of isolated fixed points of σ and let t be the trace of $\sigma | H^2(S, \mathbb{C})$. Then:*

$$k = K_S \cdot D + 4; \quad t = 2 - D^2.$$

Furthermore if X is the blow-up of the k isolated fixed points of σ, and $Y = X/ < \sigma >$ one has

$$\rho(S) + t = 2\rho(Y) - 2k$$

.

Proof. The first Fixed Point formula gives

$$k = 4 + K_S \cdot D \tag{4.1}$$

Together with the second formula we obtain

$$t := \mathrm{Trace}(\sigma | H^2(S, \mathbb{C})) = 2 - D^2. \tag{4.2}$$

For the last part notice that we have

$$e(S) + k = e(X) = 2e(Y) - 2k - e(D).$$

Since by the topological fixed point formula $e(D) = -k + 2 + t$, one has

$$e(S) + t + 2 = 2e(Y) - 2k.$$

Now $p_g = q = 0$ implies $e(S) = \rho(S) + 2, e(Y) = \rho(Y) + 2$ and we obtain

$$\rho(S) + t = 2\rho(Y) - 2k \tag{4.3}$$

$\square$

Theorem 4.3. *A surface of general type S with $p_g(S) = 0$ and $K_S^2 = 9$ has no automorphism of order 2.*

Proof. Assume otherwise. Since $\rho(S) = 1$, we have $t = 1$. Lemma 4.2 gives $D^2 = 1$. Since the canonical class is invariant for σ, we have $K_S \sim rD$ for some $r \in \mathbb{Q}$. Then $K_S^2 = 9$ yields $K_S \sim 3D$ and $K_S \cdot D = 3$. Thus Lemma 4.2 gives $k = 7$ and $2 = 2\rho(Y) - 14$, i.e. $\rho(Y) = 8$. So Y contains $\rho(Y) - 1$ disjoint nodal curves and $K_Y^2 = 2$. This is a contradiction in view of Theorem 3.3 and Proposition 4.1. $\square$

Theorem 4.4. *Let S be a minimal surface of general type with $p_g(S) = 0$, $K_S^2 = 8$ and let σ be an automorphism of S of order 2. Let D be the divisorial part of the fixed locus of σ, let k be the number of isolated fixed points of σ and let Y be a minimal resolution of the quotient $S/<\sigma>$. Then:*

$$D^2 = 0, \quad K_S \cdot D = k - 4$$

and one of the following cases occurs:

(i) *$k = 4$, $D = 0$ and Y is a minimal surface of general type with $p_g(Y) = 0$ and $K_Y^2 = 4$.*
(ii) *$k = 6$, and Y is a minimal surface of general type with $p_g(Y) = 0$ and $K_Y^2 = 2$.*
(iii) *$k = 8$, Y is a minimal surface with $p_g(Y) = q(Y) = 0$, $\kappa(Y) = 1$ and such that the elliptic fibration $Y \to \mathbb{P}^1$ with two reducible fibres of Kodaira type I_0^*.*
(iv) *$k = 10$, and Y is a rational surface from Example 1 with $\rho = 12$. The fibration with connected rational fibres $f: Y \to \mathbb{P}^1$ pulls back on S to a pencil of hyperelliptic curves of genus 5.*
(v) *$k = 12$, and Y is a rational surface from Example 1 with $\rho = 14$. The fibration with connected rational fibres $f: Y \to \mathbb{P}^1$ pulls back on S to a pencil of hyperelliptic curves of genus 3.*

Proof. Since $\rho(S)=2$, the possible values for the trace t are 0 and 2.

The case $t=0$ does not occur. Indeed, assume otherwise. By Lemma 4.2, $D^2=2$ so that $D\neq 0$. Since $t=0$, the invariant part of $H^2(S,\mathbb{Q})$ is one-dimensional and thus (because the canonical class is invariant for σ), $K_S\sim rD$ for some $r\in\mathbb{Q}$. Thus $K_S\sim 2D$ and, hence $K_S\cdot D=4$. Lemma 4.2 gives $k=8$ and $\rho(Y)=9$, and so by Noether's formula $K_Y^2=1$. Since Y contains 8 disjoint nodal curves, we have a contradiction to Theorem 3.3 and Proposition 4.1. So $t\neq 0$.

Now we consider the case $t=2$, that is, the involution σ acts identically on $H^2(S,\mathbb{Q})$. In this case $D^2=0$.

If $D=0$, we get $k=4$ and the surface Y is a surface of general type with $K_Y^2=4$ and $\rho(Y)=6$. It contains an even set of four disjoint nodal curves $C_1,\dots C_4$ and thus it is minimal by Proposition 4.1. This is case (i).

The last case to consider is $t=2$ and $D\neq 0$. Since $D^2=0$, we have $K_S\cdot D=2m$, with $m>0$. Then Lemma 4.2 gives $k=4+2m$, so that in particular k is ≥ 6 and even, and $\rho(Y)=6+2m=k+2$.

Assume that $\kappa(Y)\geq 0$. Since Y is a minimal surface by Proposition 4.1, $K_Y^2\geq 0$ and so $k=8-K_Y^2\leq 8$. So either $k=6$ or $k=8$. If $k=6$, $K_Y^2=2$ and so Y is of general type and we have case (ii). If $k=8$, then $K_Y^2=0$ and thus Y, being minimal, is not of general type. Since $p_g(Y)=q(Y)=0$, Y is either an Enriques surface or a surface of Kodaira dimension 1. The first case cannot occur. In fact since $K_Y\sim 0$ and $D^2=0$ we would have $K_S\cdot D=0$, a contradiction. So $\kappa(Y)=1$ and Y is a minimal properly elliptic surface. Denote by $f\colon Y\to\mathbb{P}^1$ the elliptic fibration and let F be a general fibre of f. Since K_Y is numerically a rational multiple of F, we have $F\cdot C_i=0$ for every i, namely the C_i are mapped to points by f. Let $\bar F$ be a fibre containing, say, $C_1\dots C_s$ and let $A_1\dots A_p$ be the remaining irreducible components of $\bar F$. It is well known that the classes of $A_1\dots A_p, C_1\dots C_r$ in $H^2(Y,\mathbb{Q})$ are independent and span a subspace U_1 on which the intersection form is seminegative. The classes of $C_{s+1}\dots C_8$ are also independent and span a subspace U_2 such that the intersection form is negative on U_2 and $U_1\cap U_2=\{0\}$. Since $\rho(Y)=10$, we see that the only possibility is $p=1$. Looking at Kodaira's list of singular elliptic fibres (see e.g. [BPV], pg.150), one sees that the possible types of singular fibres containing some of the C_i are I_2, I_0^* and III. In addition, we have $12=e(Y)=\sum_t e(F_t)$, where F_t is the fibre of f over the point $t\in\mathbb{P}^1$ and e denotes the topological Euler–Poincaré characteristic. It is easy to check that the only numerical possibility is that f has two I_0^* fibres, each containing 4 of the C_i, and that every other singular fibre is a multiple of a smooth elliptic curve. Up to a permutation we may assume that the I_0^* fibres of f are $C_1 + \ldots + C_4 + 2D_1$ and $C_5 + \ldots + C_8 + 2D_2$. So $C_1 + \ldots + C_8 \equiv 2(F - D_1 - D_2)$ is divisible by 2 in $\mathrm{Pic}(Y)$. Let $\pi\colon Y' \to Y$ be the corresponding double cover. For a general fibre F of f, π^*F is disconnected and the Stein factorization of $f \circ \pi$ gives rise to an elliptic fibration $f'\colon Y' \to \mathbb{P}^1$ "with the same fibres" as f. The inverse images of D_1, D_2 are smooth elliptic curves. The inverse images of $C_1, \ldots C_8$ are 8 $(-1)-$ curves contained in the fibres of f'. Blowing these exceptional curves down, one obtains an elliptic fibration $f''\colon Y'' \to \mathbb{P}^1$ whose only singular fibres are multiples of smooth elliptic fibres. Thus f'' has constant moduli, and therefore f' and f have constant moduli too. This is case (iii).

Finally, assume that Y is a rational surface. Since $k \geq 6$ and $\rho(Y) = k+2$ we can apply Theorem 3.3 to obtain that Y is as in the standard example. In particular there is a fibration $f : Y \to \mathbb{P}^1$ with general fibre F isomorphic to $\mathbb{P}^1$. If we write $K_S \cdot D = 2m$ (hence $k = 2m+4$), then f has precisely $m+2$ singular fibres of the form $C_{2i-1}+C_{2i}+2E_i$, with E_i a $(-1)-$curve and $E_i \cdot C_{2i-1} = E_i \cdot C_{2i} = 1$. Denote by $\bar{D}$ the image of D on Y and by L the line bundle of Y such that $2L \equiv \bar{D}+C_1+\ldots+C_k$. The intersection number $E_i \cdot \bar{D} = E_i \cdot 2L - E_i \cdot (C_1 + \ldots C_k) = 2L \cdot E_i - 2$ is even. Thus we may write $\bar{D} \cdot F = \bar{D} \cdot (2E_i + C_{2i-1} + C_{2i}) = 2\bar{D} \cdot E_i = 4d$, and the pre-image in X of the ruling on Y is a pencil of hyperelliptic curves of genus $2d-1$. Blowing down the curves E_i and then the images of the C_{2i}, we obtain a birational morphism $p\colon Y \to \mathbf{F}_e$ onto a geometrically ruled surface. Let C be the image of $\bar{D}$ on $\mathbf{F}_e$. Let F, S be the standard generators of $\mathrm{Pic}(\mathbf{F}_e)$ with $F^2 = 0, S^2 = -e \leq 0, F \cdot S = 1$. We have $C \sim aF + 4dS$. The curve C has $m+2$ singular points of type $(2d, 2d)$, that are solved by the morphism p. Since $\bar{D}^2 = D^2/2 = 0$, we get

$$0 = C^2 - (m+2)8d^2 = 8d(a - 2de - d(m+2)).$$

This gives us a first equation:

$$a = d(m+2) + 2de. \tag{4.4}$$

We also know that $\bar{D} \cdot K_Y = 2m$. On the other hand,

$$\bar{D} \cdot K_Y = C \cdot K_{\mathbf{F}_e} + 2(m+2)2d,$$

and we get the second equation

$$a = 2d(m+e) - m. \tag{4.5}$$

Comparing the two equations, we get $dm = m + 2d$. This has the solutions $(m, d) = (3, 3), (4, 2)$, which yield the cases (iv) and (v), respectively. $\square$

Remark 4. We do not know whether all the possibilities in Theorem 4.4 really occur. One can check that in the case of the bicanonical involution of the surface S of example (4.2) of [MP] the quotient is as in case (v). In addition, $\mathrm{Aut}(S) = \mathbb{Z}_2^3$ and the remaining involutions are as in case (iii). Example (4.3) of [MP] has a group Γ of automorphisms isomorphic to $\mathbb{Z}_2^4$: some elements of Γ have no $1-$dimensional fixed part, and thus are as in case (i), while the others are as in case (iii). Both examples are Beauville-type surfaces (cf [BPV], pg. 236). We intend to return to this problem in a future paper.

References

[AS] M.F. Atiyah, I.M. Singer, *The index of elliptic operators: III*, Ann. of Math. **87** (1968), 546–604.

[BPV] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **3.** Folge, Band **4**, Springer-Verlag, Berlin (1984).

[B] A. Beauville, *Sur le nombre maximum de points doubles d'une surface dans* $\mathbf{P^3}$ $(\mu(5) = 31)$, in "Journées de Géométrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry", Angers, 1979, pp. 207–215, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.

[Eb] W. Ebeling, *Lattices and Codes*, Adv. Lectures in Mathematics, Vieweg, 1994.

[Gr] M. Greenberg, *Algebraic Topology: A First Course*, W. A. Benjamin Publ., Reading, Mass. 1981.

[M] Y. Miyaoka, *The maximal number of quotient singularities on surfaces with given numerical invariants*, Math. Ann., **268** (1973), 159–171.

[MP] M. Mendes Lopes, R. Pardini, *The bicanonical map of surfaces with* $p_g = 0$ *and* $K^2 \geq 7$, to appear in Bull. L.M. S. (Eprint: AG/9910074).

[Pa] R. Pardini, *Abelian covers of algebraic varieties*, J. reine angew. Math. **417** (1991), 191–213.

[SGA5] A. Grothendieck, L. Illusie, *Exposé III* in *Cohomologie ℓ-adique et fonctions L. (Seminaire de Géom étrie Algébrique du Bois-Marie 1965-66 SGA 5).* Lecture Notes in Mathematics, 589. Springer-Verlag.

Igor Dolgachev
Department of Mathematics
University of Michigan
Ann Arbor, MI48109
USA
idolga@math.lsa.umich.edu

Margarida Mendes Lopes
CMAF
Universidade de Lisboa
Av. Prof. Gama Pinto, 2
1649–003 Lisboa, PORTUGAL
mmlopes@lmc.fc.ul.pt

Rita Pardini
Dipartimento di Matematica
Università di Pisa
Via Buonarroti, 2
56127 Pisa, ITALY
pardini@dm.unipi.it