PART 2

1. Abelian categories

A category is called *additive* if for any object A, the set $\mathcal{C}(B,A)$ has a structure of an abelian group such that the composition map is bilinear. Moreover one assumes that finite products exist in $\mathcal{C}$. Since we always have morphisms $A \to A \times B, B \to A \times B$ defined by $i_A = (1_A, 0), i_B = (0, 1_B)$. If $f : A \to Z$ and $g : B \to Z$ we define $A \times B \to Z$ by the formula $h = f \circ p_A + g \circ p_B$. Clearly,

$$h \circ i_A = (f \circ p_A + g \circ p_B) \circ i_A = (f \circ p_A) \circ i_A = f.$$

Similarly we check that $h \circ i_A = g$.

We shall denote the direct sum by $A \oplus B$.

Finally we require the existence of an object with $\mathcal{C}(A,A) = \{0\}$. Obviously, a final or cofinal object satisfies this property. We call such object a zero object. All of them are isomorphic and we choose one of them and denote it by 0.

We define the kernel of a morphism $f : A \to B$ as the kernel $ker(f)$ (if it exists) of the pair $A \rightrightarrows B$, where one arrow is f and another arrow is the zero morphism. By definition, for any object Z we have the exact sequence of abelian groups

$$0 \to Hom(Z, ker(f)) \to Hom(Z, A) \to Hom(Z, B).$$

We define the cokernel of f as the cokernel $coker(f)$ of the pair $A \rightrightarrows B$. By definiton, there is an exact sequence of abelian groups

$$0 \to Hom(coker(f), Z) \to Hom(B, Z) \to Hom(A, Z).$$

It follows from the definition that $ker(f) \to A$ is injective and $B \to coker(f)$ is a surjective morphism.

Note that the canonical morphism $coim(f) \to im(f)$ is always bijective but not necessary isomorphism.

Remark. Note that h_A is an abelian presheaf, so the Yoneda functor embeds $\mathcal{C}$ into the category $\check{\mathcal{C}}^{ab}$ of abelian presheaves on $\mathcal{C}$. In particular we can speak about kernels and cokernels of morphisms $h_A \to h_B$ in $\check{\mathcal{C}}^{ab}$. It follows from the definition that the kernel $ker(f)$ represents $ker(h_A \to h_B$. However, it is not true that the cokernel $coker(f)$ represents $coker(h_A \to h_B)$. For example, we may take $\mathcal{C} = (Ab)$ and $f : \mathbb{Z} \to \mathbb{Z}$ to be the multiplication by $d > 1$. Then $coker(f) \cong \mathbb{Z}/d$ but $T(\mathbb{Z}/d) = coker(h_A \to h_B)(\mathbb{Z}/d) = 0$. If T were representable, we would have $T(\mathbb{Z}) = \mathbb{Z}/d \cong Hom(\mathbb{Z}, A) = A$ and hence $T(\mathbb{Z}/d) = Hom(\mathbb{Z}/d, \mathbb{Z}/d) \neq 0$. This shows that T is not representable.

Example 1. Let A be an abelian group. Consider the additive category which consists of two objects 0 and e and $\mathcal{C}(e,a) = End(A)$, where $End(A)$ denotes the group of endomorphims of A. It is easy to see that, in general, this category does not contain kernels or cokernels.

We can define the image $im(f)$ as the kernel of $B \to coker(A \to B)$ and the coimage as the cokernel of $ker(f) \to A$. There is a canonical morphism $s : coim(f) \to im(f)$ such that the morphism f decomposes

$$A \xrightarrow{p} coim(f) \to im(f) \xrightarrow{i} B.$$

The morphism s is not necessary an isomorphism, as the following example shows:

Example 2. Let $\mathcal{C}$ be the category of filtered vector spaces. Its object are vector spaces together with a filtration $V = F_0(V) \supset F_1(V) \supset \ldots \supset F_n(V) \supset \ldots \supset \{0\}$. A morphism is a linear map which preserves the filtrations. For example, suppose V has two filtrations $(F_i(V))$ and $(F_i'(V))$ such that $F_i(V) \subset F_i(V)'$ and with strict inclusion for some i. Then the identity linear map is bijective but not an isomorphism.

Definition. An additive category is called *preabelian* if kernels and cokernels exist.

Example 3. The category from Example 1 is preabelian. We take the kernel K of a linear map f and define the filtration by $F_i(K) = K \cap F_i(V)$. We take the cokernel C and define the filtration by $F_i(C) = F^i/F^i \cap im(f)$.

Lemma 1. *Assume $\mathcal{C}$ is preabelian. Then finite projective and inductive limits exist.*

Proof. It is enough to show that $A \times_C B$ and $A \coprod_C B$ exist, where $a : A \to C, b : B \to C$ (resp. $C \to A, C \to B$) are morphisms. We define

$$A \times_C B = Ker(A \times B \rightrightarrows C) = Ker(A \oplus B \to C),$$

where $A \oplus B \to C$ is the difference of $(a, 0)$ and $(0, b)$ and similarly

$$A \coprod_C B = Coker(C \to A \oplus B).$$

Lemma 2. *Assume $\mathcal{C}$ is preabelian. Let $f : A \to B$. The canonical morphism $s : I = coim(f) \to I' = im(f)$ is bijective.*

Proof. Let $Z = ker(s)$. Consider $T = A \times_B Z$, where $Z \to B$ is the composition $Z \to I \to I' \to B$. There is a projection $T \to A$ such that its composition with f is zero. For any $a : X \to A$ such that the composition $X \to A \to B$ is zero, there is a morphism $X \to A \times_B Z$ defined by $(a, 0)$. Its composition with the projection $T \to A$ is equal to a. Thus T satisfies the definition of the kernel of f and, by its uniqueness, the projection $T \to A$ factors through an isomorphism $p : T \to ker(f)$. But obviously, the composition of the morphism $(0, 1_Z) : Z \to T$ with p is zero. Since p is injective, the morphism $1_Z : Z \to Z$ must be zero, i.e. $Z = 0$.

Similarly, if $Z = coker(s)$, we consider $B \oplus_A Z$.

A category is called *abelian* if it is additive and two additional properties are satisfied:
(A1) kernel and cokernel exist,
(A2) the canonical morphism from coimage to the image is an isomorphism.

Proposition 1. *A bijective morphism in an abelian category is an isomorphism.*

Proof. By definition, a morphism $f : A \to B$ is bijective if for any Z the maps $\mathcal{C}(B, Z) \to \mathcal{C}(A, Z)$ and $\mathcal{C}(Z, A) \to \mathcal{C}(Z, B)$ are injective. This implies that the kernel and cokernel of f are the zero objects. Thus $im(f) = A$ and $coim(f) = B$.

Examples 4. The category of Mod_R of modules over any ring R.
5. Let $\mathcal{C}$ be an abelian category, for any catgeory $\mathcal{C}'$, the category of functors $Funct(\mathcal{C}', \mathcal{C})$ is an abelian category. In fact, let $F, G : \mathcal{C}' \to \mathcal{C}$ be two functors and $u, v : F \to G$ be two natural transformation. We can define $(u+v)_X = u_X + v_X : F(X) \to G(X)$. Since $F \oplus G$ can be easily defined by $(F \oplus G)(X) = F(X) \oplus G(X)$, this makes $Funct(\mathcal{C}', \mathcal{C})$ an additive category. Now, if $u : F \to G$ we can define $ker(u)(X) = ker(u_X : F(X) \to G(X))$ and $coker(u)(X) = coker(u_X : F(X) \to G(X))$. Let $\alpha : coim(u) \to im(v)$. Then $\alpha_X : coim(u_X) \to im(u_X)$ is bijective, so α is an isomorphism.

In particular, we see that the category $\check{\mathcal{C}}^{ab}$ of *abelian presheaves* (= presheaves $F : \mathcal{C} \to (\mathcal{S}ets)$ which factor through the category of abelian groups) is abelian.

Let $\mathcal{C}$ be a site and $\tilde{\mathcal{C}}^{ab}$ be the full subcategory of $\check{\mathcal{C}}^{ab}$ which consists of sheaves. If we restrict the sheafification functor $a : \tilde{\mathcal{C}} \to \check{\mathcal{C}}$, we obtain a functor $a : \tilde{\mathcal{C}}^{ab} \to \check{\mathcal{C}}^{ab}$ which is left adjoint to the inclusion functor $i : \tilde{\mathcal{C}}^{ab} \to \check{\mathcal{C}}^{ab}$.

Lemma 3. *Let $u : F \to G$ be a morphism in $\tilde{\mathcal{C}}^{ab}$. Then*

$$ker(u) = ker(i(u)), \quad coker(u) = a(coker(i(u))).$$

Proof. Let $K = ker(i(u))$. For any covering sieve R of $X \in Ob(\mathcal{C})$, we have the following commutative diagram:

$$\begin{array}{ccccc} K(X) & \to & F(X) & \to & G(X) \\ \downarrow & & \downarrow & & \downarrow \\ K(R) & \to & F(R) & \to & G(R) \end{array}$$

Since the vertical arrows $F(X) \to F(R)$ and $G(X) \to G(R)$ are bijective, this implies that the vertical arrow $K(X) \to K(R)$ is bijective. Thus K is a sheaf. Then it is immediately checked that $K = ker(u)$.

Let $C = coker(i(u))$. The composition $G \to C \to a(C)$ defines a morphism $G \to a(C)$ such that the composition $F \to G \to a(C)$ is the zero morphism. By adjunction, for any sheaf Z, there is a canonical bijection $\tilde{\mathcal{C}}^{ab}(a(C), Z) \cong \check{\mathcal{C}}^{ab}(C, i(Z))$. Thus if $F \to G \to Z$ is the zero morphism of sheaves, the natural morphism of presheaves $C \to i(Z)$ defines a morphism of sheaves $a(C) \to Z$. This shows that $a(C) = coker(u)$.

Proposition 2. *Let $\mathcal{C}$ be a site. Then the category of abelian sheaves is abelian.*

Proof. By the previous lemma, we have kernels and cokernels. Let $u : F \to G$ be a morphism in $\tilde{\mathcal{C}}^{ab}$. Let $C = coker(i(u))$. By the same lemma, $coim(u) = a(coim(u(i))$ and $im(u) = ker(G \to a(C)) = ker(a(G) \to a(C))$. Since a is left exact (see Theorem 1 in section 9), we obtain $im(u) = a(im(u(i))$ and thus canonical morphism $coim(u) \to im(u)$ is equal to $a(\alpha)$, where α is the canononical morphism $coim(i(u)) \to im(i(u))$. Since $\check{\mathcal{C}}^{ab}$ is an abelian category, α is an isomorphism. Hence $a(\alpha)$ is an isomorphism.

We shall consider the following additional axioms which are often satisfied:

(A3) Arbitrary direct sums exist in $\mathcal{A}$.

For example this implies that for any family of monomorphims $U_i : A_i \to A$ there exists the supremum $\sup u_i : B \to A$. It is enough to take $\coprod A_i$ and consider the canonical monomorphism of the image of $\coprod A_i \to A$. We will allways identify a subobject with a choice of a corresponding monomorphism. So we can speak about the supremum of subobjects.

(A4) = (A3) and the direct sum of any family of monomorphisms is a monomorphism.

(A5) = (A3) and for any directed increasing family of subobjects (A_i)

$$(\sup A_i) \cap B \cong \sup(A_i \cap B).$$

Here we define for two subobjects of an object X

$$A \cap B = ker(X \to (X/A) \times (X/B)).$$

Here X/A denotes $coker(A \to X)$. Clearly $A \cap B$ serves as $\inf(A, B)$.

The axiom (AB5) is equivalent to the following:

(AB5)' = (AB3) and for any increasing set of subobjects A_i of A

$$\sup_{i \in I} A_i \to B \cong \lim_{i \in I}\text{ind}\, A_i.$$

Note that since for each A_i there is a morphism to sup we have always a canonical morphism lim ind $\to$ sup.

A set of objects G_i of a category $\mathcal{C}$ is called a family of *generators* if for any object X and its subobject Y not equal to X there exists a morphism $G_i \to X$ which does not factor through the canonical morphism $Y \to X$. If $\mathcal{C}$ is abelian and has direct sums we can replace (G_i) with $G = \coprod_i G_i$.

Definition. An abelian category is called a *Grothendieck category* if (A3) is satisfied and it contains a generator.

Examples 6. Consider the category of $\check{\mathcal{C}}$ where $\mathcal{C}$ is a small category, i.e. the class of objects is a set. Then the set of representable functors is a set of generators. Note that the objects of $\check{\mathcal{C}}$ do not form a set. For this we have to restrict ourselves with presheaves with values in a fixed universe.

7. Let $\mathcal{A} = (Ab)$ be the category of abelian groups. Take $X = \mathbb{Z}$. Then for any object A we have a homomorphism $\mathbb{Z} \to A$.

8. Let $\mathcal{A} = Mod_R$, then we take $M = R$.

Lemma 4. *Let*

$$B \overset{\alpha}{\to} A \overset{\alpha'}{\to} B' \to 0, \quad C \overset{\gamma}{\to} A \overset{\gamma'}{\to} C' \to 0$$

be two exact sequences in an abelian category $\mathcal{A}$. Suppose that the composition $\gamma' \circ \alpha$ is an epimorphism. Then $\alpha' \circ \gamma$ is an epimorphism.

Proof. The exact sequences are equivalent to the exact sequences of abelian groups

$$0 \to \mathcal{A}(B', X) \to \mathcal{A}(A, X) \to \mathcal{A}(B, X),$$

$$0 \to \mathcal{A}(C', X) \to \mathcal{A}(A, X) \to \mathcal{A}(C, X),$$

where X is any object in $\mathcal{A}$. The property that $\alpha' \circ \gamma$ is an epimorphism is equivalent to the injectivity of the composition map

$$\mathcal{A}(C', X) \to \mathcal{A}(A, X) \to \mathcal{A}(B, X)$$

and the property that $\alpha' \circ \gamma$ is an epimorphism is equivalent to the injectiviy of the composition map

$$\mathcal{A}(B', X) \to \mathcal{A}(A, X) \to \mathcal{A}(C, X).$$

Thus the assertion follows from the following easy fact: Let

$$0 \to N \to M \to N' \quad 0 \to L \to M \to L',$$

be exact sequences of abelian groups such that the composition $N \to M \to L'$ is injective. Then the composition $L \to M \to N'$ is injective.

Lemma 5. *An epimorphism in an abelian category is a universal epimorphism.*

Proof. Let $f : A \to B$ be an epimorphism and $v : C \to B$ be a morphism. We have an exact sequence

$$0 \to A \times_B C \to A \oplus C \to B \to 0.$$

where $A \oplus C \to B$ is defined as the difference $f - v$. Also we have an obvious exact sequence

$$0 \to A \to A \oplus C \to C.$$

Now the composition $A \to A \oplus C \to B$ is obviously epimorphism, hence the composition $A \times_B C \to A \oplus C \to C$ is an epimorphism as follows from the previous lemma.

We can consider presheaves with values in any category. They are just contravariant functors $F : \mathcal{C}^\circ \to \mathcal{D}$. If $\mathcal{C}$ is a site, we say that F is a sheaf with values in D if for for any $A \in Ob\mathcal{D}$) the presheaf of sets

$$F_A : X \to \mathcal{D}(A, F(X))$$

is a sheaf.

Lemma 6. *Let $u : \mathcal{D}_1 \to \mathcal{C}_2, v : \mathcal{D}_2 \to \mathcal{C}_1$ a pair of adjoint functors. Let $F : \mathcal{C} \to \mathcal{D}_2$ be a presheaf with values in $\mathcal{D}_2$ on a site $\mathcal{C}$. If F is a sheaf, then the presheaf $G = v \circ F : X \to v(F(X))$ is a sheaf with values in $\mathcal{D}_1$.*

Proof. By adjunction, for any $A \in \mathcal{D}_1$ we have $\mathcal{D}_1(A, v(F(X))) = \mathcal{D}_2(u(A), F(X))$. Thus $G_A = F_{u(A)}$ is a sheaf.

Example 9. Take $\mathcal{D}_2 = (Ab), \mathcal{D}_1 = (\mathcal{S}ets)$, let v be the forgetful functor. Then an abelian sheaf is a sheaf of sets.

Now if we take $\mathcal{D}_2 = (\mathcal{S}ets), \mathcal{D}_1 = (Ab)$ and v the adjoint of the forgetful functor, we obtain for any sheaf of sets that the presheaf $X \to \mathbb{Z}^F(X)$ is a sheaf of abelian groups.

Proposition 3. *The category $\tilde{\mathcal{C}}^{ab}$ is an abelian category satisfying properties (A5) and has generators.*

Proof. We already know that $\tilde{\mathcal{C}}^{ab}$ is an abelian category. Let us check (A3). Let $(F_i)_{i\in I}$ be an arbitrary family of sheaves. Their direct sum $\oplus_{i\in I}F_i$ obviously exists as a presheaf. Since the functor of sheafification commutes with inductive limits (since it has the right adjoint) we obtain

$$a(\oplus_{i\in I}F_i) \cong \oplus_{i\in I}a(F_i) \cong \oplus_{i\in I}F_i.$$

Thus for any sheaf G we have

$$\tilde{\mathcal{C}}^{ab}(\oplus_{i\in I}F_i, G) = \check{\mathcal{C}}(\oplus_{i\in I}F_i, i(G)) = \prod_{i\in I}\check{\mathcal{C}}(F_i, i(G)) = \prod_{i\in I}\tilde{\mathcal{C}}^{ab}(a(F_i), G) = \prod_{i\in I}\tilde{\mathcal{C}}^{ab}(F_i, G).$$

This proves that $\oplus_{i\in I}F_i$ is a sheaf and is a direct sum in $\tilde{\mathcal{C}}^{ab}$.

Let (F_i) be an increasing family of subobjects of F in $\tilde{\mathcal{C}}^{ab}$. Set $A = \cup_{i\in I}i(F_i)$. This is a subpresheaf of F. Clearly, $A = im(\oplus_{i\in I}F_i) \to F$ and since a is exact we obtain that $a(A) = \sup_{i\in I}F_i$ in the category of sheaves. But obviously $A = \lim\operatorname{ind} i(F_i)$. Since a is right exact, we are done.

Finally to get generators, we take the set $\mathbb{Z}_X, X \in Ob(\mathcal{C})$. Here the sheaf $\mathbb{Z}_X$ is defined as follows. Let $v : (\mathcal{S}ets) \to (Ab)$ be the functor "free abelian group generated by a set". Then for any sheaf $F : \mathcal{C} \to (\mathcal{S}ets)$ the composition $v \circ F : \mathcal{C} \to (Ab)$ is an abelian sheaf. Applying this construction to $F = a(h_X)$ we obtain the sheaf $\mathbb{Z}_X$. We have, for any abelian sheaf F

$$\tilde{\mathcal{C}}^{ab}(\mathbb{Z}_X, F) = \tilde{\mathcal{C}}(a(h_X), F) = \check{\mathcal{C}}(h_X, F) = F(X).$$

This easily implies that the set of sheaves $\mathbb{Z}_X$ is a set of generators.

Theorem 1 (Gabriel). *For each abelian category $\mathcal{C}$ there exists a fully faithfully additive exact functor to a Grothendieck category.*

Proof. We consider $\tilde{\mathcal{C}}^{ab}$ with respect to the canonical Grothendieck topology described in Example 6. Let $R = a\circ h : \mathcal{C} \to \tilde{\mathcal{C}}^{ab}$ be the composition of the Yoneda functor h and the sheafification functor a. Since h and a are left exact, R is left exact. To prove that R is right exact, it suffices to show that it transforms epimorphisms to epimorphisms. Let $f : A \to B$ be an epimorphism in $\mathcal{C}$ and let $H = coker(h(f))$. For any $X \in Ob(\mathcal{C})$ we have an exact sequence

$$h_A(X) \to h_B(X) \to H(X) \to 0.$$

Let $\bar{a} \in H(X)$ and $a : X \to B$ be its pre-image in $h_B(X)$. Consider the fibred product $Y = A\times_B X$ corresponding to the morphisms f and a:

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "a"] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

By Lemma 5, the projection $p : Y = A\times_B X \to X$ is an epimorphism. We want to show that the image of $\bar{a}$ in $H(Y)$ under the map $H(p) : H(X) \to H(Y)$ is zero. Since p is a covering morphism in the canonical site, this will imply that the associated sheaf $a(H)$ is zero. Consider the diagram

$$\begin{tikzcd} h_A(X) \arrow[r] \arrow[d, "h_A(p)"'] & h_B(X) \arrow[r] \arrow[d, "h_B(p)"'] & H(X) \arrow[r] \arrow[d, "H(p)"'] & 0 \\ h_A(Y) \arrow[r] & h_B(Y) \arrow[r] & H(Y) \arrow[r] & 0. \end{tikzcd}$$

Let $q : Y \to A$ be the first projection. Since $a \circ p = f \circ q$, we have $h_B(p)(a) = a \circ p = f \circ q = h(f)(Y)(q)$. Since the bottom row is exact, the image of $h_B(p)(a)$ in $H(Y)$ is equal to zero. But it is equal to $H(p)(\bar{a})$.

Example 10. Consider the example from Remark 1. Then $coker(\mathbb{Z} \to \mathbb{Z}) \cong \mathbb{Z}/d$ and hence the sheaf associated to the presheaf $coker(h_{\mathbb{Z}} \to h_{\mathbb{Z}})$ is isomorphic to $a(h_{\mathbb{Z}/d}) = h_{\mathbb{Z}/d}$. In particular, we see that $coker(h_{\mathbb{Z}} \to h_{\mathbb{Z}})$ in the category of presheaves is not isomorphic to a sheaf.

2. Cohomology

Let $\mathcal{A}$ be an additive category. A *chain complex* is a sequence of morphisms

$$K : \ldots \overset{d_n}{\to} K_n \overset{d_{n-1}}{\to} K_{n-1} \to \ldots .$$

such that $d_{n-1} \circ d_n = 0$. We can define the *homology* by

$$H_i(K) = coker(im(d_n) \to ker(d_{n-1}).$$

One can show that

$$coker(im(d_n) \to ker(d_{n-1}) \cong coker(K_n \to ker(d_{n-1}) \cong ker(coker(d_n) \to K_{n-2}).$$

More generally, if $ST(\mathcal{A})$ is the category of simplicial objects in $\mathcal{A}$. For any object K of this category, the boundary maps $d_n^i = \Delta_n^i : K_n \to K_{n-1}$ belong to the abelian group $\mathcal{A}(K_n, K_{n-1})$ hence we can form their alternating sum

$$d_n = \sum_{i=0}^{n} (-1)^i d_n^i.$$

Since $d_{n-1}^i \circ d_n^j = d_{n-1}^j \circ d_n^i$ (both of the compositions correspond to the maps $[n-2] \to [n]$ such that the image does not contain i, j) and they occur in the sum $d_{n-1} \circ d_n$ with opposite sign, we obtain

$$d_{n-1} \circ d_n = 0.$$

Now we can define the *homology groups* of a simplicial object

$$H_i(K_\bullet) = coker(im(d_n) \to ker(d_{n-1})).$$

When we replace $\mathcal{A}$ with the dual category we obtain the notions of *cochain complexes* and *cohomology groups*.

Let $\mathcal{A}$ be a category of abelian sheaves on a site $\mathcal{C}$. Let K be a simplicial object in $\mathcal{C}$. Then, for any abelian sheaf from $\mathcal{A}$ we can define a cosimplicial abelian group $(\mathcal{F}(K_n))$. Then its cohomology are called the *Čech cohomology* of $\mathcal{F}$ with respect to $K_\bullet$ and denoted by

$$\check{H}^n(K, \mathcal{F}).$$

For example, we may take $\mathcal{C} = Open(X)$ (as always with disjoint sum added). A simplicial object $K = cosk_0(U)$, where $U = \coprod_{i \in I} U_i$ corresponds to an open cover of X. Then we obtain the Čech cohomology of an open cover with coefficients in a sheaf $\mathcal{F}$.

Assume that $(U_i \to U)_{i \in I}$ is a covering family. We can define a hypercovering $cosk_0(\coprod U_i)$ with

$$K_0 = \coprod U_i, \quad K_1 = \coprod_{(i,j) \in I \times I} U_i \times_U U_j, \ldots .$$

It follows from the definition of a sheaf

$$H^0(K_\bullet, \mathcal{F}) = \mathcal{F}(U).$$

More generally $K_\bullet$ is a hypercovering in $\mathcal{C}$. Then $K_1 \to cosk_0(K_0)_1 = K_0 \times K_0$ is a covering morphism so $\mathcal{F}(K_0 \times K_0) \to \mathcal{F}(K_1)$ is injective. Since $K_0 \to f$ is a covering morphism,

$$\mathcal{F}(f) = ker(\mathcal{F}(K_0) \rightrightarrows \mathcal{F}(K_0 \times K_0)) \cong ker(\mathcal{F}(K_0) \rightrightarrows \mathcal{F}(K_1) = H^0(K_\bullet, \mathcal{F}).$$

One defines Čech cohomology of a sheaf $\mathcal{F}$ by

$$\check{H}^i(\mathcal{F}) = \liminf_{K_\bullet \in HR(\mathcal{C})} H^i(K_\bullet, \mathcal{F}).$$

3. Injective objects

A functor $F : \mathcal{C} \to \mathcal{C}'$ of additive categories is called *additive* if it defines linear maps on the groups of morphisms.

For example, the functor $\mathcal{C}(?, Y)$ and $\mathcal{C}(X, ?)$ are additive functors from $\mathcal{C}$ to (Ab).

Another example $\tilde{\mathcal{C}} \to (Ab), \mathcal{F} \to \mathcal{F}(X)$.

An additive functor is called *left exact* if $F(Ker(A \to B)) = Ker(F(A) \to F(B))$. Dual definition for right exactness. An additive functor which is left and right exact is called *exact*.

For example, $\mathcal{C}(?, A)$ is left exact and $\mathcal{C}(X, ?)$ is right exact. The functor $\mathcal{F} \to \mathcal{F}(X)$ is left exact but not right exact.

Example 1. Consider the Grothendieck topology in an additive category $\mathcal{A}$ with $Cov(X)$ equal to the set of *universal epimorphisms* $Y \to X$ (i.e. epimorphisms such that for any $Z \to X$ the projection $Y \times_X Z \to Z$ is an epimorphism). Let $f : A \to B$ be an epimorphism. Consider the fibred product $A \times_B A \to A$ and let p_1, p_2 be the projections. The morphism $p_1 - p_2 : A \times_B A \to A$ defines a morphism $u : A \times_B A \to Ker(A \to B)$. Since it admits a section $s : B \to A$ defined by $(i_A, 0)$, where $i : ker \to A$, we see that u is an epimorphism. It is also an universal epimorphism. In fact, if $\phi : C \to B$ is a morphism, we can define a section of $A \times_B C \to C$ by $s' = (s \circ \phi, 1_C)$. By definition of a sheaf, the morphism $F(u) : F(ker(f) \to F(A \times_B A)$ is injective. Thus

$$ker(F(A) \to F(ker(f)) = ker(F(A) \to F(ker(f)) \stackrel{F(u)}{\to} F(A \times_B A) =$$

$$ker(F(A) \stackrel{F(p_1 - p_2)}{\to} A \times_B A) = ker(F(A) \rightrightarrows A \times_B A).$$

This shows that F is a sheaf if and only if it is left exact.

Since h_A is left exact, we obtain that all representable presheaves are sheaves. Conversely, if $(A \to B) \in Cov(B)$ and not epimorphism, we find X such that h_X is not a sheaf.

Example 2. Let $\mathcal{C}$ be an abelian category and $R : \mathcal{C} \to \tilde{\mathcal{C}}^{ab}$ be the additive functor $a \circ h$. Then R is exact. This follows from the proof of Gabriel's theorem.

Definition. An object I of an abelian category $\mathcal{A}$ is called *injective* if the functor h_I is exact. An injective object in the dual category $\mathcal{A}^\circ$ is called *projective*.

It follows from the definition that X is injective if and only if for any monomorphism $A \to B$ and any morphism $A \to X$ there exists an extension $B \to X$.

Dually, X is projective if and only if for any epimorphism $A \to B$ and a morphism $X \to B$ there exists an extension $X \to A$. For example, a free R-module is a projective object in Mod_R.

The next result describes injective objects in any "good" abelian category.

Theorem 1. *Assume that $\mathcal{A}$ is a Grothendieck abelian category. Then an object I is injective if and only if for any subobject V of a generator U in $\mathcal{A}$ any morphism $V \to I$ extends to a morphism $U \to I$.*

Proof. The condition is obviously necessary. Let us prove its sufficiency. Let $u : B \to A$ be a monomorphism and $f : B \to X$ be a morphism. Consider the set of subobjects of (B', u') which dominate such that f can be extended to B'. We can take the supremum and try to prove that it must be A. So, we may assume that $B \neq A$ is the supremum and f does not extend to A. Let U be a generator such that there exists a morphism $j : U \to A$ with image not contained in B. Take $V = U \times_A B \to B$. The projection $j' : V \to U$ makes it a subobject of U. Let $B' = j(U) + B$. By assumption on j, $B \neq B'$. Consider the exact sequence $V \to U \times B \to B' \to 0$, where the first morphism $\phi' : V \to U \times B$ is equal to $(i, -j')$ and the second one $\phi : U \times B \to B'$ is equal to $j \oplus u$. We claim that any morphism $f : B \to X$ can be extended to $f' : B' \to X$. This will contradict the maximality of B. To define f', it suffices to define $v : U \times B \to X$ such that its composition with ϕ' is equal to 0 and the compostion with $(0, 1_B) : B \to U \times B$ is equal to f. Let s be an extension of $f \circ j' : V \to X$ to U. Define $v = (s, f) : U \to B \to X$. It is easy to see that it is what we want.

Example 3. Let $\mathcal{C} = (Ab)$. A generator is $\mathbb{Z}$. For any subobject $n\mathbb{Z}$ a morphism $f : n\mathbb{Z} \to A$ is given by the image $a = f(n) \in A$. Let $f' : \mathbb{Z} \to A$ be its extension and $b = f'(1)$. Then $f'(n) = nf'(1) = f(n) = b$. This shows that A is *divisible group*, i.e. any equation $nx = a$ in the group has a non-zero solution. Since this condition is sufficient, we see that an abelian group is injective if and only if it is a divisible group.

Example 4. Let $\mathcal{A}$ be the category of abelian sheaves on a canonical site $\mathcal{C}$, where $\mathcal{C}$ is an abelian category. We know that $U = \oplus_{X \in \mathcal{A}} h_X$ is a generator in $\mathcal{A}$. In fact, it is a projective object in $\mathcal{C}$. Let us prove it. Let $v : A \to B$ be an epimorphism of sheaves and $f : U \to B$ be a morphism. The composition of f with the canonical morphism $h_X \to U$ is a morphism $h_X \to B$. By Yoneda's Lemma it is defined by an element of $b_X \in B(X)$. In fact, we see that f is described by the famuily $(b_X)_{X \in Ob(\mathcal{C})}$. Since v is an epimorphism of sheaves, there is an extension $f' : U \to A$. In fact, let $u_X \in U(X)$ and $f(X)(u_X) = b_X \in B(X)$. Since v is an epimorphism in the category of sheaves, we can find an epimorphism $Y \to X$ such that the image b_Y of b_X in $B(Y)$ is equal to $v_Y(a_Y)$ for some $a_Y \in A(Y)$.

Proposition 1. *Let $F : \mathcal{A} \to \mathcal{A}'$ be an additive functor of abelian categories and $G : \mathcal{A}' \to \mathcal{A}$ be its right adjoint additive functor. Assume that F is exact. Then for any injective object I of $\mathcal{C}'$ the object $G(I)$ is injective.*

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathcal{A}$. We have to show that the sequence

$$0 \to Hom(A, G(I)) \to Hom(B, G(I)) \to Hom(C, G(I)) \to 0$$

is exact. By definition of the adjointeness this is equivalent to the exactness of the sequence

$$0 \to Hom(F(A), I) \to Hom(F(B), I) \to Hom(F(C), I) \to 0.$$

But it is exact since F is exact.

Example 5. Let $\mathcal{C}$ be a site and $X \in Ob(\mathcal{C})$. Consider the subcategory $\{X\}$ of $\mathcal{C}$ which consists of one object X and the set of morphisms $\{1_X\}$. Let $\mathcal{A}$ be the category of abelian presheaves on $\{X\}$. It is isomorphic to the category of abelian groups. Let $\mathcal{A}'$ be the category of abelian sheaves on $\mathcal{C}$. Let $i : \{X\} \to \mathcal{C}$ be the inclusion. Consider the functor $i^* : \mathcal{A}' \to \mathcal{A}$. We have $i^*(F) = F(A) \in Ob(Ab)$. On the other hand, for any sheaf $\underline{A}$ on $\{X\}$ defined by an abelian group A we have

$$i_!(\underline{A})(Y) = \lim_{\substack{\longrightarrow \\ Y \to X}} \underline{A}(X) = A^{h_X(Y)}.$$

Here we use that the inductive system $Y \to X$ is discrete, i.e. no morphisms between different objects. The functor $i_!$ is obviously exact (since for any set S the functor $A \to A^S$ in (Ab) is exact). Thus we obtain that the functor $F \to F(X)$ sends injective to injectives, in other words, the value of an an injective abelian presheaf are injective abelian groups.

Similarly we get that the values of an injective sheaf are injective groups.

Definition. We say that $\mathcal{A}$ has enough injective objects if for any object X there exists a monomorphism to an injective object $u : X \to I$.

Let us remind some definitions from the set theory which we shall use in the proof of the next theorem. Let (E,Γ) be a pair consisting of a set and an order relation $\Gamma \subset E\times E$ on it. We say that (E,Γ) is isomorphic to (E',Γ') if there exists a bijection $f: E\to E'$ such that $f(i)\le f(j)$ if and only if $i\le j$. The class of isomorphisms is called an *ordinal number* or just an *ordinal*. Using the axiom of choice we pick up a representative (E,Γ) in each ordinal and denote it $Ord(E)$. The cardinality of an ordinal is the cardinality of the set E. There is a natural definition of sum and product of ordinals (using the definition of an order on the sum or product of two sets). Also there is an order on the set of ordinals compatible with sums and products. The supremum of a set of ordinals exists and is an ordinal.

Theorem 2. *Every Grothendieck category $\mathcal{A}$ has enough injective objects.*

Proof. Let A be an object of $\mathcal{A}$. Let $I(A)$ be the set of monomorphisms $u_i: V_i\to A$, where V_i is a subobject of a generator U. Consider $A\times U^{\oplus I(A)}$ and define $f:\oplus_{i\in I(A)}V_i\to M_1(A)$ such that its restriction to each V_i is the morphism (u_i,α_i), where $\alpha_i: V_i\to U$ is the canonical embedding of V_i into a copy of U corresponding to i. Set $M_1(A)=coker(f)$. Then the composition $f(A)$ of $(1_A,0): A\to A\times U^{\oplus I(A)}$ and $A\times U^{\oplus I(A)}\to M_1(A)$ is a monomorphism. This follows from the fact that the morphism $\oplus_{i\in I(A)}V_i\to U^{\oplus I(A)}$ is a monomorphism (AB4). Also, any morphism $u_i: V_i\to A$ extends to a morphism $U\to A$ (take the restriction of $U^{\oplus I(A)}\to M_1(A)$ to the summand corresponding to i). Next we repeat the construction by replacing A with $M_1(A)$. Let us use now the transfinite induction. For any ordinal number i let us construct an object $M_i(A)$ and, for any two ordinals $i\le j$, a monomorphism $M_i(A)\to M_j(A)$ such that $M_i(A)$ form an inductive system for all $i<i_0$ (i_0 is a fixed ordinal). For $i=0$ we take $M_0(A)=A$ and for $i=1$ we take $M_1(A)$ and the monomorphism $M_0(A)\to M_1(A)$ as constructed in above. Suppose we have such a construction for all ordinals $j<i$. If $i=j+1$ for some j, we set $M_i(A)=M_1(M_j(A))$ and $M_j(A)\to M_{j+1}(A)$ equal to $f(M_j(A))$, where f is as above (this also defines all morphisms $M_k(A)\to M_i(A)$ for all $k\le i$). If i is the limit ordinal, we set $M_i(A)=\lim\operatorname{ind}_{j<i}M_j(A)$ and take the morphisms $M_j(A)\to M_i(A)$ to be the inductive limit of the constructed monomorphims (the inductive limit of monomorphims is a monomorphim). Let k be the smallest ordinal whose cardinality is strictly greater than the cardinality of the set of subobjects of U. Set $M(A)=M_k(A)$. It suffices to prove that $M(A)$ is injective. By Theorem 1, it is enough to show that for any subobject V of U and a morphism $v: V\to M(A)$ the image $v(V)$ is contained in some $M_i(A)$ where $i<k$. Since $M_k(A)=\sup M_i(A)$, we have $V=\sup_{i<k}v^{-1}(M_i(A))$ (here we use the axiom (AB5)). The set of subobjects of V has smaller cardinality than the cardinality of k. Also any set of ordinals $<k$ with k as its limit has cardinality equal to the cardinality of k (because the cardinality of k is not equal to the limit ordinal number). This implies that $v^{-1}(M_i(A))$ becomes constant starting from some $i_0<k$. Thus $V=v^{-1}(M_{i_0})$ and we are done.

Remark 1. Here is a direct proof of Theorem 2 in the case when $\mathcal{A}$ is the category of left R-modules. Let

$$E_R = Hom_{\mathbb{Z}}(R,\mathbb{Q}/\mathbb{Z}).$$

It has a natural structure of an R-module. Let us see that it is an injective module. Let $u: M\to M'$ be an injective homomorphism of R-modules. We have the following commutative diagram:

$$\begin{array}{ccc} Hom_R(M,E_R) & \overset{Hom_R(u,1)}{\longrightarrow} & Hom_R(M',E_R) \\ \downarrow & & \downarrow \\ Hom_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}) & \overset{Hom_R(u,1)}{\longrightarrow} & Hom_{\mathbb{Z}}(M',\mathbb{Q}/\mathbb{Z}) \end{array}.$$

Here the vertical arrows are defined naturally: if $f: M\to Hom_{\mathbb{Z}}(R,N)$ then we define $M\to N$ by $m\to f(m)(1)$. It is clear that it is an isomorphism. Since $\mathbb{Q}/\mathbb{Z}$ is a divisible abelian group, it is an injective module over $\mathbb{Z}$. Hence the bottom arrow is surjective. Thus the top arrow is surjective.

For each R-module M we set

$$I^0(M) = E_R^{Hom_R(M,E_R)}.$$

As a direct sum of injective modules it is an injective module. Let $e_M: M\to I^0(M)$ be the homomorphism defiuned by

$$e(m) = (\phi(m))_{\phi\in Hom_R(M,E_R)}.$$

I claim that e_M is injective. Suppose $e_M(m) = 0$. Then for any $\phi : M \to E_A$ we have $\phi(m) = 0$. Notice that for any abelian group G there exists a quotient isomorphic to $\mathbb{Z}/n$, where $n \neq 1$. Hence there exists a non-trivial homomorphism $G \to \mathbb{Q}/\mathbb{Z}$. This easily implies that for any module M there exists a non-trivial homomorphism from the submodule $Rm \subset M$ to $Hom_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ (since $Hom_R(Rm, Hom_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}) = Hom_{\mathbb{Z}}(Rm, \mathbb{Q}/\mathbb{Z}))$.

Theorem 3. *For every abelian category $\mathcal{A}$ there is a fully faithful exact functor to a category of left modules over an associative ring.*

Proof. First of all, using Gabriel's theorem we embed $\mathcal{A}$ into a Grothendieck category by using an exact functor. Let $U = \oplus_{X \in Ob(\mathcal{C})} h_X$ be its generator. Note that U is a projective object of $\mathcal{A}$. Let $I = \cup_{A \in Ob(\mathcal{A})} Mor_{\mathcal{A}}(U, A)$ and $P = \oplus_I U$. For any $A \in Ob(\mathcal{A})$ there is an epimorphism

$$P_A = \oplus_{Mor_{\mathcal{A}}(U,A)} U \to A.$$

For any morphism $P_A \to A$ there exists a unique morphism, and hence an epimorphism

$$s_A : P \to A.$$

Note that the functor $A \to Mor_{\mathcal{A}}(P, A)$ is exact. Let

$$R = Mor_{\mathcal{A}}(P, P).$$

This is an associative ring. For every $X \in Ob(\mathcal{A})$ the group $M(X) = Mor_{\mathcal{A}}(P, X)$ is a left module over R. Consider the functor $M : X \to M(X)$ from $\mathcal{A}$ to Mod_R. By Example 4 from the previous section, P is a projective object. Hence the functor M is exact. Also, since P is a generator, M is faithful.

Let us show that it is fully faithful. Suppose $\phi : M(A) \to M(B)$ is a homomorphism of R-modules. We want to find $f : A \to B$ such that $M(f) = \phi$. Let $u : P \to A$ be an epimorphism and $K = ker(u)$. Let $v : P \to B$ be an epimorphism. Applying M we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M(K) & \to & R & \to & M(A) & \to & 0 \\ & & & & s\downarrow & & \downarrow\phi & & \\ & & & & R & \to & M(B) & \to & 0 \end{array}.$$

Here the homomorphism s is defined by the projectivity of R as an object of Mod_R. Since any endomorphism of a ring R is of the form $x \to x \circ r$, where $p : P \to P$ is an endomorphism of P, we obtain a diagram

$$\begin{array}{ccccccccc} 0 & \to & K & \to & P & \to & A & \to & 0 \\ & & & & r\downarrow & & \downarrow f & & \\ & & & & P & \to & B & \to & 0 \end{array}.$$

Since the composition $M(K) \to R \to R \to M(B)$ is zero, the composition $K \to P \to P \to B$ is zero (because M is exact). Thus there exists a morphism $f : A \to B$ such that the diagram

$$\begin{array}{ccc} P & \to & A \\ r\downarrow & & f\downarrow. \\ P & \to & B \end{array}$$

This implies the diagram

$$\begin{array}{ccc} R & \to & M(A) \\ s\downarrow & & \downarrow M(f). \\ P & \to & B \end{array}$$

is commutative. Since $R \to M(A)$ is epimorphic, $M(f) = \phi$.

4. Derived category

Let $K(\mathcal{A})$ denote the category of cochain complexes

$$K^\bullet = (K^n, d_K^n : K^n \to K^{n+1})_{n\in\mathbb{Z}}$$

in an abelian category $\mathcal{A}$. It is a preabelian category. Its objects are cochain complexes.

We shall denote by $K(\mathcal{A})^+$ the full subcategory formed by complexes such that there exists N such that $K^i = 0$ for $i < N$. Similarly we define $K(\mathcal{A})^-$ and $K(\mathcal{A})^b = K(\mathcal{A})^+ \cap K(\mathcal{A})^-$.

Let

$$H^\bullet : K(\mathcal{A}) \to K(\mathcal{A}),$$

be the functor which is defined by taking the cohomology of a cochain complex. We consider $(H^\bullet(K^\bullet, d_K))$ as a cochain complex with zero coboundary morphisms. We shall use the following notations:

$$\tau_{\leq n}(K^\bullet)_i = \begin{cases} K^i & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases}, \quad d^i_{\tau_{\leq n}(K^\bullet)} = d^i_K, i \leq n.$$

$$\tau_{\geq n}(K^\bullet)_i = \begin{cases} K^i & \text{if } i \geq n \\ 0 & \text{if } i < n \end{cases}, \quad d^i_{\tau_{\leq n}(K^\bullet)} = d^i_K, i \geq n.$$

In particular, we set

$$\tau_+(K^\bullet) = \tau_{\geq 0}(K^\bullet), \quad \tau_-(K^\bullet) = \tau_{\leq 0}(K^\bullet).$$

$$K^\bullet[n]^i = K^{n+i}, \quad d^i_K = (-1)^n d^{n+i}_K.$$

We define the dual complex $\check{K}^\bullet$ as a complex in $K(\mathcal{A}^\circ)$ defined by

$$(\check{K}^\bullet)^i = K^{-i}, \quad d^i_{\check{K}} = d^{-i}_K.$$

Definition. A morphism $f : (K^\bullet, d_K) \to (L^\bullet, d_L)$ is called *homotopy* to zero if there exist morphisms $k^i : K^i \to L^{i-1}$ such that

$$f^i = k^{i+1} d^i_K + d^{i-1}_L k^i.$$

Two morphisms are homotopy equivalent $f \sim g$ if the difference is homotopy equivalent to zero. It is clear that the homotopy to zero morphisms form a subgroup in the group $Hom_{K(\mathcal{A})}(K^\bullet, L^\bullet)$ (the corresponding homotopy maps are the sums). Thus the homotopy equivalence is an equivalence relation. Let $Hom_{HK(\mathcal{A})}(K^\bullet, L^\bullet)$ be the quotient group by the subgroup of morphisms which are homotopy to zero.

Lemma 1. *Let $f : K^\bullet \to L^\bullet, g : L^\bullet \to M^\bullet$ be two morphisms of complexes. Assume that f is homotopy to 0. Then $g \circ f$ is homotopy to zero.*

Proof. Let (k^i) define a homotopy for f. Then $g(k^i)$ will define a homotopy for $g \circ f$.

Let $HK(\mathcal{A})$ denote the category whose objects are compexes in $\mathcal{A}$ and morphisms are equivalence classes of morphisms in $K(\mathcal{A})$ modulo the homotopy equivalence relation. The previous lemma shows that this is well-defined.

Lemma 2. *If $f \sim g$ then*

$$H^\bullet(g) = H^\bullet(f).$$

Proof. It is enough to check that $H^\bullet(f) = 0$ if $f \sim 0$. This is immediate.

Corollary. *The functor $H^\bullet : K(\mathcal{A}) \to K(\mathcal{A})$ extends to a functor $H^\bullet : HK(\mathcal{A}) \to HK(\mathcal{A})$.*

We shall identify objects of $\mathcal{C}$ with complexes $X^\bullet$ such that $X^n = 0, n \neq 0$ and $X^0 = X$ (i.e. $\tau_{\leq -1} = \tau_{\geq 1} = 0$).

Definition. Let $A \in Ob(\mathcal{A})$. A complex $I^\bullet = (I^n)_{n\geq 0}$ together with a morphism $e_A : A \to I^\bullet$ is called an *injective resolution* of an object A if

(i) $I^\bullet = \tau_+(I^\bullet)$;

(ii) each I^n is injective;

(iii) $H^n(I^\bullet) = 0, n > 0$;

(iv) $e : A \to I^0$ is a kernel of $d^0 : I^0 \to I^1$ (in particular, $A \cong H^0(I^\bullet)$.)

A *projective resolution* is a complex such that its dual complex is an injective resolution in $\mathcal{A}^\circ$.

Theorem 1. *Let $I^\bullet$ be an injective resolution of A and $J^\bullet$ be an injective resolution of B. Then any morphism $f : A \to B$ can be extended to a morphism of complexes $I^\bullet \to J^\bullet$. Two different extensions are homotopy equivalent.*

Proof. We have

$$\begin{array}{ll} A & \to I^0 \to I^1 \to \ldots \to I^{n-1} \to I^n \to \ldots \\ \downarrow & \\ B & \to J^0 \to J^1 \to \ldots \to J^{n-1} \to J^n \to \ldots \end{array}.$$

Since $e_A : A \to I^0$ is a monomorphism, and J^0 is injective, the composition $e_B \circ f : A \to J^0$ extends to a morphism $f^0 : I^0 \to J^0$ such that $f^0 \circ d_A = d_B \circ f$. Assume that we can extend f to a morphism of the truncated complexes $\tau_{\leq n}(I^\bullet) \to \tau_{\leq n}(J^\bullet)$. Let $f^i : I^i \to J^i$ be the corresponding morphisms. Since $d^n_J \circ f^n \circ d^{n-1}_I = d^n_J \circ d^{n-1}_J \circ f^{n-1} = 0$ we see that f_n defines a morphism from $im(d^n_I)$ to J^{n+1}. Since I^{n+1} is injective we can extend it to a morphism $f^{n+1} : I^{n+1} \to J^{n+1}$. This extends f to a morphism of the truncated complexes $\tau_{\leq n+1}(I^\bullet) \to \tau_{\leq n+1}(J^\bullet)$. This proves the existence of an extension.

Let us prove the second assertion. Obviously it is enough to show that an extension of the zero morphism $A \to B$ is homotopy equivalent to zero. Since $f^0 \circ e_A = 0$, the morphism f^0 factors through $coker(e_A) = im(d^0_K)$. Since I^1 is injective, the latter extends to a morphism $k^1 : I^1 \to J^0$. Clearly, $f^0 = k^1 \circ d^0 + k^0 \circ d^{-1}$, where $k^0 = 0, d^{-1} = 0$. Assume we can construct a homotopy $(k^i), i \leq n+1$, for the truncated morphism $\tau_{\leq n} : \tau_{\leq n}(I^\bullet) \to \tau_{\leq n}(J^\bullet)$. Thus $f^n = k^{n+1} \circ d^n_K + d^{n-1}_j \circ k^n$. Let $\alpha = f^{n+1} - k^{n+1} \circ d^n_K : I^{n+1} \to J^{n+1}$. It is easy to see that $d^n_J \circ \alpha = 0$. Thus α factors through $im(d^{n+1}_K)$ and then extends to $k^{n+2} : I^{n+2} \to J^{n+1}$. We have $f^{n+2} = k^{n+2} \circ d^{n+1}_K + d^n_j \circ k^{n+1}$ and, by induction, we are done.

Corollary. *For any object X there exists an injective resolution. For any two injective resolutions of X there exists a morphism between the resolution which is defined uniquely up to homotopy.*

We would like to identify objects with their injective resolutions. This would be possible if we build a category where all injective resolutions of an object define the same object. This is an idea of the derived category.

Theorem 2. *Let $\mathcal{C}$ be a small category and S be a set of its morphisms. There exists a category $\mathcal{C}[S^{-1}]$ and a functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ satisfying the following properties:*

(i) for any $f \in S$, $Q(f)$ is an isomorphism;

(i) if $F : \mathcal{C} \to \mathcal{C}'$ is a functor satisfying property (i), then there exists a unique functor $G : \mathcal{C}[S^{-1}] \to \mathcal{C}'$ such that $F = G \circ Q$.

Proof. The idea is simple we have to add formally inverses of all $s \in S$. Let us consider an oriented graph Γ' whose vertices are objects of $\mathcal{C}$ and morphisms as edges. For each $s \in S$ add the edge from $end(s)$ to $tail(s)$. Let Γ be the new graph. Now define the category $\mathcal{C}[S^{-1}]$ as follows. Its objects are vertices of Γ. Its morphisms correspond to paths with the same origin and the end modulo the following equivalence relation: two paths are equivalent if they obtained from each other by the following elementary operations:

(a) two edges can be replaced by the edge corresponding to the composition with respect to

(b) the two path corresponding to the edges s and its inverse is equivalent to the loops corresponding to the identity morphisms (of the origin and the end of the path).

Remark 1. It is easy to see that the functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ is universal with respect to morphisms $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is invertible for any $s \in S$.

Definition. The category $\mathcal{C}[S^{-1}]$ is called the *localization* of $\mathcal{C}$ with respect to the set of morphisms S.

Now we take $\mathcal{C} = HK(\mathcal{A})$ and S is a set of morphisms $f : K^\bullet \to L^\bullet$ such that $H^\bullet(f)$ is an isomorphism (such morphisms are called *quasi-isomorphims*). The obtained category is called the *derived category* of $\mathcal{A}$ and is denoted by $D(\mathcal{A})$. Replacing $HK(\mathcal{A})$ with $HK(\mathcal{A})^\pm$ or $HK(\mathcal{A})^b$ we obtain the derived categories $D^\pm(\mathcal{A})$ (of complexes bounded from below or above) and $D(\mathcal{A})^b$ of bounded complexes.

Theorem 3. *The inclusion $\mathcal{A} \subset K(\mathcal{A})$ extends to a a morphism*

$$i : \mathcal{A} \to D(\mathcal{A}).$$

The image $\bar{I}^\bullet$ of any injective resolution $I^\bullet$ of X in $D(\mathcal{A})$ is isomorphic to $i(X)$. For each $X \in Ob(\mathcal{A})$ pick one injective resolution $\tau(X)$ of X. Then $X \to \tau(X)$ is an isomorphism from $i(\mathcal{A})$ and the subcategory of $D(\mathcal{A})$ generated by $\tau(X)$.

Proof. This follows from the definitions.

Example 1. Let $\mathcal{A}$ is semi-simple, t.e. all short exact sequences split. Then $D(\mathcal{A})$ is isomoprhic to the category of cyclic complexes, i.e. complexes with zero differentials.

Definition. A set S of morphisms in a category $\mathcal{C}$ is called *localizable* if it satisfies the following properties:

(FR1) S is closed under compositions;

(FR2) for any $s : X \to Y$ from S and a morphism $f : Z \to Y$ there exists $g : W \to X$ and $t : W \to Z$ from S such that $s \circ g = f \circ t$. Also the similar property holds when we reverse the arrows;

(FR3) for any $f, g : A \rightrightarrows B$ the existence of $s \in S$ such that $s \circ f = s \circ g$ is equivalent to the existence of $t \in S$ such that $f \circ t = g \circ t$.

Condition (i) means that we can write each $s^{-1}f$ in $\mathcal{C}[S^{-1}]$ in the form gt^{-1} or can write each fs^{-1} in the form $t^{-1}g$. Let $f : X' \to Y$ be a morphism in $\mathcal{A}$ and $s : X' \to X$ belongs to S. If we denote the morphism fs^{-1} in $\mathcal{C}[S^{-1}]$ as a "roof"

$$\begin{tikzcd} X' \arrow[d, "s"'] \arrow[dr, "f"] & \\ X & Y \end{tikzcd}.$$

Two roofs define the same morphism if they can be extended to a common roof

$$\begin{tikzcd} & X' \arrow[d, "s"'] \arrow[dr, "f"] & \\ \tilde{X} \arrow[ur] \arrow[dr] & X & Y \\ & X'' \arrow[u, "s'"] \arrow[ur, "f'"'] & \end{tikzcd}.$$

To check that this is indeed an equivalence relation one has to use the third property of localizing sets. If $(X', s, t) : X \to Y$ is equivalent to (X'', t, g) by means of (Z', r, h) and (X'', t, g) is equivalent to (X''', u, e) by means of (Z'', p, i), then we first define $sr : Z' \to X, tp : Z'' \to X$, then take $v : W \to Z', k : W \to Z''$ such that $srv = tpk$. Then $f_1 = hv, f_2 = pk$ satisfy $tf_1 = tf_2$. Thus we find $w : Z''' \to W$ such that $f_1 w = f_2 w$. If we take $q = rvw : Z''' \to X'$ and $j = ikw : Z''' \to X'''$ we get the equivalence $(X', s, t) \sim (X''', u, e)$.

The composition is defined by composing the roofs:

$$\begin{tikzcd} & & \tilde{X} \arrow[dl] \arrow[dr] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & X'' \arrow[dl, "s'"] \arrow[dr, "f'"] & \\ X & & Y & & Z \end{tikzcd}.$$

Proposition 1. *Let S be a localizing set of morphisms in an additive category $\mathcal{C}$. Then $\mathcal{C}[S^{-1}]$ is an additive category.*

Proof. The idea is to reduce to common denominator. Suppose we have two morphisms $X \to Y$ represented by two roofs $fs^{-1} = (s, f)$ and $f's'^{-1} = (s', f')$. Let $U = Z \times_X Z'$ with respect to $s : Z \to X, s' : Z \to X$. Then the two projections $r : U \to Z, r' : U \to Z'$ are quasi-isomorphisms. Then we have $t = sr = s'r' : U \to X$. Let $g = fr, g' = f'r$; they are morphisms from U to Y. Then $gt^{-1} = fs^{-1}, g't^{-1} = f's'^{-1}$ and we define $fs^{-1} + f's'^{-1}$ to be equal to $t^{-1}(g + g')$.

The notion of localization is closely related to the notion of a *quotient-category.*

A subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ is called *thick* if any zero object of $\mathcal{A}$ belongs to $\mathcal{B}$, $\mathcal{B}$ is closed under taking a subobject or a quotient-object, and if $u : B \to A$ is a subobject of A and B and $\mathrm{coker}(u)$ belong to $\mathcal{B}$, then B belong to $\mathcal{B}$.

Now the factor-category $\mathcal{C}/\mathcal{B}$ is defined as follows. First of all its objects are are objects of $\mathcal{A}$. For any $A \in Ob(\mathcal{A})$ let $\mathcal{C}_A$ denote the full subcategory of $\mathcal{A}/A$ which consists of monomorphims $A' \to A$ such that $A/A' \in \mathcal{B}$. Let $\mathcal{C}_A^\circ$ denotes the same thing for the dual category $\mathcal{A}^\circ$. We have a functor $hom : \mathcal{C}_A \times \mathcal{C}_B^\circ \to (\mathcal{S}ets)$ defined by $hom(A', B') = \mathcal{A}(A', B')$. For any A, B in $\mathcal{A}$ We set

$$Mor_{\mathcal{A}/\mathcal{B}}(A, B) = \liminj_{\mathcal{C}_A \times \mathcal{C}_B^\circ} hom.$$

If we assume that $\mathcal{A}$ is small (or the class of subobjects is a set for any object) then this limit exists. For example, if A, b belong to $\mathcal{B}$, then $\mathcal{C}_A$ and $\mathcal{C}_A^\circ$ contain 0 and hence any two elements in $hom(A', B')$ are equivalent to the zero map. We define the composition of new morphisms as follows. Let $\bar{u} : A' \to B'$ be a representative of $u : A \to B$ and $\bar{v} : B'' \to C'$ is a representative of $v : B \to C$. Let $X = im(B'' \to B \to B')$. It is a subobject of B' and a factor-object of B''. Let $C'' = C' \to X \coprod_{B''} C'$. This is a quotient-object of C'. Let $A'' = A' \times'_B X \to A'$. This is a subobject of A'. We can replace A' with A'' and C' with C'' and take for the composition the morphism $A' \to X \to C'$. Notice that $\mathcal{C}_A$ is closed under taking an overobject of a subobject of A. Similarly $\mathcal{C}_A^\circ$ is closed under passing from B' to B'' where $A \to B'' \to B'$ is a composition of epimorphisms.

Notice that $\mathcal{A}(A, B)$ is mapped to the inductive limit (since A/A and $\ker(B \to B$ belong to $\mathcal{B}$). The identity functor defines a functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{B}$. It is easy to see that it is exact and $F(A) = 0$ iff $A \in \mathcal{B}$. Also each object in the factor-category is of the form $F(A)$.

5. Exact triangles

Before we show that the set of quasi-isomorphisms in $HK(\mathcal{A})$ is a localizing set we have to introduce some constructions familiar from homotopy theory.

Recall the *cone construction* from algebraic topology. Let $f : X \to Y$ be a continuous map of topological spaces. We define the cone C_f of f as the topological space

$$C_f = Y \coprod X \times [0, 1]/ \sim,$$

where $(x, 1) \sim f(x)$ and $(x, 0) \sim (x_0, 0)$ for some fixed $x_0 \in X$. In the case Y is a point, $C_f = \Sigma X$ is the suspension of X. In the case when $X \to Y$ is an inclusion, C_f is homotopy equivalent to Y/X. So in this case C_f is an analog of a cokernel. Also note that there is an inclusion $Y \to C_f$ and if we apply the cone construction to this we get that C_f/Y is homotopy equivalent to ΣX. Thus we have a sequence of morphisms in the homotopy category:

$$X \to Y \to C_f \to \Sigma X.$$

Recall that we have the suspension isomorphism:

$$\tilde{H}_{n+1}(\Sigma X) \cong \tilde{H}_{n+1}(\Sigma X).$$

Also recall that $\Sigma X = X \times [0, 1]$ with $X \times \{0\}$ and $X \times \{1\}$ identified with a point (a double cone). Since $X \times (0, 1]$ and $X \times [0, 1)$ are contractible and $X \times (0, 1)$ is homotopy equivalent to X, we can use the Mayer-Vietoris exact sequence:

$$\tilde{H}_{n+1}(X_1 \cup X_2) \to H_n(X_1 \cap X_2) \to \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \to H_n(X_1 \cup X_2) \to .$$

Thus, using the suspension isomorphism this gives, a long exact sequence

$$H_i(X) \to H_i(Y) \to H_i(C_f) \to H_{i-1}(X) \to H_{i-1}(Y) \to \dots. \qquad (*)$$

In the case when $Y \to X$ is an inclusion, $H_i(C_f) \cong H_i(Y, X)$ and we get the exact sequence of a pair (X, Y). Also, if f induces the isomorphism on homology and X, Y are simply-connected, we get $H_*(C_f) = 0$. By Hurevich's theorem, $\pi_*(C_f) = 0$, and by Whitehead theorem C_f is homotopy to a point. This implies that f is a homotopy equivalence.

There is an analogous construction for simplicial sets. Let $f : (X_n) \to (Y_n)$ be a morphism of simplicial sets. Then C_f is a simplicial set with

$$(C_f)_i = Y_i \cup \bar{X}_{i-1},$$

where $\bar{X}_{i-1} = X_{i-1} \times *$ is the cone over X_{i-1}. Passing to the cochain complexes this leads to the cochain complex

$$C_X^\bullet[1] \oplus C_Y^\bullet, \quad d = \begin{pmatrix} d_X[1] & 0 \\ f & d_Y \end{pmatrix}.$$

This can be taken for the definition of the cone C_f for any morphism of cochain complexes $f : X^\bullet \to Y^\bullet$ in $K(\mathcal{A})$.

Definition. Let $f : X^\bullet \to Y^\bullet$ be a morphism in $K(\mathcal{A})$. Define the *cone* of f as the complex

$$C_f = X^\bullet[1] \oplus Y^\bullet, \quad d = \begin{pmatrix} d_{X[1]} & 0 \\ f & d_Y \end{pmatrix}.$$

Define the *cylinder* $Cyl(f)$ as the complex

$$Cyl(f) = X^\bullet \oplus X^\bullet[1] \oplus Y^\bullet, \quad d = \begin{pmatrix} d_X & -1 & 0 \\ 0 & d_{X[1]} & 0 \\ 0 & f & d_Y \end{pmatrix}.$$

Example 1. Let $f : X \to Y$ considered as a morphism of complexes. Then $(C_f)^{-1} = X, C_f^0 = Y$ and $d = f$. Thus $H^0(C_f) = Ker(f)$ and $H^1(C_f) = Coker(f)$.

Example 2. Take $f = id : X^\bullet \to X^\bullet$. Let $k^i : X^{i+1} \oplus X^i \to X^i \oplus X^{i-1}$ defined by $(x^{i+1}, x^i) \to (x^i, 0)$. This defines the homotopy between 1_{C_f} and 0_{C_f}.

Lemma 1. *Let $f : X^\bullet \to Y^\bullet$ be a morphism of complexes. There is a commutative diagram*

$$\begin{array}{ccccccccc} & 0 & \to & Y^\bullet & \to & C_f & \to & X^\bullet[1] & \to & 0 \\ & & & a\downarrow & & || & & & & \\ 0 & \to & X^\bullet & \to & Cyl(f) & \to & C_f & \to & 0 & \end{array}. \qquad (*)$$

The first vertical arrow is a quasi-isomorphism.

Proof. We define a by $(k^i, k^{i+1}, l^i) \to (0, 0, l^i)$ and its inverse $b : (k^i, k^{i+1}, l^i) \to f(k^i) + l^i$. Now $b \circ a = id$. We define the homotopy $h^i : Cyl_i \to Cyl_{i-1}$ by $(k^i, k^{i+1}, l^i) = (0, k^i, 0)$. We have

$$a \circ b(k^i, k^{i+1}, l^i) = (0, 0, f(k^{i+1}) + l^i),$$

$$(d_{Cyl(f)}^{i-1} h^i + h^i d_{Cyl(f)}^i)(k^i, k^{i+1}, l^i) = (-k^i, -d_K^i(k^i), f(k^i)) + (0, d_K^i(k^i) - k^{i+1}, 0) =$$

$$(-k^i, -k^{i+1}, f(k^{i+1})) = (a \circ b - 1_{Cyl(f)})(k^i, k^{i+1}, l^i).$$

This checks that $a \circ b \sim id$.

Corollary. *Let $f : X^\bullet \to Y^\bullet$ be a morphism in $HK(\mathcal{A})$. Then it can be extended to a sequence*

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} C_f \xrightarrow{h} X^\bullet[1],$$

where the composition of any two morphisms is zero.

Proof. We define $g : Y \to C_f = X^\bullet[1] \oplus Y^\bullet$ as the natural inclusion and the morphism $C_f^\bullet \to X[1]$ as the natural projection. By the proof of the previous lemma, the composition $g \circ f : X^\bullet \to Y^\bullet \to C_f$ is homotopy equivalent to the composition $X^\bullet \to Cyl(f) \to C_f$ which is zero. The composition $h \circ g : Y^\bullet \to C_f \to X^\bullet[1]$ is zero because the top row in Lemma 1 is an exact sequence.

Definition. A triangle in $K(\mathcal{A})$ is a diagram of the form

$$K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1].$$

A *distinguished triangle* is a triangle which is quasi-isomorphic to the part of the diagram from the previous lemma

$$K^\bullet \to Cyl(f)^\bullet \xrightarrow{\pi} C_f^\bullet \to K^\bullet[1].$$

Lemma 2. *Any short exact sequence of complexes is quasi-isomorphic to the middle row of the diagram (*).*

Proof. Let

$$0 \to K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} M^\bullet \to 0$$

be an exact sequence in $K(\mathcal{A})$. We define $\beta : Cyl(f) \to L^\bullet$ to be equal to b from the previous lemma and $\gamma : C_f \to M^\bullet$ by composing the natural projection $C_f \to L^\bullet$ with g. We have $ker(\gamma) = K^\bullet[1] \oplus Imf = K^\bullet[1] \oplus K^\bullet = C_{1_{K^\bullet}}$. By Example 2, the latter complex has trivial cohomology. Thus, using the exact sequence $0 \to Ker(\gamma) \to C_f \to M^\bullet \to 0$ we obtain that γ is a quasi-isomorphism.

Theorem 1. *Any distinguished triangle*

$$K^\bullet \to L^\bullet \to M^\bullet \to K^\bullet[1]$$

defines an exact sequence of cohomology:

$$\ldots \to H^i(K^\bullet) \to H^i(L^\bullet) \to H^i(M^\bullet) \to H^i(K^\bullet[1]) = H^{i+1}(K^\bullet) \to \ldots.$$

Proof. It is enough to prove it for the distinguished triangle:

$$K^\bullet \to Cyl(f)^\bullet \to C_f^\bullet \xrightarrow{g} K^\bullet[1].$$

We have the exact sequence

$$0 \to K^\bullet \to Cyl(f)^\bullet \to C_f \to 0.$$

It gives the exact sequence of cohomology

$$\ldots \to H^i(K^\bullet) \to H^i(Cyl(f)) \to H^i(C_f) \xrightarrow{\delta} H^{i+1}(K^\bullet) \to \ldots.$$

It remains to identify the coboundary morphism δ with $H^i(g)$. We use that $Cyl(f)^i = K^i \oplus C_f^i$ and check the definitions.

Theorem 2. *In the category $HK(\mathcal{A})$ quasi-isomorphims form a localizing set of morphisms.*

Proof. Let $s : K^\bullet \to L^\bullet$ and $g : M^\bullet \to L^\bullet$, where s is a q.i.. We have to find a q.i. $t : N^\bullet \to M^\bullet$, and a morphism of complexes $g : N^\bullet \to K^\bullet$ making the square

$$\begin{array}{ccc} N^\bullet & \xrightarrow{t} & M^\bullet \\ \downarrow & & \downarrow g \\ K^\bullet & \xrightarrow{s} & L^\bullet \end{array}, \qquad (**)$$

commutative. Let $\pi = (0, 1_L) : L^\bullet \to C_s = K^\bullet[1] \oplus L^\bullet$, Consider the following diagram in $HK(\mathcal{A})$:

$$\begin{array}{ccccccc} C_{\pi\circ g}[-1] & \xrightarrow{t} & M^\bullet & \xrightarrow{\pi\circ g} & C_s & \to & C_{\pi\circ g} \\ h\downarrow & & \downarrow g & & || & & \downarrow \\ K^\bullet & \xrightarrow{s} & L^\bullet & \xrightarrow{\pi} & C_s & \to & K^\bullet[1] \end{array},$$

where $t = (1_M, 0, 0) : M^\bullet \to C_{\pi\circ g}[-1] = M^\bullet \oplus K^\bullet[-1] \oplus L^\bullet[-1]$ and $h(m^i, k^i, l^{i-1}) = -k^i$. We claim that the left square in the diagram is quasi-isomorphic to the square (**). We have

$$t : (m^i, k^i, l^{i-1}) \to -m^i,$$

$$g \circ t - s \circ h : (m^i, k^i, l^{i-1}) \to g(m^i) + s(k^i).$$

Let $p^i : C_{\pi\circ g}[-1]^i \to L^{i-1}$ defined by the natural projection. We have

$$(p^{i+1} d^{i-1}_{C_{\pi\circ g}[-1]} + d^i_L p^i)(m^i, k^i, l^{i-1}) = g(m^i) + s(k^i) - d^{i-1}_L(l^{i-1}) + d^i_{L^\bullet[-1]}(l^i) = g(m^i) + s(k^i).$$

This shows that $g \circ t \sim s \circ h$. Thus we see that $g \circ t \sim s \circ h$ so that the square is indeed commutative in $HK(\mathcal{A})$

Since s is a q.i., the complex C_f is *acyclic* (i.e. all cohomology are zero). Since the top row is a distinguished triangle, t must be a q.i..

Similarly,

$$\begin{array}{ccccccc} C_f[-1] & \xrightarrow{\tau} & K^\bullet & \xrightarrow{s} & L^\bullet & \to & C_f \\ || & & \downarrow g & & \downarrow h & & || \\ C_f[-1] & \xrightarrow{g\tau} & M^\bullet & \xrightarrow{t} & C_{g\circ\tau} & \to & C_f \end{array}$$

is used to check the property about fibred products.

Finally we have to check the third propery of localizing sets. Let $f : K^\bullet \to L^\bullet$ be a morphism in $HK(\mathcal{A})$. Assume $sf = 0$ for some q.i. $s : L^\bullet \to M^\bullet$. We have to show that $ft = 0$ for some q.i. $t : N^\bullet \to K^\bullet$. Consider the following commutative diagram:

$$\begin{array}{ccccc} C_s[-1] & \xrightarrow{\delta} & L^\bullet & \xrightarrow{s} & M^\bullet \\ || & & \uparrow f & & \\ C_s[-1] & \xleftarrow{g} & K^\bullet & \xleftarrow{t} & C_g[-1] \end{array}.$$

The morphism g is defined as follows

$$g^i : K^i \to C_s[-1]^i = L^i \oplus M^{i-1}, \quad k^i \to (f^i(k^i), -h^i(k^i)),$$

where h defines the homotopy $sf \sim 0$. We have $tf = tg\delta = 0$ (because $tg = 0$ in $HK(\mathcal{A})$ as follows from Corollary to Lemma 2). Since s is a q.i., t is a q.i..

6. Triangulated categories

Definition. An additive category $\mathcal{C}$ is called *triangulated* if it is equipped with the following data:

(i) An additive automorphism $T : \mathcal{C} \to \mathcal{C}$ (the *shift functor*).

(ii) A class of distinguished triangles (closed under a naturally defined isomorphism of triangles)

$$X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} T(X)$$

(we also write it as

$$\begin{tikzcd} & Z \arrow[dl, "{[1]}"'] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

to justify the name). The following axioms must be satisfied:
(TR1) $X \overset{1_X}{\to} X \to 0 \to T(X)$ is distinguished;
(TR2) (*extension axiom*) any morphism $f : X \to Y$ can be completed to a distinguished triangle;
(TR3) (*shift axiom*) a triangle $X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} T(X)$ is distinguished if and only if

$$Y \overset{v}{\to} Z \overset{w}{\to} T(X) \overset{-T(u)}{\longrightarrow} T(Y)$$

is distinguished;
(TR4) Any commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd}.$$

extends to a morphism of triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & T(X) \arrow[d, "T(f)"'] \\ X' \arrow[r, "u'"] & Y \arrow[r, "v'"] & Z \arrow[r, "w'"] & T(X') \end{tikzcd}$$

(TR5) (*octaedron axiom*) Given three triangles

$$X \overset{u}{\to} Y \overset{j}{\to} Z' \to T(X), \quad Y \overset{v}{\to} Z \to X' \overset{i}{\to} T(Y), \quad X \overset{vu}{\to} Z \to Y' \to T(X)$$

there exist two morphisms $f : Z' \to Y', g : Y' \to X'$ such that $(1_X, v, f), (u, 1_Z, g)$ are morphisms of triangles and $(Z', Y', X', f, g, T(j) \circ i)$ is a distinguished triangle. One can illustrate it by the following diagrams:

$$\begin{tikzcd} X' \arrow[dd] \arrow[dr, "{[1]}"'] & & Z \arrow[ll, dashed] \\ & Y \arrow[ur] \arrow[dl] & \\ Z' \arrow[rr, dashed, "{[1]}"] & & X \arrow[ul] \arrow[uu] \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[dd, "{[1]}"'] & & Z \arrow[ll, dashed] \arrow[dl] \\ & Y' \arrow[ul] \arrow[dr, "{[1]}"'] & \\ Z' \arrow[ur] \arrow[rr, dashed, "{[1]}"] & & X \arrow[uu] \end{tikzcd}.$$

Here the upper and the bottom triangle in the left diagram are distinguished triangles and the other two triangles are commutative. In the right diagram, the upper and the bottom triangles are commutative and the other two are distinguished triangles. It is also required that the two possible morphisms $Y \to Y'$ (factored through Z and Z') are equal. The axiom says that the left diagram can be completed to the right diagram.

Remark 1. It follows from Lemma 2 that the cone of a morphism $A \to B$ represented by an embedding of morphisms of complexes $0 \to K^\bullet \to L^\bullet$ is isomorphic to an object represented by $L^\bullet/K^\bullet$. We denote it by B/A. The octahedron axiom shows that if $A \to B$ and $B \to C$ are represented by monomorphisms of complexes, then

$$C/B \cong (C/A)/(B/A).$$

Indeed, let $X = C/B, Y = B/A$. We have

$$\begin{tikzcd} C/B \arrow[d] \arrow[dr] & & C \arrow[ll, dashed] & C/B \arrow[d] & & C \arrow[ll, dashed] \arrow[dl] \\ & B \arrow[ur] \arrow[dl] & & & C/A \arrow[ul] \arrow[dr] & \\ B/A \arrow[rr, dashed] & & A \arrow[uu] \arrow[ul] & B/A \arrow[rr, dashed] \arrow[ur] & & A \arrow[uu] \end{tikzcd}.$$

Remark 2. Using the shift axiom (TR3) allows one to extend the commutative diagram formed by g and h to a morphism of triangles by constructing a morphism f (we use the notation of (TR4)). Also it allows one to extend a morphism $f : X \to Y$ to a distinguished triangle $Z \to X \to Y \to Z[1]$.

Proposition 1. *Let*

$$X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} T(X)$$

be a distinguished triangle.
(i) The composition of any two consecutive morphisms is zero.
(ii) For any object M the sequences

$$\ldots \to Hom_{\mathcal{C}}(M, T^i(X)) \to Hom_{\mathcal{C}}(M, T^i(X)) \to Hom_{\mathcal{C}}(M, T^i(X)) \to Hom_{\mathcal{C}}(M, T^{i+1}(X)) \to \ldots,$$
$$\ldots \to Hom_{\mathcal{C}}(T^{i+1}(X), M) \to Hom_{\mathcal{C}}(T^i(Z), M) \to Hom_{\mathcal{C}}(T^i(Y), M) \to Hom_{\mathcal{C}}(T^i(X), M) \to \ldots,$$

are exact.
(iii) In axiom (TR4), if f and g are isomorphisms then h is also an isomorphism.
(iv) If u is zero, then there exists $\gamma : Z \to Y$ such that $\gamma \circ w = 1_{T(X)}$.

Proof. (i) and (ii) By axiom (TR2) it is enough to check the exactness at Y. Extend $(1_X : X \to X) \to (u : X \to Y)$ to a morphism of triangles

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "1_X"'] & X \arrow[r, "0"] \arrow[d, "u"'] & 0 \arrow[r, "0"] \arrow[d, "h"'] & T(X) \arrow[d, "1_{T(X)}"'] \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & T(X) \end{tikzcd}.$$

Since $h : 0 \to Z$ is the zero morphism we get $vu = 0$. Consider

$$\begin{tikzcd} U \arrow[r, "0"] \arrow[d, "f"'] & 0 \arrow[r, "0"] \arrow[d, "0"'] & T(U) \arrow[r, "-1"] & T(U) \arrow[d, "T(f)"'] \\ Y \arrow[r, "v"] & Z \arrow[r, "w"] & T(X) \arrow[r, "-T(u)"] & T(Y) \end{tikzcd}.$$

Each arises from the triangles

$$\begin{tikzcd}
U \arrow[r, "1_U"] \arrow[d, "g"'] & U \arrow[r, "0"] \arrow[d, "f"'] & 0 \arrow[r, "0"] \arrow[d, "0"'] & T(U) \arrow[d, "T(g)"'] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & T(X)
\end{tikzcd}.$$

By (TR2) we can find the third arrow in the second diagram $T(g) : T(U) \to T(X)$ and then reconstruct the first arrow $g : U \to X$ in the first diagram (by shifting and applying T^{-1}). From the commutativity of the first square we get $f = ug$.

(iii) Apply (ii) to $Hom_{\mathcal{C}}(Z', ?)$ to both sequences, use the five-lemma to deduce that the homomorphism $Hom(h) : Hom(Z', Z) \to Hom(Z', Z')$ is an isomorphism. This implies that ther exists $\phi \in Hom(Z', Z)$ such that $h \circ \phi = 1_{Z'}$. Similarly, using $Hom_{\mathcal{C}}(?, Z)$ we find the left inverse of h.

(iv) Assume $u = 0$. Apply (TR4) to

$$\begin{tikzcd}
X \arrow[r, "0"] \arrow[d, "1_X"'] & 0 \arrow[r, "0"] \arrow[d, "0"'] & T(X) \arrow[r, "1_{T(X)}"] \arrow[d, "\gamma"'] & T(X) \arrow[d, "1_{T(X)}"'] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & T(X)
\end{tikzcd}.$$

Here the top triangle is constructed by using (TR1) and (TR3).

Corollary 1. *Let $X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} T(X)$ extends $u : X \to Y$. Then Z is defined uniquely modulo isomorphism.*

Definition. We shall call Z *a cone* over u and denote it by C_u.

Remark 3. In the cone notation the octahedron axiom shows that

$$C_{v \circ u} = T^{-1}(C_w),$$

where $w : C_v \to T(C_u)$ is the composition $C_v \to T(Y) \to T(C_u)$.

Corollary 2. *A morphism in a triangulated category is an isomorphism if and only if its cone is a zero object.*

Proof. Assume $u : X \to Y$ is an isomorphism. Applying (TR1) and (TR4) we get that $X \to Y \to C_u \to T(X)$ is isomorphic to $X \to X \to 0 \to T(X)$. Conversely, if $C_u = 0$ we apply (TR3) to get a distinguished triangle $0 \to X \to Y \to 0$. By Proposition 1 (ii), we get that u is an isomorphism.

Corollary 3. *Let $\mathcal{C}$ be a triangulated abelian category. Then any monomorphism $u : X \to Y$ splits, i.e. u is an isomorphism to a direct summand of Y.*

Proof. Extend u to a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

Let $\alpha = T^{-1}(w) : T^{-1}(Z) \to X$. Applying (TR3) we get a distinguished triangle $T^{-1}(Z) \xrightarrow{\alpha} X \to Y \to Z$. By Proposition 1 (i), we obtain that $f \circ \alpha = 0$ and hence $\alpha = 0$ since u is a monomorphism. By Proposition 1 (iv) there exists $\gamma : Z \to Y$ such that $v \circ \gamma = 1_Z$. It remains to prove that $s = u \oplus \gamma : X \oplus Z \to Y$ is an isomorphism. By Corollary 2 this is equivalent to $C_f = 0$. We shall apply the octahedron axiom. Let $i : X \to X \oplus Z$ be the natural inclusion and $p : X \oplus Z \to Z$ be the natural projection. The triangle

$$X \xrightarrow{i} X \oplus Z \xrightarrow{p} Z \to T(X)$$

is distinguished. This follows from considering the triangle $0 \to Z \to Z \to 0$ and applying (TR3) to $f : X \to 0, g = p : X \oplus Z \to X$ to obtain a morphism $C_i \to Z$, and then applying (TR3) again to $f : 0 \to X, g : Z \to X \oplus Z$ to obtain its inverse $Z \to C_i$. Now apply (TR5). We have

$$\begin{array}{ccccccccccccc}
C_s & \longleftarrow & - - - & - - - & Y & & C_s & \longleftarrow & - - - & - - - & Y \\
| & \searrow & & \nearrow & \uparrow & & | & \nwarrow & & \swarrow & \uparrow \\
| & & X \oplus Z & & | & & | & & Z & & | \\
\downarrow & \swarrow & & \nwarrow & | & & \downarrow & \nearrow & & \searrow & | \\
Z & - - - & - - - & \longrightarrow & X & & Z & - - - & - - - & \longrightarrow & X
\end{array}.$$

Here the upper and the bottom triangle in the left diagram are distinguished triangles (with vertical arrows are of degree 1) and the other two triangles are commutative. In the right diagram, the upper and the bottom triangles are commutative and the other two are distinguished triangles. It follows that $C_s = C_{1_Z}$ and hence is zero.

Remark 4. The previous corollary says that an abelian triangulated category is semi-simple. In particular the category of abelian groups is not triangulated. Let $\mathcal{C}$ be a semi-simple abelian category. Then it can be triangulated and each distinguished triangle is isomorphic to the triangle

$$X \to Y \to Ker(f) \oplus Coker(f) \to X[1].$$

Definition. A functor $F : \mathcal{C} \to \mathcal{A}$ from triangulated category $\mathcal{C}$ to an abelian category $\mathcal{A}$ is called *cohomological* if for any distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

the sequence

$$\ldots \to F(T^i(X)) \to F(T^i(Y)) \to F(T^i(Z)) \to F(T^{i+1}(X)) \to \ldots$$

is exact in $\mathcal{A}$.

Example 1. The functors $X \to Hom_{\mathcal{C}}(U, X)$ and $X \to Hom(X, U)$ for a fixed U are cohomological. This follows from Proposition 1.

It follows from Remark 2 that the category $K(\mathcal{A})$ cannot be triangulated unless $\mathcal{A}$ is semi-simple. Instead we consider the category $HK(\mathcal{A})$. We use the same shift functor $X^\bullet \to X^\bullet[1]$ and consider the set of distinguished triangles equal to the set of triangles isomorphic in $HK(\mathcal{A})$ to a triangle $X \xrightarrow{f} Y \to C_f \to X[1]$ in $K(\mathcal{A})$.

Theorem 1. *The category $HK(\mathcal{A})$ is triangulated with respect to the functor $T = ?[1]$ and the set of distinguished triangles is equal to the set of triangles isomorphic to a triangle $X \xrightarrow{f} Y \to C_f \to X[1]$ in $K(\mathcal{A})$.*

Proof. We shall check each axiom.

(TR1) This follows from Example 3 since the identity morphism of C_{1_X} is homotopy to the zero morphism so that $C_{1_X} \cong 0$ in $HK(\mathcal{A})$.

(TR2) We know that any morphism in $HK(\mathcal{A})$ can be represented by a morphism in $K(\mathcal{A})$. We extend it to a distinguished triangle in $K(\mathcal{A})$ and then consider the corresponding triangle in $HK(\mathcal{A})$.

(TR3) Let $X \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} X^\bullet[1]$ be a distinguished triangle in $HK(\mathcal{A})$. We may assume that $Z^\bullet \cong C_u$. It suffices to check that $C_v \cong X[1]$ and $-T(u) : X[1] \to Y[1]$ is isomorphic to the canonical morphism $C_v \to Y[1]$ such that the triangle

$$Y^\bullet \to C_u \xrightarrow{s} C_v \to Y^\bullet[1]$$

is isomorphic to the triangle

$$Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} X^\bullet[1] \xrightarrow{-u[1]} Y^\bullet[1]$$

Define $\alpha : X^\bullet[1] \to C_v$ by the formula $\alpha(x^{i+1}) = (-u^{i+1}(x^{i+1}), x^{i+1}, 0)$. To prove that $(1, 1, \alpha, 1)$ defines a morphism of triangles, we have to check that $s \sim \alpha \circ w$. The homotopy is defined by $h^i(x^{i+1}, y^i) = (y^i, 0, 0)$. Finally we need to show that the morphism we have just constructed is an isomorphism. Define $\beta : C_v \to X^\bullet[1]$ as the projection. Then $\beta \circ \alpha = 1_X$ and $\alpha \circ \beta$ is given by the formula $(y^{i+1}, x^{i+1}, y^i) \to (-f^{i+1}(x^{i+1}), x^{i+1}, 0)$. The homotopy is given by $h^i : (y^{i+1}, x^{i+1}, y^i) \to (y^i, 0, 0)$.

(TR4) We may assume that $Z = C_u$, $Z' = C_{u'}$. We take $h = f[1] \oplus g$.

(TR5) It suffices to find a distinguished triangle isomorphic to

$$C_u \xrightarrow{f} C_{vu} \xrightarrow{g} C_v \xrightarrow{h} C_u[1],$$

where

$$f : C_u = X^\bullet[1] \oplus Y^\bullet \xrightarrow{(1,v)} C_{vu} = X[1] \oplus Z,$$

$$g : C_{vu} = X^\bullet[1] \oplus Z^\bullet \xrightarrow{(T(u),1)} C_v = Y^\bullet[1] \oplus Z^\bullet,$$

$$h : C_v = Y^\bullet[1] \oplus Z^\bullet \xrightarrow{(1,0)} C_u[1] = X^\bullet[2] \oplus Y^\bullet[1].$$

For this we have to find an isomorphism

$$a : C_f = C_u[1] \oplus C_{vu} = X^\bullet[2] \oplus Y^\bullet[1] \oplus X[1] \oplus Z^\bullet \cong C_v = Y^\bullet[1] \oplus Z^\bullet.$$

To check that this and the above morphisms are morphisms of complexes.

We have

$$(d^i_{C_{vu}} f^i)(x^{i+1}, y^i) = d^i_{C_{vu}}(x^{i+1}, v(y^i)) = (-d^{i+1}_X(x^{i+1}), (vu)^{i+1}(x^{i+1}) + d^i_Z(v(y^i)))$$

$$f^{i+1}(d_{C_u}(x^{i+1}, y^i)) = f^{i+1}(-d^{i+1}_X(x^{i+1}), u^{i+1}(x^{i+1}) + d^i_Y(y^i)) = (-d^{i+1}_X(x^{i+1}), (vu)^{i+1}(x^{i+1}) + v^i(d^i_Y(y^i)).$$

This checks that f is a morphism of complexes. We have

$$(d^i_{C_v} g^i)(x^{i+1}, z^i) = d^i_{C_v}(u^{i+1}(x^{i+1}), z^i) = -d^{i+1}_Y(u^{i+1}(x^{i+1}), (vu)^{i+1}(x^{i+1}) + d^i_Z(z^i)),$$

$$g^i(d^i_{C_v}(x^{i+1}, z^i)) = g^i(-d^{i+1}_X(x^{i+1}), (vu)^{i+1}(x^{i+1}) + d^i_Z(z^i)) = (u^{i+1}(x^{i+1})), (vu)^{i+1}(x^{i+1}) + d^i_Z(z^i).$$

This checks that g is a morphism of compexes. We have

$$(d^i_{C_u[1]} h^i)(y^{i+1}, z^i) = d^i_{C_u[1]}(0, y^{i+1}) = (0, d^{i+1}_Y(y^{i+1}),$$

$$h^i(d_{C_v^i}(y^{i+1}, z^i)) = h^i(d_Y^{i+1}(y^{i+1}), v^{i+1} + d_Z(z^i)) = (0, d_Y^{i+1}(y^{i+1})).$$

This checks that h is a morphism of complexes. To construct an isomorphism $a : C_f \to C_v$ we observe that $C_f = Cyl(\phi)$, where $\phi : X[1] \to Y[1] \oplus Z$, where $\phi(x^{i+1}) = (u^{i+1}(x^{i+1}), 0)$. It follows from Lemma 1 that $Cyl(\phi)$ is isomorphic to $C_v = Y[1] \oplus Z$ in $HK(\mathcal{C})$.

Definition. Let $\mathcal{A}$ be a triangulated category and S be a localizing set. We say that S is compatible with triangulation if

(FR4) $s \in S$ iff $T(s) \in S$;

(FR5) in (TR4) if $f, g \in S$ we can find $h \in S$.

Definition. A morphism between triangulated categories is called a ∂-functor if it transforms distinguished triangles to distinguished triangles and commutes with the shift functor.

Example 2. Any additive functor between abelian categories $F : \mathcal{A} \to \mathcal{B}$ defines a ∂-functor of the corresponding categories $HK(\mathcal{A}) \to HK(\mathcal{B})$. This follows from the fact that, for any morphism $f : X \to Y$,

$$F(C_f) = C_{F(f))}, \quad F(Cyl(f)) = Cyl(F(f)))$$

so that it transforms the standard triangles $X \to Cyl(f) \to C_f \to X[1]$ to a standard triangle. Since an additive functor transforms isomorphisms to isomorphisms, it is a ∂-functor.

Theorem 2. *Let $\mathcal{C}$ be a triangulated category and S be a localizing set compatible with the triangulation. The localized category $\mathcal{C}[S^{-1}]$ is a triangulated category such that the canonical functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ is a ∂-functor.*

Proof. Since objects do not change we take for T the shift in $\mathcal{C}$. We define a distinguished triangle in $\mathcal{D} = \mathcal{C}[S^{-1}]$ as a triangle isomorphic to the image of a distinguished triangle under the functor Q. We shall check the axioms one by one. First recall that

$$Hom_{\mathcal{D}}(X, Y) = \liminf_{s : X' \to Y} Hom_{\mathcal{C}}(X', Y).$$

(TR1) Obvious.

(TR2) We choose a representative $f' : X' \to Y$ of $f : X \to Y$ for some $s : X' \to X$ from S. Extend f' to a distinguished triangle $X' \to Y \to Z \overset{w}{\to} X'[1]$ in $\mathcal{C}$. Then it is isomorphic in $\mathcal{D}$ to the triangle

$$X \to Y \to Z \overset{T(s)w}{\to} X[1].$$

(TR3) Obvious.

(TR4)

$$\begin{tikzcd}
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & T(X) \\
X'' \arrow[u, "s"] \arrow[d, "\tilde{f}"'] \arrow[r, "u''"] & Y'' \arrow[u, "t"] \arrow[d, "\tilde{g}"'] \arrow[r, "v''"] & Z'' \arrow[u, "r"] \arrow[d, "\tilde{h}"'] \arrow[r, "w''"] & T(X'') \arrow[u, "T(s)"] \arrow[d, "T(\tilde{f})"'] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & T(X')
\end{tikzcd}$$

Here the morphism $X \to Y$ in the localized category is represented by a pair $(s, \tilde{f})$ and similar for the morphism $G : Y \to Z$, $w : Z \to T(X)$. We would like to construct the morphisms u'', v'', w''. We do it by first replacing f and g with equivalent roofs.

First, by property (FR2) of localizing sets, we find a morphism $\tilde{t} : \tilde{X} \to X''$ from S and a morphism $\tilde{u} : \tilde{X} \to Y''$ making the diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{u}"] \arrow[d, "\tilde{t}"'] & Y'' \arrow[d, "t"] \\ X'' \arrow[r, "us"] & Y. \end{tikzcd}$$

commutative. It is easy to check that the compositions $u'\tilde{f}\tilde{t}$ and $\tilde{g}\tilde{u}$ from $\tilde{X}$ to Y' are equal in the localized category but not in the original one. This easily implies that there exists a morphism $q : \tilde{X}' \to \tilde{X}$ from S such that the compositions $u'\tilde{f}\tilde{t}q$ and $\tilde{g}\tilde{u}q$ define the same morphism $\tilde{X} \to Y'$ in $\mathcal{C}$. Now replace X'' with $\tilde{X}'$ and $f = (s, u)$ with $(s\tilde{t}q, \tilde{f}, \tilde{t}q)$. We have a morphism $u'' : X'' \to Y''$ defined by the roof $(\tilde{q}, \tilde{u}q)$ which makes the square

$$\begin{tikzcd} X'' \arrow[r, "u''"] \arrow[d, "s"'] & Y'' \arrow[d, "\tilde{u}q"] \\ X \arrow[r, "u"] & Y. \end{tikzcd}$$

commutative. Extend u'' to a distinguished triangle $X'' \overset{u''}{\to} Y'' \overset{v''}{\to} Z'' \overset{w''}{\to} X''[1]$ in $\mathcal{C}$. By Property (TR4) we can find a morphism $r : Z'' \to Z$. By property (FR4), $r \in S$. Again we apply (TR4) to the commutative square

$$\begin{tikzcd} X'' \arrow[r, "u''"] \arrow[d, "q\tilde{t}\tilde{f}"'] & Y'' \arrow[d, "\tilde{g}\tilde{u}q"] \\ X' \arrow[r, "u'"] & Y'. \end{tikzcd}$$

to obtain a morphism $\tilde{h} : Y'' \to Y'$. The pair $h = (r, \tilde{h})$ defines a morphism $Z \to Z'$ in the localized category such that the triple (f, g, h) is a morphism of the distinguished triangles.

(TR5) We leave it as a homework exercise.

To apply the previous lemma to our situation $\mathcal{C} = HK(\mathcal{A})$ we have to check that the localizing set of quasi-isomorphisms satisfies (FR4) and (FR5). This follows from the following more general statement (we apply it by taking the functor $F = H^\bullet$).

Lemma 1. *Let $F : \mathcal{C} \to \mathcal{A}$ be a cohomological functor from a triangulated category to an abelian category. Let S be the set of morphisms in $\mathcal{C}$ such that $F(T^i(s))$ is an isomorphism for all $i \in \mathbb{Z}$. Then S is a localizing set compatible with triangulations.*

Proof. This is similar to the proof of Theorem 1 from the previous section. The first property is trivial. To check the second one, assume that $s : X \to Y$ belongs to S and $g : Y' \to Y$ is a morphism in $\mathcal{C}$. Extend s

to a distinguished triangle $X \xrightarrow{s} Y \xrightarrow{v} Z \to T(X)$ and extend $v' = g \circ v : Y' \to Z$ to a distinguished triangle $W \xrightarrow{t} Y' \to Z \to T(W)$. Applying (TR4) (and Remark 1) we find a morphism of triangles

$$\begin{array}{ccccccc} W & \xrightarrow{t} & Y' & \xrightarrow{v'} & Z & \xrightarrow{w'} & T(W) \\ \downarrow f & & \downarrow g & & \downarrow 1_Z & & \downarrow T(f) \\ X & \xrightarrow{s} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \end{array}$$

It remains to show that $t \in S$. Applying the long exact sequence to the image of the bottom triangle under F we obtain that $F(T^i(Z)) = 0$ for all i. Applying the same to the top triangle, we see that $F(T^i(t)) = 0$ for all i. Thus $t \in S$. We leave the proof of the dual statement in (FR2) to the reader.

Let us prove (FR3). Suppose $f : X \to Y$ is such that there exists a morphism $s : Y \to Y'$ from S with $s \circ f = 0$. Let $Z \xrightarrow{v} Y \xrightarrow{s} Y' \to T(Z)$ be a distinguished triangle obtained by applying (TR2) and (TR3). Under the canonical map $Hom(X, Y) \to Hom(X, Y')$ the map f goes to zero. Since Hom is a cohomological functor, there exists $g \in Hom(X, Z)$ such that $f = v \circ g$. Again we find a triangle $W \xrightarrow{t} X \xrightarrow{g} Z \to T(W)$. By Proposition 1 (i), $g \circ t = 0$, hence $f \circ t = v \circ (g \circ t) = 0$. It remains to show that $t \in S$. Since $s \in S$, $F(T^i(Z)) = 0$ for all i. From the second triangle we find that $F(T^i(t)) = 0$ for all i.

Property (FR4) is trivial and property (FR5) follows from the exact sequence of a cohomological functor and the five-lemma.

Corollary 1. *$D(\mathcal{A})$ is a triangulated category.*

Definition. A subcategory of a triangulated category is called a *triangulated subcategory* if two sides of a triangle belongs to it implies that the third side belongs to it.

Corollary 2. *The set of quasi-isomorphisms in any triangulated subcategory of $HK(\mathcal{A})$ is a localizing set compatible with triangulations.*

Examples. 1. Let $\mathcal{B}$ be a thick subcategory of an abelian category $\mathcal{A}$. Consider the subcategory $HK_{\mathcal{B}}(\mathcal{A})$ of $HK(\mathcal{A})$ formed by complexes whose cohomology belong to $\mathcal{B}$. Then $HK_{\mathcal{B}}(\mathcal{A})$ is a triangulated subcategory.

2. Let $HK(\mathcal{I})$ be the subcategoryof $HK(\mathcal{A})$ which consists of complexes whose objects are injective objects of $\mathcal{A}$. Then it is triangulated. This follows from the proof of Theorem 2 (since a direct sum of injective complexes is injective).

7. Injective complexes

The main goal is to prove the following

Theorem 1. *Assume $\mathcal{A}$ has enough injective objects. Then $D^+(\mathcal{A})$ is equivalent to the subcategory $I^+(\mathcal{A})$ of $HK(\mathcal{A})^+$ generated by complexes of injective objects bounded from below.*

Lemma 1. *Any morphism from an acyclic complex to an injective bounded from below complex is homotopic to zero.*

Proof. This proved it while proving Theorem 1 in section 6.

Lemma 2. *Any quasi-isomorphism of an injective bounded from below complex to another complex has a homotopy inverse.*

Proof. Extend $f : I^\bullet \to X^\bullet$ to a distinguished triangle $I^\bullet \to X^\bullet \to Z^\bullet \to I^\bullet[1]$. Since f and $f[1]$ are a q.i., we see that $Z^\bullet$ is acyclic. By the previous lemma, $Z \to I^\bullet[1]$ is homotopic to zero. Thus, by Prop. 1 from §6, f has the inverse in $H(\mathcal{A})$.

Lemma 3. *Let $\mathcal{B}$ be a full subcategory of a category $\mathcal{C}$ and S is a localizing set in $\mathcal{C}$. Assume that for any morphism $s : X \to X'$ from S with $X \in Ob(\mathcal{B})$ there exists a morphism $f : X' \to Z$ with $Z \in Ob(\mathcal{B})$ such that $f \circ s \in S$. Then $S' = S \cap Mor(\mathcal{B})$ is a localizing set and the natural functor $\mathcal{B}[S'^{-1}] \to \mathcal{B}[S^{-1}]$ is fully faithful.*

Proof. A morphism of $B \to B'$ of two objects in $\mathcal{C}[S^{-1}]$ is defined by morphisms $s : X \to X', g : Y \to X'$ in $\mathcal{C}$, where $s \in S$. The morphisms $f \circ s : X' \to Z$ and $f \circ g : Y \to Z$ will represent this morphism by a morphism from in $\mathcal{B}[S'^{-1}]$.

Corollary. *Let $I(\mathcal{A})^+$ be the full subcategory of $HK(\mathcal{A})^+$ formed by injective complexes and $D^+(I)$ its derived category. Then the canonical functor $D^+(I) \to D^+(\mathcal{A})$ is fully faithful.*

Proof. Apply Lemmas 2 and 3.

Lemma 4. *Let $\mathcal{A}$ be an abelian category with enough injective objects. Then every $X^\bullet \in HK^+(\mathcal{A})$ admits a quasi-isomorphism into an injective complex.*

Proof. Without loss of generality we may assume that $X^p = 0, p < 0$. Let $f^0 : X^0 \to I^0$ be an embedding of X^0 into an injective object. We have $0 = Im(d_X^{-1}) \cong Im(d_I^{-1}) = 0$, where we consider f^0 as a morphism of truncated complexes $\tau_+ X^\bullet \to \tau_+ I^\bullet$. Assume by induction that we have constructed $I^0, \ldots, I^p$ and morphisms $f^k : X^k \to I^k$ such that the morphism $f^\bullet$ of the truncated complexes $\tau_{\le p-1} X^\bullet \to \tau_{\le p-1} I^\bullet$ is a quasi-isomorphism and $f^p : Im(d_X^{p-1}) \to Im(d_I^{p-1})$ is an isomorphism. Embed

$$(I^p / Im(I^{p-1} \to I^p)) \oplus_{X^p} X^{p+1}$$

in I^{p+1} and define the morphisms $d_I^p : I^p \to I^{p+1}$ and $f^{p+1} : X^{p+1} \to I^{p+1}$ in the obvious way. It is clear that $I^{p-1} \to I^p \to I^{p+1}$ is zero. Note that

$$Ker(I^p \to I^{p+1}) = Ker(I^p \to (I^p / Im(I^{p-1} \to I^p)) \oplus_{X^p} X^{p+1}).$$

The restriction of the morphism f^p to $Ker(X^p \to X^{p+1})$ defines a morphism to $Ker(I^p \to I^p \oplus_{X^p} X^{p+1}$. This defines a morphism

$$Ker(X^p \to X^{p+1}) / Im(X^{p-1} \to X^p) \to Ker(I^p \to I^{p+1}) / Im(I^{p-1} \to I^p)$$

which as is easy to see must be an isomorphism.

Proof of Theorem 1: Since any quasi-isomorphism in $I^+(\mathcal{A})$ is an isomorphism (Lemma 4), $I^+(\mathcal{A}) = D^+(I)$. By the previous Corollary the functor $I^+(\mathcal{A}) \to D^+(\mathcal{A})$ is fully faithful. Now we apply the previous lemma to show that each complex is isomorphic in $D^+(I)$ to an injective complex.