PART I

1. Categories

Definition A category $\mathcal{C}$ is a set of *objects* $Ob(\mathcal{C})$ and for each $A, B \in Ob(\mathcal{C})$ a set of morphisms $\mathcal{C}(A,B)$ (or $Mor_{\mathcal{C}}(A,B)$) and for each A, B, C a map $\mathcal{C}(A,B) \times \mathcal{C}(B,C) \to \mathcal{C}(A,C)$ (the image of this mapat (f,g) is denoted by $f \circ g$ and is called the composition of f and g) such that the following properties hold:
(i) there exists $1_A \in \mathcal{C}(A,A)$ such that $f \circ 1_A = f, 1_A \circ f = f$ whenever the compositions are defined;
(ii) $Mor_{\mathcal{C}}(A,B) \cap Mor_{\mathcal{C}}(A',B') \neq \emptyset$ iff $A = A', B = B'$;
(iii) the composition is associative.

We shall denote $Mor_{\mathcal{C}}$ the union of the sets $Mor_{\mathcal{C}}(A,B)$. The set $Ob(\mathcal{C})$ can be identified with the subset of this set by assigning to A the morphism 1_A.

To avoid some logical problems in the future we shall assume that all sets $Ob(\mathcal{C})$ and $Mor_{\mathcal{C}}(A,B)$ are subsets of a fixed set (a universum). A *universum* is a set U satisfying certain natural axioms (e.g.with each subset X the set of its parts $\mathcal{P}(X)$ is also a subset of U). In this case say that $\mathcal{C}$ is a U-category. We shall always assume that we are dealing with some U-categories taking U large enough. Another way to solve logical problems is to admit *classes* (e.g. the set of all sets is a class but not a set). Then we define $Ob(\mathcal{C})$ to be a class, and in the case when $Ob(\mathcal{C})$ and $Mor_{\mathcal{C}}$ are sets we say that $\mathcal{C}$ is a *small category*. The problem here is that, even when $Ob(\mathcal{C})$ is a set, the union of $Mor_{\mathcal{C}}(A,B)$ may not be a set. We stick to universums.

There are some natural definitions:

a *subcatageory*: $Ob(\mathcal{C}) \subset Ob(\mathcal{C}')$, $\mathcal{C}(A,B) \subset \mathcal{C}'(A,B)$. A subcategory is *full* if $\mathcal{C}(A,B) = \mathcal{C}'(A,B)$.

The *dual category* $\mathcal{C}^\circ$: $Ob(\mathcal{C}^\circ) = Ob(\mathcal{C})$, $\mathcal{C}^\circ(A,B) = \mathcal{C}(B,A)$.

A morphism $f : A \to B$ is called *monomorphism* or *injective* if the map $Mor_{\mathcal{C}}(C,A) \to Mor_{\mathcal{C}}(C,B)$ defined by the composition is injective. A map is an *epimorphism* or *surjective* if it is injective when considered to be a morphism in the dual category. A morphism which is a mono and epi is called *bijective.*

One may also define a monomorphism $u : A \to B$ as a morphism admitting a left inverse morphism $v : B \to A$ such that $v \circ u = 1_A$. Dually we get the notion of a *epimorphism.* A morphism which is a mono and epi in this sense is called an *isomorphism.* However, the new notions are not equaivalent to the previous ones and very rarely used in the theory of general categories.

The set of morphisms $Mor_{\mathcal{C}}(A,B)$ has a natural quasi-order $f \leq g$ if $g = h \circ f$ for some $h : B \to B$. We say that two morphisms are equivalent if $f \leq g$ and $g \leq f$. A *subobject* of an object A is an equivalence class of monomorphims $B \to A$. Dually one defines a notion of a *factor-object.*

Examples 1. $(\mathcal{S}ets)$: objects = sets, morphisms = maps, compositions = compositions of maps, $1_A = id_A$. Note that each subobject of a set S contains a unique representative which is a subset of S. A factor-object is an equivalence class of surjections $A \to B$. There is no a natural choice of a representative for a factor-object.
2. Various subcategories of $(\mathcal{S}ets)$ defined by putting some structures on the sets and taking morphisms to be maps compatible with these structures:

Mod_R = modules over a ring R with linear maps as morphims

In particular, we have the category $Vect_k$ of vector spaces over a field k.

Rings = associative rings, Com = commutative rings.

Top = topological spaces, morphisms = continuos maps, so on.

$Diff$ = differential manifolds. The notion of a subobject has a natural meaning: a submodule, a subring, a subspace.

3. Let Γ be an oriented graph. Define the category $\mathcal{C}_\Gamma$ as follows. Its objects are the vertices, its morphims are paths. It is not a subcategory of $(\mathcal{S}ets)$. The dual category of $\mathcal{C}_\Gamma$ is $\mathcal{C}_{\Gamma'}$, where Γ' is obtained from the graph Γ by inversing the orientation of arcs in Γ.
4. Let M be a semigroup with the unity e (a monoid). It defines a category $< M >$ with one object e and morphisms are elemensts from M. The composition is the multiplication. For example, if M is a group, then each morphism in $< M >$ is an isomorphism. A category with this property is called a *groupoid.*
5 For experts: the category of affine schemes is dual to the category of commutative rings.

2. Functors

A (covariant) functor $F : \mathcal{C} \to \mathcal{C}'$ is a map of sets $F : Mor_{\mathcal{C}} \to Mor_{\mathcal{C}'}$ which preserves the compositions and maps $Ob(\mathcal{C})$ to $Ob(\mathcal{C}')$. It is called *faithfull* if it is injective and *fully faithfull* if it is surjective on each $Mor_{\mathcal{C}}(A, B)$. A functor $F : \mathcal{C}^\circ \to \mathcal{C}'$ is called a *contravariant functor* from $\mathcal{C}$ to $\mathcal{C}'$. From now when we say a functor we mean a covariant functor. Note that when both $\mathcal{C}$ and $\mathcal{D}$ are U-categories the class of functors $Funct(\mathcal{C}, \mathcal{D})$ is a set belonging to U. Very often we define a functor on objects only leaving to the reader to extend it to morphims in a natural way.

Examples 1. $\mathcal{C}$ is from examples 2. $F : \mathcal{C} \to (\mathcal{S}ets)$ is the *forgetful functor*, it forgets about the aditional structure and sends a morphism to the corresponding map with less structure. It is always faithfull but rarely fully faithfull.

2. Let Γ be a graph. A functor $F : \mathcal{C}_\Gamma \to \mathcal{C}$ is called a *graph in a category*. For example, a graph in $Vect_k$ is called a *quiver*.

3. A functor $\mathcal{F} : \mathcal{C}^\circ \to (\mathcal{S}ets)$ is called a *presheaf* of sets on $\mathcal{C}$. For example, assigning to a topological space the set of continuous functions on it is a presheaf on Top. Here, for $u : X \to Y$, we have $\mathcal{F}(u) = u^*$ is defined by composition with u. If $f : \mathcal{C}' \to \mathcal{C}$ is a functor, composing $\mathcal{F}$ with f we obtain a presehaf on $\mathcal{C}'$. It is denoted by $f^*(\mathcal{F})$. For example, let X be a topological space and $Op(X)$ be the category of open subsets of X with $Mor_{Op(X)}(V, Y))$ consisting of the identity map if V is a subset of U and empty otherwise. Then $Op(X)$ is a subcategory of Top, and restricting a pre-sheaf $\mathcal{F}$ to $Op(X)$ we get a definition of a presheaf of sets on X. If $\mathcal{F} : \mathcal{C}^\circ \to (\mathcal{S}ets)$ factors through a subcategory $\mathcal{C}'$ of $(\mathcal{S}ets)$ we have the notion of a $\mathcal{C}'$-presheaf on $\mathcal{C}$. for example, we can speak about the category of sheaves of abelian groups, of topological space and so on.

We can make the set of functors $Funct(\mathcal{C}, \mathcal{C}')$ into a category by defining a morphism between functors $u : F \to F'$ as a collection of morphisms $u_A : F(A) \to F'(A)$ in $\mathcal{C}'$ such that for any $f : A \to B$ in $\mathcal{C}$ we have

$$u_B \circ F(f) = u_A \circ F'(f).$$

There is the identity functor $I_{\mathcal{C}}$ defined in the obvious way. In particular, we have the notion of isomorphic functors.

Example 4. Let $(F_s)_{s \in S}$ be a subset of the ring of polynomials $\mathbb{Z}[T_1, \ldots, T_n]$ in n variables with integer coefficients. It defines a functor $\underline{X} : (Com) \to (\mathcal{S}ets)$ by assigning to a commutative ring R the set of solutions of the system of algebraic equations $F_s(T) = 0$ in R. If $\phi : R \to R'$ is a homomorphism of rings and $(r_1, \ldots, r_n) \in \underline{X}(R)$, then $(\phi(r_1), \ldots, \phi(r_n)) \in \underline{X}(R')$. This defines $\underline{X}(\phi)$. Two sets of polynomials define the same functors if and only if the ideals generated by the sets are equal. Let F be another functor which assigns to R the set of homomorphisms $\mathbb{Z}[T_1, \ldots, T_n]/(F_s, s \in S) \to R$. For every $r \in \underline{X}(R)$ we define an element from $F(R)$ by sending the coset of T_i tor_i. This defines an isomorphism of functors $\underline{X}(R) \to F$. Two systems (in different number of variables) define isomorphic functors if and only of the qiotients rings $\mathbb{Z}[T_1, \ldots, T_n]/(F_s, s \in S)$ and $\mathbb{Z}[T_1, \ldots, T_{n'}]/(F_{s'}, s' \in S')$ are isomorphic.

One can define the notion of isomorphism of categories in the obvious way but this is rarely used. A weaker notion is an *equivalence of categories*: $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories if there exists a functor $G : \mathcal{C}' \to \mathcal{C}$ and isomorphisms of functors (considered as objects of $Funct(\mathcal{C}, \mathcal{C}')$) $u : F \circ G \to I_{\mathcal{C}'}$ and $v : G \circ F \to I_{\mathcal{C}}$ such that the map $Mor_{\mathcal{C}'}(F(A), G(A)) \to Mor_{\mathcal{C}}(A, B)$ defined by applying G is the inverse to the map $F : Mor_{\mathcal{C}}(A, B) \to Mor_{\mathcal{C}'}(F(A), F(B))$.

Example 5. For any set S let $\mathcal{C}_S$ be the category with $\#Mor_{\mathcal{C}}(A, B) = 1$ for any $A, B \in Ob(\mathcal{C})$. Then for any S, S' the categories $\mathcal{C}_S$ and $\mathcal{C}_{S'}$ are equivalent. They are isomorphic if and only if S and S' have the same cardinality.

3. Representable functors

Let $\check{\mathcal{C}} = Funct(\mathcal{C}^\circ, (\mathcal{S}ets))$. It is the category of *presheaves* on $\mathcal{C}$. Note that $\check{\mathcal{C}}$ is not a U-category if $\mathcal{C}$ is a U-category because the calss of sets is a set. So, to be rigorous we have to consider U-preseheaves restricting ourselves by a subcategory of $(\mathcal{S}ets)$ formed by U-sets.

For every $X \in Ob(\mathcal{C})$ define the presheaf h_X by

$$h_X(A) = Mor_{\mathcal{C}}(A, X), \quad h_X(A \to B) = Mor_{\mathcal{C}}(B, X) \to Mor_{\mathcal{C}}(A, X),$$

where the latter map is defined by composing with $A \to B$.

If $f : X \to Y$ is a morphism in $\mathcal{C}$, then we a have a natural map $h_X(A) = Mor_{\mathcal{C}}(A, X) \to Mor_{\mathcal{C}}(A, Y)$ defined by composition with f. This defines a functor

$$h : \mathcal{C} \to \check{\mathcal{C}}.$$

It is called the *Yoneda functor.*

Lemma (Yoneda). *The Yoneda functor is fully faithfull.*

Proof. We have to show that, for any $X, Y \in Ob(\mathcal{C})$, the map

$$h_{X,Y} : Mor_{\mathcal{C}}(X, Y) \to Mor_{\check{\mathcal{C}}}(h_X, h_Y)$$

is bijective. Notice that $Mor_{\mathcal{C}}(X, Y) = h_Y(X)$. For any $\mathcal{F} \in \check{\mathcal{C}}$ let us define a bijection

$$u : \mathcal{F}(X) \to Mor_{\check{\mathcal{C}}}(h_X, \mathcal{F})$$

such that $u = h_{X,Y}$ if $\mathcal{F} = h_Y$. Then we will be done. Take $s \in \mathcal{F}(X)$, then $u(s)$ must be a morphism from h_X to $\mathcal{F}$. For any $A \in Ob(\mathcal{C})$ it must assign a map $u(s)(A) : h_X(A) \to \mathcal{F}(A)$. We define it by

$$u(s)(A)(A \to X) = \mathcal{F}(A \to X)(s) \in \mathcal{F}(A).$$

If $\mathcal{F} = h_Y$, $s : X \to Y$, then $h_Y(A \to X)(s) = h_{X,Y}(s)$. It remains to show that u is bijective. We define the inverse map

$$v : Mor_{\check{\mathcal{C}}}(h_X, \mathcal{F}) \to \mathcal{F}(X)$$

as follows. Let $F : h_X \to \mathcal{F}$ be a morphism of functors. It sends the set $h_X(X) = Mor_{\mathcal{C}}(X, X)$ to $\mathcal{F}(X)$. Define $v(F)$ to be equal to the image of 1_A.

Let us check that $v \circ u = id$. Let $s \in \mathcal{F}(X)$. Then $v \circ u(s) = u(s)(X)(1_X) : \mathcal{F}(id : X \to X)(s) = s$.

Let us check that $u \circ v = id$. Let $f : h_X \to \mathcal{F}$ be a morphism of functors. Consider the diagram

$$\begin{array}{ccc} h_X(X) & \to & \mathcal{F}(X) \\ \downarrow & & \downarrow \\ h_X(Y) & \to & \mathcal{F}(Y) \end{array}.$$

corresponding to $\alpha : Y \to X$ in th definition of functor morphisms. The image of 1_X under the top arrow is equal to $v(f)$. Its image under the right vertical arrow is $\mathcal{F}(\alpha)(v(f)) = u(v(f))(Y)(\alpha)$. Now the image of 1_X under the left vertical arrow is the composition $\alpha \circ id_X = \alpha$. Its image under the bottom horizontal arrow is $f(Y)(\alpha)$. This shows that $u \circ v(f) = f$. This finishes the proof.

Corollary 1. *Let $u : F \to G$ is a morphism of presheaves. Then it is a monomorphism if and only if for any $A \in Ob(\mathcal{C})$ the map of sets $F(A) \to G(A)$ is injective.*

Proof. $\mathcal{F}(A) \to \mathcal{F}'(A) = Mor_{\check{\mathcal{C}}}(h_A, \mathcal{F}) \to Mor_{\check{\mathcal{C}}}(h_A, \mathcal{F}')$. So, if $f(A)$ is not injective f is not injective, by definition of injective morphisms. This proves the sufficiency. The converse easily follows from the definitions.

Corollary 2. *A morphism $f : X \to Y$ in $\mathcal{C}$ is an isomorphism iff $h(f) : h_X \to h_Y$ is an isomorphism in $\check{\mathcal{C}}$.*

Definition A functor $\mathcal{F} \in \check{\mathcal{C}}$ is called representable (by an object $A \in Ob(\mathcal{C})$) if it is isomorphic to the functor h_A.

Examples 1. The functor $\underline{X}$ from section 2, Example 4 is representable by the quotient ring $k[T]/((F_s)$.
2. Let $\mathcal{C} = Mod_R^{\circ}$. Consider the functor $M \to M^{\oplus n}$. It is representable by the free module R^n. Indeed there is a natural isomorphism

$$f_M : Mor_{\mathcal{C}}(R^n, M) = Hom_R(R^n, M) \cong M^n, \phi \to (\phi(e_1), \ldots, \phi(e_n)),$$

where $(e_1, \dots, e_n)$ is the standard basis of M^n.

Let $(X_i)_{i\in I}$ be a family of objects in $\mathcal{C}$. Consider the functor $\mathcal{F}(Z) = \prod_{i\in I} Mor_{\mathcal{C}}(Z, X_i)$. Here we use the usual definition of cartesian products of sets. If this functor is representable by an object A, then we have, for any Z a bijection

$$Mor_{\mathcal{C}}(Z, A) = \prod_{i\in I} Mor_{\mathcal{C}}(Z, X_i).$$

Thus any element $(..., f_i, ..)$ of the RHS defines a unique morphism $f : Z = A$. The image of 1_A defines the maps $p_i : A \to X_i$ such that $f_i = p_i \circ f$. We denote in this case A by $\prod_{i\in I} X_i$ and call it the *direct product* of the family of objects X_i. It is defined uniquely up to isomorphism. A category is called a category with products (resp. with finite products) if products exist for any families (resp. finite families).

Dually one defines the notion of the *direct sum* $\coprod_{i\in I}$ of objects.

Examples 3. The direct products in $(\mathcal{S}ets)$ coincide with cartesian products. The direct sum correspond to disjoint unions.
4. The finite direct products and arbitrary direct sums in Mod_R coincide with usual direct sums of modules.
5. In the category of $k-algebras$ the tensor product is the direct sum. The finite direct products are the direct products of algebras.
6. In $\check{\mathcal{C}}$ direct products and direct sums exist and are defined by taking the direct products and direct sums of the value sets. This allows one to define the direct product of objects in any category as an object in $\check{\mathcal{C}}$ by considering $\mathcal{C}$ as a full subcategory of $\check{\mathcal{C}}$.
7. Let S be an object of $\mathcal{C}$. Consider the category $\mathcal{C}/S$ whose objects are morphisms $X \to S$ in $\mathcal{C}$ and morphisms $(f : X \to S) \to (g : Y \to S)$ are morphims $u : X \to Y$ such that $g \circ u = f$. The object $1_S : S \to S$ is the *final object* of $\mathcal{C}$, i.e. an object e of a category such that for any other object there exists a unique morphism to e. Direct products in $\mathcal{C}/S$ are called *fibred products* over S. If $\mathcal{C} = (\mathcal{S}ets)$, the product of $f_i : X_i \to S$ is the pre-image of the diagonal $\Delta_S \subset \prod_{i\in I} S$ under the product of the maps $\prod_{i\in I} f_i : \prod_{i\in I} X_i \to \prod_{i\in I} S$.
8. In category (Top) the direct products are usual cartesian products equipped with the product topology.

5. Structures on objects of a category.

The Yoneda functor $\mathcal{C} \to \check{\mathcal{C}}$ allows one to define a structure on an object in $\mathcal{C}$. Let $\mathcal{C}'$ be some subscategory of $(\mathcal{S}ets)$ whose objects are sets with some additional structure. We say that $X \in Ob(\mathcal{C})$ is $\mathcal{C}'$-object if h_X is a $\mathcal{C}'$-preseheaf. For example, we may speak about a group object in $\mathcal{C}$. It is an object such that all sets $h_X(A) = Mor_{\mathcal{C}}(A, X)$ are equipped with a group structure such that for any morphism $A \to B$ in $\mathcal{C}$ the corresponding map of sets $h_X(B) \to h_X(A)$ is a homomorphism of groups. Here are some examples:

Example 1. Let $\mathcal{C} = Rings^{\circ}$. Consider the functor $R \to R^*$. Then it is represented by the ring $\mathbb{Z}[t, t^{-1}]$ since

$$Mor_{\mathcal{C}}(\mathbb{Z}[t, t^{-1}], R) \to R^*, \quad \phi \to \phi(t)$$

is a functorial bijection. Thus $\mathbb{Z}[t, t^{-1}]$ is a group-object in the category $Rings^{\circ}$.

Example 2 The ring $\mathbb{Z}[t]$ is a ring-object in the category $(Rings)$. Indeed $Mor_{\mathcal{C}}(\mathbb{Z}[t, t^{-1}], R) = R$.

If $\mathcal{C}$ admit finite products and contains a final object e then one can give an equivalent definition of a group-object by requiring the existence of morphisms $\mu : X \times X \to X, \beta : X \to, \alpha : e \to X$ satisfying some natural commutative diagrams expressing the properties of associativity, the existence of inverse and the existence of the neutral element. In fact, the existence of group structure on each $h_X(A)$ defines a map $h_X(A) \times h_X(A) \to h_X(A)$. Using the Yoneda theorem, it defines a morphism $X \times X \to X$.

6. Inductive and projective limits.

Let $F : \mathcal{I} \to \mathcal{C}$ be a functor. For any $A \in Ob(\mathcal{C})$ denote by F_A the *constant functor* $F_A(i \to j) = 1_A$. Consider the functor on $\mathcal{C}$ by $I(A) = Mor_{Funct(\mathcal{I},\mathcal{C}}(F, F_A)$. An element of $I(A)$ is a set of maps $f_i : F(i) \to A$ defined for each $i \in Ob(\mathcal{I})$ such that $f_i = f_j \circ \phi$, where $\phi : i \to j$ is a morphism in $\mathcal{I}$. If $I(A)$ is representable, then the representing object is called the *inductive limit* of F. The dual notion is the *projective limit.*

In many applications, $\mathcal{I}$ is the category defined by a partially ordered set I identified with a graph with arrows corresponding to $i \le j$. A functor $F : \mathcal{I} \to \mathcal{C}$ is called in this case an *inductive system*. The inductive limit is denoted by

$$\lim_{i \in I} \operatorname{ind} F_i = \varinjlim_{i \in I} F_i.$$

The dual notion is a *projective system*. The projective limit is denoted by

$$\lim_{i \in I} \operatorname{proj} F_i = \varprojlim_{i \in I} F_i.$$

Here are examples:

Example 1. I with the trivial order $i \le j$ iff $i = j$. Then F is defined by the set $X_i = F(i)$ of objects in $\mathcal{C}$. The inductive limit is the direct sum, the projective limit is the direct product. By definition the direct sum (resp. direct product) of the empty set I is the *cofinal* or *initial* (resp. *final*) object of the category, i.e. $\#\mathcal{C}(e, X) = 1$ for any $X \in Ob(\mathcal{C})$, (resp. $\#\mathcal{C}(X, f) = 1$ for any $X \in Ob(\mathcal{C})$).

Example 2. Take I with a minimal element i_0 and no other elements are comparable by order. Then we get the notion of fibred products and fibred sums. Another special case is when $\mathcal{I}$ consists of two elements with two morphisms. The functor $F\mathcal{I} \to \mathcal{C}$ is defined by two arrows in $\mathcal{C}$: $A \rightrightarrows B$. The projective limit is called the *equalizer* of the pair of morphisms. The dual notion is the *co-equalizer*.

Example 3. Let $I = \mathbb{N}$ with the usual order. Let $\mathcal{C} = Com$. Fix a prime number p. Consider the functor $F(n) = \mathbb{Z}/p^n$ with $F(n) \to F(m), m \le n$, equal to the natural factor map $\mathbb{Z}/p^n \to \mathbb{Z}/p^m$. The direct limit is the ring $\mathbb{Z}_p$ of p-adic number. The inductive limit is the p-torsion subgroup of $\mathbb{Q}/\mathbb{Z}$.

Example 4. Inductive and projective limits of inductive systems exist if $\mathcal{C} = (\mathcal{S}ets)$. To construct the projective limit we define $\lim_{\mathcal{I}} F$ as the subset of $\prod_{i \in I} F(i)$ which consists of string $(..., a_i, ..)$ such that $(F(j) \to F(i))(a_i) = a_j$ if $j \le i$. The inductive limit is defined as the quotient of the set $\cup_{i \in I} F(i)$ by the minimal equivalence relation containing the relation $(a_i, a_j) \in R$ if the previous equality holds.

Theorem 1. *The following properties of a category $\mathcal{C}$ are equivalent:*
(i) $\mathcal{C}$ has projective limits;
(ii) $\mathcal{C}$ has direct products and equalizers;
(iii) $\mathcal{C}$ has direct products and fibred products of two objects.

Proof. Assume direct products exist. First notice that equalizers of a pair $A \rightrightarrows B$ is the fibred product $A \times_{A \times B} A$, where the two morphisms $A \to A \times B$ are $1_A \times f$ and $1_A \times g$. To show that $\lim \operatorname{proj}_I d$ exists if (ii) holds we construct a pair of morphisms

$$\prod_{i \in I} d(i) \quad \rightrightarrows \prod_{u \in Mor(\mathcal{C})} d(end(u)),$$

where the first arrow is defined by the projections $\prod \to d(end(u))$ and the second one by the composition of projection to $d(source(u))$ and the morphism $d(u) : d(source(u)) \to d(end(u))$. It is immediately seen that the the projective limit is isomorphic to the equalizer of this pair.

We leave to the reader to state the dual statement about the inductive limits.

Definition We say a $F : \mathcal{C} \to \mathcal{C}'$ is *left exact* (resp.*right exact*) if it commutes with finite projective (resp. inductive) limits. A functor which is left and right exact is called *exact*.

If $F : \mathcal{C}^\circ \to \mathcal{C}'$ is a contravariant functor. Then it is left exact if it transforms inductive limits in $\mathcal{C}$ into projective limits.

Corollary. *A functor $F : \mathcal{C} \to \mathcal{C}'$ is left exact if and only if it commutes with direct products and equalizers. In other words, if*

$$F(\prod_{i\in I} X_i) = \prod_{i\in I} F(X_i),$$
$$F(Ker(A \rightrightarrows B)) = Ker(F(A) \rightrightarrows F(B)).$$

Example 5. Let $\mathcal{F}$ be a presheaf on the category $Open(X)$. Let us add to $Open(X)$ direct sums (for example, by embedding it into the category of presheafs where direct sums exist). Let $U_i \to U$ be an open covering of an open set U. Then U is the co-equalizer of the pair

$$\coprod_{(i,j)\in I\times I} U_i \cap U_j \rightrightarrows \coprod_{i\in I} U_i \to U.$$

Here we embed $U_i \cap U_j$ in U_i and U_j to define the two arrows. A presheaf is called a sheaf if it is left exact. This means that it defines an exact sequence of sets

$$\mathcal{F}(U) \to Ker(\prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j).$$

The exactness means that $\mathcal{F}(U)$ is the equalizer of the the pair of maps. For example, the presheaf $U \to C(U)$ of continuous functions is a sheaf.

Remark 1. Commute with the direct sums of empty set of indices means that the functor sends the cofinal object to the cofinal object.

We leave to the reader to state the dual statement about the right exactnedness.

Corollary. *Suppose that $\mathcal{C}$ has inductive limits. Then $h_X : \mathcal{C} \to (\mathcal{S}ets)$ is right exact.*

Proof. It is enough to check that h_X commutes with direct sums and fibred sums. We have, by definition of the direct sum,

$$h_X(\coprod_{i\in I} Y_i) = \prod_i h_X(Y_i),$$

and similarly

$$h_X(A \coprod_C B) = h_X(A) \times_{h_X(C)} h_X(B).$$

This gives a necessary condition for representability of a presheaf of sets:

Theorem 2. *Suppose $\mathcal{F} \in \check{\mathcal{C}}$ is a representable functor and $\mathcal{C}$ has inductive limits. Then $\mathcal{F}$ is right exact.*

Let us see how to reconstruct a presheaf by its values using inductive limit. For any $F \in \check{\mathcal{C}}$ consider teh category $\mathcal{C}/F$ of objects of C over F (here we identify $\mathcal{C}$ with $\check{\mathcal{C}}$ using the Yoneda functor). Let $d : \mathcal{C}/F \to \check{\mathcal{C}}$ be the forgetful of the arrow functor $\mathcal{C}/F \to \check{\mathcal{C}}$.

Proposition 1.

$$\lim_{\mathcal{C}/F} \operatorname{ind} d \cong F.$$

(or symbolically:

$$F = \lim_{X\to F} \operatorname{ind} h_X.)$$

Proof. By definition, for any $(u : X \to F) \in Ob(\mathcal{C}/F)$ there is a unique morphism $d(u) \to F$. The set of such morphisms defines a morphism ϕ from the inductive limit to F. Suppose G is an object of $\check{\mathcal{C}}$ such that for any u as above there is a morphism $d(u) = h_X \to G$ (of course satifying the functoriality condition). Let

$x \in F(X)$ correspond to some $u : X \to F$ and let $y \in G(X)$ corresponds to $d(u)$. It is easy to check that the family of maps of sets $f_X : x \to y$ defines a morphism of presheaves $F \to G$. Taking G to be the inductive limit, we get a morphism f from G to the inductive limit. It is easy to see that this is the inverse of ϕ.

7. Adjoint functors.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor. For any $B' \in Ob(\mathcal{C}')$ consider the functor $X \to Mor_{\mathcal{C}'}(F(A), B')$. Suppose this functor is representable. Denote the representing object by $G(B')$. Then we have a bijection

$$Mor_{\mathcal{C}}(A, G(B')) \to Mor_{\mathcal{C}}(F(A), B')$$

which must be "functorial" in A and B'. It is easy to see that the correspondence $B' \to G(B')$ is functor on $\mathcal{C}'$. It is called the *right adjoint functor* of F. The functor F is called the *left adjoint* of G. To say more symmetrically, we introduce the product of the categories $\mathcal{C} \times \mathcal{C}'$. Suppose that we have two functors $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}$. Consider the functors $\mathcal{C} \times \mathcal{C}' \to (\mathcal{S}ets)$ defined by

$$Mor_{\mathcal{C}}(?, G(?)) : (A, B') \to Mor_{\mathcal{C}}(A, G(B')), \quad Mor'_{\mathcal{C}}(F(?), ?) : (A, B') \to Mor'_{\mathcal{C}}(F(A), B').$$

We require that there exists an isomorphism of these functors:

$$u : Mor_{\mathcal{C}}(?, G(?)) \to Mor'_{\mathcal{C}}(F(?), ?).$$

In particular, taking $B' = F(A)$ we have a bijective map $Mor_{\mathcal{C}}(A, G \circ F(A) \to Mor'_{\mathcal{C}}(F(A), F(A)$. The pre-image of $1_{F(A)}$ defines a morphism $A \to G \circ F(A)$. Similarly, we get a morphism $F \circ G(B') \to B'$. It is easy to see that, varying A and B' we get the morphisms of functors

$$G \circ F \to Id_{\mathcal{C}}, \quad F \circ G \to Id_{\mathcal{C}'}$$

The pair of such morphisms is called the *adjunction morphisms*. Here are some examples.

Examples 1. Let $\mathcal{C} = Mod_R$. The functor $A \to M \otimes A$ is adjoint to $B \to Hom(B, M)$.
2. Let $HTop$ bethe subcategory of Top which consists of Hausdorff spaces. Let F be the inclusion functor. The adjoint functor assigns to any topological space the maximal Hausdorff factor-object (see exercise).
3. Let $F : \mathcal{C} \to (\mathcal{S}ets)'$ is a forgetful functor. The adjoint functor gives a construction of a free object in the category. For example, $\mathcal{C} = (Groups)$ is the category of groups. The adjoint of the forgetful functor assigns to a set S the free group with tyhe set of generators S. If $\mathcal{C} = Com$, the adjoint functor assigns toa set S the polynomial ring in the set of variables indexed by S. If $\mathcal{C} = Top$, then the adjoint functor assigns to a set S the discrete topological space S.
4. Let $\mathcal{C} = HTop_c$ be the category of compact Hausdorff topological spaces. The forgetful functor $HTop_c \to HTop$ has a right adjoint. This is the Stone-Chech compactification. It is defined as follows. One embeds X in the space $\mathbb{R}^{C(X)}$ by sending x to the map $f \to f(x)$. Then one compactifies X by taking the closure of X in $\mathbb{R}^{C(X)}$, where the latter has the standard product topology.
5. Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor. For any presheaf of sets $\mathcal{F} : \mathcal{C}^\circ \to (\mathcal{S}ets)$ we can define the inverse image $F^*(\mathcal{F}) \in \check{\mathcal{C}}$ as the composition $\mathcal{F} \circ F$. For example, if $f : X' \to X$ is a continuous map of topological spaces, it defines the functor $Open(X) \to Open(X')$ by $U \to f^{-1}(U)$. If $\mathcal{F}$ is a presheaf on X' (i.e. a presheaf of sets on $Open(X')$, its inverse image is the presheaf $f_*(\mathcal{F})$ (called the direct image under f) defined by

$$f_*(F)(U) = \mathcal{F}(f^{-1}(U)).$$

This defines the functor:

$$F_* : \check{\mathcal{C}}' \to \check{\mathcal{C}} : \mathcal{F}' \to F_*(\mathcal{F}).$$

For example if $\mathcal{O}_{X'}$ is the presheaf of continuous functions (i.e. $\mathcal{O}_{X'}(U') = C(U')$), then we have a natural morphism of presheaves $\mathcal{O}_X \to f_*(\mathcal{O}_{X'})$ which is obtained by composing a continuous function $\phi : U \to \mathbb{R}$ with f.

6. A presheaf $\mathcal{F} : \mathcal{C}^\circ \to (\mathcal{S}ets)$ has the right adjoint G if $\mathcal{C}(S, \mathcal{F}(X)) = \mathcal{C} \circ (G(S), X)$. Taking S to be a singleton, we obtain $\mathcal{F}(X) = \mathcal{C}(X, G(S))$. This means that $G(S)$ represents $\mathcal{F}$. So, if $\mathcal{F}$ is not representable, $\mathcal{F}$ does not have left adjoint.

The left adjoint of the functor $\mathcal{F} \to u^*(\mathcal{F})$ is the functor

$$u_! : \check{\mathcal{C}} \to \check{\mathcal{C}}' : \mathcal{F}' \to u_!(\mathcal{F}')$$

In the example above, $u_!(\mathcal{F})$ is denoted by $f^{-1}(\mathcal{F})$ (the inverse image under f) and is constructed as follows. Let V be an open set on X'. Consider the family of open subsets U in X such that $V \subset f^{-1}(U)$. It is an inductive system with respect to inclusions. So, we can define the inductive limit $\lim \mathcal{F}(U)$. We set $f^{-1}(\mathcal{F})(V)$ equal to this limit. In the case of general categories, we do the same. Let $\mathcal{C} \to \mathcal{C}'$ be a functor. For any $X \in \mathcal{C}'$ we consider the following subcategory I_X^u of $\mathcal{C}$. Its objects are morphisms $s : X \to u(Y) \in Ob(\mathcal{C}')$, where $Y \in Ob(\mathcal{C})$. The morphisms in this category are morphisms $\phi : Y \to Y'$ in $\mathcal{C}'$ such that such $s = u \circ s'(\phi)$. Define a functor $I_X^u \to (\mathcal{S}ets)$ by $(X \to u(Y)) \to \mathcal{F}(Y)$. The inductive limit of this functor is taken for $u_!(\mathcal{F})(X)$. In other words,

$$u_!(\mathcal{F})(X) = \liminf_{X \to u(U)} F(U).$$

The fact that it is a contravariant functor follows from the observation that for any morphism $X \to X'$ in $\mathcal{C}'$ one has a natural functor $I_{X'}^u \to I_X^u$ obtained by composition $(X \to X' \to u(Y)$. Note that the inductive limit always exists, since in $(\mathcal{S}ets)$ inductive limits exist. Also observe that $F(U) = \emptyset$ if I_X^u is the empty category. This follows from understanding that the union of sets parametrized by the empty set is the empty set. Let $\mathcal{G}$ be a presheaf on C. Suppose we have a morphism $\mathcal{F} \to u^*\mathcal{G}$. Then for any $Y \to F(X)$, we have $\mathcal{F}(Y) \to \mathcal{G}(F(Y)) = u^*(\mathcal{G})(Y) \to \mathcal{G}(X)$. By definition of inductive limits, there is a unique morphism $u_!(\mathcal{F})(Y) \to \mathcal{G}(Y)$. This shows that

$$\check{\mathcal{C}}'(u_!(\mathcal{F}), \mathcal{G}) = \check{\mathcal{C}}(\mathcal{F}, u^*(\mathcal{G})).$$

Hence u^* is right adjoint to $u_!$.

Example 7. Let us compute $u_!(h_X)$, where $X \in Ob(\mathcal{C})$. Let Z be an object of $\mathcal{C}'$ and $Y \to X$ be a morphism in $\mathcal{C}$. For any $f : Z \to u(Y)$ in I_Z^u we have the composition map

$$h_X(Y) \xrightarrow{u} h_{u(X)}(u(Y)) \xrightarrow{h_{u(X)}(f)} h_{u(X)}(Z).$$

This defines a map from $u_!(h_X)(Z)$ to $h_{u(X)}(Z)$. Any morphism $Z \to u(X)$ can be considered as an object of the category I_Z^u, so that we have the inverse map $h_{u(X)}(Z) \to u_!(h_X)(Z)$. Thus we have constructed a bijection $u_!(h_X)(Z) \to h_{u(X)}(Z)$ which as is easy to see defines an isomorphism of presheaves:

$$u_!(h_X) \cong h_{u(X)}.$$

Let us construct the right adjoint functor to u^*. It is denoted by u_* and must have the property

$$\check{\mathcal{C}}(u^*(\mathcal{F}), \mathcal{G}) = \check{\mathcal{C}}'(\mathcal{F}, u_*(\mathcal{G})).$$

By setting $\mathcal{F} = h_X$, we get

$$\check{\mathcal{C}}(u^*(h_X), \mathcal{G}) = \check{\mathcal{C}}'(h_X, u_*(\mathcal{G})) = u_*(\mathcal{G})(X).$$

So, we take for the definition

$$u_*(\mathcal{F})(X) = \check{\mathcal{C}}(u^*(h_X), \mathcal{F}).$$

In the example $u = f^{-1} : Open(Y) \to Open(X)$ we have, for every $U \in Open(X)$, and any $V \in Open(Y)$,

$$u^*(h_U)(V) = h_U(u(V)) = h_U(f^{-1}(V)) = Open(X)(f^{-1}(V), U).$$

The latter is not empty if and only if $U = f^{-1}(V')$, where $V \subset V'$. Thus, $u_*(\mathcal{F})(U) = \emptyset$ if U is not as above and $u_*(\mathcal{F})(U) = \liminf_{V \subset V'} \mathcal{F}(V)$, if $U = f^{-1}(V')$.

Theorem 1. *Suppose $F : \mathcal{C} \to \mathcal{C}'$ admits a right adjoint functor $G : \mathcal{C}' \to \mathcal{C}$. Then F is right exact and G is left exact.*

Proof. Let $d : \mathcal{I} \to \mathcal{C}$ be a functor and $X = \lim_{\mathcal{I}} d$. By definition we have a morphism of functors $d \to X$. Applying F we get a morphism of functors $F \circ d \to F(X)$. Suppose we have another morphism of functors $F \circ d \to Y$. We have to show that there exists a morphism $F(X) \to Y$ making the diagram commutative. Applying G we get morphisms $d \to GFd \to G(Y)$. By definition of inductive limit, there is a morphism $X \to G(Y)$. Hence we have $F(X) \to FG(Y) \to Y$. Similarly we prove the second assertion.

Corollary 1. *The functor $u_!$ commutes with inductive limits and the functors u^*, u_* commute with projective limits.*

8. Grothendieck topologies

Let $\mathcal{C}$ be a category. A *sieve* in $\mathcal{C}$ is a full subcategory $\mathcal{D}$ of $\mathcal{C}$ such that $(X \to Y) \in \mathcal{C}(X, Y)$, where $Y \in Ob(\mathcal{C})$, implies $X \in Ob(D)$. A sieve of an object X of $\mathcal{C}$ is a sieve of the category $\mathcal{C}/X$.

Equivalently, a sieve R of X is a subobject of h_X, we have $h_X(Y)$ is the set of morphisms $Y \to X$ from the sieve. Conversely, if R is a subobject of h_X in $\mathcal{C}^\circ$ we consider the subcategory $\mathcal{C}/R$ of all arrows $Y \to R$. This is a sieve of X and the two maps from sieves of X to subobjects of h_X are inverse to each other.

If $F : \mathcal{C} \to \mathcal{C}'$ is a functor, and $\mathcal{D}$ is a subcategory of $\mathcal{C}'$ then its inverse image $F^*(\mathcal{D})$ is the full subcategory of $\mathcal{C}$ which consists of objects A such that $F(A) \in Ob(\mathcal{D})$ is a morphism in $\mathcal{D}$. If R is a sieve in $\mathcal{C}'$, then its inverse image is obviously a sieve in $\mathcal{C}$. If $F/X : \mathcal{C}/X \to \mathcal{C}'/F(X)$ the induced functor, then the inverse image of a sieve of $F(X)$ is a sieve of X.

We shall identify objects of $\mathcal{C}$ with presheaves on $\mathcal{C}$ which they represent.

Definition A topology in $\mathcal{C}$ is a set of sieves $J(X)$ of X, for each object X of $Ob(\mathcal{C})$. The following axioms must satisfy:

(T1) If $Y \to X$ is a morphism in $\mathcal{C}$ and $R \in J(X)$, then the set of morphisms $Z \to Y$ such that the composition $Z \to Y \to X$ belongs to R forms a sieve in $J(Y)$;

(T2) if $R \in J(X)$ and R' is a sieve of X such that for any $Y \to X$ in R the set of morphisms $Z \to Y$ such that the composition $Z \to Y \to X$ belongs to R' forms a sieve in $J(Y)$ then R' is a sieve in $J(X)$;

(T3) $I_X \in J(X)$.

A category with a topology is called a *site*. A functor $F : \mathcal{C} \to \mathcal{C}'$ between two sites is called *continious* or a *morphism of sites* if for every sieve $R \in J(X)$ in $\mathcal{C}$ the morphism $F_!(R) \to F_!(h_X) = h_{F(X)}$ factors through an isomorphism $F_!(R) \to R'$, where $R' \in J(F(X))$.

A way to define a topology is by considering for each $X \in Ob(\mathcal{C})$, a set of coverings $U_i \to X, i \in I$. Each covering defines a sieve, the subfunctor of h_X equal to the union of the images of the subpresheaves h_{U_i}. Thus $Y \to X$ belongs the sieve if $h_Y \to h_X$ through the union of h_{U_i}. To define a topology using families of coverings $Cov(X)$ of X we must require the following properties:

(PT1) If $Y \to X$ and $(U_i \to X)_{i \in I} \in Cov(X)$, then $U_i \times_X Y \to Y$ belongs to $Cov(Y)$;

(PT2) if $(U_i \to X)_{i \in I} \in Cov(X)$ and for each U_i we have a covering $(V_{ij} \to U_i) \in Cov(U_i)$, then $(V_{ij} \to U_i \to X)$ belongs to $Cov(X)$;

(PT3) the morphism 1_X belongs to $Cov(X)$.

A topology defines a family of coverings, i.e. a family of morphisms $U_i \to X$ such they span a sieve from $J(X)$. If $\mathcal{C}$ has fibred products, then a topology can be defined by a pre-topology.

Example 1. If $J(X)$ is the set of all sieves of X we get a discrete topology. If we take $J(X) = \{1_X\}$, we get the "chaotique" topologie.

To give the next example we have to say a liitle more about inductive limits. Let $Funct(I, \mathcal{C})$ be the category of functors. Let J be another category together with a functor $d : J \to Funct(I, \mathcal{C})$ (equivalently, we have a bifunctor $D : I \times J \to \mathcal{C}$). Suppose that projective limits of the functors $d_i : J \to \mathcal{C}, j \to D(i \times j)$ are representable. Also assume that the functor $I \to \mathcal{C}$ defined by $i \to \lim\operatorname{proj} d_i$ is representable. Then D has projective limit and

$$\lim_{I\times J}\operatorname{proj} D \cong \lim_{I}\operatorname{proj}\lim_{J}\operatorname{proj} d_i,$$

and hence

$$\lim_{I}\operatorname{proj}\lim_{J}\operatorname{proj} d_i \cong \lim_{J}\operatorname{proj} I \lim_{I}\operatorname{proj} d_j.$$

We leave the proof to the reader. A special corollary of this is the following. Suppose $A = Ker(A_1 \rightrightarrows A_2)$, $B = Ker(B_1 \rightrightarrows B_2)$. Suppose we have two morphisms between these two pairs of arrows $A_1 \rightrightarrows B_1$ and $A_2 \rightrightarrows B_2$(think about them as two functors from $i \rightrightarrows j$ to $\mathcal{C}$). Then there are two morphisms from the equalizers $A \rightrightarrows B$ and we have

$$Ker(A \rightrightarrows B) \cong Ker(Ker(A_1 \rightrightarrows B_1) \rightrightarrows Ker(A_2, B_2)).$$

Now we are ready to give our next example:

Example 2. Any category $\mathcal{C}$ with fibred products can be equipped with a *canonical topology* $T\mathcal{C}$, its covering families $(U_i \to U)$ are universal effective epimorphisms, i.e. families such that for any object X of $\mathcal{C}$

$$h_X(U) = Ker(\coprod h_X(U_i) \rightrightarrows \coprod (h_X(U_i \times U_j)).$$

We require that, for any $V \to U$ the family $(U_i \times_U V \to V)$ also satisfies this property. This checks axiom (PT3) automatically. Axiom (PT2) follows from we have said before. Axiom (PT1) is obviously checked.

One can also describe the canonical topology in terms of sieves. A canonical topology is the finest topology such that all representable presheaves are sheaves. A sieve in this topology is called a strict universal epimorphism. For any covering crible of an object X in the canonical topology we must have, for any object Z, a canonical bijection $h_Z(X) \to \check{\mathcal{C}}(R, h_Z)$. Also for any $Y \to X$ the sieve $Y \times_X R$ must be a covering sieve of Y so that there is a canonical bijection $h_Z(Y) \to \check{\mathcal{C}}(Y \times_X R, h_Z)$. One can show that the set of sieves R satisfying these two properties form the canonical topology, i.e. axiom (T2) holds (the other axioms are satisfied by definition).

Example 3. Let X be a topological space, $\mathcal{C} = Open(X)$. For every open set U let $Cov(U)$ be the set of open coverings of U. Clearly all the axioms of pre-topology are satisfied. Any continuous map $f : X \to Y$ of topological spaces defines a morphism of the corresponding sites $U \to f^{-1}(U)$.

Example 4. Let us define a topology on a set X. First consider a subcategory $\mathcal{C}$ of the category of subsets of X. It must contain X and (a final and initial object). It must be closed under finite products. As usually we extend it in $\check{\mathcal{C}}$ to contain direct sums. Then we require that it contains co-equalizers. Then it is clear that its objects can be taken as open sets in a topology on X and also as direct sums of open sets. Now if we take the canonical topology on this category, we see that the obtained topology is the same as in the previous example.

Example 5. Let X be a topological space. Assume that X is linearly connected and each point has a neighborhood with trivial fundamental group. A continuous map $Y \to X$ is called *étale* if for any point x in the image Y $x \in X$ there exists a neighborhood $U(x)$ such that $f^{-1}(U(x)) = \coprod_{i\in I} V_i$, where the restriction of f to each V_i is a homeomorphism. It is easy to see that $f(Y)$ is open in X. We define the étale topology on X as a topology on the full subcategory $Et(X)$ of Top/X formed by étale maps. This category has fibred products, and we define the pre-topology by taking $Cov(U)$ to be a set of morphisms $f_i : U_i \to U$ in $Et(X)$ such that $\coprod_{i\in I} U_i \to U$ is surjective. Note that each f_i is necessary etale. We leave to the reader to check that this defines a pre-topology. It is called the étale topology of X. For any etale map $U \to V$ and any continuous map $Z \to V$ the pull-back morphism $U \times_V Z \to Z$ is etale. This easily shows that a continuous map $f : X \to Y$ defines a morphism of the étale sites.

Example 6. Let G be a group and $G-(\mathcal{S}ets)$ be the category of G-sets. Equip it with the canonical topology. Here $Cov(U)$ are all surjective families of morphisms $U_i \to U$.

Consider the etale site $Et(X)$ of a topological space X. Define the following functor $F : Et(X) \to \pi_1(X) - (\mathcal{S}ets)$. Here $\pi_1(X)$ is the fundamental group of X. Fix a point $x_0 \in X$. The functor F assigns to an etale cover $f : Y \to X$ the fibre $f^{-1}(x_0)$. If $\gamma : [0,1] \to X$ is a loop with $\gamma(0) = \gamma(1) = x_0$ representing an element g of $\pi_1(X, x_0)$. By the theorem of covering homotopy, for any $y \in f^{-1}(x_0)$ there is a path $\tilde{\gamma} : [0,1] \to Y$ such that $\tilde{\gamma}(0) = y, \gamma(1) = y' \in f^{-1}(x_0)$. One can show that y' depends only on y and the homotopy class of γ. This defines an action of $\pi_1(X, x_0)$ on $F(Y)$. If $g \circ f : Y' \to Y \to X$ is a composition in $Et(X)$ then $(g \circ f)^{-1}(x_0)$ is mapped π_1-equivariantly to $g^{-1}(x_0)$. So the functor is covariant. It is easy to see that it maps covering families in $Et(X)$ to covering families in $\pi_1 - (\mathcal{S}ets)$. The corresponding sites are in fact isomorphic. Let $\tilde{X}$ be the universal cover of X. Given a π_1-set S we constsruct an etale cover as the quotient topological space $\tilde{X} \times_{\pi_1} S = \tilde{X} \times S / \sim$, where $(x, s) \sim (x', s')$ if $x' = gx, s' = gs$ or some $g \in \pi_1$. We equip S with discrete topology and then put the quotient topology.

Example 7. Let k be a field, and $Sep(k)$ be the full subcategory of the category of commutative k-algebras which are finite separable extensions of k or their direct sums. Equip it with canonical topology. This site is isomorphic to the canonical site of the category of finite sets on which the Galois group $Aut(\bar{k}/k)$ acts with the kernel of the action of finite index.

Example 8. Let $\mathcal{C}$ be the category dual to the category of commutative rings. A homomorphism of rings $f : A \to B$ is called *flat* if for any ideal I of A the natural homomorphism of B-modules $I \otimes_A B \to A \otimes_A B = B$ is injective. It is called strictly flat if additionally $M \otimes_A B \neq 0$ for any nonzero A-module M. Equivalently, a sequence of A-modules $N \to M \to L$ is exact in M if and only if the sequence $N \otimes_A B \to M \otimes_A B \to L \otimes_A B$ is exact in $M \otimes_A B$ (show that that

For example any homomorphism of fields is strictly flat. I claim that a strictly flat homomorphism f defines a strict epimorphism in the category Com°. In other words, if

$$A \cong Ker(B \rightrightarrows B \otimes_A B).$$

Recall that this means that for any ring C

$$Hom(C, A) = Ker(Hom(C, B) \rightrightarrows Hom(C, B) \times_{Hom(C,A)} Hom(C, B).$$

Consider the following sequence

$$0 \to A \to B \to B^{\otimes 2} \to \ldots \to B^{\otimes r} \to \ldots,$$

where $d_r : B^{\otimes r} \to B^{\otimes r+1}$ is defined as

$$d_r(b_1 \otimes \ldots \otimes b_r) = \sum_{i=1}^{r} (-1)^{r+1} b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_{i+1} \otimes \ldots b_r.$$

It is immediately checked that this is a complex, i.e. $d_{r+1} \circ d_r = 0$. Assume $A \to B$ admits a right inverse, i.e. a homomorphism $g : B \to A$ such that $g \circ f = id$. Then we can construct the homomorphisms $k_r : B^{\otimes r+1} \to B^{\otimes r}$ such that $k_{r+1} d_{r+1} + d_r k_r = 1$. Just set

$$k_r(b_1 \otimes \ldots \otimes b_{r+1}) = g(b_1) b_2 \otimes \ldots \otimes 1 \otimes b_i \otimes \ldots b_{r+1}.$$

A standard argument (used for example in the calculus of differential forms) shows that the complex is exact in this case. Now if if we extend the scalars to B (i.e. tensor the complex by B), we get the complex

$$0 \to B \to B \otimes_A B \to B^{\otimes 3} \to \ldots \to B^{\otimes r} \to \ldots,$$

Since $B \otimes B \to B, b \otimes b' \to bb'$ is the right inverse of $B \to B \otimes B, b \to 1 \otimes b$, we obtain an exact complex. Since $f : A \to B$ is strictly flat, the original complex is exact.

Now $A = ker(d_1 : B \to B \otimes_A B)$ means exactly that $A \to B$ is a strict epimorphism in the category Com°. This gives a description of canonical topology in Com°. Its covering families $A \to B_i$ are flat morphisms such that $A \to \oplus_i B_i$ is strictly flat. This is called the flat topology. If A is a commutative ring we can consider the category $(Com/A)^\circ$ and equip it with the induced flat topology. Abusing the language one speaks about the flat topology of A.

9. Sheaves

Definition. A presheaf of sets $\mathcal{F}$ on a site $(\mathcal{C}, T)$ is called a *sheaf* if for any $X \in Ob(\mathcal{C})$ and any sieve $R \in J(X)$ the canonical map

$$\mathcal{F}(X) = \check{\mathcal{C}}(h_X, \mathcal{F}) \to \check{\mathcal{C}}(R, \mathcal{F})$$

is a bijection.

Recall that $\check{\mathcal{C}}(R, \mathcal{F}$ is given by assigning to each $(Y \to X) \in R(Y) \subset h_X(Y)$ an element of $s_Y \in \mathcal{F}(Y)$ such that if $f : (Y' \to X) \to (Y \to X) \in Mor_{\mathcal{C}/X}$ the map $\mathcal{F}(f) : \mathcal{F}(Y) \to \mathcal{F}(Y')$ sends s_Y to $s_{Y'}$. So the condition of a sheaf requires that the family $\{s_Y\}$ as above originates from a unique $s \in \mathcal{F}(X)$.

Lemma 1. *Let I and $\mathcal{C}$ be two categories. Suppose $\mathcal{C}$ has direct limits of families $X_{\alpha \in U}$ where U is the chosen universe. Assume that I has a family of objects $(i_\alpha)_{\alpha \in A}$, where A is a small set, such that the products $i_\alpha \times i_\beta$ are representable for each pair and each object of I has a morphism to some i_α. Then for any functor $F : I^\circ \to \mathcal{C}$ the projective limits exist and there is a functorial isomorphism*

$$\lim \operatorname{proj} F \cong Ker(\prod_{i \in I} F(i) \rightrightarrows \prod_{(i,j) \in I \times J} F(i \times j)).$$

Proposition. *Suppose $\mathcal{C}$ has fibred products and the topology is defined covering families $Cov(U)$. Then a presheaf is a sheaf if and only if for any $(U_i \to U) \in Cov(U)$ the sequence*

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)} \mathcal{F}(U_i \times_U U_j)$$

is bijective.

Proof. By Proposition 1 from section 4 we have

$$\mathcal{F} = \lim_{\mathcal{C}/\mathcal{F}} h_X.$$

We use the following generalization of Yoneda's lemma: for any $\mathcal{F}, \mathcal{G}$ in $\check{\mathcal{C}}$

$$\check{\mathcal{C}}(F, \mathcal{G}) \cong \lim \operatorname{proj}_{(X,u) \in \mathcal{C}/\mathcal{F}} \mathcal{G}(X).$$

This follows from the fact

$$\check{\mathcal{C}}(\lim_{\mathcal{C}/\mathcal{F}} h_X, \mathcal{G}) = \lim \operatorname{proj}_{\mathcal{C}/\mathcal{F}} \check{\mathcal{C}}(h_X, \mathcal{G}) = \lim \operatorname{proj}_{\mathcal{C}/\mathcal{F}} \mathcal{G}(X).$$

Notice that $\mathcal{C}/R$ consists of morphisms $S \to R$ which are elements of $R(S)$, i.e. morphisms $S \to X$ from R. This is our category I. It contains the objects $U_i \to X$ satisfying the assumptions of the lemma. Thus

$$\check{\mathcal{C}}(R, \mathcal{F}) \cong \lim \operatorname{proj}_{U_i \to U} \mathcal{F}(U_i) = Ker(\prod_{i \in I} F(i) \rightrightarrows \prod_{(i,j) \in I \times J} F(i \times j)).$$

Examples 1. A representable presheaf on a canonical site is a sheaf.
2. A sheaf on a topological space is a sheaf on the site $Open(X)$ with its canonical topology. By definition of a canonical topology, $U \to C(U,Y)$ is a sheaf.
3. Since any representable pre-sheaf on the flat topology of Com° is a sheaf, the presheaf $A \to A^*$ is a sheaf. It is denoted by $\mathbb{G}_m$. Similarly, we have the sheaf $A \to A^+$ denoted by $\mathbb{G}_a$.
4. Let $\mathcal{C} = G - (\mathcal{S}ets)$ with canonical topology. We know that every representable presheaf is a sheaf. Conversely, let $\mathcal{F}$ be a sheaf, and $A = \mathcal{F}(G)$, where G is considered as a G-set with respect to left multiplications. The set A is a G-set, the action is by functoriality. I claim thatΞΞΞΞΞΞΞ A represents $\mathcal{F}$ in $\mathcal{C}$, i.e. for every G-set U we have a natural bijection $\mathcal{F}(U) = \mathcal{C}(U,S) := Maps_G(U,A)$. For every point $x \in U$ let $f_x : G \to U$ be the map $g \to gx$. Its image is the G-orbit of x. Then we have a cover $(f_x : G \to U)_{x \in U}$ in $Cov(U)$. Observe that $(f_x : G \to U) \times_U (f_y : G \to U) = \{(g,g') : gx = g'y\}$. Now $\prod_{x \in U} \mathcal{F}(G)$ can be identified with the set of functions $U \to \mathcal{F}(G)$. The group G acts on this product in two ways: $(g \circ \phi)(u) = \phi(g^{-1}x)$ and ${}^g\phi(x) = g \circ \phi(x)$. To say that ϕ commutes with the action is the same as $g \circ \phi = {}^g\phi$. But this can be also expressed by saying that ϕ belongs to the kernel K of the pair of maps

$$\prod_{x \in U} \mathcal{F}(G) \rightrightarrows \prod_{(x,y) \in U \times U} \mathcal{F}(G \times_U G).$$

By definition of a sheaf there is unique bijective map $\mathcal{F}(U) \to K$.

Theorem 1. *Let $\mathcal{C}$ be a site and $\tilde{\mathcal{C}}$ be the full subscategory of $\check{\mathcal{C}}$ formed by sheaves on $\mathcal{C}$. Let $i : \tilde{\mathcal{C}} \to \check{\mathcal{C}}$ be the inclusion functor. Then there exists a left adjoint functor $a : \check{\mathcal{C}} \to \tilde{\mathcal{C}}$. The functor a is left exact.*

First we set for any $\mathcal{F}, \mathcal{G} \in Ob(\check{\mathcal{C}})$

$$\mathcal{F}(\mathcal{G}) = \check{\mathcal{C}}(\mathcal{G}, \mathcal{F}).$$

By Yoneda's lemma, there is a natural bijection:

$$\mathcal{F}(h_X) \to \mathcal{F}(X).$$

Lemma 1. *Let $\mathcal{C}/\mathcal{G}$ be the full subcategory of $\check{\mathcal{C}}/\mathcal{F}$ which consists of arrows $X = h_X \to \mathcal{G}$, where $X \in Ob(\mathcal{C})$. For any $\mathcal{F} \in \check{\mathcal{C}}$ consider $\mathcal{F}$ as a functor on $\mathcal{C}/\mathcal{G}$ defined by $\mathcal{F}(X \to \mathcal{G}) = \mathcal{F}(X)$. Then*

$$\mathcal{F}(\mathcal{G}) = \lim_{\mathcal{C}/\mathcal{G}} \mathrm{proj}\, \mathcal{F} = \lim_{(X \to \mathcal{G})} \mathrm{proj}\, \mathcal{F}(X).$$

Proof. Let us show first that

$$\mathcal{G} = \lim_{X \to \mathcal{G}} \mathrm{ind}\, h_X.$$

We have to construct a functorial bijection from $\mathcal{G}(A)$ and $\lim \mathrm{ind}_{X \to \mathcal{G}} h_X(A)$ for any $A \in Ob(\mathcal{C})$. An element $u \in \mathcal{G}(A)$ defines a morphism $A \to \mathcal{G}$, an hence an element in the inductive limit of the sets $h_X(A), X \in Ob(\mathcal{C}/\mathcal{G})$ corresponding to the equivalence class of the element $1_A \in h_A(A)$. Conversely, any element from the inductive limit represented by some $a \in h_X(A)$ defines a morphism $A \to X \to \mathcal{G}$. It is easy to see that the two constructed maps are inverse to each other. Now, by definition of projective limit,

$$\mathcal{F}(\mathcal{G}) = \check{\mathcal{C}}(\mathcal{G}, \mathcal{F}) = \check{\mathcal{C}}(\lim_{(X \to \mathcal{G})} \mathrm{ind}\, h_X, \mathcal{F}) = \lim_{(X \to \mathcal{G})} \mathrm{proj}\, \check{\mathcal{C}}(h_X, \mathcal{F}) = \lim_{(X \to \mathcal{G})} \mathrm{proj}\, \mathcal{F}(X).$$

The lemma is proven.

Corollary. *Let R be a sieve of X considered as a subcategory of $\mathcal{C}/X$. Then for any presheaf $\mathcal{F}$,*

$$\mathcal{F}(R) = \lim_{(Y \to X) \in Ob(R)} \mathrm{proj}\, \mathcal{F}(Y).$$

Proof. Consider R as a presheaf, a subpresheaf of h_X. Then $(Y \to X) \in Ob(R)$ if and only if there exists a morphism $h_Y \to R$. Then we apply the previous Lemma to $\mathcal{G} = R$.

Let us consider the set of sieves $J(X)$ as an ordered set, the order is the inclusion of pre-sheaves. Thus we can consider inductive limits over the cofiltering category $J(X)$ (the fact that it is cofiltering, in sense of HW2, Problem 1, follows from the the fact that the set of covering sieves is closed under finite products). For any presheaf $\mathcal{F}$, set

$$\mathcal{F}^{\#}(X) = \liminf_{R \in J(X)} \check{C}(R, \mathcal{F}) = \liminf_{R \in J(X)} \mathcal{F}(R). \qquad (*)$$

If $Y \to X$ is a morphism in $\mathcal{C}$, we have $R \to R \times_X Y$ maps $J(X)$ to $J(Y)$, hence we have a canonical map of sets $\mathcal{F}^{\#}(X) \to \mathcal{F}^{\#}(Y)$. This makes $\mathcal{F}^{\#}$ a presheaf.

Let $R \in J(X)$. Consider the set of sieves R' of X such that each $Z \to X$ from R' factors $Z \to Y \to X$ through some $(Y \to X) \in R$ and the set of such $Z \to Y$ forms a covering sieve of Y. By axiom (T2) of a site, each such R' is covering sieve of X. We denote the set of such covering sieves of X by $J(X; R')$. Note that $R \in J(X; R)$ and any $R' \in J(X; R)$ is a subsieve of R. The following lemma generalizes the definition (*):

Lemma 2. *For any $R \in J(X)$*

$$\mathcal{F}^{\#}(R) = \liminf_{R' \in J(X;R)} \mathcal{F}(R').$$

Proof. First we check that

$$\liminf_{Y \to R} \operatorname{lim\,proj}_{R' \in J(Y)} R' = \operatorname{lim\,proj}_{P \in J(X;P)} P.$$

Since each $P \in J(X; R)$ defines a sieve $R' = P \times_Y X \in J_Y$, the projection to R' defines a map from $\operatorname{lim\,proj}_{R' \in J(Y)} R'$ to P. This gives us a map from $\operatorname{lim\,proj}_{R' \in J(Y)} R'$ to $\operatorname{lim\,proj}_{P \in J(X;P)} P$, and hence the map from the LHS to the RHS. By definition of $J(X, P)$ this map is surjective. By Lemma 1 and (*),

$$\mathcal{F}^{\#}(R) = \operatorname{lim\,proj}_{Y \to R} \mathcal{F}^{\#}(Y) = \operatorname{lim\,proj}_{Y \to R} \liminf_{R' \in J(Y)} \mathcal{F}(R').$$

On the other hand, since $\mathcal{C}(?, T)$ transforms inductive limits to projective and projective to inductive, we have

$$\mathcal{F}(\operatorname{lim\,proj}_{P \in J(X;P)} P) = \mathcal{F}(\liminf_{Y \to R} \operatorname{lim\,proj}_{R' \in J(Y)} R') = \operatorname{lim\,proj}_{Y \to R} \mathcal{F}(\operatorname{lim\,proj}_{R' \in J(Y)} R') =$$

$$\operatorname{lim\,proj}_{Y \to R} \liminf_{R' \in J(Y)} \mathcal{F}(R') = \mathcal{F}^{\#}(R).$$

It is easy to see that

$$L : \mathcal{F} \to \mathcal{F}^{\#}$$

is a functor. By definition of inductive limit, we have morphisms, for every sieve $R \in J(X)$,

$$Z_R : \mathcal{F}(R) \to \mathcal{F}^{\#}(X).$$

Taking the sieve $h_X \in J(X)$, we get a map $\mathcal{F}(X) \to \mathcal{F}^{\#}(X)$. It is easy to see that the set of such maps defines a morphism of functors

$$\ell_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^{\#} = L(\mathcal{F}).$$

Note that

$$\ell_{\mathcal{F}}(X) = Z_{h_X}.$$

Since any $Y \to X$ defines a functor $J(X) \to J(Y), R \to R \times_X Y$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(R) & \longrightarrow & \mathcal{F}^{\#}(X) \\ \downarrow & & \downarrow \\ \mathcal{F}(R \times_X Y) & \longrightarrow & \mathcal{F}^{\#}(Y) \end{array} \qquad (**)$$

Lemma 3. *Let $u \in \mathcal{F}(R)$ be considered as a morphism of presheaves $R \to \mathcal{F}$ and $Z_R(u)$ as a morphism $h_X \to \mathcal{F}^\#$. The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\ell_{\mathcal{F}}} & \mathcal{F}^\# \\ u\uparrow & & \uparrow Z_R(u) \\ R & \overset{i}{\hookrightarrow} & h_X \end{array}.$$

Proof. We have to check that two morphisms $\ell_{\mathcal{F}} \circ u$ and $Z_R \circ i$ from R to $\mathcal{F}^\#$ are equal. By Lemma 1, it is enough to check that for any $g : Y \to R$ we have $\ell_{\mathcal{F}} \circ u \circ g = Z_R \circ i \circ g$. Let $f : Y \to X$ be the composition $i \circ g$. We have $(R \times_X Y)(Z)$ is the set of morphisms $Z \to X$ in $R(Z)$ which factor through $Z \to Y \to X$. The map $R \times_X Y(Z) \to h_Y(Z), (Z \to X) \to (Z \to Y)$ is injective and has a section $Z \to Y \to X$. Thus it is bijective, and we have an isomorphism of functors $i' : R \times_X Y \to h_Y$. Consider the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\ell_{\mathcal{F}}} & \mathcal{F}^\# \\ u\uparrow & & \uparrow Z_R(u) \\ R & \xrightarrow{i} & h_X \\ g'\uparrow & \nwarrow g & \uparrow f \\ R \times_X Y & \xrightarrow{i'} & h_Y \end{array}.$$

The bottom arrow is an isomorphism. We have

$$Z_{R \times_X Y}(u \circ g') = \ell_{\mathcal{F}} \circ u \circ g.$$

By commutativity of the diagram (**), we have

$$Z_{R \times_X Y}(u \circ g') = Z_R(u) \circ f = Z_R(u) \circ i \circ g = \ell_{\mathcal{F}} \circ u \circ g.$$

This checks what we wanted.

Lemma 4. *The sheaf $\mathcal{F}^\#$ is separated.*

Proof. Suppose there are two different $a, b \in \mathcal{F}^\#(X)$ which define the same element in $\mathcal{F}^\#(R) := \check{C}(R, \mathcal{F}^\#)$. By definition of inductive limit, we can find $a' \in \mathcal{F}(R')$ and $b' \in \mathcal{F}(R'')$ such that $Z_{R'}(a') = a, Z_{R''}(b') = b$. Since the set of covering sieves is cofiltering, we may assume that $R' = R'' \subset R$. By the previous lemma (applied to $R = R'$), the composition

$$\mathcal{F}(R') \overset{Z_{R'}}{\to} \mathcal{F}^\#(X) \overset{\mathcal{F}^\#(i)}{\to} \mathcal{F}^\#(R) \to \mathcal{F}^\#(R')$$

is equal to the value of the morphism of the functors $\mathcal{F} \to \mathcal{F}^\#$ at R'. It sends a' and b' to the same element $c \in \mathcal{F}^\#(R')$. By Lemma 2, the image of a' in b' in $\lim\operatorname{ind}_{R'' \in J(X, R')} \mathcal{F}(R'')$ is equal. This means that there exists some $R'' \in J(X; R')$ such that the images of a', b' in $\mathcal{F}(R'')$ are equal. But then $a = b$ in the inductive limit $\mathcal{F}(X) = \lim\operatorname{ind}_{R \in J(X)} \mathcal{F}(R)$. This proves that $\mathcal{F}^\#$ is separated.

Lemma 5. *Assume $\mathcal{F}$ is separated. Then $\mathcal{F}^\#$ is a sheaf.*

Proof. First of all, for any $X \in Ob(\mathcal{C})$, the canonical map $\ell_{\mathcal{F}}(X) : \mathcal{F}(X) \to \mathcal{F}^\#(X)$ is always injective. From this it follows that for any $\mathcal{G} \in Ob(\check{\mathcal{C}})$ the map $\mathcal{F}(\mathcal{G}) \to \mathcal{F}^\#(\mathcal{G})$ is injective. In fact, each map $\mathcal{F}(X) \to \mathcal{F}(R)$ is injective, so $\mathcal{F}(X)$ is mapped injectively into the inductive limit $\mathcal{F}^\#(X)$. We want to show that $\mathcal{F}^\#(X) \to \mathcal{F}^\#(R)$ is surjective for any $R \in J(X)$. By Lemma 2, $\mathcal{F}^\#(R) = \lim\operatorname{ind}_{R' \in J(X;R)} \mathcal{F}(R')$. For each $R' \in J(X; R)$ the composition map $\mathcal{F}(R') \to \mathcal{F}^\#(R) \to \mathcal{F}^\#(R')$ is injective. Since $\mathcal{F}^\#$ is separated, the map $\mathcal{F}^\#(R) \to \mathcal{F}^\#(R')$ is injective. Hence the map $\mathcal{F}(R') \to \mathcal{F}^\#(R)$ is injective. Let $s \in \mathcal{F}^\#(R)$. Pick $u \in \mathcal{F}(R')$ representing s. By the above, the pair (u, R') is defined uniquely. We set $s' = Z_{R'}(u) \in \mathcal{F}^\#(X)$. It is imeediately checked that s' is a lift of s.

Proof Theorem 1. By Lemmas 4 and 5, if we apply L twice, we get a sheaf $LL(\mathcal{F})$. If $\mathcal{F}$ is a sheaf, then $\mathcal{F}^{\#} = \mathcal{F}$. In fact, $\mathcal{F}(R) = \mathcal{F}(X)$ and the definiton (*) implies that $\mathcal{F}^{\#}(X) = \mathcal{F}(X)$. If $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, then $LL(\mathcal{F}) \to LL(\mathcal{G}) = G$. This defines a morphism from $LL(\mathcal{F})$ to G. Applying L to $LL(\mathcal{F}) \to G$ we obtain $\mathcal{F} \to \mathcal{G}$. This easily checks the adjunction. The last assertion follows from the fact that, for any sieve R, the functor $F \to \check{\mathcal{C}}(R, F)$ commutes with projective limits (this follows from th definition) and the functor $F \to \lim\mathrm{ind}_{R \in J(X)} F(R)$ commutes with projective limits because the category $J(X)$ is filtering. Thus L is left exact and $a = LL$ is left exact.

Definition An initial object of a category $\mathcal{C}$ is called a *strict initial object* if any morphism $X \to e$ is an isomorphism.

Proposition 1. *Suppose a site $\mathcal{C}$ has a strict initial object e and the Grothendieck topology is finer than the canonical topology. Then for any sheaf F the set $F(e)$ is a singleton.*

Proof. We claim that the empty sieve $\underline{\emptyset} \subset h_e$ of e belongs to $J(e)$. Since the canonical topology is the finest where representable shevaes are sheaves it is enough to show that for any Z and any $Y \to e$ we have a canonical bijection

$$h_Z(Y) \to \check{\mathcal{C}}(\underline{\emptyset} \times_e Y, h_Z).(*)$$

But $Y \to e$ is an isomorphism, so that $\underline{\emptyset} \times_e Y \cong \underline{\emptyset}$ and $\check{\mathcal{C}}(\emptyset \times_e Y, h_Z) \cong \check{\mathcal{C}}(\underline{\emptyset}, h_Z)$. If Z is not isomorphic to e, then $h_Z(e) = \emptyset$ so $\check{\mathcal{C}}(\underline{\emptyset}, h_Z) = \emptyset$. So both sides of (*) are emptysets. If $Z \cong e$, then bith sides are singletons. Thus, we have checked that $\underline{\emptyset} \in J(e)$. But, then for any sheaf F we must have a bijection $F(e) \to F(\underline{\emptyset})$, the latter is obviously a singleton.

Corollary. *Under the condition of the previous proposition assume that $(U_i \to U)$ is a covering family such that $U_i \times_U U_j$ is isomorphic to a strict initial object in $\mathcal{C}$. Then for any sheaf F there is a natural bijection $F(U) \to \prod_{i \in I} F(U_i)$.*

Example 5. If $\mathcal{C} = \mathit{Open}(X)$ with the usual Grothendieck topology, then we have to consider the covering $(U_i \to U)$ with empty set I as a covering of the empty set. So, then for any sheaf F we must have $F(\emptyset)$ is a singleton.

Example 6. Let c_S be a constant pre-sheaf $X \to S$, where S is a fixed set S. If S is not a singleton, and $\mathcal{C}$ has a strict initial object then c_S is not sheaf. In fact, assume that the empty sieve is a covering sieve of e. Then the proof of Proposition 1 shows that then for any sheaf F we have $F(e)$ must be a singleton. Let $\tilde{c}_S = a(c_S)$. Then $\check{\mathcal{C}}(\underline{\emptyset}, c_S)$ is a singleton, so, by definition of the functor $L : \mathcal{F} \to \mathcal{F}^{\#}$ we must have $c_S^{\#}(e)$ is a singleton. Also, it is easy to see that $c_S^{\#}(X) = c_S(X)$ for any X not isomorphic to e. Now

$$L(c_S^{\#})(X) = \lim_{R \in J(X)} \mathrm{ind}\, \check{\mathcal{C}}(R, c_S^{\#}).$$

If R is generated by $(U_i \to X)_{i \in I}$ such that $U_i \times_X U_j \cong e$ for any $i, j \in I$, then for any Y the value of R on Y consists of the set of morphims $Y \to X$ which factor through one of $U_i \to X$. By the asumption there is only one i such that it could happen (unless $Y \cong e$). This shows that $\check{\mathcal{C}}(R, c_S)$ is defined by an element of S^I. Assume that R be the minimal covering sieve of X with the property as above (if it exists). Then we obtain $ac_S^{\#}(c_S)(X) = S^I$.

Let $u : \mathcal{C} \to \mathcal{C}'$ be a continious functor of sites. For every covering site $R \in J(X)$ in $\mathcal{C}$ the morphism $u_!(R) \to u(X)$ is isomorphic to a covering sieve in $\mathcal{C}'$. This implies that for every sheaf $\mathcal{F}$ in $\mathcal{C}'$, the natural map of sets $\mathcal{F}(u(X)) \to \tilde{\mathcal{C}}(u_!(R), \mathcal{F})$ is bijective. By adjunction, this implies that the map $\tilde{\mathcal{C}}(a(u_!(R)), \mathcal{F}) \to \tilde{\mathcal{C}}(a(h_{u(X)}, \mathcal{F})$ is bijective. This implies that the associated sheaves $a(u_!(R)$ and $a(h_{u(X)})$ are isomorphic. Since $u^*(F)(X) = F(u(X)$, we see that for any covering sieve R of X, the natural map $u^*(F)(X) \to \tilde{\mathcal{C}}(R, u^*(F)) = \tilde{\mathcal{C}}'(u_!(R), \mathcal{F})$ is bijective. This implies that $u^*(F)$ is a sheaf. The functor u^* restricted to $\tilde{\mathcal{C}}'$ is denoted by u_s.

Remark One can define a continuous functor by requiring that for any sheaf $\mathcal{F}$ on $\mathcal{C}'$ its inverse image $u^*(\mathcal{F})$ is a sheaf on $\mathcal{C}$. It can be shown that this is equivalent to the condition that for any sieve $R \in J(X)$

the associated sheaves $a(u_!(R))$ and $a(h_{u(X)})$ are isomorphic. This condition is weaker than the one we used to define a continuous functor. If the topology in $\mathcal{C}$ and $\mathcal{C}'$ is defined by a pre-topology and $u : \mathcal{C} \to \mathcal{C}'$ commutes with fibred products the latter definition is equivalent to requiring that for any covering family $(U_i \to X)$ in $\mathcal{C}$ the family $(u(U_i) \to u(X)$ is a covering family in $\mathcal{C}'$.

It is not true, in general, that $u_!(\mathcal{F})$ is a sheaf if $\mathcal{F}$ is a sheaf. So one defines the functor

$$u^s = a \circ u_! : \tilde{C} \to \tilde{C}'.$$

Once can show that it is a left adjoint to u_s.

Example 7. Let $\mathcal{C} = Open(Y), \mathcal{C}' = Open(X)$ and $u : \mathcal{C} \to \mathcal{C}'$ corresponds to a continuous map $f : X \to y$ of topological spaces. Recall that for any sheaf $\mathcal{F}$ on Y the presheaf $u_!(\mathcal{F})$ is defined by $u_!(\mathcal{F})(V) = \lim_{V \to f^{-1}(U)} \mathcal{F}(U)$. Take $\mathcal{F}$ to be the sheaf associated to the constant pre-sheaf S. Assume U is connected by $f^{-1}(U) = V_1 \coprod V_2$ where V_1, V_2 are not-empty sets, then

$$u_!(\mathcal{F})(V_1 \cup V_2) = u_!(\mathcal{F})(V_1) = u_!(\mathcal{F})(V_2) = \mathcal{F}(U) = S.$$

However, if $u_!(\mathcal{F})$ were a sheaf, we would have $u_!(\mathcal{F})(V_1 \cap V_2) = S \times S$.

In the situation of this example, the sheaf $u^s(\mathcal{F})$ is denoted by $f^{-1}(\mathcal{F})$ and is called the inverse image of the sheaf $\mathcal{F}$.

10. Simplicial objects

Let ST be the *category of simplicial types*: its objects are natural numbers and morphisms $n \to m$ are non-decreasing maps from the set $[n] = \{0, 1, \ldots, n\}$ to $[m]$. The category ST/k is called the category of k-truncated simplicial types.

The following are special morphisms from this category:
$\partial_n^i : [n-1] \to [n]$: an injective increasing map with the image $[n] \setminus \{i\}$;
$\sigma_n^i : [n+1] \to [n]$: a non-decreasing surjective map with $\#(\sigma_n^i)^{-1}(i) = 2$.

Definition A simplicial object in a category $\mathcal{C}$ is a functor $ST^\circ \to \mathcal{C}$. The category of simplicial objects is denoted by $ST(\mathcal{C})$. For every $k \in Ob(ST)$, the category of contravariant functors from ST/k to $\mathcal{C}$ is denoted by $ST(\mathcal{C})/k$. It is called the category of truncated simplicial objects in $\mathcal{C}$.

Example 1. Take $\mathcal{C} = (\mathcal{S}ets)$. Then a simplicial object is a presheaf on ST. The representable preseheaf $h_n : k \to ST(k, n)$ is denoted by $\Delta[n]$. Its value at k is the set of all non-decreasing maps $[k] \to [n]$. Its subfunctor of injective maps can be identified with ordered subsets of $[n]$. If X is a simplicial set, we denote by X_n its value at n. Its elements are called n-simplices (0-simplices are *vertices*, 1-simplices are *edges*). In particular, $\Delta[n]_n$ is the fundamental n-simplex corresponding to the identity map $[n] \to [n]$. It is denoted by I_n. By Yoneda's Lemma,

$$X_n = \check{ST}(\Delta[n], X). \tag{1}$$

In particular, for every $x \in X_n$ there exists a unique morphism which sends I_n to x.

Let $\Delta : ST \to \check{ST}$ be the functor $[n] \to \Delta[n]$.

We know that in the category $\check{ST}$ all limits exist. For example, we can define the direct product of simplicia sets, direct sums and so on. For example,

$$Coker(\Delta(\partial_1^0), \Delta(\partial_1^1) : \Delta[0] \to \Delta[1])$$

is called a *simplicial circle*. Let us explain why. We have $\Delta[0]_k$ are constant maps, $\Delta[1]_k$ are defined by $s \le k$ such that $f(i) = 0$ if $i \le s$, $f(i) = 1$ if $i > s$. The map $\Delta(\partial_1^i)$ sends $\Delta[0]_k$ to $\Delta[1]_k$ by composing with the map $\partial_1^i : [0] \to [1]$. Thus the first map sends the constant map to the constant map with value 0, and the second sends the constant map to the constant map with value 1. Thus the value of the cokernel at k is equal to the set of numbers $s \in [0, k]$ where 0 is identified with k.

2. Let $\mathcal{C} = Top$. For each $n \in \mathbb{N}$ let I^n be the direct sum of the unit interval $[0,1] \subset \mathbb{R}$ equipped with the topology induced from the Euclidean topology of $\mathbb{R}^n$. We view I^n as the set of maps $I^{[n]}$ so that $n \to I^n$ defines a contravariant functor from ST to Top. This is a simplicial object of Top.

More generally, take any category $\mathcal{C}$ with finite direct products, aand its object X. Then the functor $n \to X^n$ is a simplicial object of the category $\mathcal{C}$.

3. For each $n \in \mathbb{N}$ let

$$\Delta_n\{(x_0, \ldots, x_n) \in \mathbb{R} : 0 \leq x_i, \sum_{i=0}^{n} = 1.\}.$$

For any $X \in Ob(Top)$ let

$$X_n = Top(\Delta_n, X).$$

Elements of this set are called *singular simplices* of X. If $f : [n] \to [m]$ is a morphism in ST, we define $\Delta(f) : \Delta_n \to \Delta_m$ as the unique affine linear map sending the vertex $e_i \in \Delta_n$ to the vertex $e_{f(i)}$ of Δ_m. This makes (Δ_n) a *cosimplicial topological space.* The composition map

$$X_m \to X_n, \alpha \to \Delta(f) \circ \alpha$$

defines the functor $X : ST^\circ \to (\mathcal{S}ets)$. It is easy to see that we get ourselves a functor from (Top) to $ST((\mathcal{S}ets))$.

Let us construct the left adjoint functor $ST((\mathcal{S}ets)) \to (Top)$. It is called the *geometric realization* of a simplicial set. We define the geometric realization as follows. Let $X = (X_n)$ be a simplicail set. Consider X_n as a topological space with discrete topology. The topological space

$$\coprod_{n=0}^{\infty} \Delta_n \times X_n)/R,$$

where R is the minimal equivalence relation which identifies the points $(s,x) \in \Delta_n \times X_n$ and $(t,y) \in \Delta \times X_m$ if $y = X(f)(x), s = \Delta_f(t)$ for some morphism $f : m \to n$. The topology is the factor-topology. In other words,

$$|X| = \liminf_{ST \times ST^\circ} \Delta \times X,$$

where $\Delta \times X$ is considered as a functor on the category $ST \times ST^\circ$ with values in (Top) which assigns to $[n] \times [m]$ the topological space $\Delta_n \times X_n$. The geometric realization provides a functor $ST((\mathcal{S}ets)) \to (Top)$.

Theorem 1. *The functor of geometric realization is left adjoint of the functor $S : (Top) \to ST((\mathcal{S}ets)), X \to S(X) = (Top(\Delta_n, X))$. In particular, we have natural bijection*

$$Top(|X|, Y) \to \check{ST}(X, S(Y)).$$

Example 4 The geometric realization of $\Delta[n]$ is homeomeorphic to Δ_n. It follows from the previous theorem that the functor of geometric realization commutes with inductive limits. This implies that

$$|Coker(\Delta(\partial_1^0), \Delta(\partial_1^1) : \Delta[0] \to \Delta[1])| = Coker(|\Delta(\partial_1^0)|, |\Delta(\partial_1^1|) : |\Delta[0]| \to |\Delta[1]|) =$$

$$Coker(|\Delta(\partial_1^0)|, |\Delta(\partial_1^1|) : \Delta_0 \to \Delta_1) = S^1.$$

It is convenient to think about the set $\Delta_n \times X_n$ as the set of n-simplices Δ_n indexed by the set X_n, $(\Delta_n, x), x \in X_n$. Then $|X|$ is obtained by gluing $\#X_0$ 0-simplices, $\#X_1$ 1-simplices, and so on together with respect to the gluing defined by the morphisms $X(f) : [n] \to [m]$. A topological space together with such gluing is called a *triangulated space.* The gluing is the *triangulation.*

Example 5 Let us find a simplicial set $X = (X_n)$ such that $|X| \cong \Delta_p \times \Delta_q$. An element of X_n is a sequence of $n+1$ distinct pairs of integers $\{(i_0, j_0), \ldots, (i_n, j_n)\}$, where

$$0 \leq i_0 \leq \ldots i_n \leq p, \quad 0 \leq j_0 \leq \ldots \leq j_n \leq q.$$

If $f:[m]\to[n]$ is a morphism in (ST), then we define $\tilde{f}(\{(i_0,j_0),\ldots,(i_n,j_n)\}=\{(i'_0,j'_0),\ldots,(i'_m,j'_m),\}$ where $i'_k=i_{f(k)},j_k=j_{f(k)}$. Now we have a map

$$\phi_n:\Delta_n\times X_n\to\Delta_p\times\Delta_q$$

It assigns to $(\Delta_n,\{(i_0,j_0),\ldots,(i_n,j_n)\})$ the subset of the RHS equal to the convex hull of the points $(e_{i_s},e_{j_s}\in\mathbb{R}^{p+1}\times\mathbb{R}^{q+1}$. It is clear that this defines a map from $|X|$ to $\Delta_p\times\Delta_q$. We refer to [GM] to verify that the map is bijective. Let us check in the case $p=q=1$. Then $\Delta_1\times\Delta_1$ is the square. It is easy to see that $\Delta_0\times X_0$ consists of 4 vertices, $\Delta_1\times X_1$ consists of 5 copies of Δ_1.They are mapped to the four sides and one of the diagonals (joining (e_0,e_0) with (e_1,e_1). The set $\Delta_2\times X_2$ consists of 2 2-dimensional simplices corresponding to the sequences $\{(0,0),(0,1),(1,0)\}$ and $\{(0,0),(1,0),(1,1)\}$. These are mapped to the two triangles divided by the diagonal. The sets X_n with $n>2$ are empty.

Let ST_k be the full subcategory of ST whose objects are $[n]$ with $n\leq k$. We have a natural inclusion functor $i_k:ST_k\to ST$. If $X\in ST(\mathcal{C})$, the composition functor $X\circ i_k^*:ST_k^\circ\to\mathcal{C}$ is called the *k-truncated simplicial object.* We have the trunctaion functor

$$i_k^*:ST(\mathcal{C})\to(ST/k)(\mathcal{C}),\quad X\to i_k(X).$$

The left (resp. the right) adjoint is denoted by sk_k (resp. $cosk_k$). We also consider the compositions

$$Cosk_k:ST(\mathcal{C})\to(ST/k)(\mathcal{C})\stackrel{cosk_k}{\to}ST(\mathcal{C}),$$

$$Sk_k:ST(\mathcal{C})\to(ST/k)(\mathcal{C})\stackrel{sk_k}{\to}ST(\mathcal{C}),$$

the adjointness properties give functorial morphisms:

$$sk_kX\to X,\quad X\to cosk_kX.$$

If $\mathcal{C}=(\mathcal{S}ets)$, then $sk_k=(F_k)_!$ (resp. $cosk_k=(F_k)^!$) and is defined by

$$Sk_k(X)_n=\liminf_{[n]\to[m],m\leq k}X_m,$$

$$Cosk_k(X)_n=\check{ST}_k(i_k^*(\Delta[n]),i_k^*(X))=$$
$$\check{ST}_k(sk_k(i_k^*(\Delta[n])),X)=\check{ST}(sk_k(\Delta[n]),X).$$

Here we used that the functor i_k^* is right adjoint to sk_k.

We have

$$i_k(sk_k(X))=i_k(X),$$

So that $sk_k(X)$ has the same simplices as X up to dimension k and all simplices in the above dimension are degenerations of n-simplices with $n\leq k$(i.e. each $x\in X_m$ with $m\geq n$ is equal to $X(f)(y)$ for some $y\in X_n,n\leq k$ and some surjective morphism $m\geq n$).

More precisely, we call an n-simpex $x\in X_n$ *degenerate* if there exists a surjection $f:[n]\to[m]$ from ST such that $x=X(f)(y)$ for some $y\in X_m$. We have the following:

Lemma (Eilenberg-Zilber). *For any $x\in X_n$ as above, there exists a non-degenerate m-simpex $y\in X_m$ and a surjective morphism $f:[n]\to[m]$ such that $X(f)(y)=x$. The pair (f,y) with this property is defined uniquely.*

Proof. The existence of (f,y) is obvious. Suppose we have another pair (f',y') so that $x=X(f)(y)=X(f')(y')$. Let σ,σ' be sections of f and f', respectfully. Then

$$X(f'\circ\sigma)(y')=X(\sigma)(X(f')(y'))=X(\sigma)(x)=X(\sigma)(X(f)(y))=y.$$

. Since y is non-degenerate, this is possible only if $f' \circ \sigma$ is injective. This implies $m' \leq m$. Similarly, we show that $m \leq m'$ so that $m = m'$ and $f' \circ \sigma$ is an increasing bijection, i.e. the identity. Thus $y = y'$ and also it follows that $f = f'$.

Corollary. *Let $|X|$ be the geometric realization of a simplicial set $X = (X_n)$. Then each point of $|X|$ can be uniquely represented by $(s, x) \in \Delta_n \times X_n$, where x is a non-degenerate simplex. In particular, each element of $|sk_k(X)|$ is uniquely represented by $(s, x) \in \coprod_{n=0}^{k} \Delta_n \times X_n$, where $n \leq k$.*

By eq(1), we have

$$Cosk_k(X)_n = Mor(i_k^*(\Delta[n]), i_k^*(X)).$$

Thus the morphism

$$X_n = \check{ST}(\Delta[n], X) \to (cosk_k X)_n = \check{ST}(sk_k(\Delta[n]), X)$$

can be interpreted as follows: any simplicial map $sk_k(\Delta[n]) \to X$ can be canonically extended to a simplicial map $\Delta[n] \to X$.

Example 6 Let A be an object of $\mathcal{C}$ considered as a 0-truncated simplicial object $ST_0^\circ \to \mathcal{C}$. Then

$$cosk_0(A)_n = \check{ST}(sk_0\Delta[n], A) = A^{n+1}.$$

So, we obtain the example of a simplicail object in $\mathcal{C}$ from Example 3.

Generalizing the construction of geometric realization of a simplicial set, we can define a geometric realization of a simplicial object in (Top). It is defined in the same way, only now the topology in X_n is not necessary discrete.

Let us see what is a simplicial object in the category $ST((\mathcal{S}ets))$. By definition its set of m-simplices is a simplicial sets $(X_n)_m$. They can be interpreted as a contravariant functor on the category $ST \times ST$. Its morphims are direct products of non-deceasing pairs of maps $(f, g) : [n] \times [n'] \to [m] \times [m']$. So, we can view an object of $ST(ST((\mathcal{S}ets)))$ as *bisimplicial set*) (X_{mn}). Any two simplicial sets (X_n) and (Y_n) define a bisimplicial set $(X_n \times Y_m)$ with obvious morphisms. It is called the direct product, although the terminology is a little confusisng, since the direct product in the category $ST((\mathcal{S}ets))$ is $(X_n \times Y_n$. This is called the *diagonal* of the bisimmplical set $X = (X_{nm})$ and is denoted by DX. One can extend the notion of geometric realization to bisimplicial sets. It can be done in three different ways:

I. Define $|X|^D$ as $|DX|$.

II. For each n we can define the geometric realization of the simplicial set $(X_{nm}$. This is a simplicial object in (Top). Then we can take the geometric realization of this object. Denote it by $|X|^I$.

III. Using the second index we define $|X|^{II}$.

Theorem (Eilenberg-Zilber). *There is a homeomorphism*

$$|X|^D \cong |X|^I \cong |X|^{II}.$$

Proof. We refer for the proof to [GM]. Let us see the assertion in the case $X_{nm} = \Delta[n] \times \Delta[m]$. First we check that $|X|^D$ is homeomorphic to $\Delta_n \times \Delta_m$. For any $(f, g) \in X_{n,n}$ set $\{(i_0, j_0), \ldots, (i_k, j_k)\}$, where $i_s = f(s), j_s = g(s)$. The simplex (f, g) is non-degenerate if and only if all pairs (i_s, j_s) are distinct. This shows that $|X|$ is homeomorphic to $\Delta_p] \times \Delta_q$.

On the other hand, we know that $|\Delta_n| = \Delta_n$. This implies that $|X|^I$ is the geometric realization of the simplicial set $\Delta_n \times \Delta_m$. It is easy to see that it is homeomorphic to $\Delta_n \times \Delta_m$.

Let $\mathcal{C}$ be a category. Define a simplicial set $N(\mathcal{C}) = (N(\mathcal{C})_n)$ as follows: $S(\mathcal{C})_0 = Ob(\mathcal{C})$, $S(\mathcal{C})_1 = Mor(\mathcal{C})$, $S(\mathcal{C})_n$ are sequences of morphisms of lenghth n:

$$A_0 \overset{\phi_0}{\to} A_1 \overset{\phi_n}{\to} \ldots A_n. \qquad (*)$$

If $f = \{i_0 \leq i_1 \dots i_m\} : [m] \to [n]$ then $X(f)$ sends (*) to

$$A_{i_0} \overset{\psi_0}{\to} A_{i_1} \overset{\psi_{i_m}}{\to} \dots A_{i_m},$$

where $\psi_s : A_{i_s} \to A_{i_{s+1}}$ is the composition of $\phi_{i_{s+1}-1} \circ \dots \circ \phi_{i_s}$. If f is degenerate we insert the identity morphisms. It is clear that non-degenerate simplices correspond to sequences of morphisms (*) such that no morphism ϕ_i is the identity.

We call $N(\mathcal{C})$ the *nerv* of $\mathcal{C}$. Its geometric realization of $B(\mathcal{C}) = |N(\mathcal{C})|$ is called the *classifying space* of the category $\mathcal{C}$.

Example 6 Consider the category defined by the graph

$$B \rightrightarrows C \leftarrow A.$$

There are 3 non-degenerate simplices are in dimension 0 and 3 in dimension 1. The classifying space of the category looks like a loop with an arm: On the othe hand if $\mathcal{C}$ is

$$A \to B \rightrightarrows C$$

we have additionally two 1-simplices and 2 2-simplices. The classifying space looks as follows.

If A is equal to the kernel of $B \rightrightarrows C$, we have only one additional 1-simplex.

Example 7 Let $\mathcal{C}$ be the groupoid category define by a group G. Since $\mathcal{C}$ has only one object, its nerv nerv has only one vertex. All 1-dimensional simplices are loops corresponding to elements of G. Nondegenerate n-simplices correspond to ordered sets $(g_1, \dots, g_n)$ of elements $g_i \neq 1$ from G. The fundamental group of the nerv is equal to G and its universal covering is contractable. Thus the classifying space of $\mathcal{C}$ is $K(G, 1)$.

Here are some properties of the nerv of a category:

1) If $\#\mathcal{C}(A, B) \leq 1$ for any pair of obejects, then the classifying space is contractible. For example, the classifying space of $Open(X)$ is a simplicial complex. Recall that a simplicial complex is a topological space X together with a homeomorphism $\phi : X \to |Y|$, where $|Y|$ is a geometric realization of a topological space such that the intersection of the images of two simplices belongs to the triangulation, two simplices intersect along their common face, and the set is closed if and only if its intersection with each simplex is closed. For example, the nerv of the category $B \rightrightarrows C \leftarrow A$ is not a simplicial complex.

2) If $\mathcal{C}$ has a final object f, then classifying space is contractible. This is because the nerv of $\mathcal{C}$ is homeomorphic to the cone over the nerv of $\mathcal{C} \setminus f$.

3) The classifying spaces of the category and the dual category coincide.

Since in many interesting cases the classifying space if a category is contractible, one proceeds differently, using Grothendieck topology and the nervs of hyeprcoverings (see below).

11. Homotopy theory

Let $f, g : X \to Y$ be two morphisms of simplicial objects. A *homotopy* between f and g is a morphism of functors $h : \Delta[1] \times X \to Y$ (from ST° to $(\mathcal{S}ets) \times \mathcal{C}$ and from ST° to $\mathcal{C}$ such that

$$f = h \circ (\Delta(\delta_1^0) \times 1 : \Delta[0] \times X \to \Delta[1] \times X \to Y,$$

$$g = h \circ (\Delta(\partial_1^1) \times 1 : \Delta[0] \times X \to \Delta[1] \times X \to Y.$$

Here we identify $\Delta[0] \times X$ with X. In general the homotopy is not an equivalence relation. To define the equivalence relation we consider n-steps homotopies. Here we replace $\Delta[1]$ with the simplicial set

$$\begin{array}{ccccccccc} \Delta[1] & & & & \Delta[1] & \Delta[1] & & & \Delta[1] \\ & \nwarrow & & \nearrow & \vdots & & \nwarrow & & \nearrow \\ & & \Delta[0] & & & & & \Delta[0] & \end{array}.$$

Thus we can define the homotopy category of simplicial objects by taking for morphisms the step-homotopy equivalence classes of morphisms of simplicial objects. They form the set $[X, Y]$.

Example 1 Consider the category $\{0 < 1\}$ of two objects and one morphism between them. A functor $\mathcal{C} \times \{0 < 1\} \to \mathcal{D}$ is just a morphism between two functors $F \Longrightarrow G$ from $\mathcal{C}$ to $\mathcal{D}$. By Milnor's theorem, for any two simplicial sets

$$|X \times Y| \cong |X| \times |Y|.$$

We have $|N(\{0 < 1\}|) = |\Delta[1]| = |\Delta_1|$. $B(\mathcal{C}) \times B(\{0 < 1\}) \to B(\mathcal{D})$ defines a homotopy between two maps of the classifying spaces of the categories $B(F), B(G) : B(\mathcal{C}) \to B(\mathcal{D})$.

In particular, suppose that $F : \mathcal{C} \to \mathcal{D}$ has a left or right adjoint. Then $B(F) : B(\mathcal{C}) \to B(\mathcal{D})$ is a homotopy equivalence.

We shall consider only simplicial sets. Let $\Delta_k^n, k = 0, \ldots, n+1$, denotes the smallest simplicial subset of $\Delta[n+1]$ which contains the simplices $(\partial_n^i : [n] \to [n+1]) \in \Delta[n+1]_n$, where $i \neq k$. For example, Δ_k^1 corresponds to the union of 1-faces of the 2-simplex containing the vertex k. It is easy to see that $(\Delta_k^n)_m$ consists of all non-decreasing maps $[m] \to [n+1]$ whose image does not contain the set $[n] \setminus \{k\}$. The geometric realization of Δ_k^n is the union of n-dimensional faces of Δ_n which contain the vertex e_k.

A (k, n)-horn of X is a simplical map $\Delta_k^n \to X$. A simplicial set X is called *complete* or a *Kan complex* if each (k, n)-horn extends to a simplicial map $\Delta[n+1] \to X$. Obviously, $\Delta[n]$ is a complete simplicial set.

Example 1 The singular simplicial set $[n] \to Top(\Delta_n, X)$ associated of a topological space X is always complete.

Lemma. *Let $\check{ST}_c$ be the full subcategory of the category of simplicial sets whose objects are complete simplicial sets. Then the relation of homotopy on the set of morphisms is an equivalence relation.*

The inclusion functor $\check{ST}_c \to \check{ST}$ has the left adjoint in the homotopy categories $Ex^\infty : \check{ST} \to \check{ST}_c$. By definition, any map of a simplicial set X to a complete simplicial set Y there exists a unique (up to homotopy map) $Ex^\infty(X) \to Y$.

Now we define the simplicial category by defining morphisms

$$[X, Y] = [X, Ex^\infty Y] = [Ex^\infty X, Ex^\infty Y].$$

The notion of a Kan complex admits a relativization. One defines the notion of a Kan fibration $p : X \to Y$. This is a simplicial map satisfying a certain property of homotopy lifting. Namely we require that for any morphism $f : \Delta_k^n \to X$ such that $f \circ p : \Delta_k^n \to X$ extends to a morphism $g : \Delta[n+1] \to Y$ there exists an extension $\bar{f} : \Delta[n+1] \to X$ such that $p \circ \bar{f} = g$. It is clear that the fibres of a Kan fibration are Kan complexes. Here by a fibre we mean the following. A point of a simplicial set Y is a morphism of simplicial sets $\Delta[0] \to Y$. The fibre is the fibred product $X \times_Y \Delta[0]$. Each simpiclial map is homotopy equivalent to a Kan fibration.

It follows from the definition that the k-coskeleton $Cos_k(X)$ of a complete simplicial set can be defined as follows. We say that $x, y \in X_n$ are equivalent and write $x \sim y$, if, considered as morphisms $\Delta[n] \to X$, their composition with $Sk_k(\Delta) \to \Delta[n]$ are equal. Then $Cosk_k(X)_n = X_n/$.

One can introduce homotopy groups of simplicial sets. Two simplices $x, x' \in X_n$ are called *comparable* if $X(\partial_n^i)(x) = X(\partial_n^i)(x')$ for all $i \in [n]$. Two comparable simplices are called *homotopy equivalent* if there exists $y \in X_{n+1}$ such that $X(\partial_{n+1}^n)(y) = x, X(\partial_{n+1}^{n+1})(y) = x'$ and

$$X(\partial_{n+1}^i)(y) = X(\sigma_n^i)(X(\partial_n^{n-1})(x) = X(\sigma_n^i)(X(\partial_n^{n-1})(x'), \quad 0 \leq i \leq n.$$

When X is complete this is an equivalence relation.

We say that X is *pointed* if there is given a simplicial map $e : \Delta[0] \to X$. If $n \geq 1$, the set of equivalence classes of n-simplexes of X congruent to $e_n = e(\Delta[0]_n) \in X_n$ forms a group, the *n-homotopy group* of X, and is denoted by $\pi_n(X)$. The operation of multiplication is defined by assigning to $x, x' \in X_n$ the simplex

$X(\partial_n^n(z)$, where $z \in X_{n+1}$ extends the functik $\Delta_n^n \to X$ defined by sending the map $\partial_n^i : [n] \to [n+1] \in \Delta[n+1]_n$ to $e_n \in X_n$ for $i = 0, \ldots, n-2$, sending ∂_n^{n-1} to x and sending ∂_n^{n+1} to x'. One can show that

$$\pi_n(X) \cong \pi_n(|X|).$$

Lemma.

$$\pi_n(Cosk_k(X)) = 0, \quad n > k.$$

For each $m \geq n$, we have a natural surjective simplicial map

$$p_m^n : Cosk_m(X) = \check{ST}(Sk_n\Delta[?], X) \to \check{ST}(Sk_m\Delta[?], X) = Cosk_n(X)$$

corresponding to inclusion of functors $Sk_m(Y) \to Sk_n(Y)$. It induces an isomorphism of homotopy groups in dimension $\leq m$. In particular, the geometric realization fibre of $p_{n-1}^n : Cosk_n(X) \to Cosk_{n-1}(X)$ is $K(\pi_n(|X|), n)$. Here we use the homotopy exact sequence for Kan fibrations.

The family of maps (p_m^n) defines a projective system functor $n \to Cosk_n(X)$. This projective limit is not representable in the homotopy category of simplicial sets. One has to enlarge the category by adding projective limits. This leads to the definition of pro-simplicial objects and their morphisms using the extended notion of homotopy. Then

$$\hat{X} = \lim \operatorname{proj} Cosk_n(X).$$

exists in the enlarged category and is called the *profinite completion* of X. The sequence

$$X \to \ldots \to Cosk_{n+1}(X) \to Cosk_n(X) \to Cosk_{n-1}(X) \to \ldots \to Cosk_0(X)$$

defines a canonical morphism (in the extended category) $X \to \hat{X}$. We can also get the sequence of Kan fibrations:

$$X \to \ldots \to Cosk_{n+1}(X)' \to Cosk_n(X)' \to Cosk_{n-1}(X) \to \ldots \to Cosk_0(X)'$$

Here we replace $Y \to Cosk_n(X)$ with $Y' \to Cosk_n(X)'$ to apply the relative Ex^∞ transforming a map to a Kan fibration. We use that $f : Y \to Cosk_n$ gives

$$Cosk_{n+1}(f) : Cosk_{n+1}(Y) \to Cosk_{n+1}(Cosk_n(X)) = Cosk_n(X).$$

This replaces X with homotopy equivalent X' admiting a *Postnikov* decomposition. Each fibre of the morphism $Cosk_{n+1}(X)' \to Cosk_n(X)'$ is an Eilenberg-Maclane space $K(\pi_n(X), n)$.

12. Hypercoverings

Let $\mathcal{C}$ be a site. We assume that $\mathcal{C}$ has finite fibred products, has an initial object $\emptyset$ and the following *distributivity* property is satisfied: For every family of objects $X, Y_i, i \in I$ in $\mathcal{C}/S$ such that $\coprod_i Y_i$ is representable, the canonical morphism

$$\coprod_i X \times_S Y_i \to X \times (\coprod_i Y_i)$$

us an isomorphism. This allows to deduce that the decomposition of an object as a direct sum of connected objexts is unique. Indeed, write $A = \coprod A_i = \coprod B_j$. Then $A \times_A B_j = B_j = \coprod(A_i \times_A B_j)$ implies that for one i $A_i \times_A B_j = B_j$ and

For example, every topos satisfies this condition.

An object X is called *connected* if $X \neq \emptyset$ and X has no non-trivial coproduct decompositions. A category is locally connected if every object is a coproduct of contractible objects. It is *connected* if it is locally connected and its final object is connected. We have the functor which assigns to $\mathcal{C}$ the set of its connected components:

$$\pi_0 : \mathcal{C} \to (\mathcal{S}ets).$$

If $K_\bullet$ is a simplicial object in $\mathcal{C}$ we can extend π_0 by defining

$$\pi_0(K_\bullet) = (\pi_0(K_i)).$$

This is a simplicial set.

Every covering $(U_i \to X)$ of an object X can be viewed as a morphism $U = \coprod U_i \to X$. We can define the simplicial object $cosk_0(U) = (U \times_X \ldots \times_X U)$. Then we can apply π_0 to get a simplicial set $\pi_0(cosk_0(U))$ and then take its geometric realization as an object of the homotopy category. Denote it by $|U|$. If W is a subcovering, i.e. $W \to X$ factors $W \to U \to X$, we get $|W| \to |U|$, and we can take the projective limit object $\{|U|\}_{U \in Cov(X)}$. This could be a definition of the homotopy type of X. If $\mathcal{C}$ has a final object, we can define the homotopy type of $\mathcal{C}$ as the homotopy type of its final object. Unfortunately the definition is too rude and does not give what we expect when taling for exampl $\mathcal{C}$ the category of CW-complexes or algebraic varieties. We have to generate from $U \to X$ more refined coverings. They are called hypercoverings.

Definition A morphism $X \to Y$ is called a *covering morphism* if one of the equivalent conditions is satisfied

(i) for any $A \to Y \in Mor(\mathcal{C})$ the image of $X \times_Y A \to A$ is a covering sieve of A;

(ii) the morphism of associated sheaves $a(h_X) \to a(h_Y)$ is an epimorphism in the category of sheaves;

(iii) for any sheaf $\mathcal{F}$ the canonical map $\mathcal{F}(Y) \to \mathcal{F}(X)$ is injective.

For example, if $f : X \to Y \in Cov(Y)$, we know that, for any sheaf $\mathcal{F}$, we have

$$\mathcal{F}(Y) = Ker(\mathcal{F}(X) \rightrightarrows \mathcal{F}(X \times_Y X),$$

and hence f is a covering morphism.

Let $\mathcal{C}$ be a category with finite products. Let $p > 0$.

Definition A simplicial object $K \in ST(\mathcal{C})$ is called a *hypercovering* in $\mathcal{C}$ of type p if the following properties are satisfied:

(HR1) the canonical morphisms $K_n \to Cosk_p(K)_n$ are isomorphisms;

(HR2) for each $n \geq 0$, the canonical morphism $K_{n+1} \to Cosk_n(K)_{n+1})$ is a covering morphism;

(HR3) the canonical morphism $K_0 \to e$, where e is the final object of $\check{\mathcal{C}}$ is a covering morphism. A hypercovering is a hypercovering of some type p. A hypercovering of an object X is a hypercovering in $\mathcal{C}/X$.

Notice that in the case when the site is canonical, (HR2) says that $K_{n+1} \to Cosk_n(K)_{n+1})$ is an epimorphism in the category of sheaves $\mathcal{C}^\sim$.

Let us make some comments about the definition of a hypercovering.

1) It follows from the definition of a coskeleton that

$$cosk_n(X)_{n+1} = \{(x_0, \ldots, x_{n+1}) \in cosk_n(X)_n^{n+2} : X(\partial_n^j)(x_j) = X(\partial_n^{j-1})(x_i), i < j\}.$$

For example, when $n = 1$ this means that (x_0, x_1, x_2) are three 1-simplices which form a triangle with vertices in X_1 This can be interpreted as $cosk_n(X)_{n+1}$ fills all the simplicial holes in $cosk_n(X)_n$

Thus the map $K_{n+1} \to (cosk_n(K_\bullet))_{n+1}$ 2) Condition (HR2) means that $\pi_n(K_\bullet) = 0$ when $n > p$. For example, if $p = 0$, $|K_\bullet|$ is contractible.

3) The geometric realization of a hypercovering in $(\mathcal{S}ets)$ is always contractible. In fact $K_{n+1} \to cosk_n(K)_{n+1} = \check{ST}(sk_n(\Delta_{n+1}, X)$ is surjective implies that any map from the boundary of Δ_{n+1} to X can be extended to a map from Δ_{n+1} to X. In fact, hypercoverings in $(\mathcal{S}ets)$ are characterized by this property.

We can define the category of hypercoverings of X.

Example 2 When $\mathcal{C} = (\mathcal{S}ets)$ with canonical topology, the condition (HR2) implies that K is complete and contractible. Indeed the natural map

$$K_{n+1} = \check{ST}(\Delta[n+1]), K) \to Cosk_n(K)_{n+1} = \check{ST}(Sk_n(\Delta[n+1]), K)$$

is surjective.

For example, $Cosk_0(X) = X$ with $X_n = X^{n+1}$ is a hypercovering of X. Starting from a covering $K_0 : U \to X$, we can take $K_1 = cosk_0(K_0)_1 = U \times_X U$, then $K_2 = cosk_1(U \times_Y U \rightrightarrows U)_2 = U \times_Y U_Y U$ and so on. A general hypercovering allows one to refine such hypercoverings.

Definition A *pointed site* is a site together with a continuous morphism $p : \mathcal{C}(\mathcal{S}ets)$ from a site $\mathcal{C}$ to the canonical site $(\mathcal{S}ets)$. Recall that this means that it carries covering morphisms to surjective maps. We shall also assume that p is exact, i.e. commutes with coproducts, finite fibred products and arbitrary direct products. If X is a simplicial object in a poined site, then we call its a pointed simplicial object if we choose a point in $p(X_0)$.

An example of a pointed set is the functor from $Et(X)$ to $(\mathcal{S}ets)$ which takes any étale covering $\pi : U \to X$ to the fibre $\pi^{-1}(x_0)$, where x_0 is a fixed points of X.

Now we take the category of hypercoverings in $\mathcal{C}$ and apply π_0 to obtain

$$\Pi(\mathcal{C}) = \{\pi_0(K_\bullet)\}_{K_\bullet \in HR(\mathcal{C})}.$$

This is an object in the homotopy category $pro - \mathcal{H}$ of pro-simplicial sets. The functor $\Pi : \mathcal{C} \to pro - \mathcal{H}$ is called the *Verdier functor.*

If $\mathcal{C}$ is a pointed site we can consider pointed hypercoverings (i.e. they become pointed simplicial objects). Then $\Pi(\mathcal{C})$ becomes an object of the homotopy theory of pointed simplicial sets. So, one can define homology of $\mathcal{C}$ with coefficients in an abelian group

$$H_q(\mathcal{C}, A) = H_q(\Pi(\mathcal{C}), A)$$

as weel as the homotopy groups

$$\pi_q(\mathcal{C}) = \pi_q(\Pi(\mathcal{C})).$$

Of course, here we have to develop the theory of homotopy groups and homology in the homotopy theory of pro-objects. We refer for this to [Artin-Mazur].

There is another approach due to Saul Lubkin. Let $\mathcal{U} = (U_i)_{i\in I}$ be a covering of a topological space by non-empty open sets. We define the category $\mathcal{C}(\mathcal{U})$ as a full subcategeory generated by U_i and then take its nerv and the classifying space. Let N be the nerv. We have $N_0 = I, N_1 = \{i, j \in I : U_i \subset U_j\}$ and so on. For each $n \in \mathbb{N}$ consider the set $\mathcal{N}_n$ of functions $f : [n] \to I$ such that

$$U_f = U_{f(0)} \cap \ldots \cap U_{f(n)} \neq \emptyset.$$

Obviously $[n] \to \mathcal{N}_n$ forms a simplicial set. It is (or its geometric realization) is often called the *nerv* of the covering. Assume (U_i) is closed under the non-empty intersections. Each U_f defines an n-simplex in the nerv N of the covering by assigning to it the chain of morphisms

$$U_{f(0)} \cap \ldots \cap U_{f(n)} \to U_{f(1)} \cap \ldots \cap U_{f(n)} \to \ldots \to U_{f(n)}.$$

Conversely, for any chain of inclusions among the members of the covering

$$U_{i_0} \subset U_{i_1} \subset \ldots \subset U_{i_n}$$

we define $f : [n] \to I$ by $f(k) = i_k$, so that

$$U_{i_n} = U_{f(n)}, \quad U_{i_{n-1}} = U_{f_n} \cap U_{f(n-1)}, \ldots, U_{i_0} = U_{f(0)} \cap \ldots \cap U_{f(n)}.$$

get an isomorphism of simplicial sets $\mathcal{N} \cong N(\mathcal{U})$.

It is easy to see that if we start with a covering $U = \cup_{i\in I} U_i$, and define the hypercovering $K_\bullet = cosk_0(\coprod_{i\in I} U_i)$, then the simplicial set $\pi_0(K_\bullet)$ is isomorphic to the nerv of the covering $(U_i)_{i\in I}$.

When X is an algebraic variety with Zariski topology this simplicial set is contractible since $K_\bullet = \Delta_n$, where $n = \#I$. This leads to consider Grothendieck topologies different from the Zariski topology.

Consider the following example:

Example 3. Let $X = \mathbb{P}^1(\mathbb{C})$, $U_1 = X \setminus \{0\}, U_2 = X \setminus \{1\}, U_2$ is the universal cover of $U_1 \cap U_2$. We have $\pi_1(U_2) = \mathbb{Z}$ and so any deck transformation $U_2 \to U_2$ is a morphism in $Et(X)$. So our category can be described as

$$e \leftarrow \mathbb{Z} \to e'.$$

Here $\mathbb{Z}$ denotes the object U_2. The nerv of the subcategory formed by U_2 is equivalent to S^1. The nerv of $\{e, \mathbb{Z}\}$ is a cone over a circle. So the nerv of the whole category is the gluing of two cones, hence S^2.

More generally, let $\mathcal{U} = (U_i)_{i\in I}$ be a finite covering of X. Assume that for any $x \in U_i \cap U_j$ there exists U_k such that $x \in U_k \subset U_i \cap U_j$. Let $J \subset I$ be the subset of $j \in I$ such that there exists a point $x \in U_j$ such that U_j is a smallest open subset containing x. Let $\mathcal{U}'$ be the subcover formed by such sets. Suppose that all sets U_j are spaces $K(\pi, 1)$. Then a theorem of Lubkin asserts that covering in $Et(X)$ formed by universal coverings of U_j is defines the nerv whose geometric realization is homotopy equivalent to X.

Note that any algebraic variety over $\mathcal{C}$ admits a cover by such sets. The proof is by induction using fibrations in curves. Use that if a base and a fibre are $K(\pi, 1)$'s , then the space is $K(\pi, 1)$.