# TRANSITION MEAN VALUES OF REAL CHARACTERS

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Abstract. We evaluate the real character sum $\sum_m \sum_n \left(\frac{m}{n}\right)$ where the two sums are of approximately the same length. The answer is surprising.

## 1. Introduction and statement of results

Let

$$S(X,Y) := \sum_{\substack{m\le X\\ m \text{ odd}}} \sum_{\substack{n\le Y\\ n \text{ odd}}} \left(\frac{m}{n}\right),$$

where $\left(\frac{m}{n}\right)$ is the Jacobi symbol. Our goal is to obtain an asymptotic formula for $S(X,Y)$. We will see that this is straightforward except when $X$ and $Y$ are of comparable size.

First we give asymptotic formulas valid for $Y = o(X/\log X)$ or $X = o(Y/\log Y)$. An easy application of the Pólya–Vinagradov inequality shows that

$$\sum_{\substack{m\le X\\ m \text{ odd}}} \left(\frac{m}{n}\right) = \begin{cases} \frac{X}{2}\frac{\varphi(n)}{n} + O(X^\varepsilon) & \text{if } n = \square \\ O(n^{\frac12}\log n) & \text{if } n \ne \square \end{cases}$$

and

$$\sum_{\substack{n\le Y\\ n \text{ odd}}} \left(\frac{m}{n}\right) = \begin{cases} \frac{Y}{2}\frac{\varphi(m)}{m} + O(Y^\varepsilon) & \text{if } m = \square \\ O(m^{\frac12}\log m) & \text{if } m \ne \square, \end{cases}$$

where $\square$ represents the square of a rational integer.

It follows that

$$\begin{aligned} S(X,Y) &= \sum_{\substack{n\le Y\\ n= \text{ odd } \square}} \left(\frac{X}{2}\frac{\varphi(n)}{n} + O(X^\varepsilon)\right) + O(Y^{\frac32}\log Y) \\ &= \frac{2}{\pi^2} X Y^{\frac12} + O(Y^{\frac32}\log Y + Y^{\frac12}X^\varepsilon + X\log Y), \qquad (1.1) \end{aligned}$$

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and similarly,

$$S(X,Y) = \frac{2}{\pi^2} X^{\frac{1}{2}} Y + O(X^{\frac{3}{2}} \log X + X^{\frac{1}{2}} Y^{\varepsilon} + Y \log X). \tag{1.2}$$

Equation (1.1) provides an asymptotic formula for $S(X,Y)$ when $Y = o(X/\log X)$, and (1.2) when $X = o(Y/\log Y)$. The range when $X$ and $Y$ are of comparable size marks a transition in the behavior of $S(X,Y)$, and our object here is to understand this transitory phase.

**Theorem 1.** *Uniformly for all large $X$ and $Y$, we have*

$$S(X,Y) = \frac{2}{\pi^2} C\left(\frac{Y}{X}\right) X^{\frac{3}{2}} + O\left((XY^{\frac{7}{16}} + YX^{\frac{7}{16}}) \log XY\right),$$

*where for $\alpha \geq 0$ we define*

$$C(\alpha) = \sqrt{\alpha} + \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{\alpha} \sqrt{y} \left(1 - \cos\left(\frac{2\pi k^2}{y}\right) + \sin\left(\frac{2\pi k^2}{y}\right)\right) dy.$$

*An alternate expression for $C(\alpha)$ is*

$$C(\alpha) = \alpha + \alpha^{\frac{3}{2}} \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{1/\alpha} \sqrt{y} \sin\left(\frac{\pi k^2}{2y}\right) dy.$$

To assist the reader in understanding the function $C(\alpha)$, graphs of $C(\alpha)$ and $C'(\alpha)$ are presented in Section 6.

The first expression for $C(\alpha)$ shows, upon integrating by parts, that

$$C(\alpha) = \sqrt{\alpha} + \frac{\pi}{18} \alpha^{\frac{3}{2}} + O(\alpha^{\frac{5}{2}}) \qquad \text{as } \alpha \to 0.$$

Similarly, the second expression for $C(\alpha)$ gives the limiting behavior

$$C(\alpha) = \alpha + O(\alpha^{-1}) \qquad \text{as } \alpha \to \infty.$$

Note that in these limiting cases, the value of $C(\alpha)$ approaches that given by the $n = \square$ terms (as $\alpha \to 0$) and the $m = \square$ terms (as $\alpha \to \infty$). From these limiting behaviors (or (1.1) and (1.2)) we see that $C(Y/X)X^{\frac{3}{2}}$ is of size $XY^{\frac{1}{2}} + YX^{\frac{1}{2}}$, so that the error term of Theorem 1 is always smaller than the main term. We shall leave to the reader the problem of showing that our two expressions for $C(\alpha)$ agree: this is an exercise in the Poisson summation formula.

If quadratic reciprocity said $\left(\frac{m}{n}\right) = \left(\frac{n}{m}\right)$ for all $m$ and $n$, then we would have a functional equation $C(\alpha) = \alpha^{\frac{3}{2}} C(1/\alpha)$. Plainly this does not hold, although the above expressions do show a relationship between $C(\alpha)$ and $\alpha^{\frac{3}{2}}C(1/\alpha)$. If the sum defining $S(X,Y)$ had been restricted to $m \equiv n \equiv 1 \pmod 4$, then such a functional equation would hold.

Differentiating our second expression for $C(\alpha)$ term by term we obtain

$$\left(\alpha^{-\frac{3}{2}}C(\alpha)\right)' = -\frac{1}{2}\alpha^{-\frac{3}{2}} - \alpha^{-\frac{5}{2}} f\left(\frac{\alpha}{2}\right),$$

where

$$f(x) = \frac{1}{\pi}\sum_{k\neq 0}\frac{1}{k^2}\sin\left(\pi k^2 x\right).$$

The function $f(x)$ is commonly called "Riemann's nondifferentiable function," so named because Weierstrass reported that Riemann suggested it as an example of a continuous function which is not differentiable. A considerable amount of work has been done investigating the differentiability properties of $f(x)$. Hardy [H] showed that it is not differentiable at $x = s$ if $s$ is irrational or if $s = \frac{p}{q}$ with $p$ or $q$ even. Gerver [G] gave a long elementary proof that $f'(p/q) = -1$ if $p$ and $q$ are odd, so Riemann's assertion is not quite correct. For an interesting survey on Riemann's function, see Duistermaat [Du]. In Section 6 we show that determining the differentiability of $f(x)$ at a rational point is a straightforward exercise in the Poisson summation formula; our approach appears to be similar to that of Smith [Sm]. It seem surprising that the asymptotics of a natural object like $S(X,Y)$ should involve non–smooth functions!

In the following section we explain our motivation for studying the sum $S(X,Y)$, and we give a generalization of Theorem 1. In Section 3 we do some preliminary reductions and identify the main terms and error terms in the sum. These are evaluated and estimated in Sections 4 and 5. In Section 6 we present graphs of $C(\alpha)$ and $C'(\alpha)$, and we determine at which rationals $C'(\alpha)$ is differentiable.

## 2. Motivation: mollifying $L(\frac{1}{2},\chi_d)$

The motivation for studying $S(X,Y)$ came from the third author's proof [S] that $L(\frac{1}{2},\chi_d) \neq 0$ for a positive proportion of fundamental discriminants $d$, where $\chi_d$ is the real primitive character to the modulus $d$. Jutila [J] showed that there exist positive constants $c_1$ and $c_2$ such that

$$\sum_{|d|<X} L(\tfrac{1}{2},\chi_d) \sim c_1 X\log X$$

and

$$\sum_{|d|<X} |L(\tfrac{1}{2},\chi_d)|^2 \sim c_2 X\log^3 X,$$

where both sums range over fundamental discriminants. It follows from the above formulas and Cauchy's inequality that the number of $|d| < X$ with $L(\frac{1}{2},\chi_d) \neq 0$ exceeds $X/\log X$.

The approach used to obtain the nonvanishing of $L(\frac{1}{2},\chi_d)$ for a positive proportion of $d$ was to consider a "mollified" sum of $L(\frac{1}{2},\chi_{8d})$. Let

$$M(d) = \sum_{\ell\leq M}\lambda(\ell)\sqrt{\ell}\left(\frac{8d}{\ell}\right),$$

where $\lambda(\ell)$ is chosen so that

$$S_1 := \sum_{|8d|<X} L(\tfrac{1}{2}, \chi_{8d}) M(d) \qquad \text{and} \qquad S_2 := \sum_{|8d|<X} |L(\tfrac{1}{2}, \chi_{8d}) M(d)|^2$$

are both of size $X$. By Cauchy's inequality this implies that $L(\frac{1}{2}, \chi_{8d}) \neq 0$ for a positive proportion of $d$. The optimal choice for $\lambda(\ell)$ is determined in [S]. The answer is complicated, so suffice it to say that $\lambda(\ell)$ is supported on the odd integers, where

$$\lambda(\ell) \qquad \text{is roughly proportional to} \qquad \frac{\mu(\ell)}{\ell} \frac{\log^2(M/\ell)}{\log^2 M} \frac{\log(X^{\frac{3}{2}} M^2 \ell)}{\log M}.$$

This leads to the result that $L(\frac{1}{2}, \chi_{8d}) \neq 0$ for at least $\frac{7}{8}$ of all squarefree integers $d$.

The most difficult part of the above argument is the evaluation of a certain "off–diagonal" contribution to the main term. This involves finding an asymptotic formula for an expression of the form

$$\Sigma_\ell(X) := \sum_{m,n} F(m, n, \ell, X) \left(\frac{m}{\ell n}\right),$$

for some explicit function $F$. In this paper we supress the function $F$, and we find that the resulting sum retains the interesting features of the corresponding sum considered in [S].

Motivated by the sum $\Sigma_\ell(X)$ we also consider the slightly more general sum

$$S_\ell(X, Y) := \sum_{\substack{m \le X \\ m \text{ odd}}} \sum_{\substack{n \le Y \\ n \text{ odd}}} \left(\frac{m}{\ell n}\right).$$

**Theorem 2.** *Let $\ell$ be an odd squarefree integer. There exists a constant $C_\ell(\alpha)$ such that, uniformly for all large $X$ and $Y$,*

$$S_\ell(X, Y) \sim \frac{1}{\sigma(\ell)} \frac{2}{\pi^2} C_\ell \left(\frac{Y}{X}\right) X^{\frac{3}{2}},$$

*where $\sigma(\ell)$ is the divisor sum function. We have $C_\ell(\alpha) = C(\alpha\ell)$, where $C(\alpha)$ is given in Theorem 1.*

The proof of Theorem 2 will be omitted because it closely follows the proof of Theorem 1.

## 3. Initial reductions

When $Y \le X^{\frac{16}{17}}$ or $X \le Y^{\frac{16}{17}}$, Theorem 1 follows immediately from (1.1) and (1.2) and the limiting behaviors of $C(\alpha)$. We assume below that $Y^{\frac{16}{17}} \le X \le Y^{\frac{17}{16}}$. In place of $S(X, Y)$ it is technically easier to consider the smoothed sum

$$\mathcal{S}(X, Y) := \sum_{m \text{ odd}} \sum_{n \text{ odd}} \left(\frac{m}{n}\right) H\left(\frac{m}{X}\right) \Phi\left(\frac{n}{Y}\right).$$

Here $H$ and $\Phi$ are smooth functions supported in $(0,1)$, satisfying $H(t)=\Phi(t)=1$ for $t\in(1/U,1-1/U)$, and such that $H^{(j)}(t)$, $\Phi^{(j)}(t)\ll_j U^j$ for all integers $j\geq 0$. The parameter $U$ will later be chosen to equal $(XY)^{\frac{2}{5}}(X+Y)^{-\frac{3}{5}}$.

Using the Pólya–Vinogradov inequality in a way similar to the argument described in the Introduction, it is easy to see that

$$|S(X,Y)-\mathcal{S}(X,Y)|\ll \frac{X^{\frac{3}{2}}+Y^{\frac{3}{2}}}{U}\log XY. \tag{3.1}$$

With our choice of $U$ this is seen to be smaller than the error term.

We evaluate $\mathcal{S}(X,Y)$ by applying the Poisson summation formula to the sum over $m$ in $\mathcal{S}(X,Y)$. For all odd integers $n$ and all integers $k$, we introduce the Gauss–type sums

$$\tau_k(n):=\sum_{a \pmod n}\left(\frac{a}{n}\right)e\left(\frac{ak}{n}\right)=:\left(\frac{1+i}{2}+\left(\frac{-1}{n}\right)\frac{1-i}{2}\right)G_k(n),$$

where $e(x):=e^{2\pi i x}$ as usual. We quote Lemma 2.3 of [S] which determines $G_k(n)$.

**Lemma 1.** *If $(m,n)=1$ then $G_k(mn)=G_k(m)G_k(n)$. Suppose that $p^a$ is the largest power of $p$ dividing $k$ (put $a=\infty$ if $k=0$). Then for $b\geq 1$ we have*

$$G_k(p^b)=\begin{cases}0 & \text{if } b\leq a \text{ is odd}\\ \varphi(p^b) & \text{if } b\leq a \text{ is even}\\ -p^a & \text{if } b=a+1 \text{ is even}\\ \left(\frac{k/p^a}{p}\right)p^a\sqrt{p} & \text{if } b=a+1 \text{ is odd}\\ 0 & \text{if } b\geq a+2.\end{cases}$$

By Poisson summation we have (see section 2.4 of [S] for details):

$$\sum_{m \text{ odd}}\left(\frac{m}{n}\right)H\left(\frac{m}{X}\right)=\frac{X}{2n}\left(\frac{2}{n}\right)\sum_{k=-\infty}^{\infty}(-1)^k\tau_k(n)\widehat{H}\left(\frac{kX}{2n}\right).$$

Expressing $\tau_k$ in terms of $G_k$, using the relation $G_k(n)=\left(\frac{-1}{n}\right)G_{-k}(n)$, and recombining the $k$ and $-k$ terms we may rewrite the above as

$$\frac{X}{2n}\left(\frac{2}{n}\right)\sum_{k=-\infty}^{\infty}(-1)^k G_k(n)\,\widetilde{H}\left(\frac{kX}{2n}\right),$$

where

$$\widetilde{H}(\xi):=\frac{1+i}{2}\widehat{H}(\xi)+\frac{1-i}{2}\widehat{H}(-\xi).$$

These manipulations show that

$$\begin{aligned}\mathcal{S}(X,Y)&=\frac{X}{2}\sum_{k=-\infty}^{\infty}\sum_{n \text{ odd}}(-1)^k\left(\frac{2}{n}\right)\frac{G_k(n)}{n}\Phi\left(\frac{n}{Y}\right)\widetilde{H}\left(\frac{kX}{2n}\right)\\&=M+R,\end{aligned} \tag{3.2}$$

where $M$ isolates the terms when $k = 2\square$, and $R$ includes the remaining terms. That is,

$$\begin{aligned} M := &\frac{X}{2}\sum_{k=0}^{\infty}\sum_{n \text{ odd}} \frac{G_{2k^2}(n)}{n}\left(\frac{2}{n}\right)\Phi\left(\frac{n}{Y}\right)\widetilde{H}\left(\frac{k^2X}{n}\right) \\ = &\frac{X}{2}\sum_{k=0}^{\infty} M_k, \end{aligned}$$

say, and

$$R := \frac{X}{2}\sum_{\substack{k=-\infty \\ k\neq 2\square}}^{\infty}(-1)^k\sum_{n \text{ odd}} \frac{G_k(n)}{n}\left(\frac{2}{n}\right)\Phi\left(\frac{n}{Y}\right)\widetilde{H}\left(\frac{kX}{2n}\right).$$

We will see that $M$ gives the main term and $R$ is an error term.

## 4. The remainder term $R$

We will require some simple estimates on $\widehat{H}(t)$ and $\widetilde{H}(t)$. These follow by integration by parts and our assumptions on $H$ and $\Phi$. We have

$$|\widehat{H}(t)|, \quad |\widetilde{H}(t)|, \quad |(\widetilde{H}(t))'| \ll_j U^{j-1}|t|^{-j} \tag{4.1}$$

for all integers $j \geq 1$, and all real $t$, and

$$\widetilde{H}(\xi) = \frac{1-\cos(2\pi\xi)+\sin(2\pi\xi)}{2\pi\xi} + O\left(\frac{1}{U}\right). \tag{4.2}$$

We handle the remainder term $R$ using the following Lemma which exhibits cancellation in the sum $G_k(n)\left(\frac{2}{n}\right)$ when $2k \neq \square$.

**Lemma 2.** *If $k \neq 2\square$ then*

$$\sum_{\substack{n\leq x \\ n \text{ odd}}} \frac{G_k(n)}{\sqrt{n}}\left(\frac{2}{n}\right) \ll |k|^{\frac{1}{4}}\log(2|k|)d(k^2)x^{\frac{1}{2}},$$

*where $d(k^2)$ is the number of divisors of $k^2$.*

Before proving the Lemma we note the bound it gives for $R$. By partial summation and Lemma 2 we have

$$\begin{aligned} \sum_{n \text{ odd}} \frac{G_k(n)}{n}\left(\frac{2}{n}\right)\Phi\left(\frac{n}{Y}\right)\widetilde{H}\left(\frac{kX}{2n}\right) &\ll |k|^{\frac{1}{4}}\log(2|k|)d(k^2) \\ &\quad\times\int_0^Y \sqrt{t}\left|\left(\frac{1}{\sqrt{t}}\Phi\left(\frac{t}{Y}\right)\widetilde{H}\left(\frac{kX}{2t}\right)\right)'\right|dt, \end{aligned}$$

and using (4.1) with $j = 3$ this is

$$\ll |k|^{\frac{1}{4}} \log(2|k|) d(k^2) \frac{U^2 Y^2}{k^2 X^2}.$$

Summing over all $k \neq 2\square$ we obtain

$$R \ll \frac{U^2 Y^2}{X}. \tag{4.3}$$

*Proof of Lemma 2.* We write $n = rs$ where $r$ and $s$ are odd with $s$ coprime to $k$ and $r$ divisible only by primes dividing $k$. By Lemma 1, $G_k(n) = G_k(r)G_k(s) = G_k(r)\mu^2(s)\sqrt{s}\left(\frac{k}{s}\right)$. Further note that $G_k(r) = 0$ unless $r|k^2$, and at any rate $|G_k(r)| \leq r$. Thus our desired sum is

$$\ll \sum_{\substack{r|k^2 \\ r \leq x}} \sqrt{r} \Big| \sum_{s \leq x/r} \mu^2(s) \left(\frac{2k}{s}\right) \Big|.$$

Expressing $\mu^2(s) = \sum_{d^2|s} \mu(d)$, and using the Pólya-Vinogradov inequality (since $\left(\frac{2k}{\cdot}\right)$ is a non-principal character with conductor $\leq 8|k|$) we obtain

$$\begin{aligned} \sum_{s \leq x/r} \mu^2(s) \left(\frac{2k}{s}\right) &\ll \sum_{d \leq \sqrt{x/r}} \Big| \sum_{t \leq x/rd^2} \left(\frac{2k}{t}\right) \Big| \ll \sum_{d \leq \sqrt{x/r}} \min\left(\frac{x}{rd^2}, |k|^{\frac{1}{2}} \log(2|k|)\right) \\ &\ll |k|^{\frac{1}{4}} (\log|2k|) x^{\frac{1}{2}} r^{-\frac{1}{2}}. \end{aligned}$$

Using this in our previous display we obtain Lemma 2.

## 5. The main term $M$

First consider the case $k = 0$. It follows straight from the definition that $G_0(n) = \varphi(n)$ if $n = \square$ and $G_0(n) = 0$ otherwise. Thus

$$M_0 = \widehat{H}(0) \sum_{n = \text{ odd } \square} \frac{\varphi(n)}{n} \Phi\left(\frac{n}{Y}\right) = \widehat{H}(0) \frac{2\sqrt{Y}}{\pi^2} \int_0^\infty \frac{\Phi(y)}{\sqrt{y}} dy + O(\log Y).$$

The second step follows from

$$\sum_{\substack{n \leq x \\ n = \text{ odd } \square}} \frac{\varphi(n)}{n} = \frac{4}{\pi^2} \sqrt{x} + O(\log x),$$

and then using partial summation.

Now suppose $k \geq 1$. Since $n$ is odd, by changing variables in the sum defining $G_k$ we have $G_{2k}(n)\left(\frac{2}{n}\right) = G_k(n)$. This gives

$$M_k = \sum_{n \text{ odd}} \frac{G_{k^2}(n)}{n} \Phi\left(\frac{n}{Y}\right) \widetilde{H}\left(\frac{k^2 X}{n}\right).$$

Using Lemma 1, and a straightforward calculation, we obtain

$$\sum_{\substack{n\le x\\ n \text{ odd}}} \frac{G_{k^2}(n)}{\sqrt{n}} = \frac{4}{\pi^2}x + O(\log x).$$

Hence, by partial summation,

$$M_k = \frac{4}{\pi^2}\int_0^\infty \Phi\left(\frac{t}{Y}\right)\widetilde{H}\left(\frac{k^2X}{t}\right)\frac{dt}{\sqrt{t}} + O\Big(\log Y\int_0^Y\Big|\Big(\Phi\left(\frac{t}{Y}\right)\widetilde{H}\left(\frac{k^2X}{t}\right)\frac{1}{\sqrt{t}}\Big)'\Big|dt\Big).$$

Making the change of variable $t = yY$ and using the bounds (4.1) (with $j = 1$ or 2), we have

$$M_k = \frac{4}{\pi^2}\sqrt{Y}\int_0^\infty \Phi(y)\widetilde{H}\left(\frac{k^2X}{yY}\right)\frac{dy}{\sqrt{y}} + O\Big(\frac{U\sqrt{Y}}{k^2X}\log Y\Big).$$

Combining this with our earlier expression for $M_0$ we conclude that

$$M = \frac{X\sqrt{Y}}{\pi^2}\int_0^\infty \frac{\Phi(y)}{\sqrt{y}}\sum_{k=-\infty}^{\infty}\widetilde{H}\left(\frac{k^2X}{yY}\right)dy + O\left((X+U\sqrt{Y})\log Y\right). \tag{5.1}$$

Using (4.1) (with $j = 1$) when $|k| \ge \sqrt{UY}/\sqrt{X}$, and (4.1) for smaller $k$ we deduce that

$$\sum_{k=-\infty}^{\infty}\widetilde{H}\left(\frac{k^2X}{yY}\right) = 1 + \frac{yY}{2\pi X}\sum_{\substack{k=-\infty\\ k\ne 0}}^{\infty}\frac{1}{k^2}\Big(1-\cos\left(\frac{2\pi k^2X}{yY}\right)+\sin\left(\frac{2\pi k^2X}{yY}\right)\Big)+O\Big(\frac{\sqrt{Y}}{\sqrt{UX}}\Big).$$

Using this in (5.1) we obtain

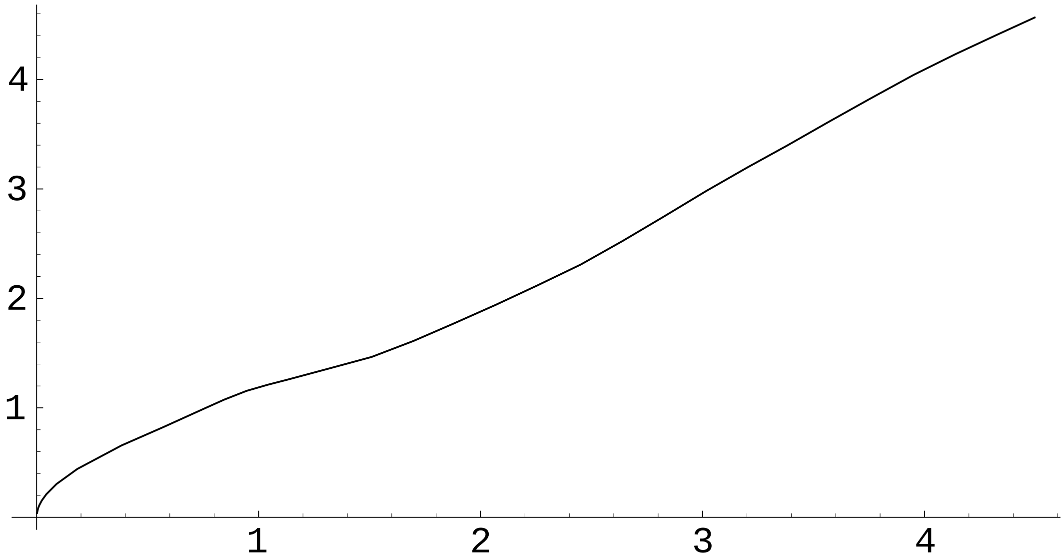
$$M = \frac{2X^{\frac32}}{\pi^2}C\left(\frac{Y}{X}\right) + O\Big(\frac{X^{\frac32}+Y^{\frac32}}{\sqrt{U}} + (X+U\sqrt{Y})\log Y\Big).$$

We combine this with (3.1), (3.2) and (4.3), and choose $U = (XY)^{\frac25}(X+Y)^{-\frac35}$ to obtain Theorem 1.

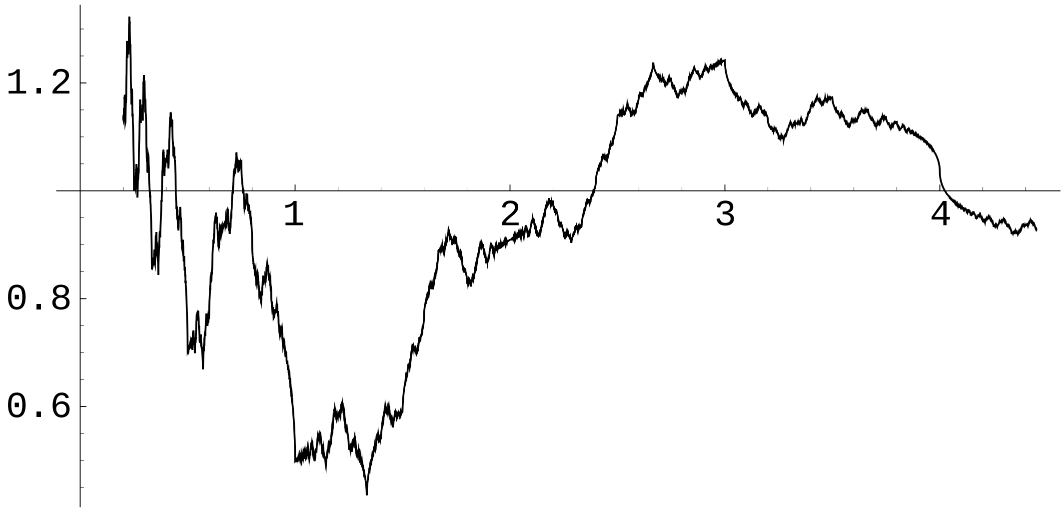
## 6. Some graphs of $C(\alpha)$

In this section we present graphs of $C(\alpha)$ and $C'(\alpha)$, and we prove that $C'(\alpha)$ is differentiable at $\alpha \in \mathbb{Q}$ if and only if $\alpha = 2p/q$ with $p$ and $q$ both odd.

The following graphs show $C(\alpha)$,




and $C'(\alpha)$:




It certainly appears from the graph that $C'(\alpha)$ is not everywhere differentiable.

**Proposition.** *$C'(\alpha)$ is differentiable at $\alpha \in \mathbb{Q}$ if and only if $\alpha = 2p/q$ with $p$ and $q$ both odd.*

The differentiability of $C'(\alpha)$ at $\alpha = 2$ can be seen in the above graph.

*Proof.* Let

$$f(\alpha) = \frac{1}{\pi} \sum_{k \neq 0} \frac{\sin(\pi k^2 \alpha)}{k^2}.$$

By the second expression for $C(\alpha)$ in Theorem 1 we have

$$(\alpha^{-\frac{3}{2}} C(\alpha))' = -\frac{1}{2}\alpha^{-\frac{3}{2}} - \alpha^{-\frac{5}{2}} f\left(\frac{\alpha}{2}\right), \tag{6.1}$$

the term–by–term differentiation being justified by the uniform absolute convergence of the resulting sum. We will show that $f(\alpha)$ is differentiable at $\frac{p}{q} \in \mathbb{Q}$ if and only if $p$ and $q$ are both odd.

The following Lemma is an exercise in the Poisson summation formula.

**Lemma.** *Suppose $(p,q)=1$. As $\alpha \to 0^{\pm}$ we have*

$$f\left(\frac{p}{q}+\alpha\right) = f\left(p/q\right) - \alpha \pm \sqrt{|2\alpha|}\left(\frac{1}{2q}\sum_{v \bmod 2q} \cos\frac{\pi p v^2}{q} \mp \sin\frac{\pi p v^2}{q}\right) + O(\alpha^{\frac{3}{2}}).$$

*In particular, $f(\alpha)$ is differentiable at $\alpha = \frac{p}{q}$ if and only if $G(p/q)=0$, where*

$$G(p/q) := \sum_{v \bmod 2q} e^{\pi i p v^2/q}.$$

The value of $G(p/q)$ is well known and can be found in Chapter 2 of Davenport [D]. We find that $G(p/q)=0$ if and only if $p$ and $q$ are both odd. Combining this with formula (6.1) completes the proof of the Proposition. Note that $C'(\alpha)$ is right– or left–differentiable at certain other $p/q$, such as the odd integers. This can be determined by considering the real and imaginary parts of $G(p/q)$.

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